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Finsler Warped Product Metrics with Relatively Isotropic Landsberg Curvature

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Abstract. In this paper, we study Finsler warped product metrics with relatively isotropic Landsberg curvature. We obtain the differential equations that characterize such metrics. Then we give some examples.

1 Introduction

Let F be a Finsler metric on manifold M. The geodesics of F are characterized locally by the equations

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where G^i are geodesic coefficients of F. In Finsler geometry, there are several very important non-Riemannian quantities [5, 18]. The Cartan torsion C is a primary quantity. Differentiating C along geodesics gives rise to the Landsberg curvature. Landsberg first studied Landsberg metrics [12, 13]. Li and Shen characterized weak Landsberg (α, β)-metrics and showed that there exist weak Landsberg metrics that are not Landsberg metrics in dimension greater than two [15]. Cheng, Wang, and Wang characterized (α, β)-metrics with relatively isotropic mean Landsberg curvature [6]. It is natural to study the relative rate of change of Landsberg curvature along geodesics, leading to the study of relatively isotropic Landsberg metrics.

A Finsler metric F is said to be spherically symmetric (orthogonally invariant in an alternative terminology in [10]) if F satisfies

$$F(Ax, Ay) = F(x, y),$$

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for all $A \in O(n)$, and equivalently, if the orthogonal group $O(n)$ acts as isometrics of F. Such metrics were first introduced by Rutz [17]. It was pointed out in [10] that a Finsler metric F on $\mathbb{B}^n(\mu)$ is a spherically symmetric if and only if there is a function $\phi : [0, \mu) \times R \to R$ such that

$$F(x, y) = |y|\phi\Big(|x|, \frac{\langle x, y\rangle}{|y|}\Big),$$

where $(x, y) \in T\mathbb{B}^n(\mu)\backslash\{0\}$, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the standard Euclidean norm and inner product, respectively. In [16], Mo, Solrzano, and Tenenblat obtained the differential equation that characterizes the spherically symmetric Finsler metrics with vanishing Douglas curvature, and by solving this equation they obtained all the spherically symmetric Douglas metrics. Guo, Liu, and Mo showed that every spherically symmetric Finsler metric of isotropic Berwald curvature is a Randers metric [7]. In [3], Chen and Song obtained a differential equation which characterizes a spherically symmetric Finsler metric with isotropic E-curvature. Zhou investigated the spherically symmetric Finsler metrics with isotropic S-curvature and obtained a characterized equation [19]. Then he proved that these metrics of Douglas type must be Randers metrics or Berwald metrics. In [4], Chen and Song obtained differential equations which characterize spherically symmetric Finsler metrics with relatively isotropic Landsberg curvature.

The warped product metric was first introduced by Bishop and O'Neil to study Riemannian manifolds of negative curvature as a generalization of Riemannian product metric [1]. The warped product metric was later extended to the case of Finsler manifolds by the work of Chen, Shen, and Zhao and Kozma, Peter, and Varga [2, 11]. These metrics are called Finsler warped product metrics. And warped product complex Finsler manifold was studied by many authors [8, 9].

Recently, Chen, Shen, and Zhao studied the warped product structures and gave the formulae of flag curvature and Ricci curvature of these metrics, and obtained the characterization of such metrics to be Einstein, and they showed that spherically symmetric Finsler metrics with warped product structure [2]. Liu and Mo obtained the differential equation that characterizes Finsler warped product metrics with vanishing Douglas curvature. By solving the equation, they obtained all the warped product Douglas metrics and constructed explicitly some new warped product Douglas metrics [14].

In this paper, we will study Finsler warped product metrics with relatively isotropic Landsberg curvature.

Consider the product manifold $M := I \times \check{M}$, where I is an interval of $\mathbb{R}$ and $\check{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric $\check{\alpha}$. Finsler metrics on M, given in the form

$$F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\Big(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\Big),$$

where $u = \big(u^1, \check{u}\big)$, $v = v^1\frac{\partial}{\partial u^1} + \check{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$, are called Finsler warped product metrics.

