Proceedings of the Edinburgh Mathematical Society (2020) **63**, 709–736
doi:10.1017/S0013091520000115

NEWMAN'S IDENTITIES, LUCAS SEQUENCES AND CONGRUENCES FOR CERTAIN PARTITION FUNCTIONS

ERNEST X.W. XIA

Department of Mathematics, Jiangsu University, Zhenjiang, Jiangsu, 212013, P. R. China (ernestxwxia@163.com)

(Received 15 August 2019)

Abstract Let r be an integer with $2 \leq r \leq 24$ and let $p_r(n)$ be defined by $\sum_{n=0}^{\infty} p_r(n)q^n = \prod_{k=1}^{\infty}(1-q^k)^r$. In this paper, we provide uniform methods for discovering infinite families of congruences and strange congruences for $p_r(n)$ by using some identities on $p_r(n)$ due to Newman. As applications, we establish many infinite families of congruences and strange congruences for certain partition functions, such as Andrews's smallest parts function, the coefficients of Ramanujan's ϕ function and p-regular partition functions. For example, we prove that for $n \geq 0$,

$$\mathrm{spt}\left(\frac{1991n(3n+1)}{2}+83\right) \equiv \mathrm{spt}\left(\frac{1991n(3n+5)}{2}+2074\right) \equiv 0 \pmod{11},$$

and for $k \geq 0$,

$$\mathrm{spt}\left(\frac{143 \times 5^{6k}+1}{24}\right) \equiv 2^{k+2} \pmod{11},$$

where $\mathrm{spt}(n)$ denotes Andrews's smallest parts function.

Keywords: congruences; partitions; Newman's identities; rank; Lucas sequences

2010 *Mathematics subject classication:* Primary 11P83
Secondary 05A17

1. Introduction

The aim of this paper is to present uniform methods to establish infinite families of congruences and strange congruences for $p_r(n)$ based on some identities due to Newman. Here r is an integer with $2 \leq r \leq 24$ and $p_r(n)$ is defined by

$$\sum_{n=0}^{\infty} p_r(n)q^n = (q;q)_\infty^r, \tag{1.1}$$

where throughout the rest of the paper we use the standard notation

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Recall that a partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. Owing to Ramanujan, it is well known that for $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Motivated by Ramanujan's work, the arithmetic properties of partitions with certain restrictions have received a great deal of attention. Recently, Cui *et al.* [3] established some congruence properties for a certain kind of partition function $a(n)$ which satisfies $\sum_{n=0}^{\infty} a(n)q^n \equiv (q;q)_\infty^k \pmod{m}$, where k is a positive integer with $1 \leq k \leq 24$ and $m \in \{2,\ 3\}$ in light of the modular equations of fifth and seventh order.

In this paper we present uniform methods for discovering congruence properties for $p_r(n)$ which is defined by (1.1). Our methods mainly rely on the ranks of (a,b)-Lucas sequences. For integers a and b, the (a,b)-Lucas sequence $S(n)$ is defined by

$$S(n) = aS(n-1) - bS(n-2) \tag{1.2}$$

with $S(0) = 0$ and $S(1) = 1$. Let $M \geq 2$ be an integer. The *rank* of (a,b)-Lucas sequence $S(n)$ modulo M is the least positive integer k such that $S(k) \equiv 0 \pmod{M}$, and we denote the rank of $S(n)$ modulo M by $R_S(M)$. For example, let $F(n)$ denote the $(1,-1)$-Lucas sequence, that is, the classic Fibonacci sequence. Since $F(0) = 0,\ F(1) = 1,\ F(2) = 1$, $F(3) = 2$ and $F(4) = 3$, it is easy to see that $R_F(2) = 3$ and $R_F(3) = 4$. We also define the *dual* (a,b)-Lucas sequence $T(n)$ of $S(n)$ as

$$T(n) = aT(n-1) - bT(n-2) \tag{1.3}$$

with initial conditions $T(0) = 1$ and $T(1) = 0$.

In order to state the main results of this paper, we introduce the Legendre symbol. Let $p \geq 3$ be a prime. The Legendre symbol $(a/p)_L$ is defined by

$$\left(\frac{a}{p}\right)_L := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ 0 & \text{if } p \mid a, \\ -1 & \text{if } a \text{ is a non} - \text{quadratic residue modulo } p. \end{cases}$$

Newman [10] established an identity (see (3.4)) for $p_r(n)$ where r is an odd integer with $3 \leq r \leq 23$. In light of (3.4), we will prove the following theorem which is employed to discover new infinite families of congruences for $p_r(n)$.

Theorem 1.1. *Let r be an odd integer with $3 \leq r \leq 23$ and let p be a prime with $p \geq 3$ if $3|r$ and $p \geq 5$ otherwise. Define*

$$\pi(p) := p_r\left(\frac{r(p^2-1)}{24}\right) + (-1)^{(p-1)(p-1-2r)/8} p^{(r-3)/2} \left(\frac{r(p^2-1)/24}{p}\right)_L. \tag{1.4}$$

Let $M \geq 2$ be an integer with $\gcd(M,p) = 1$ and let $G_p(n)$ be the $(\pi(p), p^{r-2})$-Lucas sequences.

(i) *For* $n,\ k \geq 0$, *if* $p \nmid n$, *then*

$$p_r\left(p^{2R_{G_p}(M)(k+1)-1}n + \frac{r(p^{2R_{G_p}(M)(k+1)}-1)}{24}\right) \equiv 0 \pmod{M}, \tag{1.5}$$

where $R_{G_p}(M)$ *are the ranks of* $G_p(n)$ *modulo* M. *Furthermore, for* $k \geq 0$,

$$p_r\left(\frac{r(p^{2R_{G_p}(M)k}-1)}{24}\right) \equiv H_p(R_{G_p}(M))^k \pmod{M}, \tag{1.6}$$

where $H_p(n)$ *are the dual* $(\pi(p), p^{r-2})$*-Lucas sequences of* $G_p(n)$.

(ii) *For* $n,\ k \geq 0$, *if*

$$\pi(p)\left(\frac{(r(p^2-1)/24)-n}{p}\right)_L \equiv (-1)^{(p-1)(p-1-2r)/8}p^{(r-3)/2} \pmod{M},$$

then

$$p_r\left(p^{2R_{G_p}(M)k+2}n + \frac{r(p^{2R_{G_p}(M)k+2}-1)}{24}\right) \equiv 0 \pmod{M}. \tag{1.7}$$

Remark. From Lemma 2.1 in § 2, we see that $R_{G_p}(M)$ exists since $\gcd(M,p)=1$.

Based on Newman's identity (3.4), we can deduce the following two theorems which are used to discover strange congruences for $p_r(n)$.

Theorem 1.2. *Suppose that* $M \geq 2$ *is an integer,* r *is an odd integer with* $3 \leq r \leq 23$ *and* a *is a non-negative integer such that* $p_r(a) \equiv 0 \pmod{M}$. *Suppose further that* $24a + r = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$, *with each* $\alpha_j \geq 2$, *is the prime factorization of* $24a+r$. *Then for* $n \geq 1$,

$$p_r\left(an^2 + \frac{r(n^2-1)}{24}\right) \equiv 0 \pmod{M}, \tag{1.8}$$

where $\gcd(n, 2\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$ *if* $3|r$ *and* $\gcd(n, 6\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$ *otherwise.*

Theorem 1.3. *Let* $M \geq 2$ *be an integer and define*

$$\mathfrak{S}_M := \{(i,j) | 0 \leq i \leq M-1,\ 1 \leq j \leq M-1 \text{ with } \gcd(M,j)=1\}. \tag{1.9}$$

For any pair $(i,j) \in \mathfrak{S}_M$, *let* $G_{(i,j)}(n)$ *be the* (i,j)*-Lucas sequences and let* $R_{G_{(i,j)}}(M)$ *denote the ranks of* $G_{(i,j)}(n)$ *modulo* M. *Let* $\gamma(M)$ *denote the lowest common multiple of the set* $\{R_{G_{(i,j)}}(M) | (i,j) \in \mathfrak{S}_M\}$. *If* $p_r(a) \equiv 0 \pmod{M}$, *then for* $n \geq 1$,

$$p_r\left(an^{2\gamma(M)} + \frac{r(n^{2\gamma(M)}-1)}{24}\right) \equiv 0 \pmod{M}, \tag{1.10}$$

where $\gcd(n, 2M) = 1$ *if* $3|r$ *and* $\gcd(n, 6M) = 1$ *otherwise.*

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Remark. From the definition of $\mathfrak{S}_M$, we see that $\mathfrak{S}_M$ is a finite set when $M \geq 2$ is an integer. Moreover, by Lemma 2.1, it is easy to see that $R_{(i,j)}(M)$ exists since $\gcd(M, j) = 1$. Therefore, $\gamma(M)$ exists according to the definition of $\gamma(M)$.

Newman [9] also discovered an identity (see (6.4)) on $p_r(n)$ where r is an even integer with $2 \leq r \leq 24$. In view of (6.4), we can prove the following theorem.

Theorem 1.4. *Let r be an even integer with $2 \leq r \leq 24$ and let p be a prime with $24|r(p-1)$. Let $M \geq 2$ be an integer with $\gcd(M, p) = 1$ and let $S_p(n)$ be the $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences. For $n,\ k \geq 0$, if $p \nmid (24n + r)$, then*

$$p_r\left(p^{R_{S_p}(M)(k+1)-1}n + \frac{r(p^{R_{S_p}(M)(k+1)-1} - 1)}{24}\right) \equiv 0 \pmod{M}, \tag{1.11}$$

where $R_{S_p}(M)$ are the ranks of $S_p(n)$ modulo M. Moreover, for $k \geq 0$,

$$p_r\left(\frac{r(p^{R_{S_p}(M)k} - 1)}{24}\right) \equiv T_p(R_{S_p}(M))^k \pmod{M}, \tag{1.12}$$

where $T_p(n)$ are the dual $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences of $S_p(n)$.

