

QUANTIFIED MODAL RELEVANT LOGICS

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Abstract. Here, I combine the semantics of Mares and Goldblatt [20] and Seki [29, 30] to develop a semantics for quantified modal relevant logics extending **B**. The combination requires demonstrating that the Mares–Goldblatt approach is apt for quantified extensions of **B** and other relevant logics, but no significant bridging principles are needed. The result is a single semantic approach for quantified modal relevant logics. Within this framework, I discuss the requirements a quantified modal relevant logic must satisfy to be “sufficiently classical” in its modal fragment, where frame conditions are given that work for positive fragments of logics. The roles of the Barcan formula and its converse are also investigated.

§1. Introduction. Quantified modal relevant logics (QMRLs) have received little attention over the years.¹ Moreover, development on a unified approach to relational semantics for QMRLs has been slowed by a number of some setbacks. This includes the fact that, on a more traditional approach to modeling the quantifiers, **RQ** is incomplete for a constant domain semantics [9]. Fine [8] does give a semantics for which **RQ** is complete; however, several have been trying to develop a simpler semantics for quantified relevant logics.

Mares and Goldblatt [20] give a general frame semantics for both quantified relevant logics **RQ^{of}** and **QR^{of}**, successfully constructing a semantics using models simpler than Fine’s brilliant but complex models.² Their alternative semantics for quantified logics models the universal quantifier by a new operator inspired by both an interpretation of the universal quantifier and the *functional polyadic algebra* of [15]. The resulting semantics for quantified relevant logics is fairly natural, but more importantly powerful. The same approach can be used for completeness results for a range of quantified modal classical logics as shown in Goldblatt and Mares [14], and for exploring the role of the Barcan Formula, its converse, and several other sentence schemes in [11]. The Mares–Goldblatt style semantics is thus ripe for extending to quantified modal relevant logics.

Relational semantics for modal relevant logics have been developed by Fuhrmann [10] and Mares and Meyer [22] (see also [17–19]). More recently, Seki [29, 30] has constructed general frame semantics for modal relevant logics based on the relatively

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¹ For an excellent presentation of the philosophical motivations of relevant logics, the reader is directed to [1]. Moreover, a good overview of relevant logic can be found in [2].

² The notation here correctly suggests that these are specific logics. These names, and other naming notations, will be explained later.

weak logic **B** and the usually considered extensions. Here, I develop a semantics for quantified modal relevant logics extending **B** by combining the semantics of Mares–Goldblatt and Seki, along the way demonstrating that the Mares–Goldblatt approach is apt for quantified extensions of **B** and other relevant logics. This provides a unified semantic foundation of quantified modal relevant logics.

§2. Basic quantified modal relevant logics. For quantified modal relevant logics, we begin with a denumerable set of variables, *Var*, which will be conveniently denoted by lowercase letters near the end of the alphabet (e.g., x, y, z, x_n, w_1). A possibly denumerable set \mathcal{L} of predicate symbols, and individual constants shall be called a *signature*. Each predicate symbol is of the form P^n , where n is the arity of the predicate. Often I will omit the superscript if either the arity is obvious or the arity is irrelevant. Each signature is denumerable at most, so the set of predicate symbols can be ordered. For every signature \mathcal{L} , its set of individual constants shall be denoted by *Con*. I shall denote individual constants by c , with or without subscripts.

A *term* will be denoted by τ with or without subscripts. An \mathcal{L} -*term*, relative to a signature \mathcal{L} , is defined as follows. Every variable v_n is an \mathcal{L} -*term*. Every member c of *Con* is an \mathcal{L} -*term*. No other expression is a term. A term is *closed* when it contains no variables. A term is *open* when it is not closed.

For a given signature \mathcal{L} , the atomic formulas (atomic \mathcal{L} -formulas) are those of the form $P^n(\tau_1, \dots, \tau_n)$, where $P^n \in \mathcal{L}$ and each of τ_1, \dots, τ_n is an \mathcal{L} -*term*. The set of well-formed formula of a quantified logic with signature \mathcal{L} (denoted by $wff_{\mathcal{L}}$ or simply *wff*) is defined inductively using the connectives \rightarrow (relevant implication), \wedge (extensional conjunction), \vee (extensional disjunction), \neg (negation), \circ (fusion or intensional conjunction), \mathbf{t} (intensional truth) and, for modal propositional logics, \diamond (possibility), \square (necessity) and $\forall x$ and $\exists x$, for each variable $x \in Var$. The biconditional, \leftrightarrow , is taken to be defined in the usual way. In general, I will use capital letters near the beginning of the English alphabet to denote or range over the well-formed formulas of a language.

An instance of a variable x is *bound* in the wff \mathcal{A} if either (1) the instance is the x of an expression $\forall x$ or $\exists x$ occurring in \mathcal{A} or (2) the instance of x occurs within the scope of a quantifier, $\forall x$ or $\exists x$. An instance of a variable is *free* when it is not bound. A formula with no free variables is called a *sentence*.

A term τ is *free for* (or *freely substitutable for*) x in wff \mathcal{A} if, for every variable y in τ , there are no free occurrences of x in \mathcal{A} that are within the scope of a quantifier $\forall y$ or $\exists y$.

A shorthand for substitutions will be convenient for our purposes. We shall write $\mathcal{A}[\tau/x]$ for the result of replacing every free occurrence of x in \mathcal{A} with the term τ . It will also be convenient to have a notation for the operation of several simultaneous substitutions. We will use $\mathcal{A}[\tau_0/v_0, \dots, \tau_n/v_n]$ for the result of simultaneously replacing v_0 through v_n with τ_0 through τ_n respectively.

A *variable assignment*, f , assigns to each variable an element of the domain of individuals, U , as follows. There are a denumerable number of variables which can be ordered as x_1, x_2, \dots , and a variable assignment is an ordered denumerable list of individuals. In other words, a variable assignment is a member of U^ω , where the n th individual in the ordering is the individual assigned to the variable x_n . The set of all variable assignments, relative to a domain U , is the set U^ω .

An x -variant of a variable assignment $f \in U^\omega$ for a domain of individuals U is a variable assignment that differs from f only in the individual assigned to the variable x . The set of all x -variants of f will be denoted by xf .

By convention, we write:

$$R^2xyzw =_{df} \exists v(Rxyv \wedge Rvzw),$$

$$S^2xy =_{df} \exists z(Sxz \wedge Szy).$$

§3. The logics $QB.C_{\square\Diamond}$ and $BQ.C_{\square\Diamond}$. Here, the base quantified modal relevant logic is $QB.C_{\square\Diamond}$, the implicational fragment of which is the basic affixing logic \mathbf{B} .³ The logic $QB.C_{\square\Diamond}$ extends the *regular* modal relevant logic $\mathbf{B.C}$ as defined in [30]. The logic $BQ.C_{\square\Diamond}$, as we will see, extends $QB.C_{\square\Diamond}$ so that the quantifiers are more classical.

The following axioms schemes and rules define $QB.C_{\square\Diamond}$.

- | | | |
|-------|---|----------------------------------|
| (A1) | $\mathcal{A} \rightarrow \mathcal{A}$ | Identity |
| (A2) | $\mathcal{A} \rightarrow (\mathcal{A} \vee \mathcal{B})$ | Disjunction Introduction (left) |
| (A3) | $\mathcal{B} \rightarrow (\mathcal{A} \vee \mathcal{B})$ | Disjunction Introduction (right) |
| (A4) | $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{A}$ | Conjunction Elimination (left) |
| (A5) | $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{B}$ | Conjunction Elimination (right) |
| (A6) | $\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) \rightarrow ((\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C}))$ | $\wedge\vee$ -Distribution |
| (A7) | $((\mathcal{A} \rightarrow \mathcal{B}) \wedge (\mathcal{A} \rightarrow \mathcal{C})) \rightarrow (\mathcal{A} \rightarrow (\mathcal{B} \wedge \mathcal{C}))$ | Conjunction Introduction |
| (A8) | $((\mathcal{A} \rightarrow \mathcal{C}) \wedge (\mathcal{B} \rightarrow \mathcal{C})) \rightarrow ((\mathcal{A} \vee \mathcal{B}) \rightarrow \mathcal{C})$ | Disjunction Elimination |
| (A9) | $\neg\neg\mathcal{A} \rightarrow \mathcal{A}$ | Double Negation Elimination |
| (A10) | $\forall x\mathcal{A} \rightarrow \mathcal{A}[\tau/x]$, where τ is free for x in \mathcal{A} | Universal Instantiation |
| (A11) | $\mathcal{A}[\tau/x] \rightarrow \exists x\mathcal{A}$, where τ is free for x in \mathcal{A} | Existential Introduction |
| (A12) | $(\square\mathcal{A} \wedge \square\mathcal{B}) \rightarrow \square(\mathcal{A} \wedge \mathcal{B})$ | $\square\wedge$ -Distribution |
| (A13) | $\diamond(\mathcal{A} \vee \mathcal{B}) \rightarrow (\diamond\mathcal{A} \vee \diamond\mathcal{B})$ | $\diamond\vee$ -Distribution |
| (A14) | $\forall x(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\exists x\mathcal{A} \rightarrow \mathcal{B})$ where x is not free in \mathcal{B} | |

$$\frac{\vdash \mathcal{A} \quad \vdash \mathcal{A} \rightarrow \mathcal{B}}{\vdash \mathcal{B}} \text{ (MP)}$$

$$\frac{\vdash \mathcal{A} \quad \vdash \mathcal{B}}{\vdash \mathcal{A} \wedge \mathcal{B}} \text{ (ADJ)}$$

$$\frac{\vdash \mathcal{A} \rightarrow \neg\mathcal{B}}{\vdash \mathcal{B} \rightarrow \neg\mathcal{A}} \text{ (Contraposition)}$$

$$\frac{\vdash \mathcal{A} \rightarrow \mathcal{B}}{\vdash (\mathcal{C} \rightarrow \mathcal{A}) \rightarrow (\mathcal{C} \rightarrow \mathcal{B})} \text{ (Prefix)}$$

$$\frac{\vdash \mathcal{A} \rightarrow \mathcal{B}}{\vdash (\mathcal{B} \rightarrow \mathcal{C}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C})} \text{ (Suffix)}$$

$$\frac{\vdash \mathcal{A}}{\vdash \mathbf{t} \rightarrow \mathcal{A}} \text{ (t-I)}$$

$$\frac{\vdash \mathbf{t} \rightarrow \mathcal{A}}{\vdash \mathcal{A}} \text{ (t-E)}$$

³ \mathbf{B} is a basic relevant logic that lacks the Law of Excluded Middle ($\mathcal{A} \vee \neg\mathcal{A}$). Some denote by \mathbf{B} the logic defined here extended by the Law of Excluded Middle. Here, we take \mathbf{B} to be the “weaker” logic that excludes the Law of Excluded middle.

$$\frac{\vdash \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})}{\vdash (\mathcal{A} \circ \mathcal{B}) \rightarrow \mathcal{C}} \text{ (}\circ\text{-I)}$$

$$\frac{\vdash (\mathcal{A} \circ \mathcal{B}) \rightarrow \mathcal{C}}{\vdash \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})} \text{ (}\circ\text{-E)}$$

$$\frac{\vdash \mathcal{A} \rightarrow \mathcal{B}}{\vdash \Box \mathcal{A} \rightarrow \Box \mathcal{B}} \text{ (}\Box\text{-M)}$$

$$\frac{\vdash \mathcal{A} \rightarrow \mathcal{B}}{\vdash \Diamond \mathcal{A} \rightarrow \Diamond \mathcal{B}} \text{ (}\Diamond\text{-M)}$$

$$\frac{\vdash \mathcal{A} \rightarrow \mathcal{B}}{\vdash \exists x \mathcal{A} \rightarrow \mathcal{B}} \text{ }\exists\text{-Intro}$$

$$\frac{\vdash \mathcal{A} \rightarrow \mathcal{B}}{\vdash \mathcal{A} \rightarrow \forall x \mathcal{B}} \forall\text{-Intro}^4$$

The conditions on \exists -Intro and \forall -Intro are that x is not free in \mathcal{B} for \exists -Intro and x is not free in \mathcal{A} for \forall -Intro. The rules \Box -M and \Diamond -M stand for \Box -Monotonicity and \Diamond -Monotonicity.

Note that axiom (A14) is provable from the other axioms in systems whose propositional fragment includes the axiom form of contraposition, $(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\neg \mathcal{B} \rightarrow \neg \mathcal{A})$, given the duality of the quantifiers (see Lemma 3.3). A related formula, $\forall x(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \forall x \mathcal{B})$, where x is not free in \mathcal{A} , is derivable in the defined system due to the presence of fusion. Moreover, (A14) is derivable in the system defined as $\mathbf{QB.C}_{\Box\Diamond}$ plus the left arrow governed by following rules:

$$\frac{\vdash \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})}{\vdash \mathcal{B} \rightarrow (\mathcal{C} \leftarrow \mathcal{A})} \text{ (}\leftarrow\text{-I)}$$

$$\frac{\vdash \mathcal{B} \rightarrow (\mathcal{C} \leftarrow \mathcal{A})}{\vdash \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})} \text{ (}\leftarrow\text{-E)}$$

$\mathbf{QB.C}_{\Box\Diamond}$ is defined with fusion and without the left arrow, so (A14) is adopted as an axiom, and the related $\forall x(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \forall x \mathcal{B})$ is not.

The logic $\mathbf{BQ.C}_{\Box\Diamond}$ results from adding to $\mathbf{QB.C}_{\Box\Diamond}$ the extensional confinement axiom scheme (EC \forall), or any equivalent scheme.

$$\text{EC}_{\forall} \quad \forall x(\mathcal{A} \vee \mathcal{B}) \rightarrow (\mathcal{A} \vee \forall x \mathcal{B}), \text{ where } x \text{ is not free in } \mathcal{A},$$

$$\text{EC}_{\exists} \quad (\mathcal{A} \wedge \exists x \mathcal{B}) \rightarrow \exists x(\mathcal{A} \wedge \mathcal{B}), \text{ where } x \text{ is not free in } \mathcal{A}.$$

Strictly speaking, the existential (or universal) quantifier can be taken as defined, and the resulting logic will contain the same theorems (under the usual translation). Under the usual translation, the axiom schemes and rules for the defined quantifier are derivable.

The propositional logic \mathbf{B} , the basis over which we get $\mathbf{QB.C}_{\Box\Diamond}$, is defined using a propositional language by axioms (A1)–(A9), and rules MP, AJD, Prefix, Suffix, and Contraposition. \mathbf{B}^{of} defined by extending \mathbf{B} with the rules for \mathfrak{t} and \circ .

For first order relevant logics \mathbf{QB}^{of} , we extend \mathbf{B}^{of} with (A10) and (A11) with rules \forall -Intro and \exists -Intro. The logic \mathbf{BQ}^{of} extends \mathbf{QB}^{of} with the extensional confinement axioms.

Seki uses the phrase “regular modal logic over \mathbb{L} ” to describe the modal relevant logics including the axioms and rules of \mathbb{L} , axioms (A12) and (A13), and rules \Box -M and \Diamond -M. The least regular modal logic over \mathbb{L} is denoted $\mathbb{L.C}$. Here, I will generally omit the adjective ‘regular’, opting instead for “quantified modal logic over \mathbb{L} .”