Theorem 1.1 *The warped product metric $F = \breve{\alpha}\phi(r, s)$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$ has relatively isotropic Landsberg curvature i.e., $L + cFC = 0$ if and only if ϕ satisfies*

$$\begin{cases} \phi_s \Phi_{sss} + (\phi - s\phi_s)\Psi_{sss} = \frac{1}{2} c\omega_{sss}, \\ \phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss} = \frac{1}{2} c\Xi_s, \end{cases} \tag{1.1}$$

where $c = c(u)$ is a scalar function on M, $\omega = \phi^2$, Φ, Ψ are functions which are defined by

$$\Phi = \frac{s^2(\omega_r\omega_{ss} - \omega_s\omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \tag{1.2}$$

$$\Psi = \frac{s(\omega_r\omega_{ss} - \omega_s\omega_{rs}) + \omega_r\omega_s}{2(2\omega\omega_{ss} - \omega_s^2)} \tag{1.3}$$

and Ξ_s is listed in (2.2).

Remark 1.2 Equations (1.1) are extensions of spherically symmetric metrics with relatively isotropic Landsberg curvature; see [4].

When $c(u)$ is vanished, these equations characterize the warped product Finsler metric of Landsberg type, and we obtain the following.

Theorem 1.3 *If the warped product metric $F = \breve{\alpha}\phi(r, s)$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$ is of Landsberg type, then ϕ must satisfy*

$$[f(r)s^2 + g(r)s + h(r)]\phi_{ss} + (\phi - s\phi_s)_r = 0, \tag{1.4}$$

where $f(r), g(r), h(r)$ are arbitrary differential functions of $r \in \mathbf{I}$.

Remark 1.4 The above theorems tell us the following interesting fact: having relatively isotropic Landsberg curvature or vanishing Landsberg curvature, for a Finsler warped product metric, is independent of the Riemannian metric $\breve{\alpha}$ on M. It follows that we can construct explicitly some new warped product metrics with relatively isotropic Landsberg curvature by using known spherically symmetric metrics with relatively isotropic Landsberg curvature in Section 5.

Now we briefly describe the organization of this paper. Section 2 is a quick introduction to some basic facts on Finsler geometry. Section 3 is devoted to the investigation of warped product metrics with relatively isotropic Landsberg curvature. Section 4 gives a detailed deduction of warped product metrics with vanishing Landsberg curvature. Finally, in Section 5, we give several examples of such metrics.

2 Preliminaries

In this section, we give some definitions of several geometric quantities in Finsler geometry that will be used in the proof of our main results. Throughout this paper, our index conventions are as follows:

$$1 \le A,\ B,\ C\cdots \le n, \quad 2 \le i,\ j,\ k\cdots \le n.$$

For a Finsler warped product metric $F = \check{\alpha}\phi$, the metric tensor is given by [2] :

$$\begin{pmatrix} g_{11} & g_{1j} \\ g_{i1} & g_{ij} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}w_{ss} & \frac{1}{2}\Xi_s\check{\ell}_j \\ \frac{1}{2}\Xi_s\check{\ell}_i & \frac{1}{2}\Xi\check{a}_{ij} - \frac{1}{2}s\Xi_s\check{\ell}_i\check{\ell}_j \end{pmatrix}, \tag{2.1}$$

where $\check{l}_i := \frac{\partial\check{\alpha}}{\partial v^i}$ and

$$\Xi := 2w - sw_s \quad \text{and} \quad \Xi_s := \partial_s\Xi = w_s - sw_{ss}. \tag{2.2}$$

For the geodesic coefficients G^A of F, we have

$$G^1 = \Phi\check{\alpha}^2 \quad \text{and} \quad G^i = \check{G}^i + \Psi\check{\alpha}^2\check{l}^i. \tag{2.3}$$

Let

$$C_{BCD} := \frac{1}{4}\frac{\partial^3 F^2}{\partial v^B\partial v^C\partial v^D}(u,v) = \frac{1}{2}\frac{\partial g_{BC}}{\partial v^D}. \tag{2.4}$$