As applications, employing Theorems 1.1–1.4, we establish many new infinite families of congruences and strange congruences for certain partition functions, such as Andrews's smallest parts function, the coefficients of Ramanujan's ϕ function and p-regular partition functions. For example, we prove that for $n \geq 0$,

$$\mathrm{spt}\left(\frac{1991n(3n+1)}{2} + 83\right) \equiv \mathrm{spt}\left(\frac{1991n(3n+5)}{2} + 2074\right) \equiv 0 \pmod{11}$$

and for $k \geq 0$,

$$\mathrm{spt}\left(\frac{143 \times 5^{6k} + 1}{24}\right) \equiv 2^{k+2} \pmod{11},$$

where $\mathrm{spt}(n)$ denotes Andrews's smallest parts function.

This paper is organized as follows. In § 2 we present some properties for the (a, b)-Lucas sequences. In § 3–6, using some identities due to Newman [9, 10] and some lemmas proved in § 2, we prove Theorems 1.1–1.4, respectively. In § 7–9, we will apply Theorems 1.1–1.4 to establish new infinite families of congruences and strange congruences for the coefficients of Ramanujan's ϕ function, Andrews's smallest parts function $\mathrm{spt}(n)$ and p-regular partition functions, respectively.

2. Some properties for (a, b)-Lucas sequences

The aim of this section is to prove some lemmas on properties of the (a, b)-Lucas sequences $S(n)$ and their dual (a, b)-Lucas sequences $T(n)$. These lemmas will be used to prove Theorems 1.1, 1.3 and 1.4.

The following lemma states an upper bound for the ranks of the (a, b)-Lucas sequences $S(n)$.

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Lemma 2.1. *Let $M \geq 2$ be an integer and let $S(n)$ be defined by (1.2). Suppose that $M = \prod_{i=1}^{s} p_i^{k_i}$ $(k_i \geq 1)$ is the prime factorization of M and $\gcd(M, b) = 1$. Then $R_S(M)$ exists and*

$$R_S(M) \leq \prod_{i=1}^{s} p_i^{k_i-1}(p_i + 1).$$

Proof. Let m_1 and m_2 be positive integers. Renault [**11**] proved that

$$R_S([m_1, m_2]) = [R_S(m_1), R_S(m_2)],$$

where $[c, d]$ denotes the least common multiple of c and d. Thus,

$$R_S([m_1, m_2]) \leq R_S(m_1)R_S(m_2). \tag{2.1}$$

Renault [**11**] also proved that if p_i is a prime and k_i is a positive integer, then

$$R_S(p_i^{k_i+1}) = R_S(p_i^{k_i}) \text{ or } p_i R_S(p_i^{k_i}).$$

Thus,

$$R_S(p_i^{k_i}) \leq p_i^{k_i-1} R_S(p_i). \tag{2.2}$$

In light of (2.1) and (2.2),

$$R_S(M) = R_S\left(\prod_{i=1}^{s} p_i^{k_i}\right) \leq \prod_{i=1}^{s} R_S(p_i^{k_i}) \leq \prod_{i=1}^{s} p_i^{k_i-1} R_S(p_i). \tag{2.3}$$

In order to find an upper bound for $R_S(M)$, it suffices to determine $R_S(2)$ and $R_S(p)$, where p is an odd prime. It is easy to check that

$$R_S(2) = \begin{cases} 2 & \text{if } a \text{ is even}, \\ 3 & \text{if } a \text{ is odd}, \end{cases} \tag{2.4}$$

where $S(n)$ is defined by (1.2). Lucus [**8**] proved that if p is an odd prime with $\gcd(p, b) = 1$, then

$$R_S(p) \Big| \left(p - \left(\frac{a^2 - 4b}{p}\right)_L\right). \tag{2.5}$$

Based on (2.4) and (2.5), we deduce that for any prime p,

$$R_S(p) \leq p + 1. \tag{2.6}$$

It follows from (2.3) and (2.6) that Lemma 2.1 is true. The proof is complete. □

Lemma 2.2. *Let $S(n)$ and $T(n)$ be defined by (1.2) and (1.3), respectively. For $n,\ k \geq 0$,*

$$S(n + k) = S(k)S(n + 1) + T(k)S(n) \tag{2.7}$$

and

$$T(n + k) = S(k)T(n + 1) + T(k)T(n). \tag{2.8}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Proof. Here we only prove (2.7). The proof of (2.8) is analogous to that of (2.7), so we omit it. We prove (2.7) by induction on k. The facts that $S(0) = T(1) = 0$ and $S(1) = T(0) = 1$ imply (2.7) holds when $k = 0$ and $k = 1$. Suppose that (2.7) holds when $k = m$ and $k = m + 1$, namely,

$$S(n + m) = S(m)S(n + 1) + T(m)S(n) \tag{2.9}$$

and

$$S(n + m + 1) = S(m + 1)S(n + 1) + T(m + 1)S(n). \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\begin{aligned} S(n + m + 2) &= aS(n + m + 1) - bS(n + m) \quad \text{(by (1.2))} \\ &= a(S(m + 1)S(n + 1) + T(m + 1)S(n)) - b(S(m)S(n + 1) + T(m)S(n)) \\ &= (aS(m + 1) - bS(m))S(n + 1) + (aT(m + 1) - bT(m))S(n) \\ &= S(m + 2)S(n + 1) + T(m + 2)S(n), \quad \text{(by (1.2) and (1.3))}. \end{aligned} \tag{2.11}$$

Therefore, (2.7) holds when $k = m + 2$ and this completes the proof of the lemma by induction. □

Lemma 2.3. *Let $S(n)$ be defined by (1.2) and let $R_S(M)$ denote the rank of $S(n)$ modulo M. For $k \geq 0$,*

$$S(R_S(M)k) \equiv 0 \pmod{M}. \tag{2.12}$$

Proof. We also prove this lemma by induction on k. The fact that $S(0) = 0$ implies that (2.12) is true when $k = 0$. Suppose that (2.12) is true when $k = m$, that is,

$$S(R_S(M)m) \equiv 0 \pmod{M}. \tag{2.13}$$

It follows from the definition of $R_S(M)$ that

$$S(R_S(M)) \equiv 0 \pmod{M}. \tag{2.14}$$

Taking $n = R_S(M)m$ and $k = R_S(M)$ in (2.7), we see that

$$S(R_S(M)(m + 1)) = S(R_S(M))S(R_S(M)m + 1) + T(R_S(M))S(R_S(M)m). \tag{2.15}$$

In light of (2.13)–(2.15),

$$S(R_S(M)(m + 1)) \equiv 0 \pmod{M}. \tag{2.16}$$

Therefore, (2.12) is true when $k = m + 1$ and this completes the proof by induction. □

Lemma 2.4. *Let $T(n)$ be defined by (1.3). For $k \geq 0$,*

$$T(R_S(M)k) \equiv T(R_S(M))^k \pmod{M}. \tag{2.17}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Proof. We are ready to prove this lemma by induction on k. The fact that $T(0)=1$ implies that (2.17) holds when $k=0$. Assume that (2.17) holds when $k=m$, namely,

$$T(R_S(M)m) \equiv T(R_S(M))^m \pmod{M}. \tag{2.18}$$

Putting $n=R_S(M)m$ and $k=R_S(M)$ in (2.8) yields

$$T(R_S(M)(m+1)) = S(R_S(M))T(R_S(M)m+1) + T(R_S(M))T(R_S(M)m). \tag{2.19}$$

Thanks to (2.14), (2.18) and (2.19),

$$T(R_S(M)(m+1)) \equiv T(R_S(M))^{m+1} \pmod{M}, \tag{2.20}$$

which implies that (2.17) holds when $k=m+1$. This completes the proof of the lemma by induction. □

Lemma 2.5. *Let $S(n)$ and $T(n)$ be defined by (1.2) and (1.3), respectively. For $n \geq 0$,*

$$aS(n) + T(n) = S(n+1). \tag{2.21}$$

Proof. We also prove this lemma by induction on n. It is easy to check that $S(0) = T(1) = 0$, $S(1) = T(0) = 1$ and $S(2) = a$. Therefore, (2.21) is true when $n=0$ and $n=1$. Assume that (2.21) holds when $n=m$ and $n=m+1$, namely,

$$aS(m) + T(m) = S(m+1) \tag{2.22}$$

and

$$aS(m+1) + T(m+1) = S(m+2). \tag{2.23}$$

In view of (1.2), (1.3), (2.22) and (2.23),

$$\begin{aligned} S(m+3) &= aS(m+2) - bS(m+1) \\ &= a(aS(m+1) + T(m+1)) - b(aS(m) + T(m)) \\ &= a(aS(m+1) - bS(m)) + (aT(m+1) - bT(m)) \\ &= aS(m+2) + T(m+2), \end{aligned}$$

which yields (2.21) holds when $n=m+2$. The proof of the lemma by induction is complete. □

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need to prove the following lemma.

Lemma 3.1. *Let r be an odd integer with $3 \leq r \leq 23$ and let p be a prime with $p \geq 3$ if $3|r$ and $p \geq 5$ otherwise. Let $p_r(n)$ be defined by (1.1) and let $\pi(p)$ be defined by (1.4). For $n,\ k \geq 0$,*

$$p_r\left(p^{2k}n + \frac{r(p^{2k}-1)}{24}\right) = G_p(k)p_r\left(p^2 n + \frac{r(p^2-1)}{24}\right) + H_p(k)p_r(n), \tag{3.1}$$

where $G_p(k)$ are the $(\pi(p), p^{r-2})$-Lucas sequences and $H_p(k)$ are the dual $(\pi(p), p^{r-2})$-Lucas sequences of $G_p(k)$.

