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EXPECTED TOPOLOGY OF RANDOM REAL ALGEBRAIC SUBMANIFOLDS

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Abstract Let X be a smooth complex projective manifold of dimension n equipped with an ample line bundle L and a rank k holomorphic vector bundle E. We assume that $1 \leqslant k \leqslant n$, that X, E and L are defined over the reals and denote by $\mathbb{R}X$ the real locus of X. Then, we estimate from above and below the expected Betti numbers of the vanishing loci in $\mathbb{R}X$ of holomorphic real sections of $E \otimes L^d$, where d is a large enough integer. Moreover, given any closed connected codimension k submanifold Σ of $\mathbb{R}^n$ with trivial normal bundle, we prove that a real section of $E \otimes L^d$ has a positive probability, independent of d, of containing around $\sqrt{d}^n$ connected components diffeomorphic to Σ in its vanishing locus.

Keywords: real projective manifold; ample line bundle; random polynomial; Betti numbers

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1. Introduction

Let X be a smooth complex projective manifold of positive dimension n equipped with an ample line bundle L and let E be a holomorphic vector bundle of rank k over X. From the vanishing theorem of Kodaira and Serre, we know that the dimension N_d of the complex vector space $H^0(X, E \otimes L^d)$ of global holomorphic sections of $E \otimes L^d$ grows as a polynomial of degree n in d. We will assume throughout this paper that $1 \leqslant k \leqslant n$ and that X, E and L are defined over the reals. We denote by $\mathbb{R}X$ the real locus of X and by $\mathbb{R}H^0(X, E \otimes L^d)$ the real vector space of real holomorphic sections of $E \otimes L^d$; see (5). Its dimension equals N_d. The discriminant locus $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, E \otimes L^d)$ of sections which do not vanish transversally is a codimension 1 submanifold for d large enough, and for every σ in its complement, the real vanishing locus $\mathbb{R}C_\sigma$ of σ is a smooth codimension k submanifold of $\mathbb{R}X$. The topology of $\mathbb{R}C_\sigma$ drastically depends on the choice of $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d$. When $n = k = 1$, $X = \mathbb{C}P^1$, $L = \mathcal{O}_{\mathbb{C}P^1}(1)$ and $E = \mathcal{O}_{\mathbb{C}P^1}$ for example, σ is a real polynomial of degree d in one variable and $\mathbb{R}C_\sigma$ the set of its real roots.

The space $\mathbb{R}H^0(X, E \otimes L^d)$ inherits classical probability measures. Indeed, let h_E be a Hermitian metric on E and h_L be a Hermitian metric of positive curvature on L, both h_E and h_L being real, that is invariant under the $\mathbb{Z}/2\mathbb{Z}$-Galois action of E and L. We denote by $h_{E,d} = h_E \otimes h_L^d$ the induced metric on $E \otimes L^d$. Then, the vector space $\mathbb{R}H^0(X, E \otimes L^d)$ becomes Euclidean, with the L^2-scalar product defined by

$$\forall \sigma, \tau \in \mathbb{R}H^0(X, E \otimes L^d), \quad \langle \sigma, \tau \rangle = \int_X h_{E,d}(\sigma, \tau) dx,$$

where dx denotes any chosen volume form on X (our results being asymptotic in d, they turn out not to depend on the choice of dx). It thus inherits a Gaussian probability measure $\mu_{\mathbb{R}}$ whose density at $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ with respect to the Lebesgue measure is $(1/\sqrt{\pi}^{N_d}) e^{-\|\sigma\|^2}$.

What is the typical topology of $\mathbb{R}C_\sigma$ for $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ chosen at random for $d\mu_{\mathbb{R}}$? We do not know, but can estimate its average Betti numbers. To formulate our results, let us denote, for every $i \in \{0, \dots, n-k\}$, by $b_i(\mathbb{R}C_\sigma, \mathbb{R}) = \dim H_i(\mathbb{R}C_\sigma, \mathbb{R})$ the ith Betti number of $\mathbb{R}C_\sigma$ and by

$$\mathbb{E}(b_i) = \int_{\mathbb{R}H^0(X, E\otimes L^d) \setminus \mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma, \mathbb{R}) d\mu_{\mathbb{R}}(\sigma)$$

its expected value.

1.1. Upper estimates

As in [14], for every $i \in \{0, \dots, n-k\}$, we denote by $Sym_{\mathbb{R}}(i, n-k-i)$ the open cone of real symmetric matrices of size $n-k$ and signature $(i, n-k-i)$, by $\mu_{\mathbb{R}}$ the classical Gaussian measure on the space of real symmetric matrices and by $e_{\mathbb{R}}(i, n-k-i)$ the numbers

$$e_{\mathbb{R}}(i, n-k-i) = \int_{Sym_{\mathbb{R}}(i, n-k-i)} |\det A| d\mu_{\mathbb{R}}(A); \tag{1}$$

see § 3.1. We then denote by $Vol_{h_L}(\mathbb{R}X)$ the volume of $\mathbb{R}X$ for the Riemannian metric induced by the Kähler metric g_{h_L} defined by the curvature form of h_L; see (3) and (4).

Theorem 1.1.1. *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$, $k \neq n$. Then, for every $0 \leqslant i \leqslant n-k$,*

$$\limsup_{d \to \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(b_i) \leqslant \binom{n-1}{k-1} e_{\mathbb{R}}(i, n-k-i) \frac{Vol_{h_L}(\mathbb{R}X)}{Vol_{FS}(\mathbb{R}P^k)}.$$

Moreover, when $k = n$, $(1/\sqrt{d}^n)\mathbb{E}(b_0)$ converges to $Vol_{h_L}(\mathbb{R}X)/Vol_{FS}(\mathbb{R}P^n)$ as d grows to infinity.

In fact, the right hand side of the inequality given by Theorem 1.1.1 also involves the determinant of random matrices of size $k-1$ and the volume of the Grassmann manifold of $(k-1)$ linear subspaces of $\mathbb{R}^{n-1}$ (see Theorem 3.1.2), but these can be

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computed explicitly. Note that when E is the trivial line bundle, Theorem 1.1.1 reduces to Theorem 1.1 of [14].

Theorem 1.1.1 relies on Theorem 3.1.3, which establishes the asymptotic equidistribution of clouds of critical points; see § 3.1. We obtain a similar result in a complex projective setting, for critical points of Lefschetz pencils; see Theorem 3.5.1.

1.2. Lower estimates and topology

Let Σ be a closed submanifold of codimension k of $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, which we do not assume to be connected. For every $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d$, we denote by $N_\Sigma(\sigma)$ the maximal number of disjoint open subsets of $\mathbb{R}X$ having the property that each such open subset U' contains a codimension k submanifold Σ' such that $\Sigma' \subset \mathbb{R}C_\sigma$ and (U', Σ') is diffeomorphic to $(\mathbb{R}^n, \Sigma)$. We then set

$$\mathbb{E}(N_\Sigma) = \int_{\mathbb{R}H^0(X, E\otimes L^d)\setminus\mathbb{R}\Delta_d} N_\Sigma(\sigma) d\mu_{\mathbb{R}}(\sigma) \tag{2}$$

and we associate with Σ, in fact with its isotopy class in $\mathbb{R}^n$, a constant c_Σ which is positive if and only if Σ has trivial normal bundle in $\mathbb{R}^n$; see (14) for its definition and Lemma 2.2.3. The latter measures *à la* Donaldson the amount of transversality that a polynomial map $\mathbb{R}^n \to \mathbb{R}^k$ vanishing along a submanifold isotopic to Σ may have.

Theorem 1.2.1. *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Let Σ be a closed submanifold of codimension k of $\mathbb{R}^n$ with trivial normal bundle, which does not need to be connected. Then,*

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n}\mathbb{E}(N_\Sigma) \geqslant c_\Sigma Vol_{h_L}(\mathbb{R}X).$$

In particular, when Σ is connected, Theorem 1.2.1 bounds from below the expected number of connected components diffeomorphic to Σ in the real vanishing locus of a random section $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$. The constant c_Σ does not depend on the choice of the triple $(X, (L, h_L), (E, h_E))$; it only depends on Σ. When $k = 1$ and $E = \mathcal{O}_X$, Theorem 1.2.1 coincides with Theorem 1.2 of [16]. Computing c_Σ for explicit submanifolds Σ yields the following lower bounds for the Betti numbers.

Corollary 1.2.2. *Under the hypotheses of Theorem 1.2.1, for every $i \in \{0, \ldots, n-k\}$,*

$$\liminf_{d\to\infty} \frac{1}{\sqrt{d}^n}\mathbb{E}(b_i) \geqslant \exp(-e^{84+6n}) Vol_{h_L}(\mathbb{R}X).$$

1.3. Some related results

The case where $X = \mathbb{C}P^1$, $E = \mathcal{O}_{\mathbb{C}P^1}$ and $L = \mathcal{O}_{\mathbb{C}P^1}(1)$ was first considered by M. Kac in [18] for a different measure. In this case and with our measure, Kostlan [19] and Shub and Smale [34] gave an exact formula for the mean number of real roots of a polynomial, as well as the mean number of intersection points of n hypersurfaces in $\mathbb{R}P^n$. Still in

$\mathbb{R}P^n$, Podkorytov [27] computed the mean Euler characteristics of random algebraic hypersufaces, and Bürgisser [4] extended this result to complete intersections. In [13], we proved the exponential rarefaction of real curves with a maximal number of components in real algebraic surfaces. In [14, 15], we bounded from above the mean Betti numbers of random real hypersurfaces in real projective manifolds and in [16], we gave a lower bound for them.

A similar probabilistic study of complex projective manifolds has been performed by Shiffman and Zelditch (see [2, 30, 33] for example, and also [3, 36]). In particular, the asymptotic equidistribution of critical points of random sections over a fixed projective manifold has been studied in [8, 9, 22], and also [1, 5, 11], while we studied critical points of the restriction of a fixed Morse function on random real hypersurfaces; see [14, 15].

A similar question concerns the mean number of components of the vanishing locus of eigenfunctions of the Laplacian. It has been studied on the round sphere by Nazarov and Sodin [25] (see also [35]), Lerario and Lundberg [20] and Sarnak and Wigman [28]. For a general Riemannian setting, Zelditch proved in [38] the equidistribution of the vanishing locus, whereas critical points of random eigenfunctions of the Laplacian were addressed by Nicolaescu in [26].

Section 2 is devoted to lower estimates and the proof of Theorem 1.2.1. In this proof, the L^2-estimates of Hörmander play a crucial rôle (see § 2.3), and we follow the same approach as in [16] (see also [12] for a similar construction). Section 3 is devoted to upper estimates and the proof of Theorem 1.1.1.

2. Lower estimates for the expected Betti numbers

2.1. Statement of the results

2.1.1. Framework. Let us first recall our framework. We denote by X a smooth complex projective manifold of dimension n defined over the reals, by $c_X : X \to X$ the induced Galois antiholomorphic involution and by $\mathbb{R}X = \mathrm{Fix}(c_X)$ the real locus of X which we implicitly assume to be non-empty. We then consider an ample line bundle L over X, also defined over the reals. It comes thus equipped with an antiholomorphic involution $c_L : L \to L$ which turns the bundle projection map $\pi : L \to X$ into a $\mathbb{Z}/2\mathbb{Z}$-equivariant one, so that $c_X \circ \pi = \pi \circ c_L$. We equip L in addition with a real Hermitian metric h_L, being thus invariant under c_L, which has a positive curvature form ω locally defined by

$$\omega = \frac{1}{2i\pi}\partial\bar{\partial}\log h_L(e, e) \tag{3}$$

for any non-vanishing local holomorphic section e of L. This metric induces a Kähler metric

$$g_{h_L} = \omega(\,.\,, i\,.\,) \tag{4}$$

on X, which reduces to a Riemannian metric g_{h_L} on $\mathbb{R}X$. Let finally E be a holomorphic vector bundle of rank k, $1 \leqslant k \leqslant n$, defined over the reals and equipped with an antiholomorphic involution c_E and a real Hermitian metric h_E. For every $d > 0$, we denote by

$$\mathbb{R}H^0(X, E \otimes L^d) = \{\sigma \in H^0(X, E \otimes L^d) \mid (c_E \otimes c_{L^d}) \circ \sigma = \sigma \circ c_X\} \tag{5}$$

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the space of global real holomorphic sections of $E \otimes L^d$. It is equipped with the L^2-scalar product defined by the formula

$$\forall (\sigma, \tau) \in \mathbb{R}H^0(X, E \otimes L^d), \quad \langle \sigma, \tau \rangle = \int_X h_{E,d}(\sigma, \tau)(x) dx, \tag{6}$$

where $h_{E,d} = h_E \otimes h_L^d$. Here, dx denotes any volume form of X. For instance, dx can be chosen to be the normalized volume form $dV_{h_L} = \omega^n / \int_X \omega^n$.This L^2-scalar product finally induces a Gaussian probability measure $\mu_{\mathbb{R}}$ on $\mathbb{R}H^0(X, E \otimes L^d)$ whose density with respect to the Lebesgue one at $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ is written as $(1/\sqrt{\pi}^{N_d}) e^{-\|\sigma\|^2}$, where $N_d = \dim H^0(X, E \otimes L^d)$. It is with respect to this probability measure that we consider random real codimension k submanifolds (as in the works [19] and [14–16, 34]).