Define $\mathbf{C} := C_{BCD}\mathrm{d}u^B \otimes \mathrm{d}u^C \otimes \mathrm{d}u^D$. Then $\mathbf{C}$ is called the Cartan tensor. The Landsberg curvature $\mathbf{L} := L_{BCD}\mathrm{d}u^B \otimes \mathrm{d}u^C \otimes \mathrm{d}u^D$ is defined by $L_{BCD} = C_{BCD;E}v^E$ on $TM\backslash\{0\}$, where ";" denotes the horizontal covariant derivative with respect to Chern connection of F. Thus, the Landsberg curvature can be defined as

$$L_{BCD} := -\frac{1}{2}F\frac{\partial F}{\partial v^A}\frac{\partial^3 G^A}{\partial v^B\partial v^C\partial v^D}. \tag{2.5}$$

A Finsler metric is called with the relatively isotropic Landsberg curvature if $L + cFC = 0$, *i.e.*, $L_{BCD} + cFC_{BCD} = 0$, where $c = c(u)$ is a scalar function on the manifold. When $L_{BCD} = 0$, which means that $c(u) = 0$, the Finsler metric is called the Landsberg metric.

Lemma 2.1 ([2]) *A spherically symmetric metric is a Finsler warped product metric.*

Proof Let (r, θ^i) be the polar coordinates of $\mathbb{R}^n$; then a vector y can be expressed as $y = v^1\frac{\partial}{\partial r} + v^i\frac{\partial}{\partial\theta^i}$. Thus, the Cartesian expressions can be rewritten as

$$x = r\frac{\partial}{\partial r}, \quad |x| = r, \quad \langle x,\ y\rangle = rv^1, \quad |y|^2 = v^1v^1 + r^2\check{\alpha}_+^2,$$

where $\check{\alpha}_+$ is the standard Euclidean metric on the unit sphere S^{n-1}. Now, the spherically symmetric metric can be written as

$$F(x,y) = |y|\phi\Big(|x|, \frac{\langle x,y\rangle}{|y|}\Big) = \check{\alpha}_+\widetilde{\phi}(r,s),$$

where $r := |x|, s := \frac{v^1}{\check{\alpha}_+}$ and $\tilde{\phi}(r,s) = \sqrt{r^2+s^2}\phi(r, \frac{rs}{\sqrt{r^2+s^2}})$. ■

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3 The Finsler Warped Product Metrics with Relatively Isotropic Landsberg Curvature

In this section, we are going to discuss necessary and sufficient conditions for the Finsler warped product metrics to have relatively isotropic Landsberg curvature. For the proof of Theorem 1.1, we need the following formulas:

$$
\begin{aligned}
\frac{\partial g_{11}}{\partial v^1} &= \frac{1}{2}\check{\alpha}^{-1}\omega_{sss},\\
\frac{\partial g_{1j}}{\partial v^1} &= -\frac{s}{2}\check{\alpha}^{-1}\omega_{sss}\check{l}_j,\\
\frac{\partial g_{1j}}{\partial v^k} &= \frac{s^2}{2}\check{\alpha}^{-1}\omega_{sss}\check{l}_j\check{l}_k + \frac{1}{2}\Xi_s\check{\alpha}^{-1}\check{h}_{jk},\\
\frac{\partial g_{jk}}{\partial v^l} &= \frac{s}{2}\check{\alpha}^{-1}[(3\Xi_s - s^2\omega_{sss})\check{l}_j\check{l}_k\check{l}_l - \Xi_s\check{a}_{jk}\check{l}_l(j\to k\to l\to j)],\\
\frac{\partial F}{\partial v^1} &= \phi_s, \quad \frac{\partial F}{\partial v^i} = (\phi - s\phi_s)\check{l}_i,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3}{\partial v^1\partial v^1\partial v^1}\left(\Phi\check{\alpha}^2\right) &= \check{\alpha}^{-1}\Phi_{sss},\\
\frac{\partial^3}{\partial v^1\partial v^1\partial v^j}\left(\Phi\check{\alpha}^2\right) &= -\check{\alpha}^{-1}s\Phi_{sss}\check{l}_j,\\
\frac{\partial^3}{\partial v^1\partial v^j\partial v^k}\left(\Phi\check{\alpha}^2\right) &= \check{\alpha}^{-1}[s^2\Phi_{sss}\check{l}_j\check{l}_k + (\Phi_s - s\Phi_{ss})\check{h}_{jk}],\\
\frac{\partial^3}{\partial v^j\partial v^k\partial v^l}\left(\Phi\check{\alpha}^2\right) &= -\check{\alpha}^{-1}[s^3\Phi_{sss}\check{l}_j\check{l}_k\check{l}_l + s(\Phi_s - s\Phi_{ss})\check{h}_{jk}\check{l}_l(j\to k\to l\to j)],\\
\frac{\partial^3}{\partial v^1\partial v^1\partial v^1}\left(\Psi\check{\alpha}^2\check{l}^i\right) &= \check{\alpha}^{-1}\Psi_{sss}\check{l}^i,\\
\frac{\partial^3}{\partial v^1\partial v^1\partial v^j}\left(\Psi\check{\alpha}^2\check{l}^i\right) &= \check{\alpha}^{-1}[\Psi_{ss}\check{h}^i_j - s\Psi_{sss}\check{l}^i\check{l}_j],\\
\frac{\partial^3}{\partial v^1\partial v^j\partial v^k}\left(\Psi\check{\alpha}^2\check{l}^i\right) &= \check{\alpha}^{-1}[s^2\Psi_{sss}\check{l}^i\check{l}_j\check{l}_k - s\Psi_{ss}(\check{h}^i_j\check{l}_k + \check{h}^i_k\check{l}_j + \check{h}_{jk}\check{l}^i)],\\
\frac{\partial^3}{\partial v^j\partial v^k\partial v^l}\left(\Psi\check{\alpha}^2\check{l}^i\right) &= \check{\alpha}^{-1}\Big\{[3(\Psi - s\Psi_s) - 6s^2\Psi_{ss} - s^3\Psi_{sss}]\check{l}^i\check{l}_j\check{l}_k\check{l}_l\\
&\quad + (s^2\Psi_{ss} - \Psi + s\Psi_s)\check{l}_l(\check{l}^i\check{a}_{jk} + \delta^i_j\check{l}_k)(j\to k\to l\to j)\\
&\quad + (\Psi - s\Psi_s)\delta^i_j\check{a}_{kl}(j\to k\to l\to j)\Big\},
\end{aligned}
$$

where $\check{l}^i = \frac{v^i}{\check{\alpha}}$, $\check{l}_i = \frac{\partial\check{\alpha}}{\partial v^i}$, $\check{h}_{ij} = \check{\alpha}\frac{\partial\check{l}_i}{\partial v^j}$, $\check{h}^i_j = \check{\alpha}\frac{\partial\check{l}^i}{\partial v^j}$, and $j\to k\to l\to j$ denotes cyclic permutation.

Proof of Theorem 1.1 A Finsler metric has relatively isotropic Landsberg curvature if $L + cFC = 0$, *i.e.*, $L_{BCD} + cFC_{BCD} = 0$, where $c = c(u)$ is a scalar function on M.

It can be launched that

$$\frac{\partial F}{\partial v^A}\frac{\partial^3 G^A}{\partial v^B \partial v^C \partial v^D} = c\frac{\partial g_{BC}}{\partial v^D}. \tag{3.1}$$