In the least regular modal logic over \mathbf{QB}^{of} , and over \mathbf{B}^{of} , the formulas $\Box \mathcal{A} \leftrightarrow \neg \Diamond \neg \mathcal{A}$ and $\Diamond \mathcal{A} \leftrightarrow \neg \Box \neg \mathcal{A}$ are not theorems. Thus, two additional modalities are introduced

³ This rule is called RIC for “rule of intensional confinement” by Mares and Goldblatt, due to the relation to the formula $\forall x(\mathcal{A} \vee \mathcal{B}) \rightarrow (\mathcal{A} \vee \forall x \mathcal{B})$, where x is not free in \mathcal{A} in \mathbf{RQ}^{of}

by Seki by definitions:

$$\begin{aligned} \diamond \mathcal{A} &=_{def} \neg \Box \neg \mathcal{A}, \\ \Box \mathcal{A} &=_{def} \neg \diamond \neg \mathcal{A}. \end{aligned}$$

It is worth noting that modal relevant logics, especially stronger ones, are often defined taking only one modality as primitive, and defining the other. Strictly speaking, these logics employ the modalities \Box and \diamond (or \diamond and \Box). In the frames for these logics, we typically only see a single binary relation for both modalities, which is sufficient to guarantee the kind of duality/interdefinability that the modalities share.⁴ However, in general the classical behavior of the primitive \Box and \diamond in regular modal relevant logics requires addition axioms.

The dot in the name of the logics signifies that the modal fragment is not “sufficiently classical,” which means here that there are identifiable formulas that are theorems of the modal classical logic that are not theorems (under suitable translation) of the corresponding modal relevant logic. The axiom $\Box(\mathcal{A} \vee \mathcal{B}) \rightarrow (\diamond \mathcal{A} \vee \Box \mathcal{B})$, for example, has been added to the modal relevant logic **NR** and logic of entailment **E** in order to capture modal fragment of **S4** (see [25], p. 69–70). Mares has pointed out that this axiom, first suggested by Belnap, bears a striking resemblance to the confinement axioms for the quantifiers. As we will only have means to analyze what it means for a logic to be sufficiently classical in this sense when we consider extensions to the modal fragments in Section 8, the explanation of the dot/dotless convention is left until Section 9 where I will further discuss the interactions between \Box and \diamond .

3.1. Some Proofs. Given the relative weakness of the logic **B^o**, it is worth noting a few theorems of **QB.C_{□◇}**. In particular, note the duality of the quantifiers. In contrast, as just noted, the modalities \Box and \diamond are not provably dual. The following lemmas serve to highlight features of **QB.C_{□◇}**, some of which will be used in later proofs.

The rules \forall -Intro(Con), UG(Con), and \exists -Intro(Con) are as follows:

$$\begin{aligned} \forall\text{-Intro(Con)} & \frac{\vdash \mathcal{A} \rightarrow \mathcal{B}[c/x]}{\vdash \mathcal{A} \rightarrow \forall x \mathcal{B}} \\ \text{UG(Con)} & \frac{\vdash \mathcal{A}[c/x]}{\vdash \forall x \mathcal{A}} \\ \exists\text{-Intro(Con)} & \frac{\vdash \mathcal{A}[c/x] \rightarrow \mathcal{B}}{\vdash \exists x \mathcal{A} \rightarrow \mathcal{B}} \end{aligned}$$

The rule \forall -Intro(Con) has the requirement that c is not in \mathcal{A} or \mathcal{B} and x is not free in \mathcal{A} , UG(Con) the requirement that c is not in \mathcal{A} , and \exists -Intro(Con) the requirement that c is not in \mathcal{A} or \mathcal{B} and x is not free in \mathcal{B} .

The following lemmas will be useful in completeness proofs below.

⁴ I will use ‘dual’ in multiple senses, which will be distinguishable by context. First, there is the duality of logical operators, including both modalities and quantifiers, which refers to their interdefinability in the usual way using negation. Second, is the duality of sentences, which is derivative from the duality of logical operators. Often I will refer to sentences as dual despite the underlying logic lacking the required duality between the operators. This is to highlight a relationship between the sentences, which is sometimes duality.

LEMMA 3.1. For both **QB.C**_{□◇} and **BQ.C**_{□◇},

1. $\forall y(\mathcal{A}[y/x]) \rightarrow \forall x\mathcal{A}$ is a theorem if y is not free in \mathcal{A} .
2. $\exists y(\mathcal{A}[y/x]) \rightarrow \exists x\mathcal{A}$ is a theorem if y is not free in \mathcal{A} .

LEMMA 3.2. The rules \forall -Intro(con), UG (con), and \exists -Intro(Con) are derivable.

The proof of 3.1.1 and the \forall -Intro(con), UG (con) cases for 3.2 are by the arguments given by Mares and Goldblatt Mares and Goldblatt [20] for **RQ4**^{ot} in Lemmas 6.1, 6.3, 6.5, and 6.6. (By expanding some of the condensed steps in the proofs, it is clear that the proofs apply to **QB.C**_{□◇}). The proof for 3.1.2 and \exists -Intro(Con) is by similar arguments.

LEMMA 3.3. The following are theorems of **QB.C**_{□◇}

1. $\forall x(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \forall x\mathcal{B})$, where x is not free in \mathcal{A}
2. $\forall x\mathcal{A} \rightarrow \neg\exists x\neg\mathcal{A}$
3. $\neg\exists x\neg\mathcal{A} \rightarrow \forall x\mathcal{A}$
4. $\exists x\mathcal{A} \rightarrow \neg\forall x\neg\mathcal{A}$
5. $\neg\forall x\neg\mathcal{A} \rightarrow \exists x\mathcal{A}$

Proof. A proof for 1. can be found in [20], expanding their arguments to confirm that the axioms and rules used are available in **QB.C**_{□◇}. For the remaining cases, I only show two.

1	$\mathcal{A}[\tau/x] \rightarrow \exists x\mathcal{A}$	A11	
2	$\mathcal{A}[\tau/x] \rightarrow \neg\neg\exists x\mathcal{A}$	Prefixing, MP, theorem	
3	$\neg\exists x\mathcal{A} \rightarrow \neg\mathcal{A}[\tau/x]$	2, Contra	
4	$\neg\exists x\mathcal{A} \rightarrow \forall x\neg\mathcal{A}$	\forall -Intro(Con)	
5	$\neg\exists x\mathcal{A} \rightarrow \neg\neg\forall x\neg\mathcal{A}$	Prefixing, MP, theorem	
6	$\neg\forall x\neg\mathcal{A} \rightarrow \exists x\mathcal{A}$	Contra, Suffixing, MP, theorem	
1	$\forall x\neg\mathcal{A} \rightarrow \neg\mathcal{A}[\tau/x]$	A10	
2	$\mathcal{A}[\tau/x] \rightarrow \neg\forall\neg\mathcal{A}$	1, Contra	
3	$\exists x\mathcal{A} \rightarrow \neg\forall\neg\mathcal{A}$	2, \exists -Intro(Con)	□

The Barcan formulas have been of interest in quantified modal logic. The Barcan Formula (BF), the Converse Barcan Formula (CBF), and their duals with the existential quantifier and diamonds (**BF**_{∃◇}, **CBF**_{∃◇}) are as follows:

- BF** $\forall x\Box\mathcal{A} \rightarrow \Box\forall x\mathcal{A}$,
- CBF** $\Box\forall x\mathcal{A} \rightarrow \forall x\Box\mathcal{A}$,
- BF**_{∃◇} $\Diamond\exists x\mathcal{A} \rightarrow \exists x\Diamond\mathcal{A}$,
- CBF**_{∃◇} $\exists x\Diamond\mathcal{A} \rightarrow \Diamond\exists x\mathcal{A}$.

LEMMA 3.4. The Converse Barcan Formula and its dual are theorems of **QB.C**_{□◇}.

Proof.

1. $\forall x\mathcal{A} \rightarrow \mathcal{A}$ A10
2. $\Box\forall x\mathcal{A} \rightarrow \Box\mathcal{A}$ □-M
3. $\Box\forall x\mathcal{A} \rightarrow \forall x\Box\mathcal{A}$ \forall -Intro

The proof for **CBF**_{∃◇} is similar, but using axiom A11 and rules \Diamond -M and \exists -Intro. □

The Barcan Formula and its dual are not theorems $\mathbf{QB.C}_{\Box\Diamond}$. However, this is a corollary of a result below, which demonstrates that the Barcan Formula is not a theorem of a much stronger QMRL.

§4. Semantics for $\mathbf{QB.C}_{\Box\Diamond}$ and $\mathbf{BQ.C}_{\Box\Diamond}$. To construct semantic for $\mathbf{QB.C}_{\Box\Diamond}$ and $\mathbf{BQ.C}_{\Box\Diamond}$, I combine the Mares–Goldblatt semantics for quantified relevant logics with Seki’s semantics for modal relevant logics. As such, I will first explicate general frames for propositional relevant logic $\mathbf{B}^{\circ t}$. Following that, propositions and propositional functions are explained, giving particular emphasis to the operations on propositions used by Mares and Goldblatt to model the quantifiers.

4.1. General Frame Semantics and $\mathbf{B}^{\circ t}$. First, a definition of frames for the logic \mathbf{B} is given. Given the set of primitive connectives $\rightarrow, \neg, \wedge$ and t , the rest are definable, so for this section we make the simplification to the fragment of $\mathbf{B}^{\circ t}$ containing just these connectives. From there, general frames are introduced.

DEFINITION 4.5. *A \mathbf{B} -frame is a tuple $\mathfrak{F} = \langle K, 0, R, * \rangle$, where K is a non-empty set, $0 \subseteq K$, $R \subseteq K^3$, and $*$ is a unary function on K such that a list of postulates to follow are satisfied.*

A binary relation on K is defined via 0 and R by the definition:

$$a \leq b =_{df} \exists y \in 0(Ryab).$$

Given this definition, where $a, b, c, d \in K$, the following postulates are satisfied by every \mathbf{B} -frame:

- (p1) \leq is reflexive and transitive
- (p2) 0 is closed by \leq in the upwards direction
- (p3) if $a \leq b$ and $Rbcd$ then $Racd$
- (p4) if $a \leq c$ and $Rbcd$ then $Rbad$
- (p5) if $d \leq a$ and $Rbcd$ then $Rbca$
- (p6) if $a \leq b$ then $b^* \leq a^*$
- (p7) $a^{**} = a$

Using the ternary relation R , we can define a binary operation \Rightarrow on the powerset $\wp(K)$. For every $X, Y \subseteq K$, let

$$X \Rightarrow Y = \{a \in K : \forall x \forall y (Raxy \text{ and } x \in X \text{ implies } y \in Y)\}.$$

It can be shown that if X and Y are up-sets, then so is $X \Rightarrow Y$, where up-sets are sets of *worlds* or *situations* closed under upwardly under the defined \leq relation. That is, if $a \in X$ and $a \leq b$ implies $b \in X$, then X is an up-set. A binary relation for fusion is defined such that, for every $X, Y \subseteq K$,

$$X \cdot Y = \{a \in K : \exists b \exists c (Rbca \text{ and } b \in X \text{ and } c \in Y)\}.$$

We can define a unary operation on subsets of $\wp(K)$ using the $*$ as follows. For every $X \subseteq K$

$$X^* = \{a \in K : a^* \notin X\}.$$

Similarly, if X is an up-set, then so is X^* .

DEFINITION 4.6. A **B**-model is a tuple $\mathfrak{M} = \langle K, 0, R, *, |-|^{\mathfrak{M}} \rangle$, where $\langle K, 0, R, * \rangle$ is a **B**-frame and $|-|^{\mathfrak{M}}$ is a valuation that maps each atomic proposition to a up-set of K , and is extended to all wff inductively as follows:

$$\begin{aligned} | \mathbf{t} |^{\mathfrak{M}} &= 0 \\ |\mathcal{A} \wedge \mathcal{B}|^{\mathfrak{M}} &= |\mathcal{A}|^{\mathfrak{M}} \cap |\mathcal{B}|^{\mathfrak{M}} \\ |\mathcal{A} \vee \mathcal{B}|^{\mathfrak{M}} &= |\mathcal{A}|^{\mathfrak{M}} \cup |\mathcal{B}|^{\mathfrak{M}} \\ |\neg \mathcal{A}|^{\mathfrak{M}} &= (|\mathcal{A}|^{\mathfrak{M}})^* \\ |\mathcal{A} \rightarrow \mathcal{B}|^{\mathfrak{M}} &= |\mathcal{A}|^{\mathfrak{M}} \Rightarrow |\mathcal{B}|^{\mathfrak{M}} \\ |\mathcal{A} \circ \mathcal{B}|^{\mathfrak{M}} &= |\mathcal{A}|^{\mathfrak{M}} \cdot |\mathcal{B}|^{\mathfrak{M}} \end{aligned}$$

Finally, we say that a formula \mathcal{A} is true in the model \mathfrak{M} if $0 \subseteq |\mathcal{A}|^{\mathfrak{M}}$. \mathcal{A} is valid on a frame if it is true in every model based on that frame, and \mathcal{A} is valid in a class of frames if it is valid in every frame in that class.

General frame semantics, introduced in [31] for modal logics, are a generalization of the usual Kripke-style relational frames, but are closely related to Boolean algebras with operators.⁵ General frames are interesting for a number of reasons. Many find relational semantics quite intuitive, and the general frame semantics appear much like the typical relational semantics. However, they are also natural duals of algebras. This fact can be quite useful. For example, we can obtain completeness results for many modal logics using general frames and their dual algebras.

General frame add to the typical frames a restriction on the possible valuations. This set of admissible propositions, *Prop*, can be described as “(the carrier of) a complex algebra over [the Kripke-style relation frame]” [3, p. 304]. Models are general frames with (admissible) valuations whose range is the set of admissible propositions. Validity and related notions are defined as expected. The canonical general frame can be constructed from the Kripke-style canonical frame on which it is based, taking the admissible propositions to be the sets of worlds containing each formula. We define the set of worlds containing \mathcal{A} by $||\mathcal{A}||_{\mathfrak{C}} = \{a \in K : \mathcal{A} \in a\}$, and then the admissible propositions in the canonical frame by $Prop = \{||\mathcal{A}||_{\mathfrak{C}} : \mathcal{A} \text{ is a formula}\}$. Or more simply we take $Prop_{\mathfrak{C}} = \{|\mathcal{A}| : \mathcal{A} \text{ is a wff}\}$.

Extending the general frame treatment to \mathbf{B}^{of} , we thus get the following definition.

DEFINITION 4.7. A general-**B**-frame is a tuple

$$\mathfrak{F} = \langle K, 0, R, *, Prop \rangle,$$

where $\langle K, 0, R, * \rangle$ is a **B**-frame and *Prop* is a subset of the up-sets of K called the admissible propositions which is closed under \cap, \cup, \Rightarrow and $*$ and contains 0.

A model based on a general frame is given by a valuation that assigns to each propositional variable a member of *Prop*. This is extended as before. Finally, truth in a model, a frame, and a class of frames is defined as usual.

The canonical general frame is obtained from the canonical frame as follows. We obtain $Prop_{\mathfrak{C}}$ by setting $Prop_{\mathfrak{C}} = \{|\mathcal{A}| : \mathcal{A} \text{ is a wff}\}$.