2.1.2. The lower estimates. The aim of § 2 is to prove Theorem 1.2.1. In addition to Theorem 1.2.1, we also get the following Theorem 2.1.1, which is a consequence of Proposition 2.4.2 below.

Theorem 2.1.1. *Under the hypotheses of Theorem 1.2.1, for every* $0 \leqslant \epsilon < 1$,

$$\liminf_{d \to \infty} \mu_{\mathbb{R}} \{ \sigma \in \mathbb{R}H^0(X, E \otimes L^d) \mid N_\Sigma(\sigma) \geqslant \epsilon c_\Sigma Vol_{h_L}(\mathbb{R}X) \sqrt{d}^n \} > 0.$$

In fact, the positive lower bound given by Theorem 2.1.1 can be made explicit; see (30).

Let us now denote, for every $1 \leqslant k \leqslant n$, by $\mathcal{H}_{n,k}$ the set of diffeomorphism classes of smooth closed connected codimension k submanifolds of $\mathbb{R}^n$. For every $i \in \{0, \ldots, n-k\}$ and every $[\Sigma] \in \mathcal{H}_{n,k}$, we denote by $b_i(\Sigma) = \dim H_i(\Sigma; \mathbb{R})$ its ith Betti number with real coefficients and by $m_i(\Sigma)$ its ith Morse number. This is the infimum over all Morse functions f on Σ of the number of critical points of index i of f. Then, we set $c_{[\Sigma]} = \sup_{\Sigma \in [\Sigma]} c_\Sigma$ and

$$\mathbb{E}(m_i) = \int_{\mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_d} m_i(\mathbb{R}C_\sigma) d\mu_{\mathbb{R}}(\sigma).$$

Corollary 2.1.2. *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Then, for every $i \in \{0, \ldots, n-k\}$,*

$$\liminf_{d \to \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(b_i) \geqslant \Bigg(\sum_{[\Sigma] \in \mathcal{H}_{n,k}} c_{[\Sigma]} b_i(\Sigma) \Bigg) Vol_{h_L}(\mathbb{R}X) \ \textit{and likewise} \tag{7}$$

$$\liminf_{d \to \infty} \frac{1}{\sqrt{d}^n} \mathbb{E}(m_i) \geqslant \Bigg(\sum_{[\Sigma] \in \mathcal{H}_{n,k}} c_{[\Sigma]} m_i(\Sigma) \Bigg) Vol_{h_L}(\mathbb{R}X). \tag{8}$$

Note that in Corollary 2.1.2, we could have chosen one representative Σ in each diffeomorphism class $[\Sigma] \in \mathcal{H}_{n,k}$ and obtained the lower estimates (7), (8) with constants c_Σ instead of $c_{[\Sigma]}$. But it turns out that in the proof of Corollary 2.1.2 we are free to choose the representative that we wish in every diffeomorphism class and that the higher c_Σ is, the better the estimates (7), (8) are. This is why we introduce the constant $c_{[\Sigma]}$, which is positive if and only if $[\Sigma]$ has a representative Σ with trivial normal bundle in $\mathbb{R}^n$; see (14) and Lemma 2.2.3.

2.2. Closed affine real algebraic submanifolds

We introduce here the notion of a regular pair (see Definition 2.2.1), and the constant c_Σ associated with any isotopy class of the smooth closed codimension k submanifold Σ of $\mathbb{R}^n$; see (14).

Definition 2.2.1. Let U be a bounded open subset of $\mathbb{R}^n$ and $P \in \mathbb{R}[x_1, \dots x_n]^k$, $1 \leqslant k \leqslant n$. The pair (U, P) is said to be regular if and only if

1. zero is a regular value of the restriction of P to U,
2. the vanishing locus of P in U is compact.

Hence, for every regular pair (U, P), the vanishing locus of P does not intersect the boundary of U and it meets U in a smooth compact codimension k submanifold.

In the sequel, for every integer p and every vector $v \in \mathbb{R}^p$, we denote by $|v|$ its Euclidean norm, and for every integer p and q, and every linear map $F : \mathbb{R}^p \to \mathbb{R}^q$, we denote by F^* the adjoint of F, defined by the property

$$\forall v \in \mathbb{R}^p, \forall w \in \mathbb{R}^q, \quad \langle F(v), w\rangle = \langle v, F^*(w)\rangle,$$

and denote by $\|F\|$ its operator norm, that is

$$\|F\| = \sup_{v\in\mathbb{R}^p\setminus\{0\}} |F(v)|/|v|.$$

We will also use the norm

$$\|F\|_2 = \sqrt{\mathrm{Tr}\ FF^*}.$$

These norms satisfy $\|F\| \leqslant \|F\|_2$. Finally, if $P = (P_1, \dots, P_k) \in \mathbb{R}[x_1, \dots, x_n]^k$, we denote by $\|P\|_{L^2}$ its L^2-norm defined by

$$\|P\|_{L^2}^2 = \int_{\mathbb{C}^n} |P(z)|^2 e^{-\pi|z|^2} dz = \sum_{i=1}^{k} \int_{\mathbb{C}^n} |P_i(z)|^2 e^{-\pi|z|^2} dz = \sum_{i=1}^{k} \|P_i\|_{L^2}^2. \tag{9}$$

Definition 2.2.2. For every regular pair (U, P) given by Definition 2.2.1, we denote by $\mathcal{T}_{(U,P)}$ the set of $(\delta, \epsilon) \in (\mathbb{R}_+^*)^2$ such that

1. there exists a compact subset K of U satisfying $\inf_{x\in U\setminus K} |P(x)| > \delta$,
2. for every $y \in U$, $|P(y)| < \delta \Rightarrow \forall w \in \mathbb{R}^k,\ |(d_{|y}P)^*(w)| \geqslant \epsilon|w|$.

Hence, for every regular pair (U, P) given by Definition 2.2.1, (δ, ϵ) belongs to $\mathcal{T}_{(U,P)}$ provided that the δ-sublevel of P does not intersect the boundary of U while inside this δ-sublevel, and P is in a sense ϵ-far from having a critical point. This quantifies how much transversally P vanishes in a way similar to the one used by Donaldson in [7].

Then, for every regular pair (U, P), we set $R_{(U,P)} = \max(1, \sup_{y\in U} |y|)$, so U is contained in the ball centered at the origin and of radius $R_{(U,P)}$. Finally, we set

$$\tau_{(U,P)} = 24k\rho_{R_{(U,P)}} \|P\|_{L^2}^2 \inf_{(\delta,\epsilon)\in\mathcal{T}_{(U,P)}} \left(\frac{1}{\delta^2} + \frac{\pi n}{\epsilon^2}\right) \in \mathbb{R}_+^*, \tag{10}$$

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where, for every $R > 0$,

$$\rho_R = \inf_{\mathbb{R}^+} g_R, \tag{11}$$

$$g_R : s \in \mathbb{R}_+^* \mapsto \frac{(R+s)^{2n}}{s^{2n}} e^{\pi (R+s)^2}, \tag{12}$$

and so

$$e^{\pi R^2} \leqslant \rho_R \leqslant 4^n e^{4\pi R^2}. \tag{13}$$

This constant $\tau_{(U,P)}$ is the main ingredient in the definition of c_Σ; see (14). The lower $\tau_{(U,P)}$ is, the larger c_Σ is and the better the estimates given by Theorem 1.2.1 are. Note that $\tau_{(U,P)}$ remains small whenever δ, ϵ are not too small, that is when P vanishes quite transversally in U.

Now, let Σ be a closed submanifold of codimension k of $\mathbb{R}^n$, not necessarily connected. We denote by $\mathcal{I}_\Sigma$ the set of regular pairs (U, P) given by Definition 2.2.1, such that the vanishing locus of P in U contains a subset isotopic to Σ in $\mathbb{R}^n$.

Lemma 2.2.3. *Let Σ be a closed submanifold of codimension $k > 0$ of $\mathbb{R}^n$, not necessarily connected. Then, $\mathcal{I}_\Sigma$ is non-empty if and only if the normal bundle of Σ in $\mathbb{R}^n$ is trivial.*

Proof. If $(U, P) \in \mathcal{I}_\Sigma$, then $P : \mathbb{R}^n \to \mathbb{R}^k$ contains in its vanishing locus a codimension k submanifold $\widehat{\Sigma}$ which is isotopic to Σ in $\mathbb{R}^n$. The normal bundle of Σ in $\mathbb{R}^n$ is thus trivial if and only if the normal bundle of $\widehat{\Sigma}$ in $\mathbb{R}^n$ is trivial. But the differential of P at every point of $\widehat{\Sigma}$ provides an isomorphism between the normal bundle of $\widehat{\Sigma}$ in $\mathbb{R}^n$ and the product $\widehat{\Sigma} \times \mathbb{R}^k$.

Conversely, if Σ has a trivial normal bundle in $\mathbb{R}^n$, it has been proved by Seifert [29] (see also [24]) that there exist a polynomial map $P : \mathbb{R}^n \to \mathbb{R}^k$ and a tubular neighborhood U of Σ in $\mathbb{R}^n$ such that $P^{-1}(0) \cap U$ is isotopic to Σ in U. The strategy of the proof is to first find a smooth function $U \to \mathbb{R}^k$ in a neighborhood of Σ which vanishes transversally along Σ and then to suitably approximate the coordinates of this function by some polynomial; see [24, 29]. The pair (U, P) then belongs to $\mathcal{I}_\Sigma$ by Definition 2.2.1. □

We then set $c_\Sigma = 0$ if Σ does not have a trivial normal bundle in $\mathbb{R}^n$ and

$$c_\Sigma = \sup_{(U,P) \in \mathcal{I}_\Sigma} \left(\frac{m_{\tau_{(U,P)}}}{2^n Vol(B(R_{(U,P)}))} \right) \text{ otherwise}, \tag{14}$$

where $Vol(B(R_{(U,P)}))$ denotes the volume of the Euclidean ball of radius $R_{(U,P)}$ in $\mathbb{R}^n$, and where, for every $\tau > 0$,

$$m_\tau = \sup_{[\sqrt{\tau}, +\infty[} f_\tau, \tag{15}$$

with $f_\tau : a \in [\sqrt{\tau}, +\infty[\mapsto 1/\sqrt{\pi}(1 - (\tau/a^2)) \int_a^{+\infty} e^{-t^2} dt$. For large values of m_τ, such as the ones which appear in § 2.6, the estimate

$$c_\Sigma \geqslant e^{-2\tau_{(U,P)}} \tag{16}$$

holds; compare (2.8) of [16].