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Proof. We are ready to prove this lemma by induction on k. Let $\pi(p)$ be defined by (1.4), let $G_p(k)$ be the $(\pi(p), p^{r-2})$-Lucas sequences and let $H_p(k)$ be the dual$(\pi(p), p^{r-2})$-Lucas sequences of $G_p(k)$. From the fact that $G_p(0) = H_p(1) = 0$ and $G_p(1) = H_p(0) = 1$, it is easy to check that (3.1) is true when $k = 0$ and $k = 1$. Assume that (3.1) holds when $k = m$ and $k = m + 1$ $(m \geq 0)$, namely,

$$p_r\left(p^{2m}n + \frac{r(p^{2m}-1)}{24}\right) = G_p(m)p_r\left(p^2n + \frac{r(p^2-1)}{24}\right) + H_p(m)p_r(n) \tag{3.2}$$

and

$$p_r\left(p^{2m+2}n + \frac{r(p^{2m+2}-1)}{24}\right) = G_p(m+1)p_r\left(p^2n + \frac{r(p^2-1)}{24}\right) + H_p(m+1)p_r(n). \tag{3.3}$$

Newman [**10**] proved that if p is a prime and r is an odd integer with $3 \leq r \leq 23$, then

$$p_r\left(p^2n + \frac{r(p^2-1)}{24}\right) = \chi(n)p_r(n) - p^{r-2}p_r\left(\frac{n-(r(p^2-1)/24)}{p^2}\right), \tag{3.4}$$

where

$$\chi(n) = p_r\left(\frac{r(p^2-1)}{24}\right) + (-1)^{(p-1)(p-1-2r)/8}p^{(r-3)/2} \times \left(\left(\frac{(r(p^2-1)/24)}{p}\right)_L - \left(\frac{(r(p^2-1)/24)-n}{p}\right)_L\right), \tag{3.5}$$

with $p \geq 3$ if $3|r$ and $p \geq 5$ otherwise. It is easy to verify that

$$\pi(p) = \chi\left(pn + \frac{r(p^2-1)}{24}\right) = \chi\left(p^2n + \frac{r(p^2-1)}{24}\right), \tag{3.6}$$

where $\pi(p)$ is defined by (1.4). If we replace n by $p^2n + r(p^2-1)/24$ in (3.4) and then employ (3.6), we deduce that

$$p_r\left(p^4n + \frac{r(p^4-1)}{24}\right) = \pi(p)p_r\left(p^2n + \frac{r(p^2-1)}{24}\right) - p^{r-2}p_r(n). \tag{3.7}$$

Replacing n by $p^{2m}n + r(p^{2m}-1)/24$ in (3.7) and employing (3.2) and (3.3), we have

$$p_r\left(p^{2m+4}n + \frac{r(p^{2m+4}-1)}{24}\right) = \pi(p)p_r\left(p^{2m+2}n + \frac{r(p^{2m+2}-1)}{24}\right) - p^{r-2}p_r\left(p^{2m}n + \frac{r(p^{2m}-1)}{24}\right)$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

$$= \pi(p)\left(G_p(m+1)p_r\left(p^2n + \frac{r(p^2-1)}{24}\right) + H_p(m+1)p_r(n)\right)$$
$$- p^{r-2}\left(G_p(m)p_r\left(p^2n + \frac{r(p^2-1)}{24}\right) + H_p(m)p_r(n)\right)$$
$$= (\pi(p)G_p(m+1) - p^{r-2}G_p(m))p_r\left(p^2n + \frac{r(p^2-1)}{24}\right)$$
$$+ (\pi(p)H_p(m+1) - p^{r-2}H_p(m))p_r(n). \tag{3.8}$$

Since $G_p(k)$ are the $(\pi(p), p^{r-2})$-Lucas sequences and $H_p(k)$ are the dual $(\pi(p), p^{r-2})$-Lucas sequences of $G_p(k)$,

$$\pi(p)G_p(m+1) - p^{r-2}G_p(m) = G_p(m+2) \tag{3.9}$$

and

$$\pi(p)H_p(m+1) - p^{r-2}H_p(m) = H_p(m+2). \tag{3.10}$$

Based on (3.8)–(3.10),

$$p_r\left(p^{2m+4}n + \frac{r(p^{2m+4}-1)}{24}\right) = G_p(m+2)p_r\left(p^2n + \frac{r(p^2-1)}{24}\right) + H_p(m+2)p_r(n),$$

which implies that (3.1) is true when $k = m+2$. This completes the proof of the lemma by induction.

We now turn to the proof of Theorem 1.1. □

Proof of Theorem 1.1. Combining (3.1) and (3.4) yields

$$p_r\left(p^{2k}n + \frac{r(p^{2k}-1)}{24}\right) = G_p(k)\left(\chi(n)p_r(n) - p^{r-2}p_r\left(\frac{n-(r(p^2-1)/24)}{p^2}\right)\right)$$
$$+ H_p(k)p_r(n)$$
$$= (G_p(k)\chi(n) + H_p(k))p_r(n)$$
$$- p^{r-2}G_p(k)p_r\left(\frac{n-(r(p^2-1)/24)}{p^2}\right). \tag{3.11}$$

If we replace n by $pn + (r(p^2-1))/24$ in (3.11) and employ (3.6), we find that for $n,\ k \geq 0$,

$$p_r\left(p^{2k+1}n + \frac{r(p^{2k+2}-1)}{24}\right) = (\pi(p)G_p(k) + H_p(k))p_r\left(pn + \frac{r(p^2-1)}{24}\right)$$
$$- p^{r-2}G_p(k)p_r(n/p). \tag{3.12}$$

Because of (2.21) and the fact that $G_p(k)$ are the $(\pi(p), p^{r-2})$-Lucas sequences,

$$\pi(p)G_p(k) + H_p(k) = G_p(k+1). \tag{3.13}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

It follows from (3.12) and (3.13) that

$$p_r\left(p^{2k+1}n+\frac{r(p^{2k+2}-1)}{24}\right)=G_p(k+1)p_r\left(pn+\frac{r(p^2-1)}{24}\right)-p^{r-2}G_p(k)p_r(n/p). \tag{3.14}$$

Let $M\geq 2$ be a positive integer with $\gcd(M,p)=1$. In view of Lemma 2.1, we see that the ranks of $G_p(n)$ modulo M exist. Let $R_{G_p}(M)$ denote the ranks of $G_p(n)$ modulo M. Thanks to Lemma 2.3, we deduce that for $k\geq 0$,

$$G_p(R_{G_p}(M)k)\equiv 0 \pmod{M}. \tag{3.15}$$

If we replace k by $R_{G_p}(M)(k+1)-1$ in (3.14) and use (3.15), we see that for $n,\ k\geq 0$,

$$\begin{aligned}&p_r\left(p^{2R_{G_p}(M)(k+1)-1}n+\frac{r(p^{2R_{G_p}(M)(k+1)}-1)}{24}\right)\\&\qquad\equiv -p^{r-2}G_p(R_{G_p}(M)(k+1)-1)p_r(n/p) \pmod{M}.\end{aligned} \tag{3.16}$$

If $p\nmid n$, then n/p is not an integer and

$$p_r(n/p)=0. \tag{3.17}$$

In light of (3.16) and (3.17), we arrive at (1.5).

If we set $n=0$ and replace k by $R_{G_p}(M)k$ in (3.1), then employ (3.15) and the fact that $p_r(0)=1$, we obtain

$$p_r\left(\frac{r(p^{2R_{G_p}(M)k}-1)}{24}\right)\equiv H_p(R_{G_p}(M)k) \pmod{M}. \tag{3.18}$$

In view of Lemma 2.4,

$$H_p(R_{G_p}(M)k)\equiv H_p(R_{G_p}(M))^k \pmod{M}. \tag{3.19}$$

Congruence (1.6) follows from (3.18) and (3.19).

From the fact that $\gcd(M,p)=1$, we find that if

$$\pi(p)\left(\frac{(r(p^2-1)/24)-n}{p}\right)_L\equiv(-1)^{(p-1)(p-1-2r)/8}p^{(r-3)/2} \pmod{M}, \tag{3.20}$$

then

$$\left(\frac{(r(p^2-1)/24)-n}{p}\right)_L\neq 0 \tag{3.21}$$

and

$$\pi(p)\equiv(-1)^{(p-1)(p-1-2r)/8}p^{(r-3)/2}\left(\frac{(r(p^2-1)/24)-n}{p}\right)_L \pmod{M}. \tag{3.22}$$

It follows from (1.4), (3.5) and (3.22) that if (3.20) is true, then

$$\chi(n)\equiv 0 \pmod{M}. \tag{3.23}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Thanks to (3.21), we see that $(r(p^2-1)/24-n)/p$ is not an integer. Therefore, $(n-r(p^2-1)/24)/p^2$ is not an integer and

$$p_r\left(\frac{n-(r(p^2-1)/24)}{p^2}\right)=0. \tag{3.24}$$

In view of (3.4), (3.23) and (3.24),

$$p_r\left(p^2n+\frac{r(p^2-1)}{24}\right)\equiv 0 \pmod{M}. \tag{3.25}$$

If we replace k by $R_{G_p}(M)k$ in (3.1) and employ (3.15), we have

$$p_r\left(p^{2R_{G_p}(M)k}n+\frac{r(p^{2R_{G_p}(M)k}-1)}{24}\right)\equiv H_p(R_{G_p}(M)k)p_r(n) \pmod{M}. \tag{3.26}$$

Replacing n by $p^2n+(r(p^2-1)/24)$ in (3.26) and using (3.25) yields (1.7). This completes the proof. □

4. Proof of Theorem 1.2

We prove Theorem 1.2 by induction on the total number of prime factors of n. Suppose that r is an odd integer with $3\le r\le 23$. Let $p_r(n)$ be defined by (1.1) and let $M\ge 2$ be an integer. If $n=1$ (n has no prime factors), then (1.8) states that $p_r(a)\equiv 0 \pmod{M}$, which is true by hypothesis. Define

$$S_r := \{p|p \text{ is a prime with } p\ge 3 \text{ if } 3|r \text{ and } p\ge 5 \text{ otherwise}\}. \tag{4.1}$$