⁵ Indeed, a certain class of general frames are identical in a category-theoretic sense to Boolean algebras with operators [3, p. 308].

4.2. Propositions. The propositions in the models for relevant logics are sets of situations closed under \leq , and are often called *hereditary* sets or *up-sets*. The relations between situations, such as $R, *,$ and \cap , are type-lifted to propositions. Already we have seen the operations \Rightarrow and $*$. Operations \cap and \cup are defined as expected. Here is a list of all the remaining operations of propositions corresponding to the connectives of **QB.C** $_{\square\lozenge}$. The operations \square and \lozenge responding to S_{\square} and S_{\lozenge} , given by their usual definitions in general frames, are not to be confused with the unary connectives of the logic using the same symbol. For $X, Y \in \wp(K)$

$$\begin{aligned} \square X &= \{a \in K : \forall b (S_{\square}ab \Rightarrow b \in X)\}, \\ \lozenge X &= \{a \in K : \exists b (S_{\lozenge}ab \ \& \ b \in X)\}. \end{aligned}$$

To model the quantifiers, Mares and Goldblatt introduce operations from sets of sets of situations to sets of situations. That is, from sets of propositions to propositions. This operation is determined by *Prop*. When *Prop* is a set of hereditary subsets of possible worlds, an operation \sqcap of type $\sqcap : \wp\wp K \rightarrow \wp K$ such that, for every $S \subseteq \wp K$

$$\sqcap S = \cup\{X \in Prop : X \subseteq \cap S\}.$$

Additionally, an operation \sqcup of the same type is defined by:

$$\sqcup S = \cap\{X \in Prop : \cup S \subseteq X\}$$

for every $S \subseteq \wp K$.

The \sqcap is explained as being “motivated by the intuition that the sentence $\forall x\phi$ expresses the conjunction of all the sentences $\phi[a/x]$ ” [11, p. 17]. The problem with using the arbitrary conjunction of the sentences $\phi[a/x]$ is that this conjunction is not guaranteed to be admissible. We only guarantee that the binary conjunction of two members of *Prop* is also in *Prop*. Thus, Goldblatt explains that a notion of entailment between propositions (of the sort considered) will work in this case. First, given our notion of proposition, a proposition X entails a proposition Y if $X \subseteq Y$, for whenever you have a world at which X is true, Y is also true. Goldblatt also calls Y *weaker* than X , and X *stronger* than Y [11, p. 17]. Next, the operation is similar enough to the arbitrary conjunction of a collection of elements of *Prop* is a member of *Prop* that entails every member of the collection. In particular, it is the *weakest* member of *Prop* that entails every member of the collection. That is, for a collection of members of *Prop* $\{X_i : i \in I\}$, the proposition sufficiently expressing their conjunction is the X in *Prop* such that,

- (i) $X \subseteq X_i$ for each $i \in I$ and
- (ii) if $Z \in Prop$ and $Z \subseteq X_i$ for all $i \in I$, then $Z \subseteq X$ [11, p. 17]

The key point here is that this conjunction-like operation always results in a subset of $\cap_{i \in I} X_i$, but not necessarily identical. The latter (\cap) is not guaranteed to be a member of *Prop*, while we require that the former (\sqcap) is. This conjunction-like operator is denoted by $\cap_{i \in I} X_i$.

Similarly, we can think of \sqcup as expressing the strongest member of *Prop* that is entailed by every member. More explicitly, using this to model the existential quantifier intuitively expresses that the proposition corresponding to $\exists xA$ is the strongest proposition implied by every instantiation of the formula.

The following lemma is crucial, as it entails that on the Mares–Goldblatt approach using \sqcap or \sqcup any quantified relevant logic extending **QB**^{of} is sound and complete for

the Mares–Goldblatt models. The reader is reminded that the duality of the modalities \Box and \Diamond break down over the weak \mathbf{B}^{of} , while the duality of the quantifiers is provable in \mathbf{QB}^{of} .

LEMMA 4.8. *Given only the properties of \cap, \cup , and $*$ in \mathbf{B}^{of} ,*

1. $\sqcup S = (\cap(S^*))^*$
2. $\cap S = (\sqcup(S^*))^*$

Proof. Mares and Goldblatt prove the $\sqcup S = (\cap(S^*))^*$ for \mathbf{RQ}^{of} . By expanding the details of their proof, the lemma follows by observing that their proof uses only the classicality of \cap and \cup , and properties of $*$ available to us in \mathbf{B}^{of} . In particular, the steps of the proof involving $*$ are as follows. First, demonstrating that $(\cup S)^* = \cap\{X^* : X \in S\}$, which uses the fact that $a = a^{**}$. Second, showing that $\neg X \subseteq Y$ iff $\neg Y \subseteq X$, which uses the fact that $a = b^*$ iff $a^* = b$. \square

Propositional Functions. Propositional functions, on the other hand, are functions from value assignments for variables to admissible propositions. In addition to using a set *Prop* of admissible propositions, Mares and Goldblatt also use a set *PropFun* of admissible propositional functions.

A propositional function for classical logic is a function from value assignments for the variables into the set $\{True, False\}$, where a proposition is either true or false. For logics with possible world semantics, propositional functions are functions into the set of propositions, where propositions are taken to be a subset of the powerset of worlds. For example, consider the propositional function *x is a human*. The result of applying this function to a value assignment to the variables that assigns to *x* the object *Socrates* is a proposition. In classical logic, this application of a function would result in either *True* or *False*. In the case of quantified relevant logic, this function application returns the set of upwardly closed worlds in which “Socrates is a human” is true.

A propositional function in the context of the semantics of Mares and Goldblatt is a function from value assignments of the variables to admissible propositions. That is, a propositional function ϕ is of type $\phi: U^\omega \rightarrow Prop$, where U is the domain of individuals in a model. While *admissible* propositional functions are required for the semantics of \mathbf{RQ}^{of} , motivations for adopting admissible propositions can be turned into motivations for admissible propositional functions.

DEFINITION 4.9. *For any two elements ϕ and ψ of PropFun, the functions $\phi \cap \psi, \phi \cup \psi, \phi^*, \phi \Rightarrow \psi, \phi \cdot \psi, \Box\phi, \Diamond\phi, \forall_n\phi$ and $\exists_n\phi$ of the same type are defined by, for every value assignment to the variables $f \in U^\omega$:*

$$\begin{aligned} (\phi \cap \psi)f &= \phi f \cap \psi f \\ (\phi \cup \psi)f &= \phi f \cup \psi f \\ (\phi^*)f &= (\phi f)^* \\ (\phi \Rightarrow \psi)f &= \phi f \Rightarrow \psi f \\ (\phi \cdot \psi)f &= \phi f \cdot \psi f \\ (\Box\phi)f &= \Box(\phi f) \\ (\Diamond\phi)f &= \Diamond(\phi f) \\ (\forall_n\phi)f &= \prod_{j \in I} \phi(f[j/n]) \\ (\exists_n\phi)f &= \bigsqcup_{j \in I} \phi(f[j/n]) \end{aligned}$$

4.3. First order relevant logics. The Mares–Goldblatt semantics has yet been shown to apply to propositional relevant logics weaker than \mathbf{QR}^{ot} . While Goldblatt and Kane’s use of this semantic approach for propositionally quantified relevant logics in [13] is highly suggestive, here I show that the Mares–Goldblatt semantics works for \mathbf{QB}^{ot} and \mathbf{BQ}^{ot} , and their extensions.

DEFINITION 4.10. A \mathbf{QB}^{ot} frame is a tuple

$$\mathfrak{F} = \langle K, 0, R, *, U, Prop, PropFun \rangle,$$

where $\langle K, 0, R, * \rangle$ is an \mathbf{B} -frame, U is a non-empty set, $Prop$ is a subset of the up-sets of K which contains 0, $PropFun$ is a subset of the functions of type $U^\omega \rightarrow Prop$, and the following conditions are satisfied:

- (c1) if $a \in 0$ and $a \leq b$ then $b \in 0$;
- (c2) \leq is reflexive and transitive;
- (c3) if $a \leq b$ and $Rbcd$, then $Racd$;
- (c4) if $a \leq c$ and $Rbcd$, then $Rbad$;
- (c5) if $d \leq a$ and $Rbcd$, then $Rbca$;
- (c6) if $a \leq b$ then $b^* \leq a^*$;
- (c7) $a^{**} = a$;
- (c8) if $X, Y \in Prop$, then $X \cup Y, X \cap Y, X \Rightarrow Y, X \cdot Y, X^* \in Prop$, where these operations are defined as before;
- (c9) the constant function ϕ_0 is in $PropFun$;
- (c10) if $\phi, \psi \in PropFun$, then $\phi \cup \psi, \phi \cap \psi, \phi \Rightarrow \psi, \phi^* \in PropFun$; and
- (c11) if $\phi \in PropFun$, then $\exists_n \phi, \forall_n \phi \in PropFun$, for every $n \in \omega$.

Using condition c4, and the fact that 0 is an up-set, we can derive the transitivity requirements of \leq . A \mathbf{QB}^{ot} -frame is called *full* if the set $Prop$ contains every hereditary subset of K , and $PropFun$ contains every function from U^ω to $Prop$.

DEFINITION 4.11. A pre-model for \mathbf{QB}^{ot} is a tuple

$$\mathfrak{M} = \langle K, 0, R, *, U, Prop, PropFun, |\cdot|^\mathfrak{M} \rangle,$$

where $\langle K, 0, R, *, U, Prop, PropFun \rangle$ is a \mathbf{QB}^{ot} -frame and $|\cdot|^\mathfrak{M}$ is a value assignment that assigns

1. an element $|c|^\mathfrak{M} \in U$ to each constant symbol c ;
2. a function $|P|^\mathfrak{M} : U^n \rightarrow \wp(K)$ to each n -ary predicate symbol P ; and
3. a propositional function $|\mathcal{A}|^\mathfrak{M} : U^\omega \rightarrow \wp(K)$ to each formula \mathcal{A} such that, when \mathcal{A} is the atomic $P\tau_1, \dots, \tau_n$, the propositional function assigned to it is given by, for each $f \in U^\omega$,

$$|P\tau_1, \dots, \tau_n|^\mathfrak{M} f = |P|^\mathfrak{M} (|\tau_1|^\mathfrak{M} f, \dots, |\tau_n|^\mathfrak{M} f).$$

Further, when \mathcal{A} is not atomic, the function assigned to the formula is given by the following:

$$\begin{aligned} |\mathbf{t}|^\mathfrak{M} &= \phi_0 \\ |\mathcal{A} \wedge \mathcal{B}|^\mathfrak{M} &= |\mathcal{A}|^\mathfrak{M} \cap |\mathcal{B}|^\mathfrak{M} \\ |\mathcal{A} \vee \mathcal{B}|^\mathfrak{M} &= |\mathcal{A}|^\mathfrak{M} \cup |\mathcal{B}|^\mathfrak{M} \end{aligned}$$

$$\begin{aligned} |\neg \mathcal{A}|^{\mathcal{M}} &= (|\mathcal{A}|^{\mathcal{M}})^* \\ |\mathcal{A} \rightarrow \mathcal{B}|^{\mathcal{M}} &= |\mathcal{A}|^{\mathcal{M}} \Rightarrow |\mathcal{B}|^{\mathcal{M}} \\ |\mathcal{A} \circ \mathcal{B}|^{\mathcal{M}} &= |\mathcal{A}|^{\mathcal{M}} \cdot |\mathcal{B}|^{\mathcal{M}} \\ |\forall x_n \mathcal{A}|^{\mathcal{M}} &= \forall_n |\mathcal{A}|^{\mathcal{M}} \\ |\exists x_n \mathcal{A}|^{\mathcal{M}} &= \exists_n |\mathcal{A}|^{\mathcal{M}}. \end{aligned}$$

DEFINITION 4.12. A model for \mathbf{QB}^{ot} is a pre-model for \mathbf{QB}^{ot} that assigns a member of $PropFun$ to each atomic formula.

The following theorems are provable using the arguments from later sections of the paper, ignoring the modal machinery. The proofs are only given in later sections to avoid unnecessary repetition.

THEOREM 4.13 (Soundness). All of the theorems of \mathbf{QB}^{ot} (\mathbf{BQ}^{ot}) are valid in every \mathbf{QB}^{ot} -model (\mathbf{BQ}^{ot} -model).

THEOREM 4.14 (Completeness for \mathbf{QB}^{ot} (\mathbf{BQ}^{ot})). If \mathcal{A} is valid in every \mathbf{QB}^{ot} -model (\mathbf{BQ}^{ot} -model), then \mathcal{A} is a theorem of \mathbf{QB}^{ot} (\mathbf{BQ}^{ot}).

Let \mathbb{L} be a propositional relevant logic extending \mathbf{B}^{ot} . We may define \mathbb{LQ} and \mathbf{QL} by adding frame conditions for the relevant logic fragment.

COROLLARY 4.15. The logic \mathbb{LQ} (\mathbf{QL}) is sound and complete for the \mathbb{LQ} -models (\mathbf{QL} -models).

Table 1 contains a list of some common axioms for propositional relevant logics and their corresponding frame conditions.

4.4. Frames and Models for $\mathbf{QB.C}_{\square\Diamond}$ and $\mathbf{BQ.C}_{\square\Diamond}$.

DEFINITION 4.16. A $\mathbf{QB.C}_{\square\Diamond}$ frame is a tuple

$$\mathfrak{F} = \langle K, 0, R, *, S_{\square}, S_{\Diamond}, U, Prop, PropFun \rangle,$$

Table 1. Frame conditions for axioms of relevant logics

Axiom	Frame condition ^a
(A16) $\mathcal{A} \vee \neg \mathcal{A}$	if $a \in 0$ then $a^* \leq a$
(A17) $(\mathcal{A} \rightarrow \neg \mathcal{B}) \rightarrow (\mathcal{B} \rightarrow \neg \mathcal{A})$	if $Rabc$ then Rac^*b^*
(A18) $(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow ((\mathcal{B} \rightarrow \mathcal{C}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C}))$	if R^2abcd then $Rb(Rac)d$
(A19) $(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow ((\mathcal{C} \rightarrow \mathcal{A}) \rightarrow (\mathcal{C} \rightarrow \mathcal{B}))$	if R^2abcd then $Ra(Rbc)d$
(A20) $(\mathcal{A} \rightarrow (\mathcal{A} \rightarrow \mathcal{B})) \rightarrow (\mathcal{A} \rightarrow \mathcal{B})$	if $Rabc$ then R^2abc
(A21) $\mathcal{A} \rightarrow ((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{B})$	if $Rabc$ then there is an $x \in K$ such that $a \leq x$ and $Rbxc$
(A22) $(\mathcal{A} \rightarrow \neg \mathcal{A}) \rightarrow \neg \mathcal{A}$	Raa^*a

^a A more complete treatment of axioms and their corresponding frame conditions can be found in [27], particularly in Sections 4.1 and 4.4.

where $\langle K, 0, R, *, U, Prop, PropFun \rangle$ is an \mathbf{QB}^{ot} -frame, $S_{\square}, S_{\diamond} \subseteq K^2$, U is a non-empty set, $Prop$ is a subset of the up-sets of K which contains 0 , $PropFun$ is a subset of the functions of type $U^\omega \rightarrow Prop$, and the following conditions are satisfied:

- (c12a) if $a \leq b$ and $S_{\square}bc$ then $S_{\square}ac$;
- (c12b) if $a \leq b$ and $S_{\diamond}ac$ then $S_{\diamond}bc$;
- (c13) if $X \in Prop$, then $\square X, \diamond X \in Prop$; and
- (c14) if $\phi \in PropFun$, then $\square\phi, \diamond\phi \in PropFun$.