Substituting the above formulas into (3.1) and letting index (B, C, D) to be $(1,1,1)$, $(1,1,j)$, $(1,j,k)$, and (j,k,l), we obtain the following equations:

$$\text{(i)}\ \frac{\partial F}{\partial v^A}\frac{\partial^3 G^A}{\partial v^1 \partial v^1 \partial v^1} = c\frac{\partial g_{11}}{\partial v^1} \quad\Leftrightarrow\quad \check{\alpha}^{-1}[\phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss}] = \frac{1}{2}c\check{\alpha}^{-1}\omega_{sss},$$

$$\Leftrightarrow \phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss} = \frac{1}{2}c\omega_{sss}. \tag{3.2}$$

$$\begin{aligned}\text{(ii)}\ \frac{\partial F}{\partial v^A}\frac{\partial^3 G^A}{\partial v^1 \partial v^1 \partial v^j} = c\frac{\partial g_{1j}}{\partial v^1} &\Longleftrightarrow -\check{\alpha}^{-1}s[\phi_s\Phi_{sss}\check{l}_j + (\phi - s\phi_s)\Psi_{sss}\check{l}_j] \\ &= -\frac{1}{2}cs\check{\alpha}^{-1}\omega_{sss}\check{l}_j.\end{aligned}$$

Contracting the above equation with $\check{l}^j$ yields

$$\phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss} = \frac{1}{2}c\omega_{sss},$$

which is the same as (3.2).

$$\begin{aligned}\text{(iii)}\ \frac{\partial F}{\partial v^A}\frac{\partial^3 G^A}{\partial v^1 \partial v^j \partial v^k} &= c\frac{\partial g_{1j}}{\partial v^k} \\ \Longleftrightarrow\ & s^2[\phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss}]\check{l}_j\check{l}_k \\ &+ [\phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss}]\check{h}_{jk} \\ &= \frac{1}{2}c[s^2\omega_{sss}\check{l}_j\check{l}_k + \Xi_s\check{h}_{jk}].\end{aligned}$$

Contracting the above equation with $\check{l}^j\ \check{l}^k$ and $\check{a}^{jk}$, respectively; we get

$$\begin{cases}\phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss} = \frac{1}{2}c\omega_{sss}, \\ \phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss} = \frac{1}{2}c\Xi_s.\end{cases} \tag{3.3}$$

$$\begin{aligned}\text{(iv)}\ \frac{\partial F}{\partial v^A}\frac{\partial^3 G^A}{\partial v^j \partial v^k \partial v^l} &= c\frac{\partial g_{jk}}{\partial v^l} \\ \Leftrightarrow\ & [-s^3(\phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss}) \\ &+ 3s(\phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss})]\check{l}_j\check{l}_k\check{l}_l \\ &+ s[-\phi_s(\Phi_s - s\Phi_{ss}) + s(\phi - s\phi_s)\Psi_{ss}]\check{a}_{jk}\check{l}_l(j \to k \to l \to j) \\ &= \frac{1}{2}cs[(3\Xi_s - s^2\omega_{sss})\check{l}_j\check{l}_k\check{l}_l - \Xi_s\check{a}_{jk}\check{l}_l(j \to k \to l \to j)].\end{aligned}$$

Using the same method, we obtain the following equations

$$\begin{cases} l\phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss} = \frac{1}{2}c\omega_{sss}, \\ \phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss} = \frac{1}{2}c\Xi_s, \end{cases}$$

which are the same as equations (3.3).

Above all, we get the equations (1.1). And the converse is obvious. Thus we have completed the proof of Theorem 1.1. ■

4 Finsler Warped Product Metrics of Landsberg Type

In this section, we will discuss the Finsler warped product metrics of Landsberg type. A Finsler metric is called the relatively isotropic Landsberg metric if $L + cFC = 0$, *i.e.*, $L_{BCD} + cFC_{BCD} = 0$, where $c = c(u)$ is a scalar function on the manifold. When $L_{BCD} = 0$, which means that $c(u) = 0$, the Finsler metric is called the Landsberg metric.

By Theorem 1.1, we obtain the following lemma.