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2.3. Hörmander sections

Our key tool for proving Theorems 1.1.1 and 1.2.1 has been developed by L. Hörmander. We introduce in this part, § 2.3, the material that we need. For every positive d and every $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$, we set

$$\|\sigma\|^2_{L^2(h_L)} = \int_X \|\sigma\|^2_{h_{E,d}} dV_{h_L},$$

where $dV_{h_L} = \omega^n / \int_X \omega^n$; compare (6). Let us choose a field of h_L-trivializations of L on $\mathbb{R}X$ given by Definition 3.2 of [16]. It provides in particular, for every $x \in \mathbb{R}X$, a local holomorphic chart $\psi_x : (W_x, x) \subset X \to (V_x, 0) \subset \mathbb{C}^n$ isometric at x, and a non-vanishing holomorphic section e of L defined over W_x such that $\phi = -\log h_L(e, e)$ vanishes at x and is positive elsewhere. Moreover, there exists a positive constant α_1 such that

$$\forall y \in V_x, |\phi \circ \psi_x^{-1}(y) - \pi |y|^2| \leqslant \alpha_1 |y|^3. \tag{17}$$

Restricting W_x if necessary, we choose a holomophic trivialization $(e_1, \dots, e_k)$ of $E_{|W_x}$ which is orthonormal at x. This provides a trivialization $(e_1 \otimes e^d, \dots, e_k \otimes e^d)$ of $E \otimes L^d_{|W_x}$. In this trivialization, the restriction of σ to W_x is written as

$$\sigma = \sum_{j=1}^{k} f^j_\sigma e_j \otimes e^d \tag{18}$$

for some holomorphic functions $f^j_\sigma : W_x \to \mathbb{C}$. We write $f_\sigma = (f^1_\sigma, \dots, f^k_\sigma)$ and we set

$$|\sigma| = |f_\sigma|, \tag{19}$$

so on W_x, $\|\sigma\|^2_{h_{E,d}} = \| \sum_{j=1}^k f^k_\sigma e_j \|^2_{h_E} e^{-d\phi}$ and $\|\sigma(x)\|^2_{h_{E,d}} = |\sigma(x)|^2$ since the frames $(e_1, \dots, e_k)$ and e are orthonormal at the point x, and so in particular $\phi(x) = 0$. For every $z \in W_x$, we define

$$\|d_{|z}\sigma\|_2 = \|d_{|y}(f_\sigma \circ \psi_x^{-1})\|_2, \tag{20}$$

$$\|d_{|z}\sigma\| = \|d_{|y}(f_\sigma \circ \psi_x^{-1})\|, \tag{21}$$

and

$$(d_{|z}\sigma)^* = (d_{|y}(f_\sigma \circ \psi_x^{-1}))^*, \tag{22}$$

where $y = \psi_x(z)$. Finally, we denote, for every small enough $r > 0$, by $B(x, r) \subset W_x$ the ball centered at x and of radius r for the flat metric of V_x pulled back by ψ_x, so

$$B(x, r) = \psi_x^{-1}(B(0, r)). \tag{23}$$

Proposition 2.3.1. *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. We choose a field of h_L-trivializations on $\mathbb{R}X$. Then, for every regular pair (U, P), every large enough integer d, every x in $\mathbb{R}X$ and every local trivialization of E orthonormal at x, there exist $\sigma_{(U,P)} \in \mathbb{R}H^0(X, E \otimes L^d)$ and an open subset U_d of $B(x, R_{(U,P)}/\sqrt{d}) \cap \mathbb{R}X$ such that*

1. $\|\sigma_{(U,P)}\|_{L^2(h_L)}$ *becomes equivalent to* $((\|P\|_{L^2})/\sqrt{\delta_L})$ *as* d *grows to infinity, where* $\|P\|_{L^2}$ *is defined by (9) and* $\delta_L = \int_X \omega^n$,
2. $(U_d, \sigma_{(U,P)}^{-1}(0) \cap U_d)$ *is diffeomorphic to* $(U, P^{-1}(0) \cap U) \subset \mathbb{R}^n$,
3. *for every* $(\delta, \epsilon) \in \mathcal{T}_{(U,P)}$ *given by Definition 2.2.2, there exists a compact subset* $K_d \subset U_d$ *such that*
$$\inf_{U_d \setminus K_d} |\sigma_{(U,P)}| > \frac{\delta}{2}\sqrt{d}^n,$$
while for every y *in* U_d,
$$|\sigma_{(U,P)}(y)| < \frac{\delta}{2}\sqrt{d}^n \Rightarrow \forall w \in \mathbb{R}^k,\ |(d_{|y}\sigma_{(U,P)})^*(w)| \geqslant \frac{\epsilon}{2}\sqrt{d}^{n+1}|w|. \tag{24}$$

Proof. We proceed as in the proof of Proposition 3.4 of [16]. Let (U, P) be a regular pair, $x \in \mathbb{R}X$ and d large enough. We set $U_d = \psi_x^{-1}((1/\sqrt{d})U) \subset B(x, R_{(U,P)}/\sqrt{d})$ and $K_d = \psi_x^{-1}((1/\sqrt{d})K)$. Let $\chi : \mathbb{C}^n \to [0, 1]$ be a smooth function with compact support in $B(0, R_{(U,P)})$, which equals 1 in a neighborhood of the origin. Then, let σ be the global smooth section of $E \otimes L^d$ defined by $\sigma_{|X\setminus W_x} = 0$ and
$$\sigma_{|W_x} = (\chi \circ \psi_x)\left(\sum_{j=1}^{k} P_j(\sqrt{d}\psi_x)e_j \otimes e^d\right),$$
where $P = (P_1, \ldots, P_k)$ is now considered as a function $\mathbb{C}^n \to \mathbb{C}^k$. From the L^2-estimates of Hörmander (see [17] or [21]), there exists a global section τ of $E \otimes L^d$ such that $\bar\partial\tau = \bar\partial\sigma$ and $\|\tau\|_{L^2(h_{E,d})} \leqslant \|\bar\partial\sigma\|_{L^2(h_{E,d})}$ for d large enough. This section τ can be chosen orthogonal to holomorphic sections and is then unique, and in particular real. Moreover, there exist positive constants c_1 and c_2, which do not depend on x, such that $\|\tau\|_{L^2(h_{E,d})} \leqslant c_1 e^{-c_2 d}$ and $\sup_{U_d}(|\tau| + \|\tau\|_2) \leqslant c_2 e^{-c_2 d}$; see Lemma 3.5 of [16]. We then set $\sigma_{(U,P)} = \sqrt{d}^n(\sigma - \tau)$. It has the desired properties, as can be checked along the same lines as in the proof of Proposition 3.4 of [16] and thanks to Lemma 2.3.2. □

Lemma 2.3.2. *Let* U *be an open subset of* $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, $f : U \to \mathbb{R}^k$ *be a function of class* C^1 *and* $(\delta, \epsilon) \in (\mathbb{R}_+^*)^2$ *be such that*

1. *there exists a compact subset* K *of* U *such that* $\inf_{U\setminus K} |f| > \delta$,
2. *for every* y *in* U, $|f(y)| < \delta \Rightarrow \forall w \in \mathbb{R}^k,\ |(d_{|y}f)^*(w)| \geqslant \epsilon|w|$.

Then, for every function $g : U \to \mathbb{R}^k$ *of class* C^1 *such that* $\sup_U |g| < \delta$ *and* $\sup_U \|dg\| < \epsilon$, *zero is a regular value of* $f + g$ *and* $(f + g)^{-1}(0)$ *is compact and isotopic to* $f^{-1}(0)$ *in* U.

Proof. The proof is analogous to that of Lemma 3.6 of [16], since $\|(dg)^*\| = \|dg\|$. □

The following Lemma 2.3.3 establishes the existence of peak sections for higher rank vector bundles.

Lemma 2.3.3 (Compare Lemma 1.2 of [37])**.** *Let* X *be a smooth real projective manifold of dimension* n, (L, h_L) *be a real holomorphic Hermitian line bundle of positive curvature*

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over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Let $x \in \mathbb{R}X$, $(p_1, \dots, p_n) \in \mathbb{N}^n$, $i \in \{1, \dots, k\}$ and $p' > p_1 + \cdots + p_n$. There exists $d_0 \in \mathbb{N}$ independent of x such that for every $d > d_0$, there exists $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ with the property that $\|\sigma\|_{L^2(h_L)} = 1$ and if $(y_1, \dots, y_n)$ are local real holomorphic coordinates in the neighborhood of x and $(e_1, \dots e_k)$ is a local real holomorphic trivialization of E orthonormal at x, we can assume that in a neighborhood of x,

$$\sigma(y_1, \dots, y_n) = \lambda y_1^{p_1} \cdots y_n^{p_n} e_i \otimes e^d (1 + O(d^{-2p'})) + O(\lambda |y|^{2p'}), \tag{25}$$

where $\lambda^{-2} = \int_{B(x, (\log d/\sqrt{d}))} |y_1^{p_1} \cdots y_n^{p_n}|^2 \|e^d\|^2_{h_L^d} dV_{h_L}$, with $dV_{h_L} = \omega^n / \int_X \omega^n$ and where e is a local trivialization of L whose potential $-\log h_L(e, e)$ reaches a local minimum at x with Hessian $\pi\omega(., i.)$.

Proof. The proof goes along the same lines as that of Lemma 1.2 of [37]. Let η be a cutoff function on $\mathbb{R}$ with $\eta = 1$ in a neighborhood of 0, and

$$\psi = (n + 2p')\eta\left(\frac{d\|z\|^2}{\log^2 d}\right) \log\left(\frac{d\|z\|^2}{\log^2 d}\right)$$

in the coordinates z on X. Then, $i\partial\bar{\partial}\psi$ is bounded from below by $-C\omega$, where C is some uniform constant independent of d and x. Let $s \in C^\infty(X, E \otimes L^d)$ be the real section defined by

$$s = \eta\left(\frac{d\|z\|^2}{\log^2 d}\right) y_1^{p_1} \cdots y_n^{p_n} e_i \otimes e^d.$$

Then, from Theorem 5.1 of [6], for d large enough and not depending on x, there exists a real section $u \in C^\infty(X, E \otimes L^d)$ such that $\bar{\partial}u = \bar{\partial}s$, and satisfying the Hörmander L^2-estimates

$$\int_X \|u\|^2_{h_{E,d}} e^{-\psi} dV_{h_L} \leqslant \int_X \|\bar{\partial}s\|^2_{h_{E,d}} e^{-\psi} dV_{h_L}.$$

The presence of the singular weight $e^{-\psi}$ forces the jets of u to vanish up to order $2p'$ at x. As in Lemma 1.2 of [37], we conclude that the real holomorphic section $\sigma = (s - u)/\|s - u\|_{L^2(h_{E,d})}$ satisfies the required properties. □

In this first section we will only need peak sections given by Lemma 2.3.3 with $\sum_{i=1}^n p_i = 0$, whereas in the second one we will need those given with $\sum_{i=1}^n p_i \leqslant 2$.

Definition 2.3.4. For $i \in \{1, \dots, k\}$, let σ_0^i be the section given by Lemma 2.3.3 with $p' = 3$ and $p_1 = \cdots = p_n = 0$. Likewise, for every $j \in \{1, \dots, n\}$, let σ_j^i be a section given by (25) with $p' = 3$, $p_j = 1$ and $p_l = 0$ for $l \in \{1, \dots, n\} \setminus \{j\}$. Finally, for every $1 \leqslant l \leqslant m \leqslant n$, let σ_{lm}^i be a section given by (25) with $p' = 3$, $p_j = 0$ for every $j \in \{1, \dots, n\} \setminus \{l, m\}$ and $p_l = p_m = 1$ if $l \neq m$, while $p_l = 2$ otherwise.

The asymptotic values of the constants λ in (25) are given by Lemma 2.3.5 (compare Lemma 2.1 of [37]).

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Lemma 2.3.5. *For every $i \in \{1, \dots, k\}$, the sections given by Definition 2.3.4 satisfy*

$$\sigma_0^i/\sqrt{\delta_L d^n} \underset{d\to\infty}{\sim} e_i \otimes e^d + O(\|y\|^6), \tag{26}$$

$$\forall j \in \{1, \dots, n\}, \sigma_j^i/\sqrt{\pi\delta_L d^{n+1}} \underset{d\to\infty}{\sim} y_j e_i \otimes e^d + O(\|y\|^6), \tag{27}$$

$$\forall l, m \in \{1, \dots, n\}, l \neq m, \sigma_{lm}^i/\big(\pi\sqrt{\delta_L d^{n+2}}\big) \underset{d\to\infty}{\sim} y_l y_m e_i \otimes e^d + O(\|y\|^6), \tag{28}$$

$$\text{and } \forall l \in \{1, \dots, n\}, \sigma_{ll}^i/\big(\pi\sqrt{\delta_L d^{n+2}}\big) \underset{d\to\infty}{\sim} \frac{1}{\sqrt{2}} y_l^2 e_i \otimes e^d + O(\|y\|^6). \tag{29}$$

Moreover, these sections are asymptotically orthonormal as d grows to infinity, as follows from Lemma 2.3.6.