A $\mathbf{QB.C}_{\square\diamond}$ -frame is called *full* if the set $Prop$ contains every hereditary subset of K , and $PropFun$ contains every function from U^ω to $Prop$.

DEFINITION 4.17. A pre-model for $\mathbf{QB.C}_{\square\diamond}$ is a tuple

$$\mathfrak{M} = \langle K, 0, R, *, S_{\square}, S_{\diamond}, U, Prop, PropFun, |\cdot|^{\mathfrak{M}} \rangle,$$

where $\langle K, 0, R, *, S_{\square}, S_{\diamond}, U, Prop, PropFun \rangle$ is a $\mathbf{QB.C}_{\square\diamond}$ -frame and where $|\cdot|^{\mathfrak{M}}$ is a value assignment as in definition 4.11, with the exceptions/additions that

$$\begin{aligned} |\square\mathcal{A}|^{\mathcal{M}} &= \square|\mathcal{A}|^{\mathcal{M}}, \\ |\diamond\mathcal{A}|^{\mathcal{M}} &= \diamond|\mathcal{A}|^{\mathcal{M}}. \end{aligned}$$

DEFINITION 4.18. A model for $\mathbf{QB.C}_{\square\diamond}$ is a pre-model for $\mathbf{QB.C}_{\square\diamond}$ that assigns a member of $PropFun$ to each atomic formula.

As shown by Seki [30], binary relations for the dual modalities \diamond and \square can be defined by, for every $a, b \in K$:

$$\begin{aligned} S_{\diamond}ab &\text{ iff } S_{\square}a^*b^*, \\ S_{\square}ab &\text{ iff } S_{\diamond}a^*b^*. \end{aligned}$$

Given the functions determined by a (pre-)model, we can then consider the truth sets of each function as it is applied to a variable assignment. These truth sets are given, for each $f \in U^\omega$, as follows:

$$\begin{aligned} |t|^{\mathcal{M}}f &= 0 \\ |\mathcal{A} \wedge \mathcal{B}|^{\mathcal{M}}f &= |\mathcal{A}|^{\mathcal{M}}f \cap |\mathcal{B}|^{\mathcal{M}}f \\ |\mathcal{A} \vee \mathcal{B}|^{\mathcal{M}}f &= |\mathcal{A}|^{\mathcal{M}}f \cup |\mathcal{B}|^{\mathcal{M}}f \\ |\neg\mathcal{A}|^{\mathcal{M}}f &= (|\mathcal{A}|^{\mathcal{M}}f)^* \\ |\mathcal{A} \rightarrow \mathcal{B}|^{\mathcal{M}}f &= |\mathcal{A}|^{\mathcal{M}}f \Rightarrow |\mathcal{B}|^{\mathcal{M}}f \\ |\mathcal{A} \circ \mathcal{B}|^{\mathcal{M}}f &= |\mathcal{A}|^{\mathcal{M}}f \cdot |\mathcal{B}|^{\mathcal{M}}f \\ |\square\mathcal{A}|^{\mathcal{M}}f &= \square(|\mathcal{A}|^{\mathcal{M}}f) \\ |\diamond\mathcal{A}|^{\mathcal{M}}f &= \diamond(|\mathcal{A}|^{\mathcal{M}}f) \\ |\forall x\mathcal{A}|^{\mathcal{M}}f &= \prod_{g \in xf} |\mathcal{A}|^{\mathcal{M}}g \\ |\exists x\mathcal{A}|^{\mathcal{M}}f &= \sqcup_{g \in xf} |\mathcal{A}|^{\mathcal{M}}g. \end{aligned}$$

Finally, a familiar looking $\models_{\mathfrak{M}}$ relation — or simply \models for convenience — is determined as follows:

- (i) $a, f \models P\tau_1, \dots, \tau_n$ iff $a \in |P\tau_1, \dots, \tau_n|^{\mathfrak{M}} f$
- (ii) $a, f \models \mathbf{t}$ iff $a \in 0$
- (iii) $a, f \models \mathcal{A} \wedge \mathcal{B}$ iff $a, f \models \mathcal{A}$ and $a, f \models \mathcal{B}$
- (iii) $a, f \models \mathcal{A} \vee \mathcal{B}$ iff $a, f \models \mathcal{A}$ or $a, f \models \mathcal{B}$
- (iv) $a, f \models \neg \mathcal{A}$ iff $a^*, f \not\models \mathcal{A}$
- (v) $a, f \models \mathcal{A} \rightarrow \mathcal{B}$ iff $\forall b, c ((Rabc \text{ and } b, f \models \mathcal{A}) \Rightarrow c, f \models \mathcal{B})$
- (v) $a, f \models \mathcal{A} \circ \mathcal{B}$ iff $\exists b, c (Rbca \text{ and } b, f \models \mathcal{A} \text{ and } c, f \models \mathcal{B})$
- (vi) $a, f \models \Box \mathcal{A}$ iff $\forall b (S_{\Box} ab \Rightarrow b, f \models \mathcal{A})$
- (vii) $a, f \models \Diamond \mathcal{A}$ iff $\exists b (S_{\Diamond} ab \ \& \ b, f \models \mathcal{A})$
- (viii) $a, f \models \forall x \mathcal{A}$ iff there is an $X \in Prop$ such that $X \subseteq \bigcap_{g \in x_n f} |\mathcal{A}|^{\mathfrak{M}} g$ and $a \in X$
- (ix) $a, f \models \exists x \mathcal{A}$ iff, for every $X \in Prop$ such that $a^* \in X$, there is a $b \in X$ and x -variant such that $b^* \in |\mathcal{A}|^{\mathfrak{M}} g$

Cases (viii) and (ix) are as they are in [20].

A formula \mathcal{A} is *satisfied* by a variable assignment f in a model \mathfrak{M} if $a, f \models \mathcal{A}$, for every $a \in 0$. A formula \mathcal{A} is *valid* in the model \mathfrak{M} , if it is satisfied by every variable assignment in that model. A formula \mathcal{A} is valid in a frame, if it is valid in every model based on the frame. Further, a formula is valid in a class of frames, if it is valid in every frame in the class.

DEFINITION 4.19. For the logic **BQ.C $_{\Box\Diamond}$** , frames and models are defined as above, with the addition for the following frame condition:

$$c15 \ X - Y \subseteq \bigcap_{a \in U} \phi(f[a/n]) \text{ implies } X - Y \subseteq (\forall_n \phi)f.$$

As the extensional confinement axioms are equivalent in **QB of** , and this single frame condition taken from [20] is sufficient for the validity (and for completeness results).

The following lemmas are useful in establishing soundness results. When unspecified, the lemmas apply to both **QB.C $_{\Box\Diamond}$** and **BQ.C $_{\Box\Diamond}$** .

LEMMA 4.20 (Semantic Entailment). In a model \mathfrak{M} , a formula $\mathcal{A} \rightarrow \mathcal{B}$ is satisfied by the variable assignment f iff for every $a \in K$, if $a, f \models \mathcal{A}$, then $a, f \models \mathcal{B}$.

Proof. The original statement of this lemma (for a propositional relevant logic) and its proof can be found as Lemmas 2 and 3 in [26]. □

LEMMA 4.21. For any formula \mathcal{A} , if f and g agree on all free variable of \mathcal{A} , then $|\mathcal{A}|f = |\mathcal{A}|g$.

Proof. The proof is by induction on the complexity of \mathcal{A} . The arguments of Mares and Goldblatt may be used for most cases. The remaining cases are shown here.

If $\mathcal{A} = \exists x_n \mathcal{B}$, then a variable in \mathcal{B} is free if either it is free in \mathcal{A} or it is the variable x_n . It follows that for every $j \in U$, $f[j/n]$ and $g[j/n]$ agree on all free variables of \mathcal{B} . By the induction hypothesis, we get that $|\mathcal{B}|f[j/n] = |\mathcal{B}|g[j/n]$, and thus

$$|\mathcal{A}|f = \sqcup_{j \in I} |\mathcal{B}|f[j/n] = \sqcup_{j \in I} |\mathcal{B}|g[j/n] = |\mathcal{A}|g,$$

as required.

If $\mathcal{A} = \Box \mathcal{B}$, then $|\Box \mathcal{B}|f = \Box |\mathcal{B}|f$. By the induction hypothesis $|\mathcal{B}|f = |\mathcal{B}|g$, so $\Box |\mathcal{B}|f = \Box |\mathcal{B}|g$. The case of $\mathcal{A} = \Diamond \mathcal{B}$ is similar to the case for \Box . □

LEMMA 4.22. *Let τ be free for x in a formula \mathcal{A} . If $g \in xf$ and $|x|g = |\tau|f$, then $|\mathcal{A}[\tau/x]|f = |\mathcal{A}|g$ in a model \mathfrak{M} .*

Proof. The proof is by induction on the complexity of \mathcal{A} . The arguments of Mares and Goldblatt [20, p. 177] can be used for all cases except for $\mathcal{A} = \Box\mathcal{B}$, $\mathcal{A} = \Diamond\mathcal{B}$, and $\mathcal{A} = \exists x\mathcal{B}$. Here I show this remaining case.

Let $\mathcal{A} = \Box\mathcal{B}$ and assume the result holds for \mathcal{B} . If x is free for τ in $\Box\mathcal{B}$, then x is free for τ in \mathcal{B} . By the induction hypothesis, $|\mathcal{B}[\tau/x]|f = |\mathcal{B}|g$. Thus, $|\Box\mathcal{B}[\tau/x]|f = |\Box\mathcal{B}|g$. From this we get that $|\Box\mathcal{B}[\tau/x]|f = |\Box\mathcal{B}|g$, as required.

The case for $\mathcal{A} = \Diamond\mathcal{B}$ is similarly straightforward.

For $\mathcal{A} = \exists x\mathcal{B}$, the proof is obtained by modifying the Mares–Goldblatt proof for the universal quantifier case. Assume that the result holds for \mathcal{B} as the induction hypothesis. Either x is free in \mathcal{A} or it isn't. If it isn't, then $\mathcal{A}[\tau/x] = \mathcal{A}$. Since g is an x -variant of f , the result is immediate.

On the other hand, if x is free in \mathcal{A} , then $x \neq y$. Further, $\mathcal{A}[\tau/x] = \exists y\mathcal{B}[\tau/x]$. Because x is free for τ , it follows that $y \neq \tau$ and x is free for τ in \mathcal{B} . Let y be the variable x_n . We have the following identities:

$$|\mathcal{A}[\tau/x]|^{\mathfrak{M}} f = \sqcup_{i \in U} |\mathcal{B}[\tau/x]|^{\mathfrak{M}} f[i/n],$$

$$|\mathcal{A}|^{\mathfrak{M}} g = \sqcup_{i \in U} |\mathcal{B}|^{\mathfrak{M}} g[i/n].$$

To complete the proof we show that the two right sides are identical, as in the previous case.

Again, note that for any $i \in U$, the assignment $f[i/n]$ is an x -variant of $g[i/n]$, as they are x -variants before the substitution, and the substitution is applied to each of them. From $x_n \neq \tau$ and $x \neq x_n$ it follows that $|\tau|f[i/n] = |\tau|f = |x|g = |x|g[i/n]$. Using the induction hypothesis we get that

$$|\mathcal{B}[\tau/x]|^{\mathfrak{M}} f[i/n] = |\mathcal{B}|^{\mathfrak{M}} g[i/n].$$

The proof is completed as in the previous case, thus concluding the induction. □

§5. Soundness

LEMMA 5.23. *The axioms of \mathbf{B}^{ot} are valid, and the rules of \mathbf{B}^{ot} preserve validity, in the class of $\mathbf{QB.C}_{\Box\Diamond}$ and $\mathbf{BQ.C}_{\Box\Diamond}$ frames.*

The proof of this is standard using the typical methods. The interesting cases for soundness are the axioms and rules with quantifiers and modalities.

LEMMA 5.24. *Axioms*

- (A10) $\forall x\mathcal{A} \rightarrow \mathcal{A}[\tau/x]$, where τ is free for x in \mathcal{A} ,
- (A11) $\mathcal{A}[\tau/x] \rightarrow \exists x\mathcal{A}$, where τ is free for x in \mathcal{A} , and
- (A14) $\forall x(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\exists x\mathcal{A} \rightarrow \mathcal{B})$ where x is not free in \mathcal{B}

are valid in every $\mathbf{QB.C}_{\Box\Diamond}$ -model.

Proof. The proofs for (A10) and (A11) are similar. The arguments from Mares and Goldblatt [20] can be used for the case of (A10). That is, their proof is does not use any requirements beyond what is available for for \mathbf{QB}^{ot} -models. Thus, I will only show the case for (A11), which is similar.

Let $a, f \models \mathcal{A}[\tau/x]$ and let τ be free for x in \mathcal{A} . That is, that $a \in |\mathcal{A}[\tau/x]|^{\mathfrak{M}} f$. Let g be an x -variant of f such that $|x|g = |\tau|f$. It follows that $a, g \models \mathcal{A}$. That is, $|\mathcal{A}[\tau/x]|f = |\mathcal{A}|g$. Thus, $a \in |\mathcal{A}|g$. It follows that $a \in \cup_{g \in xf} |\mathcal{A}|g$. Thus, a is in every proposition X such that $\cup_{g \in xf} |\mathcal{A}|g \subseteq X$. Therefore $a \in \sqcup_{g \in xf} |\mathcal{A}|g$. By Semantic Entailment the axiom (A11) is valid.