Furthermore, assume that $24a+r=\prod_{i=1}^{u} f_i\prod_{j=1}^{v} g_j^{\alpha_j}$, with each $\alpha_j\ge 2$, is the prime factorization of $24a+r$. Let p_1 be a prime with $p_1\in S_r$ and $\gcd(p_1,\prod_{j=1}^{v} g_j^{\alpha_j})=1$. If we replace (n,p) by (a,p_1) in (3.4) and utilize the hypothesis that $p_r(a)\equiv 0 \pmod{M}$ and the fact that $\chi(a)$ is an integer, we find that

$$p_r\left(ap_1^2+\frac{r(p_1^2-1)}{24}\right)\equiv -p_1^{r-2}p_r\left(\frac{a-(r(p_1^2-1)/24)}{p_1^2}\right) \pmod{M}. \tag{4.2}$$

From the hypothesis $\gcd(p_1,\prod_{j=1}^{v} g_j^{\alpha_j})=1$, it follows that

$$\frac{a-(r(p_1^2-1)/24)}{p_1^2}=\frac{24a+r-rp_1^2}{24p_1^2}=\frac{\prod_{i=1}^{u} f_i\prod_{j=1}^{v} g_j^{\alpha_j}-rp_1^2}{24p_1^2}$$

is not an integer. Therefore,

$$p_r\left(\frac{a-(r(p_1^2-1)/24)}{p_1^2}\right)=0. \tag{4.3}$$

Thanks to (4.2) and (4.3),

$$p_r\left(ap_1^2+\frac{r(p_1^2-1)}{24}\right)\equiv 0 \pmod{M}.$$

Therefore, (1.8) holds when $n=p_1$ (n has only one prime factor). Suppose that (1.8) is true for all integers with not more than k prime factors. In order to prove

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Theorem 1.2, it suffices to prove that (1.8) is true when n has $k+1$ prime factors. We can write n as $n = p_1p_2\cdots p_kp_{k+1}$ where $3 \le p_1 \le p_2 \le \cdots \le p_k \le p_{k+1}$ with $\gcd(p_1\cdots p_{k-1}p_kp_{k+1}, 2\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$ if $3|r$ and $\gcd(p_1\cdots p_{k-1}p_kp_{k+1}, 6\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$ otherwise.

By hypothesis, (1.8) holds for all integers with not more than k prime factors. Hence,

$$p_r\left(ap_1^2p_2^2\cdots p_{k-1}^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2 - 1)}{24}\right) \equiv 0 \pmod{M} \tag{4.4}$$

and

$$p_r\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - 1)}{24}\right) \equiv 0 \pmod{M}. \tag{4.5}$$

Replacing n by

$$ap_1^2p_2^2\cdots p_{k-1}^2p_k^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - 1)}{24}$$

and replacing p by p_{k+1} in (3.4), then utilizing (4.5) and the fact that

$$\chi\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - 1)}{24}\right)$$

is an integer, we deduce that

$$\begin{aligned} &p_r\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2 - 1)}{24}\right) \\ &\quad\equiv -p_{k+1}^{r-2}p_r\left(\frac{ap_1^2p_2^2\cdots p_{k-1}^2p_k^2 + r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - p_{k+1}^2)/24}{p_{k+1}^2}\right) \pmod{M}. \end{aligned} \tag{4.6}$$

Now we break our proof into two cases: $p_{k+1} = p_k$ and $p_{k+1} > p_k$. If $p_{k+1} = p_k$, based on (4.4), we can rewrite (4.6) as

$$\begin{aligned} &p_r\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2 - 1)}{24}\right) \\ &\quad\equiv -p_{k+1}^{r-2}p_r\left(ap_1^2p_2^2\cdots p_{k-1}^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2 - 1)}{24}\right) \equiv 0 \pmod{M}. \end{aligned} \tag{4.7}$$

If $p_{k+1} > p_k$, then $p_{k+1} \notin \{p_1, p_2, \ldots, p_k\}$. From the fact that $\gcd(p_{k+1}, \prod_{j=1}^{v} g_j^{\alpha_j}) = 1$, we see that

$$\begin{aligned} &\frac{ap_1^2p_2^2\cdots p_{k-1}^2p_k^2 + r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - p_{k+1}^2)/24}{p_{k+1}^2} \\ &= \frac{(24a + r)p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - rp_{k+1}^2}{24p_{k+1}^2} \\ &= \frac{p_1^2p_2^2\cdots p_{k-1}^2p_k^2\prod_{i=1}^{u} f_i\prod_{j=1}^{v} g_j^{\alpha_j} - rp_{k+1}^2}{24p_{k+1}^2} \end{aligned}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

is not an integer. Therefore,

$$p_r\left(\frac{ap_1^2p_2^2\cdots p_{k-1}^2p_k^2 + r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - p_{k+1}^2)/24}{p_{k+1}^2}\right) = 0. \tag{4.8}$$

Combining (4.6)–(4.8) yields

$$p_r\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2 + \frac{r(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2 - 1)}{24}\right) \equiv 0 \pmod{M}. \tag{4.9}$$

Therefore, in any case, (1.8) is true when $n = p_1p_2\cdots p_kp_{k+1}$. Theorem 1.2 is proved by induction and the proof is complete.

5. Proof of Theorem 1.3

In order to prove Theorem 1.3, we first prove the following lemma.

Lemma 5.1. *Let r be an odd integer with $3 \le r \le 23$ and let S_r be defined by (4.1). Suppose that $p \in S_r$ and $M \ge 2$ is an integer with* $\gcd(M, p) = 1$. *Let $G_p(k)$ be defined in Lemma 3.1. Then*

$$G_p(\gamma(M)) \equiv 0 \pmod{M}, \tag{5.1}$$

where $\gamma(M)$ is defined in Theorem 1.3.

Proof. For any fixed $p \in S_r$ (defined by (4.1)), let $G_p(n)$ be a $(\pi(p), p^{r-2})$-Lucas sequence. Assume that $\pi(p) \equiv i \pmod{M}$ with $0 \le i \le M-1$ and $p^{r-2} \equiv j \pmod{M}$ with $1 \le j \le M-1$. It is easy to see that for $n \ge 0$,

$$G_p(n) \equiv W_{(i,j)}(n) \pmod{M}, \tag{5.2}$$

where $W_{(i,j)}(n)$ denotes an (i,j)-Lucas sequence. Let $R_{W_{(i,j)}}(M)$ denote the rank of $W_{(i,j)}(n)$ modulo M. By Lemma 2.1, $R_{W_{(i,j)}}(M)$ exists. Thanks to (5.2),

$$R_{G_p}(M) = R_{W_{(i,j)}}(M). \tag{5.3}$$

Furthermore, define

$$\begin{aligned} H_{(M,p,r)} := \{(i,j) | \pi(p) \equiv i \pmod{M} \text{ with } 0 \le i \le M-1,\\ p^{r-2} \equiv j \pmod{M} \text{ with } 1 \le j \le M-1,\ p \in S_r \text{ and } \gcd(M,p) = 1\}. \end{aligned} \tag{5.4}$$

Let $\mu(M)$ denote the lowest common multiple of the set $\{R_{W_{(i,j)}}(M) | (i,j) \in H_{(M,p,r)}\}$. Thus,

$$R_{W_{(i,j)}}(M) | \mu(M). \tag{5.5}$$

Note that $H_{(M,p,r)}$ is a non-empty subset of $\mathfrak{S}_M$, where $\mathfrak{S}_M$ is defined in Theorem 1.3. Therefore,

$$\mu(M) | \gamma(M), \tag{5.6}$$

where $\gamma(M)$ is defined in Theorem 1.3. Thanks to (5.3), (5.5) and (5.6),

$$R_{G_p}(M) | \gamma(M). \tag{5.7}$$

Congruence (5.1) follows from Lemma 2.3 and (5.7). This completes the proof. □

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. We also prove Theorem 1.3 by induction on the total number of prime factors of n. By hypothesis, it is easy to see that (1.10) is true when $n = 1$. Let $\gamma(M)$ be defined in Theorem 1.3 and let $p_1 \in S_r$ with $\gcd(M, p_1) = 1$, where S_r is defined by (4.1). If we set $(p, n, k) = (p_1, a, \gamma(M))$ in (3.1) and use the hypothesis that $p_r(a) \equiv 0 \pmod{M}$, we have

$$p_r\left(p_1^{2\gamma(M)}a + \frac{r(p_1^{2\gamma(M)} - 1)}{24}\right) \equiv G_{p_1}(\gamma(M))p_r\left(p_1^2 a + \frac{r(p_1^2 - 1)}{24}\right) \pmod{M}. \tag{5.8}$$

In view of (5.1) and (5.8),

$$p_r\left(p_1^{2\gamma(M)}a + \frac{r(p_1^{2\gamma(M)} - 1)}{24}\right) \equiv 0 \pmod{M}, \tag{5.9}$$

which implies that (1.10) is true when $n = p_1$. Assume that (1.10) is true when $n = p_1 p_2 \cdots p_\nu$, namely,

$$p_r\left((p_1p_2\cdots p_\nu)^{2\gamma(M)}a + \frac{r((p_1p_2\cdots p_\nu)^{2\gamma(M)} - 1)}{24}\right) \equiv 0 \pmod{M}. \tag{5.10}$$

Since $\gcd(n, 2M) = 1$ if $3|r$ and $\gcd(n, 6M) = 1$ otherwise, $p_i \in S_r$ and $\gcd(p_i, M) = 1$ $(1 \leq i \leq \nu)$. Let $p_{\nu+1}$ be a prime with $p_{\nu+1} \in S_r$ and $\gcd(p_{\nu+1}, M) = 1$. Setting $(p, k) = (p_{\nu+1}, \gamma(M))$ in (3.1) and employing (5.1) yields

$$p_r\left(p_{\nu+1}^{2\gamma(M)}n + \frac{r(p_{\nu+1}^{2\gamma(M)} - 1)}{24}\right) \equiv H_{p_{\nu+1}}(\gamma(M))p_r(n) \pmod{M}. \tag{5.11}$$

Replacing n by

$$(p_1p_2\cdots p_\nu)^{2\gamma(M)}a + \frac{r((p_1p_2\cdots p_\nu)^{2\gamma(M)} - 1)}{24}$$

in (5.11) and then using (5.10) yields

$$p_r\left((p_1p_2\cdots p_\nu p_{\nu+1})^{2\gamma(M)}a + \frac{r((p_1p_2\cdots p_\nu p_{\nu+1})^{2\gamma(M)} - 1)}{24}\right) \equiv 0 \pmod{M},$$

which implies that (1.10) is true when $n = p_1p_2\cdots p_\nu p_{\nu+1}$. This completes the proof of the theorem by induction. □

6. Proof of Theorem 1.4

The aim of this section is to present a proof of Theorem 1.4. We first prove the following lemma.

