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ON THE REDUCED PRODUCT CONSTRUCTION

Dedicated to the memory of Peter Fantham

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ABSTRACT. Under certain conditions the reduced product space JX of a space X has the same homotopy type as $\Omega\Sigma X$, the loop-space on the suspension of X. Several proofs can be found in the literature. The original proof [6] made unnecessarily strong assumptions. Later, in the last chapter of [3], tom Dieck, Kamps and Puppe gave a proof under much weaker conditions and showed that they could not be further weakened. The question then arose as to whether the reduced product construction could be generalized to provide combinatorial models not only of $\Omega\Sigma X$ but also of $\Omega^2\Sigma^2 X$, $\Omega^3\Sigma^3 X$ and so on. This was answered in the affirmative by May [10], using ideas of Boardman and Vogt [1], and the construction was further developed by Segal[1] [11] and others.

The present note, however, is not concerned with these generalizations. Its purpose is to give a simple proof of the original result, without striving for maximum generality, and to show that the same method can be used to prove an equivariant version of the reduced product theorem and hence a fibrewise version. Fibrewise versions of the reduced product theorem have previously been given by Eggar [5] and, more recently, by one of us [8] but it may be useful to have a relatively simple treatment which is adequate for the majority of applications.

To begin with we work in the category of pointed spaces. Unless otherwise stated maps and homotopies are basepoint-preserving. Suspension always means reduced suspension. Later we shall turn to the equivariant theory, but here again the action is to be basepoint-preserving.

Let X be a space with basepoint x_0, which we assume to be closed. As a set, the reduced product space JX may be described as the free monoid on X, with x_0 acting as neutral element. Thus a point of JX may be represented by a finite sequence of points of X. Sequences with not more than n terms form a subset $J^nX \subset JX$ $(n = 0, 1, \ldots)$. We topologize J^nX as a quotient space of the n-fold topological product $\prod^n X$. Then we obtain a sequence of spaces

$$J^0X \subset J^1X \subset \cdots \subset J^nX \subset \cdots,$$

where each member of the sequence is contained in the next as a closed subspace. Finally we topologize JX itself as the colimit[2] of the sequence; this does not mean (see (17.10)

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[1] Some of the results of [11], including those which are relevant here, need an additional numerability condition.

[2] A convenient reference for results about colimits is Vogt [12].

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of [3]) that the multiplication on JX is continuous. We prove

THEOREM 1. *Let X be a well-pointed compact Hausdorff space. Suppose that X can be covered by open sets each of which is contractible*[3] *in X to the basepoint. Then JX has the same homotopy type as $\Omega\Sigma X$.*

In the proof which follows we disregard basepoints. We shall show that JX and $\Omega\Sigma X$ have the same homotopy type in the non-pointed sense. For reasons given in (17.3) of [3] this will imply that they have the same homotopy type in the pointed sense.

First observe that the natural projection from $\prod^n X$ to $J^n X$ ($n = 0, 1, \ldots$) is proper, since X is compact, and hence the function

$$X \times J^n X \longrightarrow J^{n+1} X$$

defined by the multiplication on JX is continuous. Furthermore $X \times JX$ is the colimit of the sequence

$$X \times J^0 X \subset X \times J^1 X \subset \cdots \subset X \times J^n X \subset \cdots,$$

and so the function

$$T\colon X \times JX \longrightarrow JX$$

thus defined is continuous.

The first step in the proof of the theorem is to show that, for each n, the homotopy push-out of the cotriad

$$J^{n+1} X \longleftarrow X \times J^n X \xrightarrow{\pi_2} J^n X$$

is contractible, where the left-hand arrow is given by the restriction of T. This is trivial when $n = 0$; make the inductive hypothesis that it is true for some $n \geq 0$.
Write

$$X \rhd J^n X = x_0 \times J^n X \cup X \times J^{n-1} X \subset X \times J^n X$$

and consider the diagram shown below.

$$\begin{array}{ccccc}
X \times J^n X & & \xrightarrow{\pi_2} & & J^n X \\
\downarrow & & & & \downarrow \\
X \rhd J^{n+1} X & \longrightarrow & x_0 \times J^{n+1} X & \longrightarrow & J^{n+1} X \\
\downarrow & & \downarrow & & \downarrow \\
J^{n+1} X & \longrightarrow & J^{n+2} X & \longrightarrow & x_0
\end{array}$$

The outer square is a homotopy push-out, by the inductive hypothesis. Also the top half of the diagram is a push-out in the topological sense, hence a homotopy push-out (since the inclusion $J^n X \longrightarrow J^{n+1} X$ is a cofibration so is the inclusion $X \times J^n X \longrightarrow X \rhd J^{n+1} X$). Similarly the bottom left-hand square is a homotopy push-out. Hence it follows from

[3] Here and in Theorem 2 contractions are not required to preserve basepoints.