Lemma 4.1 *The warped product metric $F = \breve{\alpha}\phi(r,s)$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$ is of Landsberg type if and only if ϕ satisfies*

$$\begin{cases} (*) \quad \phi_s\Phi_{sss} + (\phi - s\phi_s)\Psi_{sss} = 0, \\ (\star) \quad \phi_s(\Phi_s - s\Phi_{ss}) - s(\phi - s\phi_s)\Psi_{ss} = 0. \end{cases} \tag{4.1}$$

We call the first equation of (4.1) $(*)$, and the second equation $(\star)$. Then by solving (4.1), we will find all Finsler warped product Landsberg metrics.

Proof Differentiating $(\star)$ with respect to the variable s, we obtain

$$\phi_{ss}(\Phi_s - s\Phi_{ss}) - s\phi_s\Phi_{sss} - (\phi - s\phi_s - s^2\phi_{ss})\Psi_{ss} - s(\phi - s\phi_s)\Psi_{sss} = 0. \tag{4.2}$$

Plugging $(*)$ into (4.2) yields

$$\phi_{ss}(\Phi_s - s\Phi_{ss}) - (\phi - s\phi_s - s^2\phi_{ss})\Psi_{ss} = 0. \tag{4.3}$$

Multiplying both sides of (4.3) by ϕ_s yields

$$\phi_{ss}\phi_s(\Phi_s - s\Phi_{ss}) = \phi_s(\phi - s\phi_s - s^2\phi_{ss})\Psi_{ss}. \tag{4.4}$$

Plugging $(\star)$ into (4.4), we have

$$s\phi_{ss}(\phi - s\phi_s)\Psi_{ss} = \phi_s(\phi - s\phi_s - s^2\phi_{ss})\Psi_{ss}. \tag{4.5}$$

Case I. $\Psi_{ss} \neq 0$. By (4.5), we have

$$s\phi_{ss}(\phi - s\phi_s) = \phi_s(\phi - s\phi_s - s^2\phi_{ss}). \tag{4.6}$$

Differentiating (4.6) with respect to the variable s yields

$$s\phi\phi_{ss} = \phi_s(\phi - s\phi_s) \Longleftrightarrow \frac{\phi_s}{\phi} = \frac{s\phi_{ss}}{\phi - s\phi_s}. \tag{4.7}$$

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So, we get

$$\phi = \sqrt{c(r)s^2 + c_1(r)}, \tag{4.8}$$

where c, c_1 are arbitrary differentiable real functions of r. Meanwhile, plugging (4.8) into (1.3) yields $\Psi_{ss} = 0$, which contradicts to $\Psi_{ss} \neq 0$.*Case II.* $\Psi_{ss} = 0$. Then

$$\Psi_{sss} = 0. \tag{4.9}$$

Plugging (4.9) into $(\star)$, we get $\Phi_{sss} = 0$ or $\phi_s = 0$.

If $\phi_s = 0$, then $\phi = c(r)$, where c is arbitrary differentiable real functions of r. So $F = \check{\alpha}\phi$ is a Riemannian metric.

If $\phi_s \neq 0$, we have

$$\Phi_{sss} = 0. \tag{4.10}$$

From $(\star)$, when $\phi_s \neq 0$ and $\Psi_{ss} = 0$, we have

$$\Phi_s - s\Phi_{ss} = 0. \tag{4.11}$$

Combining (4.10) and (4.11), we obtain $\Phi = a(r)s^2 + b(r)$.

For $\Psi_{ss} = 0$, $\Psi = \tilde{a}(r)s + \tilde{b}(r)$, where $\tilde{a}$ and $\tilde{b}$ are arbitrary functions of r. So

$$\Phi - s\Psi = f(r)s^2 + g(r)s + h(r), \tag{4.12}$$

where $f(r) = a(r) - \widetilde{a(r)}$, $g(r) = -\tilde{b}(r)$, and $h(r) = b(r)$.