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Lemma 6.1. *Let r be an even integer with $2 \leq r \leq 24$ and let p be a prime with $24|r(p-1)$. For $n,\ k \geq 0$,*

$$p_r\left(p^k n + \frac{r(p^k-1)}{24}\right) = S_p(k) p_r\left(pn + \frac{r(p-1)}{24}\right) + T_p(k) p_r(n), \tag{6.1}$$

where $p_r(n)$ is defined by (1.1), $S_p(k)$ are the $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences and $T_p(k)$ are the dual $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences of $S_p(k)$.

Proof. We are ready to prove this lemma by induction on k. Since $S_p(0) = T_p(1) = 0$ and $S_p(1) = T_p(0) = 1$, (6.1) holds when $k = 0$ and $k = 1$. Suppose that (6.1) is true when $k = m$ and $k = m+1$ $(m \geq 0)$, namely,

$$p_r\left(p^m n + \frac{r(p^m-1)}{24}\right) = S_p(m) p_r\left(pn + \frac{r(p-1)}{24}\right) + T_p(m) p_r(n) \tag{6.2}$$

and

$$p_r\left(p^{m+1} n + \frac{r(p^{m+1}-1)}{24}\right) = S_p(m+1) p_r\left(pn + \frac{r(p-1)}{24}\right) + T_p(m+1) p_r(n). \tag{6.3}$$

Newman [9] proved that

$$p_r\left(pn + \frac{r(p-1)}{24}\right) = p_r\left(\frac{r(p-1)}{24}\right) p_r(n) - p^{(r-2)/2} p_r\left(\frac{n-(r(p-1)/24)}{p}\right), \tag{6.4}$$

where $p_r(n)$ is defined by (1.1), p is a prime with $24|r(p-1)$ and r is an even integer with $2 \leq r \leq 24$. If we replace n by $pn + r(p-1)/24$ in (6.4), then

$$p_r\left(p^2 n + \frac{r(p^2-1)}{24}\right) = p_r\left(\frac{r(p-1)}{24}\right) p_r\left(pn + \frac{r(p-1)}{24}\right) - p^{(r-2)/2} p_r(n). \tag{6.5}$$

Replacing n by $p^m n + r(p^m-1)/24$ in (6.5) and then utilizing (6.2) and (6.3), we deduce that

$$\begin{aligned}
&p_r\left(p^{m+2} n + \frac{r(p^{m+2}-1)}{24}\right) \\
&\quad = p_r\left(\frac{r(p-1)}{24}\right) p_r\left(p^{m+1} n + \frac{r(p^{m+1}-1)}{24}\right) \\
&\qquad - p^{(r-2)/2} p_r\left(p^m n + \frac{r(p^m-1)}{24}\right) \\
&\quad = p_r\left(\frac{r(p-1)}{24}\right)\left(S_p(m+1) p_r\left(pn + \frac{r(p-1)}{24}\right) + T_p(m+1) p_r(n)\right) \\
&\qquad - p^{(r-2)/2}\left(S_p(m) p_r\left(pn + \frac{r(p-1)}{24}\right) + T_p(m) p_r(n)\right)
\end{aligned}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

$$= \left(p_r\left(\frac{r(p-1)}{24}\right)S_p(m+1) - p^{(r-2)/2}S_p(m)\right)p_r\left(pn+\frac{r(p-1)}{24}\right)$$
$$+ \left(p_r\left(\frac{r(p-1)}{24}\right)T_p(m+1) - p^{(r-2)/2}T_p(m)\right)p_r(n). \tag{6.6}$$

Since $S_p(k)$ are the $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences and $T_p(k)$ are the dual $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences of $S_p(k)$,

$$p_r\left(\frac{r(p-1)}{24}\right)S_p(m+1) - p^{(r-2)/2}S_p(m) = S_p(m+2) \tag{6.7}$$

and

$$p_r\left(\frac{r(p-1)}{24}\right)T_p(m+1) - p^{(r-2)/2}T_p(m) = T_p(m+2). \tag{6.8}$$

In view of (6.6)–(6.8), we deduce that (6.1) holds when $k = m+2$. This completes the proof of the lemma by induction. □

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. Substituting (6.4) into (6.1) yields

$$\begin{aligned} p_r\left(p^k n + \frac{r(p^k-1)}{24}\right) &= S_p(k)\left(p_r\left(\frac{r(p-1)}{24}\right)p_r(n) - p^{(r-2)/2}p_r\left(\frac{n-(r(p-1)/24)}{p}\right)\right) \\ &\quad + T_p(k)p_r(n) \\ &= \left(p_r\left(\frac{r(p-1)}{24}\right)S_p(k) + T_p(k)\right)p_r(n) \\ &\quad - p^{(r-2)/2}S_p(k)p_r\left(\frac{n-(r(p-1)/24)}{p}\right). \end{aligned} \tag{6.9}$$

By Lemma 2.5 and the fact that $S_p(k)$ are the $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences,

$$p_r\left(\frac{r(p-1)}{24}\right)S_p(k) + T_p(k) = S_p(k+1), \tag{6.10}$$

where $T_p(k)$ are the dual $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences of $S_p(k)$. With the aid of (6.9) and (6.10),

$$p_r\left(p^k n + \frac{r(p^k-1)}{24}\right) = S_p(k+1)p_r(n) - p^{(r-2)/2}S_p(k)p_r\left(\frac{n-(r(p-1)/24)}{p}\right). \tag{6.11}$$

Let $M \geq 2$ be an integer with $\gcd(M,p) = 1$. Thanks to Lemma 2.1, the ranks $R_{S_p}(M)$ of $S_p(n)$ modulo M exist. Based on Lemma 2.3, we deduce that for $k \geq 0$,

$$S_p(R_{S_p}(M)k) \equiv 0 \pmod{M}. \tag{6.12}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

If we replace k by $R_{S_p}(M)(k+1)-1$ in (6.11) and then use (6.12), we see that for $k \geq 0$,

$$p_r\left(p^{R_{S_p}(M)(k+1)-1}n + \frac{r(p^{R_{S_p}(M)(k+1)-1}-1)}{24}\right) \equiv -p^{(r-2)/2}S_p(R_{S_p}(M)(k+1)-1)p_r\left(\frac{n-(r(p-1)/24)}{p}\right) \pmod{M}. \tag{6.13}$$

If $p \nmid (24n+r)$, then $(n-r(p-1)/24)/p$ is not an integer and

$$p_r\left(\frac{n-(r(p-1)/24)}{p}\right) = 0. \tag{6.14}$$

Congruence (1.11) follows from (6.13) and (6.14).

If we replace k by $R_{S_p}(M)k$ in (6.1) and use (6.12), we get

$$p_r\left(p^{R_{S_p}(M)k}n + \frac{r(p^{R_{S_p}(M)k}-1)}{24}\right) \equiv T_p(R_{S_p}(M)k)p_r(n) \pmod{M}. \tag{6.15}$$

In view of Lemma 2.4 and the fact that $T_p(k)$ are the dual $(p_r(r(p-1)/24), p^{(r-2)/2})$-Lucas sequences of $S_p(k)$, we deduce that for $\alpha \geq 0$,

$$T_p(R_{S_p}(M)k) \equiv T_p(R_{S_p}(M))^k \pmod{M}. \tag{6.16}$$

If we set $n=0$ in (6.15), then use (6.16) and the fact that $p_r(0)=1$, we find that for $k \geq 0$,

$$p_r\left(\frac{r(p^{R_{S_p}(M)k}-1)}{24}\right) \equiv T_p(R_{S_p}(M))^k \pmod{M},$$

which is nothing more than (1.12). This completes the proof. □

7. Congruences modulo 16 for Ramanujan's ϕ function

Recently, Chan [2] established a number of congruences for the coefficients $a_\phi(n)$ of Ramanujan's ϕ function $\phi(q)$, which is defined by

$$\phi(q) = \sum_{n=0}^{\infty} a_\phi(n)q^n := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(q;q^2)_{n+1}^2}. \tag{7.1}$$

Moreover, Chan [2] proved some congruences for $a_\phi(n)$. For example, he proved that for $n \geq 0$,

$$\begin{aligned} a_\phi(9n+4) &\equiv 0 \pmod{2}, \\ a_\phi(18n+10) &\equiv 0 \pmod{4}, \\ a_\phi(25n+14) &\equiv a_\phi(25n+24) \equiv 0 \pmod{4}. \end{aligned}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