For (A14), the proof is made simpler using the duality of the quantifiers, and prove that $\models \forall x(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\neg \forall x \neg \mathcal{A} \rightarrow \mathcal{B})$ where x is not free in \mathcal{B} . Assume that this is not the case. Then (i) $a, f, \models \forall x(\mathcal{A} \rightarrow \mathcal{B})$, (ii) $Rabc$, (iii) $b, f \models \neg \forall x \neg \mathcal{A}$ and (iv) $c, f \not\models \mathcal{B}$. Thus, (v) $a \in |\mathcal{A} \rightarrow \mathcal{B}|g$ for all $g \in xf$. From (ii), we get that

$$\begin{aligned} b, f \models \neg \forall \neg \mathcal{A} \text{ iff } b^*, f \not\models \forall x \neg \mathcal{A} \\ \text{iff } b^* \notin \sqcap_{g \in xf} |\neg \mathcal{A}|g \\ \text{iff } b^* \notin |\neg \mathcal{A}|g' \text{ for some particular } g' \in xf \\ \text{iff } b^*, g' \not\models \neg \mathcal{A} \\ \text{iff } b, g' \models \mathcal{A}. \end{aligned}$$

Given the last fact, (ii), and (v), we get $c, g' \models \mathcal{B}$. Moreover, since there are no free occurrences of x in \mathcal{B} , we obtain our contradiction with $c, f \models \mathcal{B}$ and (iv). Thus, the result follows by Semantic Entailment. □

LEMMA 5.25. *The rules \forall -Intro and \exists -I preserve validity in every $\mathbf{QB.C}_{\square\Diamond}$ -model.*

Proof. For \forall -Intro, the proof is basically as in [20], but adapted where needed to fit $\mathbf{QB.C}_{\square\Diamond}$. Suppose that $\mathcal{A} \rightarrow \mathcal{B}$ is valid in the model \mathfrak{M} and that x does not occur free in \mathcal{A} . It follows that for every variable assignment g , and from Semantic Entailment, that $|\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}g$. Now take any variable assignment f . If g is an x -variant of f , then they agree on the free variables of \mathcal{A} , given that x is not free in \mathcal{A} . From this we get that $|\mathcal{A}|^{\mathfrak{M}}f = |\mathcal{A}|^{\mathfrak{M}}g$, and that $|\mathcal{A}|^{\mathfrak{M}}f \subseteq |\mathcal{B}|^{\mathfrak{M}}g$. Since g is an x -variant of f , considering all such x -variants gives us $|\mathcal{A}|^{\mathfrak{M}}f \subseteq \sqcap_{g \in xf} |\mathcal{B}|^{\mathfrak{M}}g$. This is the case for every f , therefore, by Semantic Entailment, we get that $\mathcal{A} \rightarrow \forall x \mathcal{B}$ is valid in this model, showing that \forall -Intro preserves validity.

For \exists -I, first suppose that $\mathcal{A} \rightarrow \mathcal{B}$ is valid in the model \mathfrak{M} and that x does not occur free in \mathcal{B} . It follows that for every variable assignment g , and from Semantic Entailment, that $|\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}g$. Now take any variable assignment f . If g is an x -variant of f , then they agree on the free variables of \mathcal{B} , given that x is not free in \mathcal{B} . From this we get that $|\mathcal{B}|^{\mathfrak{M}}f = |\mathcal{B}|^{\mathfrak{M}}g$, and that $|\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}f$.

Since g is an x -variant of f , considering all such x -variants gives us $\sqcup_{g \in xf} |\mathcal{A}|^{\mathfrak{M}}g \subseteq |\mathcal{B}|^{\mathfrak{M}}f$. As $\sqcup_{g \in xf} |\mathcal{A}|^{\mathfrak{M}}g = |\exists x \mathcal{A}|^{\mathfrak{M}}f$, and our choice of f was arbitrary, the result follows by Semantic Entailment. □

LEMMA 5.26. *The confinement axioms*

$$\begin{aligned} EC_{\forall} \quad \forall x(\mathcal{A} \vee \mathcal{B}) \rightarrow (\mathcal{A} \vee \forall x \mathcal{B}), \text{ where } x \text{ is not free in } \mathcal{A} \\ EC_{\exists} \quad (\mathcal{A} \wedge \exists x \mathcal{B}) \rightarrow \exists x(\mathcal{A} \wedge \mathcal{B}), \text{ where } x \text{ is not free in } \mathcal{A} \quad (EC_{\exists}) \end{aligned}$$

are valid in every $\mathbf{BQ.C}_{\square\Diamond}$ -model.

Proof. The case for EC_{\forall} is covered by the arguments of Mares and Goldblatt [20], so I show only the case for axiom EC_{\exists} . First, suppose that $a, f \models \mathcal{A} \wedge \exists x \mathcal{B}$, and that x is not free in \mathcal{A} . It follows that $a, f \models \mathcal{A}$ and $a, f \models \exists x \mathcal{B}$.

For reductio, suppose that $a, f \not\models \exists x(\mathcal{A} \wedge \mathcal{B})$. That is, via convenient dualities (which would have made this lemma redundant if we were to have taken the existential quantifier as defined), $a, f \not\models \neg \forall x \neg (\mathcal{A} \wedge \mathcal{B})$. Therefore, $a^*, f \models \forall x (\neg \mathcal{A} \vee \neg \mathcal{B})$. By the previous lemma, $a^*, f \models \neg \mathcal{A} \vee \forall x \neg \mathcal{B}$. However, if $a^*, f \models \neg \mathcal{A}$, then $a \not\models \mathcal{A}$, a contradiction. On the other hand, if $a^*, f \models \forall x \neg \mathcal{B}$, then $a, f \not\models \neg \forall x \neg \mathcal{B}$, which is $a, f \not\models \exists x \mathcal{B}$, a contradiction. Finally, by Semantic Entailment the result follows. \square

LEMMA 5.27. *The axioms*

- (A12) $(\Box \mathcal{A} \wedge \Box \mathcal{B}) \rightarrow \Box(\mathcal{A} \wedge \mathcal{B})$
- (A13) $\Diamond(\mathcal{A} \vee \mathcal{B}) \rightarrow (\Diamond \mathcal{A} \vee \Diamond \mathcal{B})$

are valid in the class of **QB.C $_{\Box\Diamond}$** -frames.

Proof. For A12, begin by supposing that $a, f \models \Box \mathcal{A} \wedge \Box \mathcal{B}$. It follows that $a, f \models \Box \mathcal{A}$ and $a, f \models \Box \mathcal{B}$. For reductio, let $a, f \not\models \Box(\mathcal{A} \wedge \mathcal{B})$. Then there exists a b such that $S_{\Box}ab$ and $b, f \not\models \mathcal{A} \wedge \mathcal{B}$. Thus, either $b, f \not\models \mathcal{A}$ or $b, f \not\models \mathcal{B}$. However, given $a, f \models \Box \mathcal{A}$ and $a, f \models \Box \mathcal{B}$ and $S_{\Box}ab$, we get that both $b, f \models \mathcal{A}$ and $b, f \models \mathcal{B}$, producing our contradiction. Thus $a, f \models \Box(\mathcal{A} \wedge \mathcal{B})$. Finally, by Semantic Entailment we get our result.

For A13, begin by supposing that $a, f \models \Diamond(\mathcal{A} \vee \mathcal{B})$. Then there is a b such that $S_{\Diamond}ab$ and $b, f \models \mathcal{A} \vee \mathcal{B}$. That is, either $b, f \models \mathcal{A}$ or $b, f \models \mathcal{B}$. If the former, then $a, f \models \Diamond \mathcal{A}$ and also $a, f \models \Diamond \mathcal{A} \vee \Diamond \mathcal{B}$. If the latter, then $a, f \models \Diamond \mathcal{B}$ and also $a, f \models \Diamond \mathcal{A} \vee \Diamond \mathcal{B}$. Either way we have $a, f \models \Diamond \mathcal{A} \vee \Diamond \mathcal{B}$, and by Semantic Entailment we have our result. \square

LEMMA 5.28. *The rules of \Box -Monotonicity and \Diamond -Monotonicity preserve validity in every **QB.C $_{\Box\Diamond}$** -model.*

Proof. For the former, suppose that $\mathcal{A} \rightarrow \mathcal{B}$ is valid in a model. For some f and a , let $a, f \models \Box \mathcal{A}$. For reductio, let $a, f \not\models \Box \mathcal{B}$. Then there is a b such that $S_{\Box}ab$ and $b, f \not\models \mathcal{B}$. Also, we have that $b, f \models \mathcal{A}$. Given the validity of $\mathcal{A} \rightarrow \mathcal{B}$, $b, f \models \mathcal{A}$, and Semantic Entailment, we have that $b, f \models \mathcal{B}$, which gives us our contradiction. Therefore $a, f \models \Box \mathcal{B}$. The result follows by Semantic Entailment.

For the latter, the proof is just as straightforward. \square

In summary, I record that:

THEOREM 5.29 (Soundness for **QB.C $_{\Box\Diamond}$** and **BQ.C $_{\Box\Diamond}$**).

1. All of the theorems of **QB.C $_{\Box\Diamond}$** are valid in every **QB.C $_{\Box\Diamond}$** -model.
2. All of the theorems of **BQ.C $_{\Box\Diamond}$** are valid in every **BQ.C $_{\Box\Diamond}$** -model.

§6. Theories. For notational convenience, let $\Gamma \gg_{\mathbb{L}} \Delta$ mean that there are some $\mathcal{A}_1, \dots, \mathcal{A}_n \in \Gamma$ and $\mathcal{B}_1, \dots, \mathcal{B}_m \in \Delta$ such that $\vdash_{\mathbb{L}} (\mathcal{A}_1 \wedge \dots \wedge \mathcal{A}_n) \rightarrow (\mathcal{B}_1 \vee \dots \vee \mathcal{B}_m)$, where Γ and Δ are sets of formulas and \mathbb{L} is a logic.

DEFINITION 6.30. *A pair (Γ, Δ) is \mathbb{L} -independent if and only if $\Gamma \not\gg_{\mathbb{L}} \Delta$*

DEFINITION 6.31. *An \mathbb{L} -theory is a set of formulas Γ such that if $\Gamma \gg_{\mathbb{L}} \mathcal{A}$, then $\mathcal{A} \in \Gamma$. A theory Γ is prime if and only if, if $\mathcal{A} \vee \mathcal{B} \in \Gamma$, then either $\mathcal{A} \in \Gamma$ or $\mathcal{B} \in \Gamma$. A theory Γ is regular if and only if it contains every theorem of \mathbb{L} .*

LEMMA 6.32. *A set of sentences Γ is a theory if and only if both:*

1. *If $\mathcal{A} \in \Gamma$ and $\vdash_{\mathbb{L}} \mathcal{A} \rightarrow \mathcal{B}$, then $\mathcal{B} \in \Gamma$*
2. *If $\mathcal{A} \in \Gamma$ and $\mathcal{B} \in \Gamma$, then $\mathcal{A} \wedge \mathcal{B} \in \Gamma$*

Finally, the extension lemma, is needed to show that, if a formula is not a theorem, then there is a regular prime theory that does not contain the formula.

LEMMA 6.33 (Extension). *If (Γ, Δ) is \mathbb{L} -independent, then there is some prime theory Γ' such that $\Gamma \subseteq \Gamma'$ and (Γ', Δ) , for $\mathbb{L} = \mathbf{QB.C}_{\square\Diamond}$ or $\mathbf{BQ.C}_{\square\Diamond}$.*

The proof of this lemma for the logic **RQ** is due to Belnap (but the result is unpublished) [7, p. 41]. Roughly, one way to prove this lemma is to take the closure of Δ under disjunction, and consider all theories extending Γ that do not include any element of the closure of Δ . We apply Zorn’s lemma to get a maximal theory, and then prove that it is prime.

DEFINITION 6.34. *Where $K'_{\mathbb{L}}$ is the set of \mathbb{L} -theories and $a, b, c \in K'_{\mathbb{L}}$, the following relations are defined:*

- R' is defined by $R'abc$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \ \& \ \mathcal{B} \in b\} \subseteq c$.*
- S_{\square} is defined by $S_{\square}ab$ iff $\{\mathcal{A} : \square\mathcal{A} \in a\} \subseteq b$.*
- S_{\Diamond} is defined by $S_{\Diamond}ab$ iff $\{\diamond\mathcal{A} : \mathcal{A} \in b\} \subseteq a$.*

§7. Completeness. The proof of completeness for **QB.C**_{□◇} and **BQ.C**_{□◇} in this section will again follow Mares and Goldblatt, making adjustments where appropriate for the added necessity operator. I will take the usual detour through theories in what Mares and Goldblatt call a “Henkin–Lemmon–Scott–Routley–Meyer canonical model construction” [20], p. 178. For the construction of a canonical model, we either assume that the set of constants *Con* is denumerable, or we add a denumerable number of constants to the set *Con* in the usual way.

Let $K'_{\mathbb{L}}$ be the set of all **QB.C**_{□◇}-theories, and let R' , S_{\square} , and S_{\Diamond} be defined on $K'_{\mathbb{L}}$ as in definition 6.34, with $\mathbb{L} = \mathbf{QB.C}_{\square\Diamond}$.

DEFINITION 7.35. *A canonical frame for **QB.C**_{□◇} is a tuple,*

$$\mathfrak{F}_{\mathcal{E}} = \langle K_{\mathcal{E}}, 0_{\mathcal{E}}, R_{\mathcal{E}}, *_{\mathcal{E}}, S_{\square\mathcal{E}}, S_{\Diamond\mathcal{E}}, U_{\mathcal{E}}, Prop_{\mathcal{E}}, PropFun_{\mathcal{E}} \rangle$$

where

- $K_{\mathcal{E}} \subseteq K'_{\mathbb{L}}$ is the set of all prime **QB.C**_{□◇}-theories.
- $0_{\mathcal{E}}$ is the set of all regular prime **QB.C**_{□◇}-theories.
- $R_{\mathcal{E}}$ is the relation R' restricted to $K_{\mathcal{E}}$.
- $S_{\square\mathcal{E}}$ is the relation S_{\square} restricted to $K_{\mathcal{E}}$.
- $S_{\Diamond\mathcal{E}}$ is the relation S_{\Diamond} restricted to $K_{\mathcal{E}}$.
- $*_{\mathcal{E}}$ is defined by $a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$.
- $U_{\mathcal{E}}$ is the infinite set of constants *Con*.
- For every closed formula \mathcal{A} , $\|\mathcal{A}\|_{\mathcal{E}}$ is defined to be the set $\{a \in K : \mathcal{A} \in a\}$.
- $Prop_{\mathcal{E}}$ is defined as the set $\{\|\mathcal{A}\|_{\mathcal{E}} : \mathcal{A} \text{ is a closed formula}\}$.
- Given a variable assignment f , the value f_n is a constant. Substituting each variable in a formula \mathcal{A} with the constant assigned to it by a variable assignment f results in a closed formula which will be denoted \mathcal{A}^f . Therefore $\mathcal{A}^f = \mathcal{A}[f_0/x_0, \dots, f_n/x_n, \dots]$.

- To each formula \mathcal{A} , there is a corresponding function $\phi_{\mathcal{A}}$ of type $U^\omega \rightarrow K$ given by $\phi_{\mathcal{A}}f = \|\mathcal{A}^f\|_{\mathfrak{E}}$. $\text{PropFun}_{\mathfrak{E}}$ is the set of all functions $\phi_{\mathcal{A}}$, where \mathcal{A} is a formula.

DEFINITION 7.36. A canonical model for $\mathbf{QB.C}_{\square\lozenge}$ is a tuple,

$$\mathfrak{M}_{\mathfrak{E}} = \langle K_{\mathfrak{E}}, 0_{\mathfrak{E}}, R_{\mathfrak{E}}, *_{\mathfrak{E}}, S_{\square\mathfrak{E}}, S_{\lozenge\mathfrak{E}}, U_{\mathfrak{E}}, \text{Prop}_{\mathfrak{E}}, \text{PropFun}_{\mathfrak{E}}, \vDash_{\mathfrak{E}}^{\mathfrak{M}} \rangle, \text{ where}$$

- $\langle K_{\mathfrak{E}}, 0_{\mathfrak{E}}, R_{\mathfrak{E}}, S_{\square\mathfrak{E}}, *_{\mathfrak{E}}, U_{\mathfrak{E}}, \text{Prop}_{\mathfrak{E}}, \text{PropFun}_{\mathfrak{E}} \rangle$ is the canonical frame.
- $\vDash_{\mathfrak{E}} c = c$, for every constant symbol c .
- $\vDash_{\mathfrak{E}} P(c_0, \dots, c_n) = \|\mathcal{P}(c_0, \dots, c_n)\|_{\mathfrak{E}}$.
- The valuation is extended to every wff as before.