On the other hand, we have

$$\Phi - s\Psi = \frac{2s\omega\omega_{rs} - 2\omega\omega_r - s\omega_r\omega_s}{2(2\omega\omega_{ss} - \omega_s^2)} = -\frac{(\phi - s\phi_s)_r}{2\phi_{ss}}. \tag{4.13}$$

Thus, we have

$$2[f(r)s^2 + g(r)s + h(r)]\phi_{ss} + (\phi - s\phi_s)_r = 0. \tag{4.14}$$

It can be rewritten as

$$[f(r)s^2 + g(r)s + h(r)]\phi_{ss} + (\phi - s\phi_s)_r = 0, \tag{4.15}$$

which completes the proof of Theorem 1.3. ■

5 Examples of The Finsler Warped Product Metrics with Relatively Isotropic Landsberg Curvature

In this section, we obtain several examples of warped product metrics with relatively isotropic Landsberg curvature.

Example 5.1 For any differentiable function λ, p, q of $r = |x|$, such that

$$\phi(r, \tilde{s}) = \frac{2c(x)\tilde{s} + \sqrt{(4c(x)^2 + \lambda p(r))\,\tilde{s}^2 + \frac{1}{2}\lambda q(r)}}{\lambda},$$

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where $\tilde{s} = \frac{\langle x,y\rangle}{|y|}$, we have the following spherically symmetric Finsler metrics with relatively isotropic Landsberg curvature [4]

$$F = \frac{2c(x)\langle x,y\rangle + \sqrt{(4c(x)^2 + \lambda p(r))\langle x,y\rangle^2 + \frac{1}{2}\lambda q(r)|y|^2}}{\lambda}.$$

Applying Lemma 2.1, their Finsler warped product forms are

$$F = \check{\alpha}_+ \frac{2c(x)rs + \sqrt{(4c(x)^2 + \lambda p(r))\, r^2 s^2 + \frac{1}{2}\lambda q(r)(r^2 + s^2)}}{\lambda},$$

where $\check{\alpha}_+$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$. Let $\hat{\alpha}$ be a Riemannian metric on $\mathbb{S}^{n-1}$. Then

$$\hat{F} = \hat{\alpha} \frac{2c(x)r\widehat{s} + \sqrt{(4c(x)^2 + \lambda p(r))\, r^2 \hat{s}^2 + \frac{1}{2}\lambda q(r)(r^2 + \hat{s}^2)}}{\lambda},$$

where $\hat{s} := \frac{v^1}{\hat{\alpha}(\check{u},\check{v})}$, are new warped product Finsler metrics with relatively isotropic Landsberg curvature.

Example 5.2 For any arbitrary constant ε with $0 < \varepsilon \le 1$, we have a Randers metric $F = \alpha + \beta$ that is of the following form:

$$\alpha(x,y) := \frac{\sqrt{\varepsilon|y|^2\left(1 + \varepsilon|x|^2\right) + (1-\varepsilon^2)\langle x,y\rangle^2}}{1 + \varepsilon|x|^2},$$

$$\beta(x,y) := \frac{\sqrt{1-\varepsilon^2}\langle x,y\rangle}{1 + \varepsilon|x|^2}.$$

Then F has relatively isotropic Landsberg curvature [20].

Applying Lemma 2.1, its Finsler warped product form is

$$F = \check{\alpha}_+ \frac{\sqrt{\varepsilon(s^2 + r^2)(1 + \varepsilon r^2) + (1-\varepsilon^2)\, r^2 s^2} + rs\sqrt{1-\varepsilon^2}}{1 + \varepsilon r^2},$$

where $\check{\alpha}_+$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$, $r = |x|$ and $s = \frac{v^1}{\check{\alpha}_+}$. Let $\widehat{\alpha}$ be a Riemannian metric on $\mathbb{S}^{n-1}$. Then

$$\hat{F} = \hat{\alpha} \frac{\sqrt{\varepsilon(\hat{s}^2 + r^2)(1 + \varepsilon r^2) + (1-\varepsilon^2)\, r^2 \hat{s}^2} + r\hat{s}\sqrt{1-\varepsilon^2}}{1 + \varepsilon r^2},$$

where $\hat{s} := \frac{v^1}{\hat{\alpha}(\check{u},\check{v})}$, is a new warped product Finsler metric with relatively isotropic Landsberg curvature.

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