In this paper we prove congruences modulo 16 for $a_\phi(n)$ based on Theorems 1.1–1.3. In order to state congruences modulo 16 for $a_\phi(n)$, define

$$\pi_1(p) := p_9\left(\frac{3(p^2-1)}{8}\right) + (-1)^{(p-1)(p-19)/8} p^3 \left(\frac{3(p^2-1)/8}{p}\right)_L, \tag{7.2}$$

where $p \geq 3$ is a prime. Moreover, define

$$\alpha_1(p) := \begin{cases} 2 & \text{if } \pi_1(p) \equiv 0 \pmod{16}, \\ 3 & \text{if } \pi_1(p)^2 \equiv p^7 \pmod{16}, \\ 4 & \text{if } \pi_1(p) \equiv 8 \pmod{16}, \\ 6 & \text{if } \pi_1(p)^2 \equiv tp^7 \pmod{16} \text{ with } t \in \{3,9,11\}, \\ 8 & \text{if } \pi_1(p) \equiv 4 \pmod{8}, \\ 12 & \text{if } \pi_1(p)^2 \equiv tp^7 \pmod{16} \text{ with } t \in \{5,7,13,15\}, \\ 16 & \text{if } \pi_1(p) \equiv 2 \pmod{4}, \end{cases} \tag{7.3}$$

and

$$g_1(p) = f_1(\pi_1(p), p^7, \alpha_1(p)) \tag{7.4}$$

with

$$f_1(x,y,\alpha_1(p))$$
$$:= \begin{cases} -y & \text{if } \alpha_1(p) = 2, \\ -xy & \text{if } \alpha_1(p) = 3, \\ -y(x^2-y) & \text{if } \alpha_1(p) = 4, \\ -y(x^4-3x^2y+y^2) & \text{if } \alpha_1(p) = 6, \\ -y(x^6-5x^4y+6x^2y^2-y^3) & \text{if } \alpha_1(p) = 8, \\ -y(x^{10}-9x^8y+28x^6y^2-35x^4y^3+15x^2y^4-y^5) & \text{if } \alpha_1(p) = 12, \\ -y(x^2-y)(x^4-3x^2y+y^2)(x^8-9x^6y+26x^4y^2-24x^2y^3+y^4) & \text{if } \alpha_1(p) = 16. \end{cases}$$

Theorem 7.1. *Let $p \geq 3$ be a prime and let $a_\phi(n)$ be defined by (7.1).*

(i) *For n, $k \geq 0$, if $p \nmid n$, then*

$$a_\phi\left(2p^{2\alpha_1(p)(k+1)-1}n + \frac{3p^{2\alpha_1(p)(k+1)}+1}{4}\right) \equiv 0 \pmod{16}. \tag{7.5}$$

Moreover, for $k \geq 0$,

$$a_\phi\left(\frac{3p^{2\alpha_1(p)k}+1}{4}\right) \equiv [g_1(p)]^k \pmod{16}. \tag{7.6}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

(ii) *For* $n,\ k \geq 0$, *if*

$$\pi_1(p)\left(\frac{(3(p^2-1)/8)-n}{p}\right)_L \equiv (-1)^{(p-1)(p-19)/8}p^3 \pmod{16},$$

then

$$a_\phi\left(2p^{2\alpha_1(p)k+2}n + \frac{3p^{2\alpha_1(p)k+2}+1}{4}\right) \equiv 0 \pmod{16}. \tag{7.7}$$

(iii) *If* a *is a non-negative integer such that* $p_9(a) \equiv 0 \pmod{16}$ *and* $24a+9 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$, *with each* $\alpha_j \geq 2$, *is the prime factorization of* $24a+9$. *Then for* $n \geq 1$,

$$a_\phi\left(2an^2 + \frac{3n^2+1}{4}\right) \equiv 0 \pmod{16}, \tag{7.8}$$

where $\gcd(n, 2\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$.

(iv) *If* a *is a non-negative integer such that* $p_9(a) \equiv 0 \pmod{16}$, *then for* $n \geq 0$,

$$a_\phi\left(2a(2n+1)^{96} + \frac{3(2n+1)^{96}+1}{4}\right) \equiv 0 \pmod{16}. \tag{7.9}$$

Example. If we set $p = 3$ in the above theorem, we get $\pi_1(3) \equiv 4 \pmod 8$ and $g_1(3) \equiv 1 \pmod{16}$. Therefore $\alpha_1(3) = 8$. In view of (7.5) and (7.6), we see that for $n,\ k \geq 0$,

$$a_\phi\left(2\times 3^{16k+15}(3n+1) + \frac{3^{16k+17}+1}{4}\right) \equiv a_\phi\left(2\times 3^{16k+15}(3n+2) + \frac{3^{16k+17}+1}{4}\right)$$
$$\equiv 0 \pmod{16}$$

and

$$a_\phi\left(\frac{3^{16k+1}+1}{4}\right) \equiv 1 \pmod{16}.$$

It is easy to check that $16 | p_9(142)$. Therefore, if we set $a = 142$ in (7.8), we deduce that for $n \geq 0$,

$$a_\phi(1139n^2 + 1139n + 285) \equiv 0 \pmod{16}.$$

Proof. Chan [2] proved that

$$\sum_{n=0}^{\infty} a_\phi(2n+1)q^n = \frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^7}. \tag{7.10}$$

By the binomial theorem, for any $k \geq 0$,

$$(q;q)_\infty^{2^k} \equiv (q^2;q^2)_\infty^{2^{k-1}} \pmod{2^k}. \tag{7.11}$$

With the aid of (7.10) and (7.11),

$$\sum_{n=0}^{\infty} a_\phi(2n+1)q^n \equiv (q;q)_\infty^9 \pmod{16},$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

which implies that

$$a_{\phi}(2n+1) \equiv p_9(n) \pmod{16}. \tag{7.12}$$

Setting $r = 9$ and $M = 16$ in (1.5), we see that for $n,\ k \geq 0$ with $p \nmid n$,

$$p_9\left(p^{2R_{G_p}(16)(k+1)-1}n + \frac{3(p^{2R_{G_p}(16)(k+1)}-1)}{8}\right) \equiv 0 \pmod{16}, \tag{7.13}$$

where $p \geq 3$ is a prime, $G_p(n)$ are the $(\pi_1(p), p^7)$-Lucas sequences, $R_{G_p}(16)$ are the ranks of $G_p(n)$ modulo 16 and $\pi_1(p)$ is defined by (7.2). With the help of a computer, we can check that

$$R_{G_p}(16) = \alpha_1(p), \tag{7.14}$$

where $\alpha_1(p)$ is defined by (7.3). Therefore, from (7.13) and (7.14), we see that if $p \nmid n$, then

$$p_9\left(p^{2\alpha_1(p)(k+1)-1}n + \frac{3(p^{2\alpha_1(p)(k+1)}-1)}{8}\right) \equiv 0 \pmod{16}. \tag{7.15}$$

Congruence (7.5) follows from (7.12) and (7.15).

Moreover, by (1.6) and (7.14), it follows that for $k \geq 0$,

$$p_9\left(\frac{3(p^{2\alpha_1(p)k}-1)}{8}\right) \equiv [H_p(\alpha_1(p))]^k \pmod{16}, \tag{7.16}$$

where $H_p(n)$ are the dual $(\pi_1(p), p^7)$-Lucas sequences of $G_p(n)$. It is easy to check that

$$H_p(\alpha_1(p)) \equiv g_1(p) \pmod{16}, \tag{7.17}$$

where $g_1(p)$ is defined by (7.4). Combining (7.12), (7.16) and (7.17), we arrive at (7.6).

In view of (1.7) and (7.14), we deduce that for $n,\ k \geq 0$, if

$$\pi_1(p)\left(\frac{(3(p^2-1)/8)-n}{p}\right)_L \equiv (-1)^{(p-1)(p-19)/8}p^3 \pmod{16},$$

then

$$p_9\left(p^{2\alpha_1(p)k+2}n + \frac{3(p^{2\alpha_1(p)k+2}-1)}{8}\right) \equiv 0 \pmod{16}. \tag{7.18}$$

Congruence (7.7) follows from (7.12) and (7.18).

By Theorem 1.2, we see that if a is a non-negative integer such that $p_9(a) \equiv 0 \pmod{16}$ and $24a + 9 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$, with each $\alpha_j \geq 2$, is the prime factorization of $24a+9$, then for $n \geq 1$,

$$p_9\left(an^2 + \frac{3(n^2-1)}{8}\right) \equiv 0 \pmod{16}, \tag{7.19}$$

where $\gcd(n, 2\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$. Congruence (7.8) follows from (7.12) and (7.19).