Lemma 2.3.6 (compare Lemma 3.1 of [37]). *For every $x \in \mathbb{R}X$, the sections $(\sigma_j^i)_{\substack{1\leqslant i\leqslant k\\ 0\leqslant j\leqslant n}}$ and $(\sigma_{lm}^i)_{\substack{1\leqslant i\leqslant k\\ 1\leqslant l\leqslant m\leqslant n}}$ given by Definition 2.3.4 have L^2-norm equal to 1 and their pairwise scalar products are dominated by an $O(d^{-1})$ term which does not depend on x. Likewise, their scalar product with every section of $\mathbb{R}H^0(X, E\otimes L^d)$ of L^2-norm equal to 1 and whose 2-jet at x vanishes is dominated by an $O(d^{-3/2})$ term which does not depend on x.*

Proof. The proof goes along the same lines as that of Lemma 3.1 of [37]. □

Lemma 2.3.7. *Denote by v the density of $dV_{h_L} = \omega^n/\int_X \omega^n$ with respect to the volume form dx chosen in (6), such that $dV_{h_L} = v(x)dx$. Then the sections given by Definition 2.3.4 times $\sqrt{v(x)}$ are still asymptotically orthonormal for (6).*

Proof. This is a direct consequence of Lemmas 2.3.3 and 2.3.6 and the asymptotic concentration of the support of the peak sections near x. □

Remark 2.3.8. The complex analogues of Lemmas 2.3.3, 2.3.5 and 2.3.6 hold; compare [37].

2.4. Proof of Theorem 1.2.1

We first compute the expected local C^1-norm of sections.

Proposition 2.4.1. *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. We equip $\mathbb{R}X$ with a field of h_L-trivializations; see § 2.3. Then, for every positive R,*

$$\limsup_{d\to\infty} \sup_{x\in\mathbb{R}X} \frac{1}{d^n} E\bigg(\sup_{B(x,\frac{R}{\sqrt{d}})} \frac{|\sigma|^2}{v(x)}\bigg) \leqslant 6k\delta_L\rho_R \text{ and}$$

$$\limsup_{d\to\infty} \sup_{x\in\mathbb{R}X} \frac{1}{d^{n+1}} E\bigg(\sup_{B(x,\frac{R}{\sqrt{d}})} \frac{\|d\sigma\|_2^2}{v(x)}\bigg) \leqslant 6\pi nk\delta_L\rho_R,$$

where v is given by Lemma 2.3.7 and ρ_R is given by (11); see (19) and (20) for the definitions of $|\sigma|$ and $\|d\sigma\|_2$.

Note that a global estimate on the sup norm of L^2 random holomorphic sections is given by Theorem 1.1 of [32].

Proof. The proof goes along the same lines as the proof of Proposition 3.7 of [16]. We first establish from the mean value inequality that for every $x \in \mathbb{R}X$, $R > 0$ and $s > 0$,

$$E\left(\sup_{B(x,\frac{R}{\sqrt{d}})} |\sigma|^2\right) \leqslant \frac{1}{Vol\left(B\left(\frac{s}{\sqrt{d}}\right)\right)} \int_{B(x,\frac{R+s}{\sqrt{d}})} \mathbb{E}(|\sigma|^2)\psi_x^* dy$$

for d large enough, not depending on x. Then, for every $z \in B(x, (R+s)/\sqrt{d}) \cap \mathbb{R}X$, we write $\sigma = \sum_{i=1}^k a_i \sigma_0^i + \tau$, where $\tau \in \mathbb{R}H^0(X, E \otimes L^d)$ vanishes at z and $(\sigma_0^i)_{i=1,\dots k}$ are the peak sections at z given by Definition 2.3.4. In particular, by Lemma 2.3.5, at the point z, for every $i = 1, \dots, k$, $\|\sigma_0^i\|_{h_{E,d}} \underset{d\to\infty}{\sim} \sqrt{\delta_L d^n}$. Moreover, since $(e_1, \dots, e_n)$ is orthonormal at x,

$$\begin{aligned} |\sigma_0^i(z)|^2 &= \|\sigma_0^i(z)\|^2_{h_{E,d}} (1 + O(|z-x|)e^{d\phi(z)} \\ &\leqslant \delta_L d^n e^{\pi(R+s)^2}(1+o(1)) \end{aligned}$$

from the inequalities (17), where the $o(d^n)$ term can be chosen not to depend on $x \in \mathbb{R}X$. Suppose that $dy = dV_{h_L}$. Then, by Lemma 2.3.6, the peak sections are asymptotically orthogonal to each other for the scalar product defined by (6), and asymptotically orthogonal to the space of sections τ vanishing at x. We deduce that

$$\begin{aligned} \mathbb{E}(|\sigma(z)|^2) &= \mathbb{E}\left(\left|\sum_{i=1}^k a_i\sigma_0^i\right|^2\right)(1+o(1)) \\ &= \left(\sum_{i=1}^k |\sigma_0^i(z)|^2\right)\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} a^2 e^{-a^2} da\,(1+o(1)) \\ &\leqslant \frac{1}{2}k\delta_L d^n e^{\pi(R+s)^2}(1+o(1)). \end{aligned}$$

When $z \notin B(x, (R+s/\sqrt{d})) \cap \mathbb{R}X$, the space of real sections vanishing at z becomes of real codimension $2k$ in $\mathbb{R}H^0(X, E\otimes L^d)$. Let $\langle\theta_1^i, \theta_2^i,\ i \in \{1,\dots,k\}\rangle$ be an orthonormal basis of its orthogonal complement. From Remark 2.3.8, for every $i \in \{1,\dots,k\}$, $j \in \{1,2\}$,

$$\limsup_{d\to\infty} \frac{1}{d^n}|\theta_j^i(z)|^2 \leqslant 2\delta_L e^{\pi(R+s)^2},$$

an upper bound which does not depend on z. We deduce that

$$\begin{aligned} \mathbb{E}(|\sigma(z)|^2) &= \int_{\mathbb{R}^{2k}} \left|\sum_{i=1}^k (a_{01}^i\theta_1^i(z) + a_{02}^i\theta_2^i(z))\right|^2 e^{-\sum_{i=1}^k (a_{01}^i)^2 + (a_{02}^i)^2} \frac{1}{\pi^k}\Pi_{i=1}^k da_{01}^i da_{02}^i \\ &\leqslant 2\delta_L d^n e^{\pi(R+s)^2}(1+o(1))\sum_{i=1}^k \int_{\mathbb{R}^2} \left((a_{01}^i)^2 + (a_{02}^i)^2 + 2|a_{01}^i||a_{02}^i|\right)\dots \end{aligned}$$

$$\dots \frac{1}{\pi} e^{-(a^i_{01})^2-(a^i_{02})^2} da^i_{01} da^i_{02}$$
$$\leqslant 6\delta_L d^n e^{\pi(R+s)^2}(1+o(1)).$$

We deduce the first part of Proposition 2.4.1 by taking the supremum over $\mathbb{R}X$, choosing s which minimizes $g_{R_{(U,P)}}$ and taking the $\limsup$ as d grows to infinity.

In general, the Bergman section at x for the L^2-product (6) associated with the volume form dx is equivalent to the Bergman section σ_0 at x for dV_h times $\sqrt{v(x)}$; see Lemma 2.3.7. The same holds true for the σ_j, and the result follows on replacing δ_L with $v(x)\delta_L$.

The proof of the second assertion goes along the same lines; see the proof of Proposition 3.7 of [16] (and [31] for similar results). □

As in [16], we then compute the probability of the presence of closed affine real algebraic submanifolds, inspired by an approach of Nazarov and Sodin [25]; see also [20]. Let (U, P) be a regular pair given by Definition 2.2.1 and $\Sigma = P^{-1}(0) \subset U$. Then, for every $x \in \mathbb{R}X$, we set $B_d = B(x, R_{(U,P)}/\sqrt{d}) \cap \mathbb{R}X$ (see (23)), and denote by $Prob_{x,\Sigma}(E \otimes L^d)$ the probability that $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$ has the property that $\sigma^{-1}(0) \cap B_d$ contains a closed submanifold Σ' such that the pair (B_d, Σ') is diffeomorphic to $(\mathbb{R}^n, \Sigma)$. That is,

$$Prob_{x,\Sigma}(E \otimes L^d) = \mu_{\mathbb{R}}\{\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \mid (\sigma^{-1}(0) \cap B_d) \supset \Sigma',\ (B_d, \Sigma') \sim (\mathbb{R}^n, \Sigma)\}.$$

We then set $Prob_\Sigma(E \otimes L^d) = \inf_{x \in \mathbb{R}X} Prob_{x,\Sigma}(E \otimes L^d)$.

Proposition 2.4.2. *Let X be a smooth real projective manifold of dimension n, (L, h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E, h_E) be a rank k real holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Let (U, P) be a regular pair given by Definition 2.2.1 and $\Sigma = P^{-1}(0) \subset U$. Then,*

$$\liminf_{d\to\infty} Prob_\Sigma(E \otimes L^d) \geqslant m_{\tau_{(U,P)}};$$

see (15).

Proof. The proof is the same as that of Proposition 3.8 of [16] and is not reproduced here. □

The proof of Theorem 1.2.1 (resp. Corollary 2.1.2) then just goes along the same lines as that of Theorem 1.2 (resp. Corollary 1.3) of [16].

2.5. Proof of Theorem 2.1.1

Let (U, P) be a regular pair given by Definition 2.2.1. For every $d > 0$, let Λ_d be a maximal subset of $\mathbb{R}X$ with the property that two distinct points of Λ_d are at distance greater than $(2R_{(U,P)})/\sqrt{d}$. The balls centered at points of Λ_d and of radius $R_{(U,P)}/\sqrt{d}$ are disjoint, whereas those of radius $(2R_{(U,P)})/\sqrt{d}$ cover $\mathbb{R}X$. Note that if we use the local flat metric given by a trivial h_L-trivialization, then the associated lattice has asymptotically the same number of balls as Λ_d as d grows to infinity, so we can suppose from now on that the balls are defined for this local metric. For every

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$\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$, denote by $N_\Sigma(\Lambda_d, \sigma)$ the number of $x \in \Lambda_d$ such that the ball $B_d = B(x, (R_{(U,P)})/\sqrt{d}) \cap \mathbb{R}X$ contains a codimension k submanifold Σ' with $\Sigma' \subset \sigma^{-1}(0)$ and (B_d, Σ') diffeomorphic to $(\mathbb{R}^n, \Sigma)$. By definition of $N_\Sigma(\sigma)$, $N_\Sigma(\Lambda_d, \sigma) \leqslant N_\Sigma(\sigma)$ (see § 1.2), while from Proposition 2.4.2, for every $0 < \epsilon < 1$,

$$\begin{aligned}|\Lambda_d| m_{\tau_{(U,P)}} &\leqslant \sum_{x \in \Lambda_d} Prob_{x,\Sigma}(E \otimes L^d)\\ &\leqslant \sum_{j=1}^{|\Lambda_d|} j \mu_{\mathbb{R}}\{\sigma | N_\Sigma(\Lambda_d, \sigma) = j\}\\ &\leqslant \epsilon m_{\tau_{(U,P)}} |\Lambda_d| \mu_{\mathbb{R}} \left\{\sigma | N_\Sigma(\Lambda_d, \sigma) \leqslant \epsilon m_{\tau_{(U,P)}} |\Lambda_d|\right\}\\ &\quad + |\Lambda_d| \mu_{\mathbb{R}} \left\{\sigma | N_\Sigma(\Lambda_d, \sigma) \geqslant \epsilon m_{\tau_{(U,P)}} |\Lambda_d|\right\}.\end{aligned}$$

We deduce that

$$(1-\epsilon) m_{\tau_{(U,P)}} \leqslant \mu_{\mathbb{R}} \left\{\sigma |\, N_\Sigma(\sigma) \geqslant \epsilon m_{\tau_{(U,P)}} |\Lambda_d|\right\} \tag{30}$$

and the result follows on choosing a sequence $(U_p, P_p)_p \in \mathcal{I}_\Sigma$ such that

$$\lim_{p \to \infty} m_{\tau_{(U_p,P_p)}} |\Lambda_d| = c_\Sigma Vol_{h_L}(\mathbb{R}X) \sqrt{d}^n;$$

see (14). □

2.6. Proof of Corollary 1.2.2

In this paragraph, for every positive integer p, S^p denotes the unit sphere in $\mathbb{R}^{p+1}$. Corollary 1.2.2 is a consequence of Theorem 1.2.1 and the following Propositions 2.6.1 and 2.6.3.