Of the following squeeze lemmas for $R_{\mathfrak{E}}$, $S_{\square\mathfrak{E}}$, and $S_{\lozenge\mathfrak{E}}$, the former has a fairly standard proof. The proof of the latter two can be found in [30]. Here, the proofs are omitted.

LEMMA 7.37. If a and b are theories, c is a prime theory, and $R'abc$, then there are prime theories a' and b' extending a and b respectively such that $R_{\mathfrak{E}}a'b'c$.

LEMMA 7.38. If a is a prime theory, b is a theory, and $S_{\square\mathfrak{E}}ab$, and $\mathcal{A} \notin b$, then there is a prime theory b' extending b such that $S_{\square\mathfrak{E}}ab'$ and $\mathcal{A} \notin b'$.

LEMMA 7.39. If a is a prime theory, b is a theory, and $S_{\lozenge\mathfrak{E}}ab$, then there is a prime theory b' extending b such that $S_{\lozenge\mathfrak{E}}ab'$.

LEMMA 7.40. Conditions c1–c7 are satisfied by the canonical model.

The conditions c1–c7 can be shown to hold by the usual arguments, which will also demonstrate that $\langle K_{\mathfrak{E}}, 0_{\mathfrak{E}}, R_{\mathfrak{E}}, *_{\mathfrak{E}} \rangle$ is a \mathbf{B} -frame and that $a \leq b$ iff $a \subseteq b$.

LEMMA 7.41. Conditions c12a and c12b are satisfied by the canonical model. That is, if $a \leq b$, $S_{\square\mathfrak{E}}bc$, and $S_{\lozenge\mathfrak{E}}ad$, then $S_{\square\mathfrak{E}}ac$ and $S_{\lozenge\mathfrak{E}}bd$.

Proof. Assume that $a \leq b$, which is $a \subseteq b$. Further, let $S_{\square\mathfrak{E}}bc$. From the latter we have that $\{\mathcal{A} : \square\mathcal{A} \in b\} \subseteq c$, but also that if $\square\mathcal{A} \in a$, then $\square\mathcal{A} \in b$. Thus, $S_{\square\mathfrak{E}}ac$. For the other condition, suppose in addition that $S_{\lozenge\mathfrak{E}}ad$. Then $\{\lozenge\mathcal{A} : \mathcal{A} \in d\} \subseteq a \subseteq b$. Thus $S_{\lozenge\mathfrak{E}}bd$. □

LEMMA 7.42. Prop is closed under $\cup, \cap, \Rightarrow, *, \cdot$. That is, the canonical model satisfies c8.

Proof. It is sufficient to show that the following equalities hold.

$$\begin{aligned} \|\neg\mathcal{A}^f\|_{\mathfrak{E}} &= \|\mathcal{A}^f\|_{\mathfrak{E}}^* \\ \|\mathcal{A} \vee \mathcal{B}^f\|_{\mathfrak{E}} &= \|\mathcal{A}^f\|_{\mathfrak{E}} \cup \|\mathcal{B}^f\|_{\mathfrak{E}} \\ \|\mathcal{A} \wedge \mathcal{B}^f\|_{\mathfrak{E}} &= \|\mathcal{A}^f\|_{\mathfrak{E}} \cap \|\mathcal{B}^f\|_{\mathfrak{E}} \\ \|\mathcal{A} \circ \mathcal{B}^f\|_{\mathfrak{E}} &= \|\mathcal{A}^f\|_{\mathfrak{E}} \cdot \|\mathcal{B}^f\|_{\mathfrak{E}} \\ \|\mathcal{A} \rightarrow \mathcal{B}^f\|_{\mathfrak{E}} &= \|\mathcal{A}^f\|_{\mathfrak{E}} \Rightarrow \|\mathcal{B}^f\|_{\mathfrak{E}}. \end{aligned}$$

The proof of these equalities is straightforward by the standard arguments Routley and Meyer. □

LEMMA 7.43. Prop is closed under \square and \lozenge . That is, the canonical model satisfies c13.

Proof. For \Box , it is sufficient to show that if $\|\mathcal{A}\|_e \in Prop$, then so is $\Box\|\mathcal{A}\|_e$. It is enough here to show that $\Box\|\mathcal{A}\|_e = \|\Box\mathcal{A}\|_e$. For every f ,

- (1) $\quad \Box\|\mathcal{A}^f\|_e = \{a : \forall b(S_{\Box e}ab \Rightarrow b \in \|\mathcal{A}^f\|_e)\}$
- (2) $\quad \|\Box\mathcal{A}^f\|_e = \{a : \Box\mathcal{A}^f \in a\}$

We show that the left-hand of each equation is equal by showing the right-hand of the equations are equal. The direction from (2) to (1) is fairly trivial, given the definition of $S_{\Box e}$. For the other direction, assume that $c \in \{a : \forall b(S_{\Box e}ab \Rightarrow b \in \|\mathcal{A}^f\|_e)\}$. For reductio, further assume that $\Box\mathcal{A}^f \notin c$. Let's construct the theory d by defining it as $\{\mathcal{A} : \Box\mathcal{A} \in c\}$. Clearly $S_{\Box e}cd$. By the squeeze lemma, there is a prime theory d' extending d such that both $S_{\Box e}cd'$ and d' does not contain \mathcal{A}^f . However, by our first assumption, d' does contain \mathcal{A}^f , giving us a contradiction, so $\Box\mathcal{A}^f \in c$.

For \Diamond , I am required to show that $\Diamond\|\mathcal{A}^f\|_e = \|\Diamond(\mathcal{A}^f)\|_e$. We first note the following equalities given by the definition of the canonical frame and the \Diamond operator.

$$\begin{aligned} \Diamond\|\mathcal{A}^f\|_e &= \{a : \exists b(S_{\Diamond e}ab \ \& \ b \in \|\mathcal{A}^f\|_e)\}, \\ \|\Diamond(\mathcal{A}^f)\|_e &= \{a : \Diamond\mathcal{A}^f \in a\}. \end{aligned}$$

For one direction, let $c \in \{a : \exists b(S_{\Diamond e}ab \ \& \ b \in \|\mathcal{A}^f\|_e)\}$. Then $\exists b(S_{\Diamond e}cb \ \& \ b \in \|\mathcal{A}^f\|_e)$. It follows that $\{\Diamond\mathcal{A} : \mathcal{A} \in b\} \subseteq c$. Thus, $\Diamond\mathcal{A}^f \in c$, as required.

For the other direction, let $c \in \{a : \Diamond\mathcal{A}^f \in a\}$. Thus, $\Diamond\mathcal{A}^f \in c$. We are required to show that there is a b such that $S_{\Diamond e}cb$ and $\mathcal{A}^f \in b$.

Let $e = \{\mathcal{B} \mid \vdash \mathcal{A}^f \rightarrow \mathcal{B}\}$, where \mathcal{A}^f is the formula in question. It follows that e is a theory. For adjunction, assume that $\mathcal{B}, \mathcal{C} \in e$. Then $\vdash \mathcal{A}^f \rightarrow \mathcal{B}$ and $\vdash \mathcal{A}^f \rightarrow \mathcal{C}$, and so $\vdash \mathcal{A}^f \rightarrow (\mathcal{B} \wedge \mathcal{C})$. From this last fact, $(\mathcal{B} \wedge \mathcal{C}) \in e$. For closure under conditional theorems, suppose that $\mathcal{B} \in e$ and $\vdash \mathcal{B} \rightarrow \mathcal{C}$. Then $\vdash \mathcal{A}^f \rightarrow \mathcal{B}$, and so $\vdash \mathcal{A}^f \rightarrow \mathcal{C}$, as required.

Further, $S_{\Diamond e}e$, which is straightforward from the definition of S_{\Diamond} and the fact that, using $\Diamond\mathcal{A}^f \in e$ and $(\Diamond\text{-M})$, we have that every $\mathcal{B} \in e$ is such that $\Diamond\mathcal{B} \in e$. Using the appropriate squeeze lemma, we obtain e' , an prime extension of e , such that $S_{\Diamond e}ce'$. Moreover, $\mathcal{A}^f \in e'$ due to $\vdash \mathcal{A}^f \rightarrow \mathcal{A}^f$. □

LEMMA 7.44. *The constant function ϕ_0 is in PropFun, and PropFun is closed under $\cup, \cap, \Rightarrow, *, \cdot, \Box$, and \Diamond . That is, the canonical model satisfies c9, c10, and c14.*

Proof. The arguments given by Mares and Goldblatt [20], Lemma 9.2 can be used for every case except the \Box and \Diamond cases. For \Box , we have to show that $\Box\phi_{\mathcal{A}} = \phi_{\Box\mathcal{A}}$. For every f ,

$$\begin{aligned} \phi_{\Box\mathcal{A}^f} &= \|\Box\mathcal{A}^f\|_e && \text{by definition of } \phi_{\mathcal{B}} \\ &= \Box\|\mathcal{A}^f\|_e && \text{lem 7.43} \\ &= \Box\phi_{\mathcal{A}^f} && \text{by definition of } \phi_{\mathcal{B}} \end{aligned}$$

For \Diamond , the argument is similar and establishes the more general equality $\Diamond\phi_{\mathcal{A}} = \phi_{\Diamond\mathcal{A}}$. For every f ,

$$\begin{aligned} \phi_{\Diamond\mathcal{A}^f} &= \|\Diamond\mathcal{A}^f\|_e && \text{By definition of } \phi_{\mathcal{B}} \\ &= \Diamond\|\mathcal{A}^f\|_e && \text{lem 7.43} \\ &= \Diamond\phi_{\mathcal{A}^f} && \text{by definition of } \phi_{\mathcal{B}} \end{aligned} \quad \square$$

LEMMA 7.45. *PropFun is closed under \exists_n and \forall_n . That is, the canonical model satisfies c11.*

To show that PropFun is closed under \forall_n , the following sub-lemma proved by Mares and Goldblatt is convenient.

LEMMA 7.46. *If $\forall x \mathcal{A}$ is a sentence, then, for every prime theory a , $\forall x \mathcal{A} \in a$ iff there is an $X \in Prop$ such that, for ever constant c , $a \in X$ and $X \subseteq \|\mathcal{A}[c/x]\|_{\mathfrak{C}}$. That is,*

$$\|\forall x \mathcal{A}\|_{\mathfrak{C}} = \sqcap_{c \in con} \|\mathcal{A}[c/x]\|_{\mathfrak{C}}.$$

The proof of this sub-lemma is in [20, Lemma 9.3], and as such is omitted. The remaining detail of Lemma 7.45 is included here, which includes the introduction of some notation.

Here we will show that, for every $n \in \omega$, for any \mathcal{A} , $\forall_n \phi_{\mathcal{A}} = \phi_{\forall_{x_n} \mathcal{A}}$. Mares and Goldblatt introduce another notation for this proof. We will write $\mathcal{A}^{f \setminus n}$ for the formula

$$\mathcal{A}[f0/x_0, \dots, f(n-1)/x_{n-1}, x_n/x_n, f(n+1)/x_{n+1}, \dots].$$

This formula is the result of applying the substitution determined by f with the exception of x_n . Given this, it follows that $\mathcal{A}^{f \setminus n}[c/x_n] = \mathcal{A}^f c/n$. It can also be seen that $\forall_{x_n}(\mathcal{A}^{f \setminus n}) = (\forall_{x_n} \mathcal{A})^f$. Given these two facts, the following derivation from Mares and Goldblatt is possible.

$$\begin{aligned} (\forall_n \phi_{\mathcal{A}})f &= \phi_{\mathcal{A}}(f[c/n]) && \text{by the definition of } \forall_n \\ &= \sqcap_{c \in U} \|\mathcal{A}^{f[c/n]}\|_{\mathfrak{C}} && \text{by definition of } \phi_{\mathcal{A}} \\ &= \sqcap_{c \in U} \|\mathcal{A}^{f \setminus n}[c/x_n]\|_{\mathfrak{C}} && \text{by an equality just established} \\ &= \|\forall_{x_n}(\mathcal{A}^{f \setminus n})\|_{\mathfrak{C}} && \text{by our sub-lemma} \\ &= \|(\forall_{x_n} \mathcal{A})^f\|_{\mathfrak{C}} && \text{by an equality just established} \\ &= \phi_{\forall_{x_n} \mathcal{A}} f && \text{by definition of } \phi_{\forall_{x_n} \mathcal{A}} \end{aligned}$$

Thus PropFun is closed under \forall_n for every $n \in \omega$.

A similar proof can be given for the definable \exists_n . Thus the proof of Lemma 7.45 is completed.

LEMMA 7.47. *In the canonical model for **BQ.C** $_{\square \diamond}$, condition*

$$c15 \ X - Y \subseteq \sqcap_{a \in U} \phi(f[a/n]) \text{ implies } X - Y \subseteq (\forall_n \phi)f$$

is satisfied.

The proof is omitted as it is by the arguments of Mares and Goldblatt [20], which are applicable for **BQ.C** $_{\square \diamond}$.

To show that this canonical model is in fact a model for **QB.C** $_{\square \diamond}$ (**BQ.C** $_{\square \diamond}$), the next lemma suffices.

LEMMA 7.48. *For every n -ary predicate symbol P , every variable assignment, and every set of terms τ_1, \dots, τ_n ,*

1. $P(\tau_1, \dots, \tau_n)^f = P(|\tau_1|f, \dots, |\tau_n|f)$
2. $|P(\tau_1, \dots, \tau_n)|_{\mathfrak{C}} = \phi_{P(\tau_1, \dots, \tau_n)}$

Mares and Goldblatt’s arguments may be used here.

COROLLARY 7.49. *The canonical model for $\mathbf{QB.C}_{\square\Diamond}$ ($\mathbf{BQ.C}_{\square\Diamond}$) is a $\mathbf{QB.C}_{\square\Diamond}$ -($\mathbf{BQ.C}_{\square\Diamond}$ -) model.*

Our final step before completeness is again a truth lemma.

LEMMA 7.50 (Truth Lemma for $\mathbf{QB.C}_{\square\Diamond}$ ($\mathbf{BQ.C}_{\square\Diamond}$)). *For any formula \mathcal{A} , $\mathcal{A} = \phi_{\mathcal{A}}$. That is, for all f , $|\mathcal{A}|_{\mathcal{E}f} = ||\mathcal{A}^f||_{\mathcal{E}}$. In other words, $a, f \models \mathcal{A}$ iff $\mathcal{A}^f \in a$.*

Proof. The proof is by induction on the complexity of \mathcal{A} . Again, the arguments for the cases covered by Mares and Goldblatt can be verified to work in $\mathbf{QB.C}_{\square\Diamond}$ ($\mathbf{BQ.C}_{\square\Diamond}$). The new cases are those of \square and \Diamond . For \square we have the following:

The following argument is enough for the \Diamond case.