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

From the definition of $\gamma(M)$ given in Theorem 1.3, we find that

$$\gamma(16) = 48. \tag{7.20}$$

Thanks to (1.10) and (7.20), we see that if $p_9(a) \equiv 0 \pmod{16}$, then

$$p_9\left(an^{96} + \frac{3(n^{96}-1)}{8}\right) \equiv 0 \pmod{16}, \tag{7.21}$$

where $n \geq 1$ is an odd integer. In view of (7.12) and (7.21), we arrive at (7.9). This completes the proof. □

8. New congruences modulo 11 for Andrews's spt-function

In [1], Andrews introduced the spt-function $\mathrm{spt}(n)$ which counts the number of smallest parts in the partitions of a positive integer n. For example, one sees that $\mathrm{spt}(4) = 10$ by examining the partitions of 4 (with the smallest parts underlined):

$$\underline{4},\ 3+\underline{1},\ \underline{2}+\underline{2},\ 2+\underline{1}+\underline{1},\ \underline{1}+\underline{1}+\underline{1}+\underline{1}.$$

The generating function for $\mathrm{spt}(n)$ is

$$\sum_{n=0}^{\infty} \mathrm{spt}(n) q^n := \frac{1}{(q;q)_\infty}\left(\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{(1-q^n)^2}\right). \tag{8.1}$$

Andrews [1] also found the following surprising congruences for $\mathrm{spt}(n)$:

$$\begin{aligned} \mathrm{spt}(5n+4) &\equiv 0 \pmod{5}, \\ \mathrm{spt}(7n+5) &\equiv 0 \pmod{7}, \\ \mathrm{spt}(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

Garvan [5] established congruences modulo powers for 5, 7 and 13 for $\mathrm{spt}(n)$. Moreover, in another paper [4], Garvan showed that for $n \geq 0$,

$$\mathrm{spt}(11n+6) \equiv 4p_{13}(n) \pmod{11}. \tag{8.2}$$

Based on Theorems 1.1–1.3 and (8.2), we can get new congruences modulo 11 for $\mathrm{spt}(n)$. In order to state the main results, define

$$\pi_2(p) := p_{13}\left(\frac{13(p^2-1)}{24}\right) + (-1)^{(p-1)(p-27)/8} p^5 \left(\frac{13(p^2-1)/24}{p}\right)_L, \tag{8.3}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

where $p \geq 5$ is a prime with $p \neq 11$. Furthermore, define

$$\alpha_2(p) := \begin{cases} 2 & \text{if } \pi_2(p) \equiv 0 \pmod{11}, \\ 3 & \text{if } \pi_2(p)^2 \equiv p \pmod{11}, \\ 4 & \text{if } \pi_2(p)^2 \equiv 2p \pmod{11}, \\ 5 & \text{if } \pi_2(p)^2 \equiv 5p \pmod{11} \text{ or } \pi_2(p)^2 \equiv 9p \pmod{11}, \\ 6 & \text{if } \pi_2(p)^2 \equiv 3p \pmod{11}, \\ 10 & \text{if } \pi_2(p)^2 \equiv 6p \pmod{11} \text{ or } \pi_2(p)^2 \equiv 8p \pmod{11}, \\ 11 & \text{if } \pi_2(p)^2 \equiv 4p \pmod{11}, \\ 12 & \text{if } \pi_2(p)^2 \equiv 7p \pmod{11} \text{ or } \pi_2(p)^2 \equiv 8p \pmod{11}, \end{cases} \tag{8.4}$$

and

$$g_2(p) = f_2(\pi_2(p), p^{11}, \alpha_2(p)) \tag{8.5}$$

with

$$f_2(x, y, \alpha_2(p)) := \begin{cases} -y & \text{if } \alpha_2(p) = 2, \\ -xy & \text{if } \alpha_2(p) = 3, \\ -y(x^2 - y) & \text{if } \alpha_2(p) = 4, \\ -xy(x^2 - 2y) & \text{if } \alpha_2(p) = 5, \\ -y(x^4 - 3x^2y + y^2) & \text{if } \alpha_2(p) = 6, \\ -y(x^2 - y)(x^6 - 6x^4y + 9x^2y^2 - y^3) & \text{if } \alpha_2(p) = 10, \\ -xy(x^4 - 3x^2y + y^2)(x^4 - 5x^2y + 5y^2) & \text{if } \alpha_2(p) = 11, \\ -y(x^{10} - 9x^8y + 28x^6y^2 - 35x^4y^3 + 15x^2y^4 - y^5) & \text{if } \alpha_2(p) = 12. \end{cases}$$

Theorem 8.1. *Let $p \geq 5$ be a prime with $p \neq 11$ and let $\operatorname{spt}(n)$ be defined by (8.1).*

(i) *For $n,\ k \geq 0$, if $p \nmid n$, then*

$$\operatorname{spt}\left(11p^{2\alpha_2(p)(k+1)-1}n + \frac{143p^{2\alpha_2(p)(k+1)} + 1}{24}\right) \equiv 0 \pmod{11}. \tag{8.6}$$

Moreover, for $k \geq 0$,

$$\operatorname{spt}\left(\frac{143p^{2\alpha_2(p)k} + 1}{24}\right) \equiv 4[g_2(p)]^k \pmod{11}. \tag{8.7}$$

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

(ii) *For* $n,\ k \geq 0$, *if*

$$\pi_2(p)\left(\frac{(13(p^2-1)/24)-n}{p}\right)_L \equiv (-1)^{(p-1)(p-27)/8}p^5 \pmod{11},$$

then

$$\operatorname{spt}\left(11p^{2\alpha_2(p)k+2}n + \frac{143p^{2\alpha_2(p)k+2}+1}{24}\right) \equiv 0 \pmod{11}. \tag{8.8}$$

(iii) *If* a *is a non-negative integer such that* $p_{13}(a) \equiv 0 \pmod{11}$ *and* $24a+13 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$, *with each* $\alpha_j \geq 2$, *is the prime factorization of* $24a+13$. *Then for* $n \geq 1$,

$$\operatorname{spt}\left(11an^2 + \frac{143n^2+1}{24}\right) \equiv 0 \pmod{11}, \tag{8.9}$$

where $\gcd(n, 6\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$.

(iv) *If* a *is a non-negative integer such that* $p_{13}(a) \equiv 0 \pmod{11}$, *then for* $n \geq 0$,

$$\operatorname{spt}\left(11an^{1320} + \frac{143n^{1320}+1}{24}\right) \equiv 0 \pmod{11}, \tag{8.10}$$

where $\gcd(n, 66) = 1$.

Example. We can verify that $\pi_2(5) \equiv 4 \pmod{11}$ and $g_2(5) \equiv 2 \pmod{11}$. Thus, $\alpha_2(5) = 3$. If we set $p = 5$ in the above theorem, we see that for $n,\ k \geq 0$,

$$\operatorname{spt}\left(11 \times 5^{6(k+1)-1}(5n+t) + \frac{143 \times 5^{6(k+1)}+1}{24}\right) \equiv 0 \pmod{11},$$

where t is an integer with $1 \leq t \leq 4$ and for $k \geq 0$,

$$\operatorname{spt}\left(\frac{143 \times 5^{6k}+1}{24}\right) \equiv 2^{k+2} \pmod{11}.$$

If we set $a = 7$ in (8.9) and using the fact that $11|p_{13}(7)$, we deduce that for $n \geq 0$,

$$\operatorname{spt}\left(\frac{1991n(3n+1)}{2} + 83\right) \equiv \operatorname{spt}\left(\frac{1991n(3n+5)}{2} + 2074\right) \equiv 0 \pmod{11}.$$

Proof. Setting $r = 13$ and $M = 11$ in Theorem 1.1, we see that for $n,\ k \geq 0$, if $p \nmid n$, then

$$p_{13}\left(p^{2R_{G_p}(11)(k+1)-1}n + \frac{13(p^{2R_{G_p}(11)(k+1)}-1)}{24}\right) \equiv 0 \pmod{11}, \tag{8.11}$$

where $p \geq 5$ is a prime with $p \neq 11$, $\pi_2(p)$ is defined by (8.3), $G_p(n)$ are the $(\pi_2(p), p^{11})$-Lucas sequences and $R_{G_p}(11)$ are the ranks of $G_p(n)$ modulo 11. We can

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

verify that

$$R_{G_p}(11) = \alpha_2(p), \tag{8.12}$$

where $\alpha_2(p)$ is defined by (8.4). Therefore, by (8.11) and (8.12),

$$p_{13}\left(p^{2\alpha_2(p)(k+1)-1}n + \frac{13(p^{2\alpha_2(p)(k+1)} - 1)}{24}\right) \equiv 0 \pmod{11}. \tag{8.13}$$

Replacing n by $p^{2\alpha_2(p)(k+1)-1}n + 13(p^{2\alpha_2(p)(k+1)} - 1)/24$ in (8.2) and using (8.13), we arrive at (8.6).

Moreover, from (1.6) and (8.12), we see that for $k \geq 0$,

$$p_{13}\left(\frac{13(p^{2\alpha_2(p)k} - 1)}{24}\right) \equiv T_p(\alpha_2(p))^k \pmod{11}, \tag{8.14}$$

where $T_p(n)$ are the dual $(\pi_2(p), p^{11})$-Lucas sequences of $G_p(n)$. It is easy to verify that

$$T_p(\alpha_2(p)) \equiv g_2(p) \pmod{11}, \tag{8.15}$$

where $g_2(p)$ is defined by (8.5). Replacing n by $13(p^{2\alpha_2(p)k} - 1)/24$ in (8.2) and employing (8.14) and (8.15), we obtain (8.7).

Thanks to (1.7) and (8.12), we deduce that if

$$\pi_2(p)\left(\frac{(13(p^2-1)/24) - n}{p}\right)_L \equiv (-1)^{(p-1)(p-27)/8}p^5 \pmod{11},$$

then

$$p_{13}\left(p^{2\alpha_2(p)k+2}n + \frac{13(p^{2\alpha_2(p)k+2} - 1)}{24}\right) \equiv 0 \pmod{11}. \tag{8.16}$$

Replacing n by

$$p^{2\alpha_2(p)k+2}n + \frac{13(p^{2\alpha_2(p)k+2} - 1)}{24}$$

in (8.2) and utilizing (8.16), we arrive at (8.8).

Setting $r = 13$ and $M = 11$ in Theorem 1.2, we find that if a is a non-negative integer such that $p_{13}(a) \equiv 0 \pmod{11}$ and $24a + 13 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$, with each $\alpha_j \geq 2$, is the prime factorization of $24a + 13$, then for $n \geq 1$,

$$p_{13}\left(an^2 + \frac{13(n^2-1)}{24}\right) \equiv 0 \pmod{11}, \tag{8.17}$$

where $\gcd(n, 6\prod_{j=1}^{v} g_j^{\alpha_j}) = 1$. Replacing n by $an^2 + 13(n^2 - 1)/24$ in (8.2) and using (8.17), we get (8.9).