Proposition 2.6.1. *For every* $1 \leqslant k \leqslant n$, $c_{S^{n-k}} \geqslant \exp(-e^{54+5n})$.

Recall the following.

Lemma 2.6.2 (Lemma 2.2 of [16]). *If* $P = \sum_{(i_1,\dots,i_n) \in \mathbb{N}^n} a_{i_1,\dots,i_n} z_1^{i_1} \cdots z_n^{i_n} \in \mathbb{R}[z_1, \dots, z_n]$, *then*

$$\|P\|_{L^2}^2 = \int_{\mathbb{C}^n} |P(z)|^2 e^{-\pi|z|^2} dz = \sum_{(i_1,\dots,i_n) \in \mathbb{N}^n} |a_{i_1,\dots,i_n}|^2 \frac{i_1! \cdots i_n!}{\pi^{i_1+\cdots+i_n}}.$$

Proof of Proposition 2.6.1. For every $n > 0$, we set $P_k(x_1, \dots, x_n) = \sum_{j=k}^n x_j^2 - 1$. For every $x \in \mathbb{R}^n$ and $\delta > 0$,

$$|P_k(x)| < \delta \Leftrightarrow 1 - \delta < \sum_{i=k}^n x_i^2 < 1 + \delta \Rightarrow \|d_{|x} P_k\|_2^2 = 4 \sum_{i=k}^n x_i^2 > 4(1-\delta).$$

Moreover from Lemma 2.6.2,

$$\|P_k\|_{L^2}^2 = 1 + \frac{2(n-k+1)}{\pi^2} \leqslant n - k + 2.$$

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Now set $P_S = (P_1, \ldots, P_k)$ with $P_j(x) = x_j$ for $1 \leqslant j \leqslant k-1$, so

$$\|P_S\|^2_{L_2} \leqslant (k-1)/\pi + (n-k+2) \leqslant n+1 \leqslant 2n.$$

Since for every $w = (w_1, \ldots, w_k) \in \mathbb{R}^k$ and every $x \in \mathbb{R}^n$,

$$|d_{|x} P_S^*(w)|^2 = \sum_{i=1}^{k-1} w_i^2 + w_k^2 \|d_{|x} P_k\|_2^2,$$

we get that $\|d_{|x} P_S^*\|^2 \geqslant \min\big(1, 4(1-\delta)\big)$ if $|P_k(x)| < \delta$. Choose

$$U_S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j^2 < 4\}.$$

Then if $0 < \delta < 1$,

$$K_\delta = \left\{ x \in U_S \mid 1-\delta \leqslant \sum_{i=k}^{n} x_i^2 \leqslant 1+\delta \text{ and } \sum_{i=1}^{k-1} x_k^2 \leqslant 1 - \frac{1}{2}(1+\delta)^2 \right\}$$

is compact in U_S and taking $R^2_{(U_S, P_S)} = 4$, we see that the pair (U_S, P_S) is regular in the sense of Definition 2.2.1. The submanifold $P_S^{-1}(0) \subset U_S$ is isotopic in $\mathbb{R}^n$ to the unit sphere S^{n-k}. We deduce that $(3/4, 1) \in \mathcal{T}_{(U_S, P_S)}$. From (10) and (13) we deduce

$$\tau_{(U_S, P_S)} \leqslant 24k4^n e^{16\pi} 2n(2+\pi n) \leqslant e^{53+5n}.$$

The estimate $c_{S^{n-1}} \geqslant \exp(-e^{54+5n})$ follows then from (16).

Proposition 2.6.3. *For every* $1 \leqslant k \leqslant n$ *and every* $0 \leqslant i \leqslant n-k$, $c_{S^i \times S^{n-i-k}} \geqslant \exp(-e^{82+6n})$.

Proof. For every $1 \leqslant k \leqslant n$ and every $0 \leqslant i \leqslant n-k$, we set

$$Q_k((x_1, \ldots, x_{i+1}), (y_1, \ldots, y_{n-i-1})) = (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 - 1.$$

For every $(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}$ and $0 < \delta < 1/2$,

$$|Q_k(x, y)| < \delta \Leftrightarrow 1-\delta < (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 < 1+\delta$$

$$\Rightarrow \|d_{|(x,y)} Q_k\|_2^2 = 4 \sum_{j=1}^{n-k-i} y_j^2 + 16|x|^2(|x|^2 - 2)^2,$$

with $|x|^2 > 2 - \sqrt{1+\delta} > 1/2$. Thus $\|d_{|(x,y)} Q_k\|_2^2 > 4(1-\delta)$; compare Lemma 2.6 of [16]. Moreover from Lemma 2.6.2, $\|Q_k\|^2_{L^2} \leqslant 13n^2$; compare § 2.3.2 of [16]. Now set $Q = (Q_1, \ldots, Q_k)$ with $Q_j(x, y) = y_{n-i-j}$ for $1 \leqslant j \leqslant k-1$, so

$$\|Q\|^2_{L_2} \leqslant (k-1)/\pi + 13n^2 \leqslant 13(n+1)^2.$$

For every $w = (w_1, \dots, w_k) \in \mathbb{R}^k$ and every $(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}$,

$$|d_{|(x,y)} Q^*(w)|^2 = \sum_{i=1}^{k-1} w_i^2 + w_k^2 \|d_{|(x,y)} Q_k\|_2^2$$
$$> \min(1, 4(1-\delta))|w|^2$$

if $|Q_k(x, y)| \leqslant \delta < 1/2$. We choose

$$U = \{(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1} \mid |x|^2 + |y|^2 < 6\},$$
$$K_\delta = \left\{(x, y) \in U \mid 1 - \delta \leqslant (|x|^2 - 2)^2 + \sum_{j=1}^{n-k-i} y_j^2 \leqslant 1 + \delta \text{ and } \sum_{j=1}^{k-1} y_{n-i-j}^2 \leqslant 1 - \delta\right\},$$

and $R^2_{(U,Q)} = 6$. The pair (U, Q) is regular in the sense of Definition 2.2.1 and $Q^{-1}(0) \subset U$ is isotopic in $\mathbb{R}^n$ to the product $S^i \times S^{n-i-k}$ of unit spheres in $\mathbb{R}^{i+1}$ and $\mathbb{R}^{n-i-k+1}$. We deduce that for every positive ϵ, $(1/2 - \epsilon, 1) \in \mathcal{T}_{(U,Q)}$, and from (10) and (13), that

$$\tau_{(U,Q)} \leqslant 24k4^n e^{24\pi} 13(n+1)^2(4 + \pi n) \leqslant e^{81+6n}.$$

The estimate $c_{S^i \times S^{n-i-k}} \geqslant \exp(-e^{82+6n})$ follows then from (16). □

3. Upper estimates for the expected Betti numbers

3.1. Statement of the results

For every $1 \leqslant k \leqslant n$, we denote by $\mathrm{Gr}(k-1, n-1)$ the Grassmann manifold of $(k-1)$-dimensional linear subspaces of $\mathbb{R}^{n-1}$. The tangent space of $\mathrm{Gr}(k-1, n-1)$ at every $H \in \mathrm{Gr}(k-1, n-1)$ is canonically isomorphic to the space of linear maps $L(H, H^\perp)$ from H to its orthogonal $H^\perp$ and we equip it with the norm

$$A \in L(H, H^\perp) \mapsto \|A\|_2 = \sqrt{Tr(A^*A)} \in \mathbb{R}^+.$$

The total volume of $\mathrm{Gr}(k-1, n-1)$ for this Riemannian metric is denoted by $Vol(\mathrm{Gr}(k-1, n-1))$ and we set

$$V_{k-1,n-1} = \frac{1}{\sqrt{\pi}^{(k-1)(n-k)}} Vol(\mathrm{Gr}(k-1, n-1))$$

as its volume for the rescaled metric $A \in L(H, H^\perp) \mapsto (1/\sqrt{\pi})\|A\|_2$. Likewise, we equip $M_{k-1}(\mathbb{R})$ with the Euclidean norm $A \in M_{k-1}(\mathbb{R}) \mapsto \|A\|_2 = \sqrt{Tr(A^*A)}$ and set $d\mu(A) = (1/\pi^{k-1})e^{-\|A\|_2^2} dA$ as the associated Gaussian measure on $M_{k-1}(\mathbb{R})$. Then, we set

$$E_{k-1}(|\det|^{n-k+2}) = \int_{M_{k-1}(\mathbb{R})} |\det A|^{n-k+2} d\mu(A).$$

Remark 3.1.1.

1. The orthogonal group $\mathrm{O}_{n-1}(\mathbb{R})$ acts transitively on the Grassmannian $\mathrm{Gr}(k-1, n-1)$ with fixators isomorphic to $\mathrm{O}_{k-1}(\mathbb{R}) \times \mathrm{O}_{n-k}(\mathbb{R})$. We deduce that

$$\begin{aligned} Vol(\mathrm{Gr}(k-1,n-1)) &= Vol(\mathrm{O}_{n-1}(\mathbb{R}))/(Vol(\mathrm{O}_{k-1}(\mathbb{R}))\times Vol\big(\mathrm{O}_{n-k}(\mathbb{R})\big)) \\ &= \binom{n-1}{k-1}\sqrt{\pi}^{(k-1)(n-k)}\frac{\prod_{j=1}^{k-1}\Gamma(1+j/2)}{\prod_{j=n-k+1}^{n-1}\Gamma(1+j/2)}, \end{aligned}$$

where Γ denotes the Gamma function of Euler; see for example Lemma 3.4 of [14].

2. From formula (15.4.12) of [23] it follows that

$$E_{k-1}(|\det|^{n-k+2}) = \prod_{j=1}^{k-1}\frac{\Gamma(\frac{n-k+2+j}{2})}{\Gamma(\frac{j}{2})},$$

so $V_{k-1,n-1}E_{k-1}(|\det|^{n-k+2}) = \frac{(n-1)!}{(n-k)!2^{k-1}}$.

We now keep the framework of § 2.1. Let us denote, for every $i\in\{0,\ldots,n-k\}$, by $b_i(\mathbb{R}C_\sigma,\mathbb{R}) = \dim H_i(\mathbb{R}C_\sigma,\mathbb{R})$ the ith Betti number of $\mathbb{R}C_\sigma$ and by

$$m_i(\mathbb{R}C_\sigma) = \inf_{f\text{ Morse on }\mathbb{R}C_\sigma}|Crit_i(f)|$$

its ith Morse number, where $|Crit_i(f)|$ denotes the number of critical points of index i of f. We then denote by

$$\mathbb{E}(b_i) = \int_{\mathbb{R}H^0(X,E\otimes L^d)\setminus\mathbb{R}\Delta_d} b_i(\mathbb{R}C_\sigma,\mathbb{R})d\mu_{\mathbb{R}}(\sigma)$$

and

$$\mathbb{E}(m_i) = \int_{\mathbb{R}H^0(X,E\otimes L^d)\setminus\mathbb{R}\Delta_d} m_i(\mathbb{R}C_\sigma)d\mu_{\mathbb{R}}(\sigma)$$

their expected values. The aim of § 3 is to prove the following Theorem 3.1.2; see (1) for the definition of $e_{\mathbb{R}}(i,n-k-i)$.