$ \mathcal{A} = \Diamond\mathcal{B} $	Case Hyp
$= \Diamond \mathcal{B} $	Definition of $ -$
$= \Diamond\phi_{\mathcal{B}}$	Inductive Hypothesis
$= \phi_{\Diamond\mathcal{B}}$	lem 7.44
$= \phi_{\mathcal{A}}$	Case Hyp

Similarly, and condensed, the case for \square is as follows. Let $\mathcal{A} = \square\mathcal{B}$, and assume the inductive hypothesis. That is, $|\mathcal{B}| = \phi_{\mathcal{B}}$. It follows that $|\mathcal{A}| = \square|\mathcal{B}| = \square\phi_{\mathcal{B}} = \phi_{\square\mathcal{B}}$. □

THEOREM 7.51 (Completeness for $\mathbf{QB.C}_{\square\Diamond}$ ($\mathbf{BQ.C}_{\square\Diamond}$)). *If \mathcal{A} is valid in every $\mathbf{QB.C}_{\square\Diamond}$ -model ($\mathbf{BQ.C}_{\square\Diamond}$ -model), then \mathcal{A} is a theorem of $\mathbf{QB.C}_{\square\Diamond}$ ($\mathbf{BQ.C}_{\square\Diamond}$).*

Proof. Let \mathcal{A} be valid in every $\mathbf{QB.C}_{\square\Diamond}$ -model including the canonical model. (The same argument may be used for $\mathbf{BQ.C}_{\square\Diamond}$.) It follows that every regular prime theory includes \mathcal{A}^f for every f . For every free variable in \mathcal{A} , replace it with a different constant not in \mathcal{A} . This new formula belongs to every regular prime theory, and is therefore a $\mathbf{QB.C}_{\square\Diamond}$ theorem. Repeated but finite applications of UG(Con) followed by repeated by finite applications of the axiom $\forall x\mathcal{A} \rightarrow \mathcal{A}[t/x]$, with τ free for x in \mathcal{A} , will produce a proof of \mathcal{A} . □

The semantics of Mares and Goldblatt for \mathbf{QR} and \mathbf{RQ} improves on Fine’s more complicated semantics, and provides a natural way of interpreting the quantifiers.⁶ Further, the semantics of Goldblatt and Mares [14] for quantified modal (classical) logics, in which the universal quantifier is treated similarly, provides completeness results for a wide range of quantified modal logics. The canonical models don’t require ω -complete theories, and the usual incompleteness (which results from conflicting requirements in the truth lemma for \square and $\forall x$) is bypassed. This kind of semantics for quantified modal logics is powerful, as seen in here, and in [11, 14]. However, despite how natural and useful the semantics are, it is an open question how it can be used to model quantified relevant logics with non-dual quantifiers. In particular, there is work to be done developing an constructivist or intuitionistic approach to both quantifiers and modalities in relevant logics.

⁶ Logan [16] gives a more approachable presentation of Fine’s stratified semantics for quantified relevant logics. Logan addresses the complaints that the stratified semantics is too complex (e.g., see Section 2.1 of [4]) and difficult to interpret philosophically (e.g., see [24], p. 5).

Table 2. Frame conditions for modal axioms

	Axiom	Frame condition
<i>T</i>	$\Box A \rightarrow A$	$\forall a(S_{\Box aa})$
<i>T</i> \diamond	$A \rightarrow \diamond A$	$\forall a(S_{\diamond aa})$
<i>D</i>	$\Box A \rightarrow \diamond A$	$\forall a \exists b(S_{\Box ab} \ \& \ S_{\diamond ab})$
<i>B</i>	$A \rightarrow \Box \Box A$	$\forall a, b(S_{\Box ab} \Rightarrow S_{\Box ba})$
<i>B</i> \diamond	$\diamond \Box A \rightarrow A$	$\forall a, b(S_{\diamond ab} \Rightarrow S_{\Box ba})$
<i>4</i>	$\Box A \rightarrow \Box \Box A$	$S_{\Box^2 ab} \Rightarrow S_{\Box ab}$
<i>4</i> \diamond	$\diamond \diamond A \rightarrow \diamond A$	$S_{\diamond^2 ab} \Rightarrow S_{\diamond ab}$
<i>5</i>	$\diamond A \rightarrow \Box \diamond A$	$\forall a, b, c(S_{\diamond ab} \ \& \ S_{\Box ac} \Rightarrow S_{\diamond cb})$
<i>5</i> \diamond	$\diamond \Box A \rightarrow \Box A$	$\forall a, b, c(S_{\diamond ab} \ \& \ S_{\Box ac} \Rightarrow S_{\Box bc})$
<i>K</i> \Box	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	If <i>Rbcf</i> and $S_{\Box fd}$, there exist $b', c' \in K$ such that $S_{\Box bb'}$, $S_{\Box cc'}$, and $Rb'c'd$.
<i>K</i> \diamond	$\Box(A \rightarrow B) \rightarrow (\diamond A \rightarrow \diamond B)$	If <i>Rbcd</i> and $S_{\diamond cf}$, then there exists $b', d' \in K$ such that $Rb'fd'$, $S_{\Box bb'}$, and $S_{\diamond dd'}$.
<i>SC</i>	$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$	If $S_{\Box ab}$, <i>Rbcd</i> , $S_{\Box ab'}$, and $Rb'c'd'$, then $S_{\Box cd'}$ or $S_{\Box c'd}$.
<i>CON</i>	$\Box((A \wedge \Box A) \rightarrow B) \vee \Box((B \wedge \Box B) \rightarrow A)$	If $S_{\Box ab}$, <i>Rbcd</i> , $S_{\Box ab'}$, and $Rb'c'd'$, then $c \leq d'$ or $c' \leq d$ or $S_{\Box cd'}$ or $S_{\Box c'd}$.
<i>Alt</i> _{<i>n</i>}	$\Box A_1 \vee \Box(A_1 \rightarrow A_2) \vee \dots \vee \Box(A_1 \wedge \dots \wedge A_n \rightarrow A_{n+1})$	If $S_{\Box ad_0}$, $\bigwedge_{i=1}^n (S_{\Box ab'_i} \ \& \ Rb_i c_i d_i)$ and $d_n \neq u$ then $\bigvee_{j=0}^{n-1} \bigvee_{i=j+1}^n c_j \leq d_i$.
	Rule	Frame condition
<i>NEC</i>	$\frac{A}{\Box A}$	If $b \in 0$ and $S_{\Box bc}$, then $c \in 0$.

§8. Extensions of QB.C $\Box\diamond$ and BQ.C $\Box\diamond$. The extensions of **QB.C $\Box\diamond$** and **BQ.C $\Box\diamond$** I consider here extend either the modal or implicational fragments of the logics.

Finally, we will show that the canonical model corresponding to various extensions of **QB.C $\Box\diamond$** and **BQ.C $\Box\diamond$** satisfies the conditions corresponding to the additional axioms or rules. For the axioms of the relevant fragment, the proof is standard. Thus, we turn to the modal axioms. In Section 4.7, the Barcan formula is briefly given some attention.

Let \mathbb{L} be a quantified relevant logic and \mathbb{A} be a set of axioms and rules from Table 2:

1. the logic $\mathbb{L}.C$ is the regular modal logic over \mathbb{L} and
2. the logic $\mathbb{L}.\mathbb{A}$ is the quantified modal relevant logic that results from adding the axioms and rules of \mathbb{A} to \mathbb{L} .

We say that \mathbb{A} *corresponds with duality* to a named modal classical logic \mathbb{M} when both (1) if $\mathcal{A} \in \mathbb{A}$, then the dual of \mathcal{A} is also in \mathbb{A} , and (2) \mathbb{M} can be defined by extending classical propositional logic with the set \mathbb{A} . In this case, we have

3. the logic $\mathbb{L.M}$ is the quantified modal relevant logic that results from adding the axioms and rules of \mathbb{A} , where \mathbb{A} corresponds with duality to \mathbb{M} .

For example, **KT** can be defined using necessitation, plus the K , T , K_\diamond and T_\diamond axioms. It follows that $\mathbb{L.KT}$ is defined by adding necessitation, K , T , K_\diamond and T_\diamond to \mathbb{L} . The logics denoted by \mathbb{LM} (with no dot) will be defined in Section 9

This notation for the names of logics is not particularly pretty, especially when considering each has a corresponding name without the dot, but it take advantage of the known names for relevant logics and modal logics to give a somewhat intuitive grasp of the axiomatization. Moreover, because the dot/dotless notation is in keeping with the literature on modal relevant logics, the author has chosen to keep this notation (and make the convention more precise) rather than introduce a new notation in order to reduce alternative notations.

For any logic \mathbb{L} that extends **QB.C \square_\diamond** (or **BQ.C \square_\diamond**) with a set of axioms (with out without the necessitation rule) from Table 2, we define \mathbb{L} -models to be the **QB.C \square_\diamond** -models (or **BQ.C \square_\diamond** -models) that satisfy the corresponding conditions from Table 2. For these logics, the set of which we will denote by \mathfrak{B} , we record soundness and completeness.

LEMMA 8.52. *A formula \mathcal{A} is valid in every \mathbb{L} -model if and only if \mathcal{A} is a theorem of \mathbb{L} , for every $\mathbb{L} \in \mathfrak{B}$.*

Proofs are as usual, and as in [30].

Given this list of correspondences, I will now isolate a couple of interesting extensions of **B.C \square_\diamond** . As described by Seki [30], p. 408, the logic **R.K** is obtained from **B.C \square_\diamond** by deleting \mathbf{t} and \diamond (and using \diamond as a replacement) from the logic's signature, removing the axioms \mathbf{t} and $\diamond(\mathcal{A} \vee \mathcal{B}) \rightarrow (\diamond\mathcal{A} \vee \diamond\mathcal{B})$, and the rule \diamond -monotonicity, extending the relevant logic fragment from **B** to **R**, and adding the axiom ($\mathbf{K}\square$) and the rule of necessitation. The logic **RK** results from adding the axiom $(\diamond\mathcal{A} \wedge \square\mathcal{B}) \rightarrow \diamond(\mathcal{A} \wedge \mathcal{B})$ to **R.K**. Finally, **R4** is obtained by adding to **RK** the axioms $\square\mathcal{A} \rightarrow \mathcal{A}$ and $\square\mathcal{A} \rightarrow \square\square\mathcal{A}$.

There are two interesting types of axioms to consider. The first are axioms with both of a non-dual pair \square and \diamond . Some sentences of this kind are highlighted in the next section. The other kind of axiom of interest are axioms in which both quantifiers and modalities appear, such as the Barcan Formulas. The next after next will deal with these.

§9. Dropping the Dot. The dot/dotless notation for a logics name is used in the modal relevant logic literature, but it is not entirely precise. Sometimes, the dot is removed when the modal fragment is sufficiently similar to the modal fragment of the corresponding modal classical logic (as in, for example, in [29, 30]). This convention is not entirely standard. Mares and Tanaka [23] use the name **R.K $^-$** , with the minus symbol indicating the lack of the axiom $\mathbf{K}\square$, for a logic that here we would denote without the dot. It have also been common to use the name **NR** for our **R.4** without primitive \diamond . Here, however, my aim is to both make rigorous the dotting convention as used consistently with Seki, and to motivate interest in the distinction picked out by the convention. My hope is that this interest will standardize once and for all the

dotting convention. As mentioned above, the dot is removed from a logic's name when it is sufficiently classical. We are now in a position to explain what is meant by this, which is roughly that the diamond and box to behave sufficiently like they do in modal classical logic.

Relevant logicians have been interested in when a relevant modal logic contains the theorems of a modal logic whose base is classical logic. We shall often call modal logics with a classical base *modal classical logics* (cf. modal relevant logics), which are not to be confused with the established term 'classical modal logics.'⁷ That is, when the theorems of a modal logic with a classical base, written in \sim , \wedge , and \Box , are theorems of the modal relevant logic. Mares and Meyer have been at the forefront of research in this area. In [18], it is proved that a number of modal relevant logics are conservative extensions of modal logics. Similar results, including conservative extension by Boolean negation, can be found in [21–23]. There is therefore an interest the 'classicality' of the modal fragments of relevant logics.

The dot notation identifies a certain degree of classicality in the modal fragment of relevant logics, and it is this degree which is necessary, though not sufficient, to ensure that a relevant modal logic is a conservative extension of its classical counterpart. As this fragment includes negation, weaker logics such as **B** have no hope of containing the theorems of a modal classical logic. On the other hand, the axioms below required for 'dropping to dot' are exactly the postulates Dunn [6] identifies as required for positive modal logics (without implication). Without these postulates, a positive modal logic would lack theorems (in \wedge , \vee , \Box , \Diamond) that are theorems of the corresponding modal classical logic. Therefore, for weaker relevant logics there are interesting results for be found in the positive modal fragments, and it is only fitting that the frame conditions given below are suitable for positive fragments of the logics considered. So let us make rigorous the dotting convention.

To warrant the removal of the dot from a logic's name, we must add $\Box(\mathcal{A} \vee \mathcal{B}) \rightarrow (\Box\mathcal{A} \vee \Diamond\mathcal{B})$ or the dual $(\Diamond\mathcal{A} \wedge \Box\mathcal{B}) \rightarrow \Diamond(\mathcal{A} \wedge \mathcal{B})$, and we must have the duality of the modalities \Box and \Diamond . For a (classical) regular modal logic to be contained in a corresponding relevant modal logic, all the theorems must be included under translation. The duality of the quantifiers is a theorem of classical regular modal logics. Moreover, $(\Diamond\mathcal{A} \wedge \Box\mathcal{B}) \supset \Diamond(\mathcal{A} \wedge \mathcal{B})$ is a theorem of all classical regular modal logics (as in [5], Theorem 8.12). Thus, only the dotless logics can contain their classical counterparts.

Here we establish frame conditions for these axioms.

LEMMA 9.53. *The axiom scheme $\Box(\mathcal{A} \vee \mathcal{B}) \rightarrow (\Box\mathcal{A} \vee \Diamond\mathcal{B})$ is valid in all $\mathbf{QB.C}_{\Box\Diamond}$ -models that satisfy the following condition:*

$$(c16) \text{ if } S_{\Box}ab, \text{ then } \exists x \leq b(S_{\Box}ax \ \& \ S_{\Diamond}ax)$$

Proof. Suppose that $a, f \models \Box(\mathcal{A} \vee \mathcal{B})$. For reductio, let $a, f \not\models \Box\mathcal{A}$ and $a, f \not\models \Diamond\mathcal{B}$. From the former we get that $S_{\Box}ab$ and $b, f \not\models \mathcal{A}$. From the condition above, $S_{\Box}ac$ and $S_{\Diamond}ac$ for $c \leq b$.