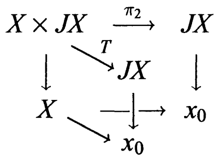
Theorem 10(ii) of Mather [9] that the bottom right-hand square is also a homotopy push-out, which proves the inductive step.

Now the colimit of this sequence of contractible homotopy push-outs is just the homotopy push-out of the cotriad

$$JX \xleftarrow{T} X \times JX \xrightarrow{\pi_2} JX.$$

We conclude, therefore, that this space is contractible.[4] So far we have only used the assumptions that X is well-pointed and compact Hausdorff.

Now consider the diagram shown below, which depicts two adjacent faces of a cube.

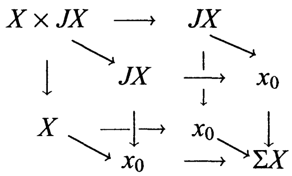


Obviously the back face in the diagram is a homotopy pull-back. We assert that the other face is also a homotopy pull-back. This amounts to showing that the "shearing map"

$$\xi: X \times JX \to X \times JX,$$

with components (π_1, T), is a fibrewise homotopy equivalence, where $X \times JX$ is regarded as a fibre space over X through the first projection. However X is path-connected and of finite category, and so this follows from Dold's theorem (6.3) of [4].

Thus both faces are homotopy pull-backs and so, completing the diagram of the cube as shown below, we can apply Theorem 11 of Mather [9] and conclude that the front face is a homotopy pull-back, from which Theorem 1 follows at once.



We turn now to the equivariant version of the theorem, which does not seem to have been treated in the literature. Specifically let X be a (pointed) G-space, where G is a compact group. Then JX and $\Omega\Sigma X$ are also G-spaces, and we state

THEOREM 2. *Let X be an equivariantly well-pointed compact Hausdorff G-space, where G is a compact Lie group. Suppose that X can be covered by invariant open sets each of which is G-contractible to the basepoint. Then JX has the same G-homotopy type as $\Omega\Sigma X$.*

The proof of Theorem 2 proceeds on the same lines as the proof of Theorem 1. An equivariant version of Mather's theory is needed, but this is completely routine. An

[4] Here and in Theorem 2 contractions are not required to preserve basepoints.

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equivariant version of Dold's theorem is also needed, but this poses no problems.[5] It seems unnecessary, therefore, to go through the details.

The supposition is equivalent to the assumption that the G-category of X is defined, in the pointed sense, and is therefore finite, by compactness. This is true, for example, if X is a finite G-complex such that the fixed point set X^H is connected for all closed subgroups H of G, as is the case when X is a double suspension. Thus take G to be the orthogonal group $O(n-1)$ acting on the sphere S^n in the usual way, where $n \geq 1$, so that the poles are left fixed and a point x_0 on the equator is also left fixed. Cover S^n by the enlarged hemispheres, which are $O(n-1)$ contractible to their respective poles, and then deform the poles into x_0, which we take as basepoint, along the line of longitude. We see that the condition is fulfilled in this case.

Finally we turn to the fibrewise version of the reduced product theorem. We work in the category of sectioned fibrewise spaces over a given base space B and require the standard section to be closed. Then the fibrewise reduced product space J_BX is defined as in [8] for each fibrewise space X over B. The fibres of J_BX are just the reduced product spaces of the corresponding fibres of X. We prove

THEOREM 3. *Let X be a (sectioned) fibre bundle over the base space B with compact structure Lie group G and fibre the (pointed) G-space A. Suppose that A satisfies the conditions of Theorem 2. Then J_BX has the same fibrewise homotopy type as $\Omega_B\Sigma_BX$.*

Let P be the principal G-bundle over B associated with X. Then (see [7]) P determines a functor $P_\#$ which transforms each G-space into the associated bundle with that fibre, and similarly with G-maps and G-homotopies. Moreover, $P_\#$ transforms the equivariant reduced product space JA into the fibrewise reduced product space J_BX, and similarly with the suspension and loop-space. Since JA has the same equivariant homotopy type as $\Omega\Sigma A$, by Theorem 2, it follows at once that J_BX has the same fibrewise homotopy type as $\Omega_B\Sigma_BX$.

The original version of the fibrewise reduced product theorem is due to Eggar [5] but conditions are imposed which are rather inconvenient in practice. More recently one of us [8] has published a fibrewise version of the proof given by tom Dieck, Kamps and Puppe [3] in the ordinary case. This does not assume local triviality. However, Theorem 3 seems adequate for most applications and its proof is a good deal simpler than the alternatives. For example it applies in the case of an orthogonal sphere-bundle which admits a pair of mutually orthogonal sections.

DEDICATION. This paper is dedicated to the memory of Peter Fantham, who died on June 10th, 1992. Peter was an editor of the Canadian Journal of Mathematics from 1971 to 1976.

[5] See, for example, the last exercise in (7.5) of tom Dieck [2].

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