Theorem 3.1.2. *Let X be a smooth real projective manifold of dimension n, (L,h_L) be a real holomorphic Hermitian line bundle of positive curvature over X and (E,h_E) be a rank k real holomorphic Hermitian vector bundle, with $1\leqslant k\leqslant n-1$. Then, for every $0\leqslant i\leqslant n-k$,*

$$\limsup_{d\to\infty}\frac{1}{\sqrt{d}^n}\mathbb{E}(m_i) \leqslant \frac{1}{\Gamma(\frac{k}{2})}V_{k-1,n-1}E_{k-1}(|\det|^{n-k+2})e_{\mathbb{R}}(i,n-k-i)Vol_{h_L}(\mathbb{R}X).$$

Note that the case $k=n$ is covered by Theorems 1.1.1 and 3.1.3. When $k=1$ and $E=\mathcal{O}_X$, $Vol_{FS}(\mathbb{R}P^k)=\sqrt{\pi}$ (see Remark 2.14 of [14]), so Theorem 3.1.2 reduces to Theorem 1.0.1 of [14] in this case. The proof of Theorem 3.1.2 actually goes along the same lines as that of Theorem 1.1 of [14]. The strategy goes as follows. We fix a Morse function $p:\mathbb{R}X\to\mathbb{R}$. Then, almost surely on $\sigma\in\mathbb{R}H^0(X,E\otimes L^d)$, the restriction of p to $\mathbb{R}C_\sigma$ is itself a Morse function. For $i\in\{0,\ldots,n-k\}$, we denote by $Crit_i(p_{|\mathbb{R}C_\sigma})$ the set of critical points of index i of this restriction and set

$$\nu_i(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n}\sum_{x\in Crit_i(p_{|\mathbb{R}C_\sigma})}\delta_x$$

if $n > k$ and $\nu_0(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n}\sum_{x\in\mathbb{R}C_\sigma}\delta_x$ if $k = n$. We then set

$$\mathbb{E}(\nu_i) = \int_{\mathbb{R}H^0(X,E\otimes L^d)} \nu_i(\mathbb{R}C_\sigma)d\mu_\mathbb{R}(\sigma)$$

and prove the following equidistribution result (compare Theorem 1.2 of [14]).

Theorem 3.1.3. *Under the hypotheses of Theorem 3.1.2, let $p : \mathbb{R}X \to \mathbb{R}$ be a Morse function. Then, for every $i \in \{0, \dots, n-k\}$, the measure $\mathbb{E}(\nu_i)$ weakly converges to*

$$\frac{1}{\Gamma(\frac{k}{2})} V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) e_\mathbb{R}(i, n-k-i) dvol_{h_L}$$

as d grows to infinity. When $k = n$, $\mathbb{E}(\nu_0)$ converges weakly to $1/\sqrt{\pi}\,\Gamma((n+1)/2)dvol_{h_L}$.

In Theorem 3.1.3 $dvol_{h_L}$ denotes the Lebesgue measure of $\mathbb{R}X$ induced by the Kähler metric. Theorem 3.1.2 is deduced from Theorem 3.1.3 by integration of 1 over $\mathbb{R}X$. The following paragraphs are devoted to the proof of Theorem 3.1.3.

Proof of Theorem 1.1.1. It follows from Theorem 3.1.2, the Morse inequalities, Remark 3.1.1 and the computation $Vol_{FS}\mathbb{R}P^n = \sqrt{\pi}/\Gamma(n+1/2)$ (see Remark 2.14 of [14]) when $k \leqslant n-1$ and from Theorem 3.1.3 when $k = n$.

3.2. Incidence varieties

Under the hypotheses of Theorem 3.1.3, we set

$$\mathbb{R}\Delta_p^d = \{\sigma \in \mathbb{R}H^0(X, E\otimes L^d) \,|\, \sigma \in \mathbb{R}\Delta_d \text{ or } p_{|\mathbb{R}C_\sigma} \text{ is not Morse}\}$$

and

$$\mathcal{I}_i = \{(\sigma, x) \in (\mathbb{R}H^0(X, E\otimes L^d)\setminus \mathbb{R}\Delta_p^d) \times (\mathbb{R}X \setminus Crit(p)) \,|\, x \in Crit_i(p_{|\mathbb{R}C_\sigma})\}.$$

We set

$$\pi_1 : (\sigma, x) \in \mathcal{I}_i \mapsto \sigma \in \mathbb{R}H^0(X, E\otimes L^d) \text{ and} \tag{31}$$

$$\pi_2 : (\sigma, x) \in \mathcal{I}_i \mapsto x \in \mathbb{R}X. \tag{32}$$

Then, for every $(\sigma_0, x_0) \in ((\mathbb{R}H^0(X, E\otimes L^d)\setminus\mathbb{R}\Delta_p^d) \times (\mathbb{R}X \setminus Crit(p)))$, π_1 is invertible in a neighborhood $\mathbb{R}U$ of σ_0, defining an evaluation map at the critical point

$$ev_{(\sigma_0,x_0)} : \sigma \in \mathbb{R}U \mapsto \pi_2 \circ \pi_1^{-1}(\sigma) = x \in Crit_i(p_{|\mathbb{R}C_\sigma}) \cap \mathbb{R}V,$$

where $\mathbb{R}V$ denotes a neighborhood of x_0 in $\mathbb{R}X$; compare § 2.4.2 of [14]. We denote by $d_{|\sigma_0}ev^\perp_{(\sigma_0,x_0)}$ the restriction of its differential map $d_{|\sigma_0}ev_{(\sigma_0,x_0)}$ at σ_0 to the orthogonal complement of $\pi_1(\pi_2^{-1}(x_0))$ in $\mathbb{R}H^0(X, E\otimes L^d)$.

Proposition 3.2.1. *Under the hypotheses of Theorem 3.1.3,*

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{d}^n}(\pi_2)_*(\pi_1^* d\mu_\mathbb{R}).$$

Moreover, at every point $x \in \mathbb{R}X \setminus Crit(p)$,

$$(\pi_2)_*(\pi_1^* d\mu_\mathbb{R})_{|x} = \frac{1}{\sqrt{\pi}^n}\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^\perp_{(\sigma,x)}|^{-1} d\mu_\mathbb{R}(\sigma) dvol_{h_L}.$$

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Proof. The proof is the same as that of Proposition 2.10 of [14] and is not reproduced here. □

Fix $x \in \mathbb{R}X \setminus Crit(p)$. Then $\pi_1(\pi_2^{-1}(x))$ is open in a subspace of $\mathbb{R}H^0(X, E \otimes L^d)$. Namely,

$$\pi_1(\pi_2^{-1}(x)) = \{\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_p^d \mid \sigma(x) = 0 \text{ and} \tag{33}$$

$$\exists \lambda \in \mathbb{R}(E \otimes L^d)^*_{|x},\ \lambda \circ \nabla_{|x}\sigma = d_{|x}p\}, \tag{34}$$

where $\mathbb{R}((E \otimes L^d)^*_{|x})$ is the real part of the fiber $(E \otimes L^d)^*_{|x}$. We deduce a well-defined map

$$\rho_x : \pi_1(\pi_2^{-1}(x)) \to \mathrm{Gr}(n-k, \ker d_{|x}p) \times (\mathbb{R}(E \otimes L^d)^*_{|x} \setminus \{0\}) \tag{35}$$

$$\sigma \mapsto (\ker \nabla_{|x}\sigma, \lambda). \tag{36}$$

For every $\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \setminus \mathbb{R}\Delta_p^d$, the tangent space of $\pi_1(\pi_2^{-1}(x))$ at σ reads

$$T_\sigma \pi_1(\pi_2^{-1}(x)) = \{\dot\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \mid \dot\sigma(x) = 0 \text{ and}$$
$$\exists \dot\lambda \in \mathbb{R}(E \otimes L^d)^*_{|x} \mid \dot\lambda \circ \nabla_{|x}\sigma + \lambda \circ \nabla\dot\sigma_{|x} = 0\}.$$

Likewise, for every $\lambda \in \mathbb{R}(E \otimes L^d)^*_{|x} \setminus \{0\}$, the tangent space of $\rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\})$ at σ reads

$$T_\sigma \rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\}) = \{\dot\sigma \in \mathbb{R}H^0(X, E \otimes L^d) \mid \dot\sigma(x) = 0 \text{ and } \lambda \circ \nabla_{|x}\dot\sigma = 0\}.$$

Finally, for every $K \in \mathrm{Gr}(n-k, \ker d_{|x}p)$, the tangent space of $\rho_x^{-1}(K, \lambda)$ at σ reads

$$T_\sigma \rho_x^{-1}(K, \lambda) = \{\dot\sigma \in \mathbb{R}H^0(X, E \otimes L^d) | \dot\sigma(x) = 0, \nabla_{|x}\dot\sigma_{|K} = 0 \text{ and } \lambda \circ \nabla_{|x}\dot\sigma = 0\}.$$

Let us choose local real holomorphic coordinates $(x_1, \dots, x_n)$ of X near x such that $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ is orthonormal at x, with $d_{|x}p$ being collinear to dx_1 and such that $K = \ker \nabla_{|x}\sigma = \langle \partial/\partial x_{k+1}, \dots, \partial/\partial x_n \rangle$. Let us choose a local real holomorphic trivialization $(e_1, \dots, e_k)$ of E near x that is orthonormal at x and is such that $\ker \lambda_{|x} = \langle e_2 \otimes e^d, \dots, e_k \otimes e^d \rangle_{|x}$. For d large enough, we define the following subspaces of $\mathbb{R}H^0(X, E \otimes L^d)$:

$$H_x = \langle (\sigma_0^i)_{1 \leqslant i \leqslant k}, (\sigma_j^1)_{k+1 \leqslant j \leqslant n} \rangle \tag{37}$$

$$H_\lambda = \langle (\sigma_j^1)_{1 \leqslant j \leqslant k} \rangle \tag{38}$$

$$H_K = \langle (\sigma_j^i)_{\substack{2 \leqslant i \leqslant k \\ k+1 \leqslant j \leqslant n}} \rangle, \tag{39}$$

where the sections $(\sigma_0^i)_{1 \leqslant i \leqslant k}$ and $(\sigma_j^i)_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n}}$ of $\mathbb{R}H^0(X, E \otimes L^d)$ are given by Lemma 2.3.3 and Definition 2.3.4.

H_K is a complement of $T_\sigma \rho_x^{-1}(K, \lambda)$ in $T_\sigma \rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\})$, H_λ is a complement of $T_\sigma \rho_x^{-1}(\mathrm{Gr}(n-k, \ker d_{|x}p) \times \{\lambda\})$ in $T_\sigma \pi_1(\pi_2^{-1}(x))$ and H_x is a complement of $T_\sigma \pi_1(\pi_2^{-1}(x))$ in $\mathbb{R}H^0(X, E \otimes L^d)$. Then, from Lemmas 2.3.6 and 2.3.7, up to a uniform rescaling by $\sqrt{v(x)}$, these complements are asymptotically orthogonal and their given basis orthonormal. Hence, we can assume from now on that $v = 1$.

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Lemma 3.2.2. *Under the hypotheses of Theorem 3.1.3, let* $(\sigma, x) \in \mathcal{I}_i$ *and* $\lambda \in \mathbb{R}(E \otimes L^d)^*_{|x} \setminus \{0\}$ *be such that* $\lambda \circ \nabla_{|x}\sigma = d_{|x}p$. *Then,* $\lambda \circ \nabla^2\sigma_{|K_x} = \nabla^2(p_{|\mathbb{R}C_\sigma})_{|x}$, *so the quadratic form* $\lambda \circ \nabla^2\sigma_{|K_x}$ *is non-degenerate of index* i.