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

Furthermore, it follows from the definition of $\gamma(M)$ given in Theorem 1.3 that

$$\gamma(11) = 660. \tag{8.18}$$

In view of (1.10) and (8.18), we see that if $p_{13}(a) \equiv 0 \pmod{11}$, then

$$p_{13}\left(an^{1320} + \frac{13(n^{1320}-1)}{24}\right) \equiv 0 \pmod{11}, \tag{8.19}$$

where $\gcd(n, 66) = 1$. Congruence (8.10) follows from (8.2) and (8.19). This completes the proof. □

9. Nonlinear congruences modulo p for p-regular partitions

Let $l \geq 2$ be an integer. A partition is called an l-regular partition if there is no part divisible by l. Let $b_l(n)$ denote the number of l-regular partitions of n. As usual, set $b_l(0) = 1$. The generating function of $b_l(n)$ is

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{(q^l;q^l)_\infty}{(q;q)_\infty}. \tag{9.1}$$

Recently, numerous congruence properties for l-regular partitions have been proved; see, for example, [**6**, **7**, **12**].

Using Theorem 1.4, we can also find congruences modulo p for $b_p(n)$ where p is an odd prime with $p \leq 23$. In the following, we only present congruences modulo 23 for $b_{23}(n)$.

In order to state our main results, define

$$\pi_3(p) = p_{22}\left(\frac{11(p-1)}{12}\right), \tag{9.2}$$

where p is a prime with $12|(p-1)$. Furthermore, define

$$\alpha_3(p) := \begin{cases} 2 & \text{if } \pi_3(p) \equiv 0 \pmod{23}, \\ 3 & \text{if } \pi_3(p)^2 \equiv p^{10} \pmod{23}, \\ 4 & \text{if } \pi_3(p)^2 \equiv 2p^{10} \pmod{23}, \\ 6 & \text{if } \pi_3(p)^2 \equiv 3p^{10} \pmod{23}, \\ 8 & \text{if } \pi_3(p)^2 \equiv tp^{10} \pmod{23} \text{ with } t \in \{7, 20\}, \\ 11 & \text{if } \pi_3(p)^2 \equiv tp^{10} \pmod{23} \text{ with } t \in \{6, 8, 12, 13, 16\}, \\ 12 & \text{if } \pi_3(p)^2 \equiv tp^{10} \pmod{23} \text{ with } t \in \{9, 18\}, \\ 22 & \text{if } \pi_3(p)^2 \equiv tp^{10} \pmod{23} \text{ with } t \in \{11, 14, 15, 19, 21\}, \\ 23 & \text{if } \pi_3(p)^2 \equiv 4p^{10} \pmod{23}, \\ 24 & \text{if } \pi_3(p)^2 \equiv tp^{10} \pmod{23} \text{ with } t \in \{5, 10, 17, 22\}, \end{cases} \tag{9.3}$$

and

$$g_3(p) = f_3(\pi_3(p), p^{10}, \alpha_3(p)) \tag{9.4}$$

with

$$f_3(x,y,\alpha_3(p)) := \begin{cases} -y & \text{if } \alpha_3(p)=2, \\ -xy & \text{if } \alpha_3(p)=3, \\ -y(x^2-y) & \text{if } \alpha_3(p)=4, \\ -y(x^4-3x^2y+y^2) & \text{if } \alpha_3(p)=6, \\ -y(x^6-5x^4y+6x^2y^2-y^3) & \text{if } \alpha_3(p)=8, \\ -xy(x^4-3x^2y+y^2)(x^4-5x^2y+5y^2) & \text{if } \alpha_3(p)=11, \\ -y(x^{10}-9x^8y+28x^6y^2-35x^4y^3+15x^2y^4-y^5) & \text{if } \alpha_3(p)=12, \\ -y(x^2-y)(x^6-5x^4y+6x^2y^2-y^3)(x^{12}-13x^{10}y \\ \quad +64x^8y^2-146x^6y^3+148x^4y^4-48x^2y^5+y^6) & \text{if } \alpha_3(p)=22, \\ -xy(x^{10}-11x^8y+44x^6y^2-77x^4y^3+55x^2y^4-11y^5) \\ \quad \times(x^{10}-9x^8y+28x^6y^2-35x^4y^3+15x^2y^4-y^5) & \text{if } \alpha_3(p)=23, \\ -y(x^{22}-21x^{20}y+190x^{18}y^2-969x^{16}y^3+3060x^{14}y^4 \\ \quad -6188x^{12}y^5+8008x^{10}y^6-6435x^8y^7+3003x^6y^8 \\ \quad -715x^4y^9+66x^2y^{10}-y^{11}) & \text{if } \alpha_3(p)=24. \end{cases}$$

Theorem 9.1. *Let p be a prime with $p \equiv 1 \pmod{12}$. For $n,\ k \geq 0$, if $p \nmid (12n+11)$, then*

$$b_{23}\left(p^{\alpha_3(p)(k+1)-1}n + \frac{11(p^{\alpha_3(p)(k+1)-1}-1)}{12}\right) \equiv 0 \pmod{23}, \tag{9.5}$$

where $\alpha_3(p)$ is defined by (9.3). Moreover, for $k \geq 0$,

$$b_{23}\left(\frac{11(p^{\alpha_3(p)k}-1)}{12}\right) \equiv g_3(p)^k \pmod{23}, \tag{9.6}$$

where $g_3(p)$ is defined by (9.4).

Example. It is easy to check that $\pi_3(13) \equiv 10 \pmod{23}$. Hence $\alpha_3(13) = 11$ and $g_3(13) \equiv 1 \pmod{23}$. If we set $p = 13$ in the above theorem, we see that for $n,\ k \geq 0$, if

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press

$13 \nmid (12n+11)$, then

$$b_{23}\left(13^{11k+10}n + \frac{11(13^{11k+10}-1)}{12}\right) \equiv 0 \pmod{23}$$

and

$$b_{23}\left(\frac{11(13^{11k}-1)}{12}\right) \equiv 1 \pmod{23}.$$

Proof. Setting $r = 22$ and $M = 23$ in Theorem 1.4, we see that for $n,\ k \geq 0$, if $p \nmid (12n+11)$, then

$$p_{22}\left(p^{R_{S_p}(23)(k+1)-1}n + \frac{11(p^{R_{S_p}(23)(k+1)-1}-1)}{12}\right) \equiv 0 \pmod{23}, \tag{9.7}$$

where p is a prime with $p \equiv 1 \pmod{12}$, $S_p(n)$ are the $(\pi_3(p), p^{10})$-Lucas sequences and $R_{S_p}(23)$ are the ranks of $S_p(n)$ modulo 23. It is easy to check that

$$R_{S_p}(23) = \alpha_3(p), \tag{9.8}$$

where $\alpha_3(p)$ is defined by (9.3). Combining (9.7) and (9.8) yields

$$p_{22}\left(p^{\alpha_3(p)(k+1)-1}n + \frac{11(p^{\alpha_3(p)(k+1)-1}-1)}{12}\right) \equiv 0 \pmod{23}. \tag{9.9}$$

Based on (1.1), (7.11) and (9.1),

$$\sum_{n=0}^{\infty} b_{23}(n)q^n = \frac{(q^{23};q^{23})_\infty}{(q;q)_\infty} \equiv (q;q)_\infty^{22} = \sum_{n=0}^{\infty} p_{22}(n)q^n \pmod{23},$$

which implies

$$b_{23}(n) \equiv p_{22}(n) \pmod{23}. \tag{9.10}$$

Congruence (9.5) follows from (9.9) and (9.10).

Furthermore, it follows from (1.12) and (9.8) that for $k \geq 0$,

$$p_{22}\left(\frac{11(p^{\alpha_3(p)k}-1)}{12}\right) \equiv T_p(\alpha_3(p))^k \pmod{23}, \tag{9.11}$$

where $T_p(n)$ are the dual $(\pi_3(p), p^{10})$-Lucas sequences of $S_p(n)$. It is easy to verify that

$$T_p(\alpha_3(p)) \equiv g_3(p) \pmod{23}, \tag{9.12}$$

where $g_3(p)$ is defined by (9.4). In view of (9.10)–(9.12), we arrive at (9.6). This completes the proof. □

Acknowledgements. The author would like to thank the anonymous referee for his/her helpful comments. This work was supported by the National Science Foundation of China (no. 11971203, 11571143) and the Natural Science Foundation of Jiangsu Province of China (no. BK20180044).

References

1. G. E. Andrews, The number of smallest parts in the partitions of n, *J. Reine Angew. Math.* **624** (2008), 133–142.
2. S. H. Chan, Congruences for Ramanujan's ϕ function, *Acta Arith.* **153** (2012), 161–189.
3. S. P. Cui, S. S. Gu and A. X. Huang, Congruence properties for a certain kind of partition functions, *Adv. Math.* **290** (2016), 739–772.
4. F. Garvan, Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank, *Int. J. Number Theory* **6** (2010), 281–309.
5. F. Garvan, Congruences for Andrews' spt-function modulo powers of 5, 7 and 13, *Trans. Amer. Math. Soc.* **364** (2012), 4847–4873.
6. M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of partitions with odd parts distinct, *Ramanujan J.* **22** (2010), 273–284.
7. W. J. Keith, Congruences for 9-regular partitions modulo 3, *Ramanujan J.* **35** (2014), 157–164.
8. E. Lucas, Théorie des fonctions numériques simplement périodiques, *Amer. J. Math.* **1** (1878), 184–240.
9. M. Newman, Remarks on some modular identities, *Trans. Amer. Math. Soc.* **73** (1952), 313–320.
10. M. Newman, Further identities and congruences for the coefficients of modular forms, *Canad. J. Math.* **10** (1958), 577–586.
11. M. Renault, The period, rank, and order of the (a, b)-Fibonacci sequence mod m, *Math. Magazine* **86** (2013), 372–380.
12. O. X. M. Yao, New congruences modulo powers of 2 and 3 for 9-regular partitions, *J. Number Theory* **142** (2014), 89–101.

https://doi.org/10.1017/S0013091520000115 Published online by Cambridge University Press