So in particular, it follows that $c, f \models \mathcal{A} \vee \mathcal{B}$. If $c, f \models \mathcal{B}$, then we get a contradiction, as it entails $a, f \models \Diamond\mathcal{B}$. On the other hand, if $c, f \models \mathcal{A}$, then $b, f \models \mathcal{A}$, which gives us

⁷ The term 'classical modal logic,' introduced by Segerberg [28], refers to modal logics (generally with a classical base) that have both the duality of the quantifiers and are closed under the rule of replacement of equivalents: from $\vdash \mathcal{A} \leftrightarrow \mathcal{B}$ infer $\vdash \Box\mathcal{A} \leftrightarrow \Box\mathcal{B}$.

another contradiction. Thus the reductio assumption is wrong and $a, f \models \Box A \vee \Diamond B$. The result follows by Semantic Entailment. \square

LEMMA 9.54. *The axiom scheme $(\Diamond A \wedge \Box B) \rightarrow \Diamond(A \wedge B)$ is valid in all **QB.C** $_{\Box\Diamond}$ -models that satisfy the following condition:*

$$(c17) \text{ if } S_{\Diamond}ab, \text{ then } \exists x \leq b(S_{\Box}ax \ \& \ S_{\Diamond}ax)$$

Proof. Let $a, f \models \Diamond A \wedge \Box B$. From this we get $S_{\Diamond}ab$ and $b, f \models A$. By our condition, there is a $c \leq b$ such that $S_{\Box}ac$ and $S_{\Diamond}ac$. Thus, $c, f \models B$, which entails $b, f \models B$. From here, we know that $b, f \models A \wedge B$, and so $a, f \models \Diamond(A \wedge B)$, as required. The result follows by Semantic Entailment. \square

The condition from [22]—if $S_{\Diamond}ab$, then $\exists x \leq b(Tax)$, with $Tab =_{def} Sab \ \& \ Sa^*b^*$ —is used to validate $\Box(A \vee B) \rightarrow (\Box A \vee \Diamond B)$ in a modal relevant logic in which the box is primitive, and the diamond is a defined dual. The T relation here is used in particular because of the essential role of negation in defining the diamond. The proof requires looking at star worlds. However, the condition I give above can be used in both positive fragments and in logics without the duality of the modalities. The conditions above are essentially the Mares–Meyer condition restricted to the relations for two possibly non-dual modalities. The conditions above becomes the Mares–Meyer condition when $S_{\Diamond}ab$ iff $S_{\Box}a^*b^*$. Moreover, the duality of the modalities collapses the corresponding relations, as recorded below.

LEMMA 9.55. *If the modalities \Box and \Diamond are dual, then $S_{\Diamond}ab$ iff $S_{\Box}a^*b^*$.*

Proof. From the duality of \Box and \Diamond , and the duality of \Box and \Diamond , it follows that $\Diamond A$ iff $\neg\Box\neg A$ iff $\Diamond A$. Thus, \Diamond and \Diamond must be modeled by the same binary relation. Given this and the fact that $S_{\Diamond}ab$ iff $S_{\Box}a^*b^*$, we get that $S_{\Diamond}ab$ iff $S_{\Box}a^*b^*$. \square

THEOREM 9.56. *Let \mathbb{L} be a logic extending **QB.C** $_{\Box\Diamond}$ or **BQ.C** $_{\Box\Diamond}$ with any set of axioms and rules from Tables 1 and 2, and let the class of \mathbb{L} -models as defined above. The logic $\mathbb{L} + (\Diamond A \wedge \Box B) \rightarrow \Diamond(A \wedge B)$ is sound and complete w.r.t the class of all \mathbb{L} -models satisfying (c16). Further, the logic $\mathbb{L} + \Box(A \vee B) \rightarrow (\Box A \vee \Diamond B)$ is sound and complete w.r.t the class of all \mathbb{L} -models satisfying (c17).*

Proof. Lemmas 9.53 and 9.54 demonstrate the soundness claim. For completeness, we define the canonical modal as earlier, but for the appropriate logic. The completeness proofs, of which we will only show that for the axiom $\Box(A \vee B) \rightarrow (\Box A \vee \Diamond B)$, are essentially the proof given by Mares and Meyer [22, Lemma 4.5], with obvious modifications to handle the star-less condition given above.

First, assume that $S_{\Box}ab$. That is, if $\Box A \in a$, then $A \in b$. We show that there is a prime theory $x \subseteq b$ such that $S_{\Box}ax$ and $S_{\Diamond}ax$. Let c be the set of formulas A such that $\Diamond A \notin a$, and let a' be the set of formulas A such that $\Box A \in a$. Thus, $a' \subseteq b$, and is a theory. Further suppose that d is the set of formulas not in b . We have here that the pair $(a', c \cup d)$ is independent by the following:

Suppose for reductio that the pair is not independent. Then there are $\mathcal{A}_1, \dots, \mathcal{A}_n \in a'$, $\mathcal{B}_1, \dots, \mathcal{B}_m \in d$ and $\mathcal{D}_1, \dots, \mathcal{D}_l \in c$ such that

$$\vdash (\mathcal{A}_1 \wedge, \dots, \wedge \mathcal{A}_n) \rightarrow (\mathcal{B}_1 \vee, \dots, \vee \mathcal{B}_m \vee \mathcal{D}_1 \vee, \dots, \vee \mathcal{D}_l)$$

from which we can infer

$$\vdash \Box(\mathcal{A}_1 \wedge, \dots, \wedge \mathcal{A}_n) \rightarrow \Box(\mathcal{B}_1 \vee, \dots, \vee \mathcal{B}_m \vee \mathcal{D}_1 \vee, \dots, \vee \mathcal{D}_l).$$

Then, given the axiom in question, we can derive

$$\vdash \Box(\mathcal{A}_1 \wedge, \dots, \wedge \mathcal{A}_n) \rightarrow \Box(\mathcal{B}_1 \vee, \dots, \vee \mathcal{B}_m) \vee \Diamond(\mathcal{D}_1 \vee, \dots, \vee \mathcal{D}_l).$$

The primeness of the element a further implies either the $\Box(\mathcal{B}_1 \vee, \dots, \vee \mathcal{B}_m) \in a$ or that $\Diamond(\mathcal{D}_1 \vee, \dots, \vee \mathcal{D}_l) \in a$. If the latter, then $\Diamond \mathcal{D}_i \in a$ for some $i \leq l$, which entails $\mathcal{D}_i \in c$, a contradiction. On the other hand, $\Box(\mathcal{B}_1 \vee, \dots, \vee \mathcal{B}_m) \in a$, then $\mathcal{B}_1 \vee, \dots, \vee \mathcal{B}_m \in b$, which also leads to contradiction. Thus, $(a', c \cup d)$ is independent. Using the extension lemma, There is a prime theory x extending a' such that the pair $(x, c \cup d)$ is independent. It is obvious that $S_{\Box}ax$ by our definition of a' . To show that $S_{\Diamond}ax$, suppose that it did not hold. Then there is a formula \mathcal{A} such that $\mathcal{A} \in x$ but $\Diamond \mathcal{A} \notin a$. Such formulas make up the set c , as defined above. All such formulas were excluded from x in the extension lemma. So $S_{\Diamond}ax$. □

This brings us to the title of the current section. In the naming scheme of the logics here, my convention is to drop the dot in the name of the logic when (1) the logic contains both the diamond and the box (with possibly one being defined), (2) the diamond and box are duals, and (3) both the formula schemes discussed in this section are theorems. The result is that logics whose name has dropped the dot have a certain degree of classicality.

From here, further research is motivated into exactly what degree of classicality this convention picks out. One line of investigation is to see which of these logics can be proven to admit γ — from $\neg \mathcal{A} \vee \mathcal{B}$ and \mathcal{A} to infer \mathcal{B} . Following the arguments of Mares [18], γ -admissibility may lead to conservative extension results over modal relevant logics. Moreover, for quantified modal logics, a natural starting point is to consider the quantified extensions of the logics in [18], and to determine whether they conservatively extend their corresponding quantified modal classical logics.

§10. Barcan formula. Here we briefly discuss the role of the Barcan formula in modal relevant logics. In the logics defined above, using UI, the Barcan formula is not derivable. However, as we say earlier, the Converse Barcan Formula is derivable in our basic logic **QB.C** $_{\Box\Diamond}$.

LEMMA 10.57. *The Barcan formula is not a theorem of **RQ4**^{ot}.*

Proof. All of the theorems of **RQ4**^{ot} are theorems of quantified S4 with UI. One cannot prove the Barcan Formula in quantified S4 with UI [11, p. 82]. Therefore, the Barcan Formula cannot be a theorem of **RQ4**^{ot}. □

In classical logics, the Barcan formula is related to models in which *Prop* and *PropFun* are the set of all subsets of worlds and set of all functions from variable assignments into *Prop*, respectively. These models are called *Kripkean* or *full* (due to *Prop* and *PropFun* being the largest sets they can be). Here, $\Box S = \cap S$. Consequently, the verification conditions for the quantifiers for *Kripkean* models collapse into the Tarskian verification conditions. For the universal quantifier, namely

$$a, f, \vDash \forall x A \text{ iff, for every } g \in xf, g, a \vDash A$$

since $\Box S = \cap S$ and $|\forall x \mathcal{A}|^M f = \bigcap_{g \in xf} |\mathcal{A}|^M g$.

In [14], a *Tarskian General Frame* is defined as a frame for *Prop* is closed under which arbitrary (infinite) intersection. In these frames, the Tarskian verification condition is

derivable. Moreover, they show that, for a class of frames \mathbb{C} of modal classical logics, a formula \mathcal{A} is valid in all standard constant domain models based on members of \mathbb{C} iff \mathcal{A} is valid in all Tarskian general frames based on members of \mathbb{C} iff \mathcal{A} is valid in all full general frames based on members of \mathbb{C} [20, Theorem 7]. They go on to show that models that are not Tarskian can validate the Barcan Formula, showing that “the Tarskian condition is sufficient to ensure validity of BF, but it is not necessary” [14, p. 14].

The question then remains as to what the role of the Barcan formula is in modal relevant logics.

LEMMA 10.58. *If \mathfrak{M} is a Kripkean/full model, then \mathfrak{M} validates BF.*

Proof. The proof is much like in [11, Lemma 2.2.7]. Let $a, f \models \forall x \Box \mathcal{A}$. Then suppose that $S_{\Box} ab$. The domains are constant, or at least universal, in the sense that there is a single domain for the entire model. So, if $a \in U$, then $a, f[a/x] \models \Box \mathcal{A}$, which implies that $b, f[a/x] \models \mathcal{A}$.

So we have that, for every $a \in U$, $b, f[a/x] \models \mathcal{A}$. From this, and the assumption that the model is Kripkean and therefore Tarskian, we get that $b, f \models \forall x \mathcal{A}$. As this holds for every b such that $S_{\Box} ab$, it follows that $a, f \models \Box \forall x \mathcal{A}$. The result follows by Semantic Entailment. \square

Note that, in [12], it is shown that Kripkean models also validate commuting quantifiers (CQ— $\forall x \forall y \mathcal{A} \rightarrow \forall y \forall x \mathcal{A}$) in modal classical logics. In the relevant logics we consider, we can prove that \vdash CQ using UI (and \forall -Intro) without the Barcan Formula [20, Lemma 6.3 (d)].

What is left to do is prove, of certain classes of models, that the relevant logics considered plus the Barcan Formula are characterized by these models. That is, give soundness and completeness results for whichever classes of models we are able. Here, we must be careful that the class of models we choose does not imply incompleteness for the quantified relevant fragment of the logic by, for example, becoming vulnerable to Fine’s incompleteness proof as in [9].

Although Goldblatt [11], Goldblatt and Mares [14], and Mares and Goldblatt [20] prove many relations between logics with the Barcan Formula (and CQ), the tightest characterization so far is not the most illuminating. That characterization, for relevant logics, is given in the following theorem. While related to Kripkean models with Tarskian conditions, the following fact about relevant logics with the Barcan formula is the furthest we will venture into the Barcan formula in this work.

THEOREM 10.59. *Where \mathbb{L} is a quantified modal relevant logic without BF, the logic $\mathbb{L} + BF$ is characterized by the class of all admissible function models that satisfy*

$$\forall x \Box \phi \subseteq \Box \forall x \phi,$$

and are based on a general \mathbb{L} -frame. (Where \Box in the condition above is the type-lifted operation on propositional functions.)

Proof. The proof is essentially unaltered from [11]. For soundness, we have the following:

$$\begin{aligned} |\forall x \Box \mathcal{A}|f &= \forall x \Box |\mathcal{A}|f \\ &\subseteq \Box \forall x |\mathcal{A}|f \\ &= |\Box \forall x \mathcal{A}|f. \end{aligned}$$

For completeness, the goal is to show that $\phi_{\forall x \Box \mathcal{A}} \subseteq \phi_{\Box \forall x \mathcal{A}}$ using the fact that $\vdash \forall x \Box \mathcal{A} \rightarrow \Box \forall x \mathcal{A}$. For this, it is sufficient to show that, if $\vdash \mathcal{A} \rightarrow \mathcal{B}$, then $\phi_{\mathcal{A}} \subseteq \phi_{\mathcal{B}}$. Take an arbitrary $a \in K$ and $f \in U^\omega$, and suppose that $a \in \phi_{\mathcal{A}} f$. By definition, this is $a \in \|\mathcal{A}^f\|_c$, and so $\mathcal{A}^f \in a$. But a is a theory, and so $\mathcal{B}^f \in a$. This gives us $a \in \phi_{\mathcal{B}}$, as required. \square

This theorem might not seem very insightful for the role the Barcan formula plays. Indeed, the condition appears as the axiom written in a semantic dialect. Similarly, we could offer the condition $\Box \phi \subseteq \phi$ for the axiom T , and give a similar proof. While we have a deeper understanding of the T axiom because of its corresponding condition of reflexivity, such an insight is hidden in the condition $\Box \phi \subseteq \phi$. As such, conditions written in a semantic dialect, such as for the Barcan formula above, often fall short of providing the philosophical explanations that we are after. The proof above is quite modular, and is applicable to any axiom of a conditional form, taking advantage of the fact that the axiom being provable implies the corresponding subset relation holds. Consequently, it is a powerful formal technique that appears to offer no philosophical insight into the nature of the Barcan formula.

Goldblatt proves a similar theorem [11, Theorem 4.5.4]. Goldblatt explains the importance of this and related theorems.

It might be thought that characterizations using the condition [defined above] do little to advance our understanding of the Barcan Formula, since this is essentially a translation of BF into structural/algebraic form. On the other hand, for [some logics considered], the results given here would appear to be the first characterizations of any kind that are based on possible-worlds style relational semantics. [11, p. 157]

Thus, proving this theorem will at least be a good starting point in exploring characterizations of relevant logics with the Barcan Formula.

The Barcan formula's dual, with \exists and \Diamond can be given a similar characterization. Both results follow from a more general fact about the relation of propositional functions and provable conditionals. What is desired, ideally, is a semantic condition with more explanatory power. The relation, for example, of the Barcan formulas to the contraction and expansion of domains provides some explanatory insight which is missing in the straightforward algebraic translation (perhaps transliteration?) of the axiom. This motivates the following lines of investigation for future work:

1. Find a semantic condition for the Barcan formula that provides more insight into the nature of the Barcan formula in relevant logics.
2. Develop systems for quantified modal relevant logics in which both the Barcan formula and its converse fail. For example, using the restricted universal elimination axiom $\forall y (\forall x \mathcal{A} \rightarrow \mathcal{A}[y/x])$ instead of the unrestricted $\forall x \mathcal{A} \rightarrow \mathcal{A}[\tau/x]$, with other axiomatic adjustments, can ensure that the Converse Barcan formula is not a theorem.

Goldblatt [11] provides an excellent starting point for pursuing the latter project, as a large portion of the book is dedicated to logics defined using the restricted universal elimination axiom.

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