Proof. The proof is similar to that of Lemma 2.9 of [14]. □

3.3. Computation of the Jacobian determinants

3.3.1. The Jacobian determinant of ρ_x. Under the hypotheses of Theorem 3.1.3, let $(\sigma, x) \in \mathcal{I}_i$. We set $(K, \lambda) = \rho_x(\sigma)$ and denote by $d_{|\sigma}\rho^H_x$ the restriction of $d_{|\sigma}\rho_x$ to $H_K \oplus H_\lambda$. We then denote by $\det(d_{|\sigma}\rho^H_x)$ the Jacobian determinant of $d_{|\sigma}\rho^H_x$ computed in the given basis of H_λ and H_K (see (38), (39)) and in the orthonormal basis of $T_K \operatorname{Gr}(n-k, \ker d_{|x}p) \times \mathbb{R}(E \otimes L^d)^*_{|x}$. By assumption, the operator $\nabla_{|x}\sigma$ does not depend on the choice of a connection ∇ on $E \otimes L^d$ and is onto. We denote by $\nabla_{|x}\sigma^\perp$ its restriction to the orthogonal $K^\perp$ of $K = \ker \nabla_{|x}\sigma$,

$$\nabla_{|x}\sigma^\perp : K^\perp \to \mathbb{R}(E \otimes L^d)_{|x}.$$

Likewise, for every $(\dot\sigma_K, \dot\sigma_\lambda) \in H_K \oplus H_\lambda$, the operators $\nabla_{|x}\dot\sigma_K$ and $\nabla_{|x}\dot\sigma_\lambda$ do not depend on the choice of a connection ∇ on $E \otimes L^d$. Finally, we write at a point $y \in \mathbb{R}X$ near x

$$\sigma(y) = \sum_{i=1}^k \left(a^i_0\sigma^i_0 + \sum_{j=1}^n a^i_j\sigma^i_j + \sum_{1 \leqslant l \leqslant m \leqslant n} a^i_{lm}\sigma^i_{lm} \right)(y) + o(|y|^2),$$

where (a^i_0), (a^i_j) and (a^i_{lm}) are real numbers and (σ^i_0), (σ^i_j) and (σ^i_{lm}) are given by Definition 2.3.4. From Lemma 2.3.5 and (33), we deduce that $a^i_0 = 0 = a^1_j$ for $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$, and that

$$\|\lambda\| \sqrt{\pi\delta_L}\sqrt{d}^{n+1} |a^1_1| = \|d_{|x}p\| + o(1), \tag{40}$$

where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$.

Lemma 3.3.1. *Under the hypotheses of Theorem 3.1.3, let* $(\sigma, x) \in \mathcal{I}_i$ *and* $(K, \lambda) = \rho_x(\sigma)$. *Then,* $d_{|x}\rho^H_x$ *is written as*

$$\begin{aligned} H_K \oplus H_\lambda &\to T_K\, Gr(n-k, \ker d_{|x}p) \times \mathbb{R}(E \otimes L^d)^*_{|x} \\ (\dot\sigma_K, \dot\sigma_\lambda) &\mapsto \left(-(\nabla_{|x}\sigma^\perp)^{-1}_{|\ker\lambda} \circ \nabla_{|x}\dot\sigma_{K|K}, -\lambda \circ \nabla_{|x}\dot\sigma_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1} \right). \end{aligned}$$

Moreover, $|\det d_{|\sigma}\rho^H_x|^{-1} = (|a^1_1|/\|\lambda\|^k)|\det(a^i_j)_{2 \leqslant i,j \leqslant k}|^{n-k+1}(1 + o(1))$, *where the* $o(1)$ *term is uniformly bounded over* $\mathbb{R}X$.

Proof. Let $(\dot\sigma_K, \dot\sigma_\lambda) \in H_K \oplus H_\lambda$ and $(\sigma_s)_{s \in]-\epsilon,\epsilon[}$ be a path of $\pi_1(\pi_2^{-1}(x))$ such that $\sigma_0 = \sigma$ and $\dot\sigma_0 = \dot\sigma_K + \dot\sigma_\lambda$. Then, for every $s \in]-\epsilon, \epsilon[$ and every $v_s \in \ker \nabla_{|x}\sigma_s$, there exists $\lambda_s \in \mathbb{R}(E \otimes L^d)^*_{|x}$ such that

$$\begin{cases} \nabla_{|x}\sigma_s(v_s) = 0 & \text{and} \\ \lambda_s \circ \nabla_{|x}\sigma_s = d_{|x}p. \end{cases}$$

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By derivation, we deduce

$$\begin{cases} \nabla_{|x}\dot{\sigma}_0(v_0) + \nabla_{|x}\sigma(\dot{v}_0) = 0 \quad \text{and} \\ \dot{\lambda}_0 \circ \nabla_{|x}\sigma + \lambda \circ \nabla_{|x}\dot{\sigma}_0 = 0. \end{cases}$$

By setting $\dot{v}$ as the orthogonal projection of $\dot{v}_0$ onto $K^\perp$, we deduce that

$$\begin{cases} \dot{v} = -(\nabla_{|x}\sigma^\perp)^{-1} \circ \nabla_{|x}\dot{\sigma}_K(v_0) \quad \text{and} \\ \dot{\lambda}_0 = -\lambda \circ \nabla_{|x}\dot{\sigma}_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1}. \end{cases}$$

The first part of Lemma 3.3.1 follows. Now, recall that $d_{|x}p$ is collinear to dx_1, that K is equipped with the orthonormal basis $(\partial/\partial x_{k+1}, \dots, \partial/\partial x_n)$, $K^\perp$ with the orthonormal basis $(\partial/\partial x_1, \dots, \partial/\partial x_k)$, and that $\ker \lambda_{|x}$ is spanned by the orthonormal basis $(e_2, \dots, e_k)_{|x}$. From Lemma 2.3.3, the map

$$\dot{\sigma}_K \in H_K \mapsto \nabla_{|x}\dot{\sigma}_{K|K} \in L(K, \ker \lambda)$$

just dilates the norm by the factor $\sqrt{\pi \delta_L d^{n+1}}(1 + o(1))$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. Now, since the matrix of the restriction of $\nabla_{|x}\sigma^\perp$ to $K^\perp \cap \ker d_{|x}p$ in the given basis of $K^\perp \cap \ker d_{|x}p$ and $\ker \lambda$ equals

$$\sqrt{\pi \delta_L d^{n+1}}(a^i_j)_{2 \leqslant i,j \leqslant k} + o(\sqrt{d}^{n+1}),$$

where the $o(\sqrt{d}^{n+1})$ term is uniformly bounded over $\mathbb{R}X$, we deduce that the Jacobian determinant of the map

$$M \in L(K, \ker \lambda) \mapsto (\nabla_{|x}\sigma^\perp_{|\ker \lambda})^{-1} \circ M \in L(K, K^\perp \cap \ker d_{|x}p))$$

equals

$$((\sqrt{\pi \delta_L d^{n+1}})^{k-1} |\det(a^i_j)_{2 \leqslant i,j \leqslant k}|(1 + o(1)))^{k-n}.$$

The Jacobian determinant of the map

$$\dot{\sigma}_K \in H_K \mapsto (\nabla_{|x}\sigma^\perp)^{-1}_{|\ker \lambda} \circ \nabla_{|x}\dot{\sigma}_{K|K} \in T_K \mathrm{Gr}(n-k, \ker d_{|x}p)$$

thus equals $|\det(a^i_j)_{2 \leqslant i,j \leqslant k}|^{k-n} + o(1)$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. Likewise, from Lemma 2.3.3, the map

$$\dot{\sigma}_\lambda \in H_\lambda \mapsto \lambda \circ \nabla_{|x}\dot{\sigma}_\lambda \in (K^\perp)^*$$

just dilates the norm by a factor $\sqrt{\pi \delta_L d^{n+1}}\|\lambda\| + o(\sqrt{d}^{n+1})$, where the $o(\sqrt{d}^{n+1})$ term is uniformly bounded over $\mathbb{R}X$, while the Jacobian determinant of the map

$$M \in (K^\perp)^* \mapsto M \circ (\nabla_{|x}\sigma^\perp)^{-1} \in \mathbb{R}(E \otimes L^d)^*_{|x}$$

equals $(\sqrt{\pi \delta_L}\sqrt{d}^{n+1})^{-k} |\det(a^i_j)_{1 \leqslant i,j \leqslant k}|^{-1}(1 + o(1))$, so the Jacobian determinant of the map

$$\dot{\sigma}_\lambda \in H_\lambda \mapsto \lambda \circ \nabla_{|x}\dot{\sigma}_\lambda \circ (\nabla_{|x}\sigma^\perp)^{-1} \in \mathbb{R}(E \otimes L^d)^*_{|x}$$

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equals $\|\lambda\|^k|\det(a^i_j)_{1\leqslant i,j\leqslant k}|^{-1}+o(1)$, with an $o(1)$ term uniformly bounded over $\mathbb{R}X$. As a consequence,

$$|\det d_{|\sigma}\rho^H_x|^{-1} = \|\lambda\|^{-k}|\det(a^i_j)_{2\leqslant i,j\leqslant k}|^{n-k+1}|a^1_1|(1+o(1)),$$

with an $o(1)$ term uniformly bounded over $\mathbb{R}X$, since the relation $\lambda\circ\nabla_{|x}\sigma = d_{|x}p$ implies that a^1_j vanishes for $2\leqslant j\leqslant n$. □

3.3.2. Jacobian determinant of the evaluation map. Again, under the hypotheses of Theorem 3.1.3 and for $(\sigma,x)\in\mathcal{I}_i$, we set for every y in a neighborhood of x,

$$\sigma(y) = \sum_{i=1}^k\left(a^i_0\sigma^i_0+\sum_{j=1}^n a^i_j\sigma^i_j+\sum_{1\leqslant l\leqslant m\leqslant n}a^i_{lm}\sigma^i_{lm}\right)(y)+o(|y|^2), \tag{41}$$

where a^i_0, a^i_j and a^i_{lm} are real numbers. We then set, for $1\leqslant l,m\leqslant n$, $\tilde{a}^1_{ll}=\sqrt{2}a^1_{ll}$, $\tilde{a}^1_{lm}=a^1_{lm}$ if $l<m$ and $\tilde{a}^1_{lm}=a^1_{ml}$ if $l>m$. We denote by $d_{|\sigma}ev^H_{(\sigma,x)}$ the restriction of $d_{|\sigma}ev_{(\sigma,x)}$ to H_x (see (37)) and by $\det d_{|\sigma}ev^H_{(\sigma,x)}$ its Jacobian determinant computed in the given basis of H_x and orthonormal basis of $T_x\mathbb{R}X$.

Lemma 3.3.2. *Under the hypotheses of Theorem 3.1.3, let $(\sigma,x)\in\mathcal{I}_i$. Then,*

$$|\det d_{|\sigma}ev^H_{(\sigma,x)}|^{-1} = \sqrt{\pi^n d^n}|a^1_1||\det(a^i_j)_{2\leqslant i,j\leqslant k}||\det(\tilde{a}^1_{lm})_{k+1\leqslant l,m\leqslant n}|(1+o(1)),$$

where the $o(1)$ term has poles of order at most $n-k$ near the critical points of p.

Remark 3.3.3. In Lemma 3.3.2, a function f is said to have a pole of order at most $n-k$ near a point x if $r^{n-k}f$ is bounded near x, where r denotes the distance function to x. Such a function thus belongs to $L^1(\mathbb{R}X, dvol_h)$.

Proof. We choose a torsion free connection ∇^{TX} (resp. a connection $\nabla^{E\otimes L^d}$) on $\mathbb{R}X\setminus Crit(p)$ (resp. on $E\otimes L^d$) such that $\nabla^{TX}dp=0$. They induce a connection on $T^*X\otimes E\otimes L^d$ which makes it possible to differentiate twice the elements of $\mathbb{R}H^0(X,E\otimes L^d)$. The tangent space of $\mathcal{I}_i$ then reads

$$T_{(\sigma,x)}\mathcal{I}_i = \big\{(\dot\sigma,\dot x)\in\mathbb{R}H^0(X,E\otimes L^d)\times T_x\mathbb{R}X \mid \dot\sigma(x)+\nabla_{\dot x}\sigma=0 \text{ and} \tag{42}$$
$$\exists\dot\lambda\in\mathbb{R}(E\otimes L^d)^*_{|x},\ \dot\lambda\circ\nabla_{|x}\sigma+\lambda\circ\nabla_{|x}\dot\sigma+\lambda\circ\nabla^2_{\dot x,.}\sigma=0\big\}. \tag{43}$$

Recall that $T_x\mathbb{R}X$ is the direct sum $K\oplus K^\perp$, where $K=\ker\nabla_{|x}\sigma$. We write $\dot x=(\dot x_K,\dot x_{K^\perp})$, the coordinates of $\dot x$ in this decomposition. From the first equation we deduce, keeping the notation of § 3.3.1, that $\dot x_{K^\perp}=-(\nabla_{|x}\sigma^\perp)^{-1}(\dot\sigma(x))$. From Lemma 2.3.3, the evaluation map at x,

$$\dot\sigma\in\langle(\sigma^i_0)_{1\leqslant i\leqslant k}\rangle\mapsto\dot\sigma(x)\in E\otimes L^d_{|x},$$

just dilates the norm by a factor $\sqrt{\delta_L d^n}(1+o(1))$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$, while

$$|\det(\nabla_{|x}\sigma^\perp)| = (\sqrt{\pi\delta_L d^{n+1}})^k|\det(a^i_j)_{1\leqslant i,j\leqslant k}|(1+o(1)).$$

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We deduce by composition that the Jacobian of the map

$$\dot{\sigma} \in \langle(\sigma_0^i)_{1\leqslant i\leqslant k}\rangle \mapsto \dot{x}_{K^\perp} = -(\nabla_{|x}\sigma^\perp)^{-1}(\dot{\sigma}(x))$$

equals $\left(\sqrt{\pi^k d^k}|\det(a_j^i)_{2\leqslant i,j\leqslant k}||a_1^1|\right)^{-1}(1+o(1))$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. Now, equation (43) restricted to K reads

$$\lambda\circ\nabla^2_{\dot{x}_K,.}\sigma_{|K} = -\lambda\circ\nabla_{|x}\dot{\sigma}_{|K}.$$

From Lemma 2.3.3, the map

$$\dot{\sigma} \in \langle(\sigma_j^1)_{k+1\leqslant j\leqslant n}\rangle \mapsto -\lambda\circ\nabla_{|x}\dot{\sigma}_{|K} \in K^*$$

just dilates the norm by a factor $\|\lambda\|\sqrt{\pi\delta_L d^{n+1}}(1+o(1))$, where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. Likewise, from Lemma 2.3.3, the Jacobian of the map $\lambda\circ\nabla^2\sigma_{|K} : K \to K^*$ equals

$$(\|\lambda\|\pi\sqrt{\delta_L d^{n+2}})^{n-k}|\det(\tilde{a}_{lm}^1)_{k+1\leqslant l,m\leqslant n}|(1+o(1)). \tag{44}$$

Here, the $o(1)$ term is no longer uniformly bounded over $\mathbb{R}X$, though. Indeed, from Lemma 2.3.5 and (41),

$$\lambda\circ\nabla^2\sigma_{|K} = a_1^1(\|\lambda\|\sqrt{\pi\delta_L d^{n+1}})(\nabla^{TX}dx_1) + \sum_{1\leqslant l\leqslant m\leqslant n}\tilde{a}_{lm}^1(\|\lambda\|\sqrt{\pi\delta_L d^{n+2}})dx_l\otimes dx_m,$$

since the relation $\lambda\circ\nabla_{|x}\sigma = d_{|x}p$ imposes that a_j^1 vanishes for $j>1$. Moreover, since $dp = \sum_{i=1}^n \alpha_i dx_i$, with $\alpha_2(x) = \cdots = \alpha_n(x) = 0$ and $|\alpha_1(x)| = \|d_{|x}p\|$, we get that

$$0 = \nabla^{TX}(dp)_{|K} = \alpha_1(\nabla^{TX}dx_1)_{|K} + \sum_{i=1}^n (d\alpha_i\otimes dx_i)_{|K},$$

so $\|\nabla^{TX}dx_{1|K}\| = \frac{1}{\|d_{|x}p\|}\|\sum_{i=1}^n d\alpha_i\otimes dx_i\|$ has a pole of order 1 at x. In formula (44), the $o(1)$ term has thus a pole of order at most $n-k$ near the critical points of p.

We deduce that the Jacobian determinant of the map

$$\dot{\sigma} \in \langle(\sigma_j^1)_{k+1\leqslant j\leqslant n}\rangle \mapsto \dot{x}_K = -(\lambda\circ\nabla^2\sigma_{|K})^{-1}\circ(\lambda\circ\nabla_{|x}\dot{\sigma}_{|K}) \in K$$

equals $(\sqrt{\pi^{n-k}d^{n-k}}|\det(\tilde{a}_{lm}^1)_{k+1\leqslant l,m\leqslant n}|)^{-1}(1+o(1))$, up to sign, where the $o(1)$ term has a pole of order at most $n-k$ near the critical points of p. The result follows. □

3.4. Proof of Theorem 3.1.3

3.4.1. The case $k<n$. From Proposition 2.4.1 we know that

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{\pi^n d^n}}\left(\int_{\pi_1(\pi_2^{-1}(x))}|\det d_{|\sigma}ev^\perp_{(\sigma,x)}|^{-1}d\mu_{\mathbb{R}}(\sigma)\right)dvol_{h_L}.$$

From the coarea formula (see [10]), we likewise deduce that

$$\mathbb{E}(\nu_i) = \frac{1}{\sqrt{\pi^n d^n}}\left(\int_{\mathrm{Gr}(n-k,\ker d_{|x}p)\times\mathbb{R}(E\otimes L^d)^*_{|x}\setminus\{0\}} e^{-(a_1^1)^2}\frac{dK\wedge d\lambda}{\sqrt{\pi}^{(n-k)(k-1)+k}}\cdots\right.$$

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$$\dots \int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-1} |\det d_{|\sigma} \rho_x^{\perp}|^{-1} d\mu_{\mathbb{R}}(\sigma)\Big) dvol_{h_L},$$

since with the notation (41), $\sigma \in \rho_x^{-1}(K,\lambda)$ if and only if $\forall i \in \{1,\dots,k\}$ and $\forall j \in \{k+1,\dots,n\}$, $a_0^i = 0 = a_j^i$, while $\forall j \geqslant 2$, $a_j^1 = 0$ and $|a_1^1| = \|d_{|x}p\|/(\|\lambda\|\sqrt{\pi\delta_L}\sqrt{d}^{n+1})$. From Lemma 2.3.6 and the relation (40), we deduce that for every $x \in \mathbb{R}X \setminus Crit(p)$ and every $(K,\lambda) \in \mathrm{Gr}(n-k, \ker d_{|x}p) \times \mathbb{R}(E\otimes L^d)^*_{|x} \setminus \{0\}$,

$$\int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-1} |\det d_{|\sigma} \rho_x^{\perp}|^{-1} d\mu_{\mathbb{R}}(\sigma)$$
$$\underset{d\to\infty}{\sim} \int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{H}_{(\sigma,x)}|^{-1} |\det d_{|\sigma} \rho_x^{H}|^{-1} d\mu_{\mathbb{R}}(\sigma).$$

Thus, from Lemmas 3.3.1, 3.3.2 and 3.2.2, $\mathbb{E}(\nu_i)$ converges to

$$\int_{M_{k-1}(\mathbb{R})} |\det (a_j^i)_{2\leqslant i,j\leqslant k}|^{n-k+2} d\mu(a_j^i) \int_{Sym_{\mathbb{R}}(i,n-k-i)} |\det(\tilde{a}^1_{lm})_{k+1\leqslant l,m\leqslant n}| d\mu(\tilde{a}^1_{lm}) \dots$$
$$\dots \int_{\mathrm{Gr}(n-k,\ker d_{|x}p)\times\mathbb{R}(E\otimes L^d)^*_{|x}\setminus\{0\}} \frac{(a_1^1)^2 e^{-(a_1^1)^2}}{\|\lambda\|^k} \frac{dK\wedge d\lambda}{\sqrt{\pi}^{(n-k)(k-1)+k}},$$

where the convergence is dominated by a function in $L^1(\mathbb{R}X, dvol_{h_L})$; see Remark 3.3.3. We deduce that $\mathbb{E}(\nu_i)$ becomes equivalent to

$$\frac{\|d_{|x}p\|^2}{\delta_L d^{n+1}\sqrt{\pi}^{k+2}} V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) e_{\mathbb{R}}(i,n-k-i) \left(\int_{\mathbb{R}(E\otimes L^d)^*_{|x}\setminus\{0\}} \frac{e^{-(a_1^1)^2}}{\|\lambda\|^{k+2}} d\lambda\right) dvol_{h_L}.$$

Now,

$$\frac{\|d_{|x}p\|^2}{\pi\delta_L d^{n+1}} \int_{\mathbb{R}(E\otimes L^d)^*_{|x}\setminus\{0\}} \frac{e^{-(a_1^1)^2}}{\|\lambda\|^{k+2}} d\lambda = \frac{Vol(S^{k-1})\|d_{|x}p\|^2}{\pi\delta_L d^{n+1}} \int_0^{+\infty} \frac{e^{-(a_1^1)^2}}{\|\lambda\|^3} d\|\lambda\|$$
$$= Vol(S^{k-1}) \int_0^{+\infty} e^{-r^2} r dr = \frac{1}{2} Vol(S^{k-1}).$$

Since $Vol(S^{k-1}) = 2\sqrt{\pi}^k/\Gamma(k/2)$, we finally deduce that $\mathbb{E}(\nu_i)$ weakly converges to

$$\frac{1}{\Gamma(k/2)} V_{k-1,n-1} E_{k-1}(|\det|^{n-k+2}) e_{\mathbb{R}}(i,n-k-i) dvol_{h_L},$$

where the convergence is dominated by a function in $L^1(\mathbb{R}X, dvol_{h_L})$. □

3.4.2. The case $k = n$. When the rank of E equals the dimension of X, the vanishing locus of a generic section σ of $\mathbb{R}H^0(X, E\otimes L^d)$ is a finite set of points. We set $\nu = (1/\sqrt{d}^n)\sum_{x\in\mathbb{R}C_\sigma} \delta_x$, and define the incidence variety as

$$\mathcal{I} = \{(\sigma,x) \in (\mathbb{R}H^0(X,E\otimes L^d)\setminus\mathbb{R}\Delta_d)\times\mathbb{R}X \mid \sigma(x) = 0\}.$$

The projections π_1 and π_2 are defined by (31) and (32). As before, for every $(\sigma_0,x_0) \in (\mathbb{R}H^0(X,E\otimes L^d)\setminus\mathbb{R}\Delta_d)\times\mathbb{R}X$, π_1 is invertible in a neighborhood $\mathbb{R}U$ of σ_0, defining an evaluation map at the critical point

$$ev_{(\sigma_0,x_0)} : \sigma \in \mathbb{R}U \mapsto \pi_2\circ\pi_1^{-1}(\sigma) = x \in \mathbb{R}C_\sigma \cap \mathbb{R}V,$$

where $\mathbb{R}V$ denotes a neighborhood of x_0 in $\mathbb{R}X$; compare § 2.4.2 of [14]. We denote by $d_{|\sigma_0} ev^{\perp}_{(\sigma_0,x_0)}$ the restriction of its differential map $d_{|\sigma_0} ev_{(\sigma_0,x_0)}$ at σ_0 to the orthogonal complement of $\pi_1(\pi_2^{-1}(x_0))$ in $\mathbb{R}H^0(X, E \otimes L^d)$. Then, from Proposition 3.2.1,

$$\mathbb{E}(\nu) = \frac{1}{\sqrt{d}^n} (\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}})_{|x} = \frac{1}{\sqrt{\pi d}^n} \int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-1} d\mu_{\mathbb{R}}(\sigma) dvol_{h_L}.$$

The space $H_x = \langle (\sigma_0^i)_{1 \leqslant i \leqslant k} \rangle$ is a complement to $T_\sigma \pi_1(\pi_2^{-1}(x))$ in $\mathbb{R}H^0(X, E \otimes L^d)$ and in the decomposition (41), $a_0^i = 0$ for every $i = 1, \ldots, k$. The tangent space of $\mathcal{I}$ at (σ, x) reads

$$T_{(\sigma,x)}\mathcal{I} = \{(\dot\sigma, \dot x) \in \mathbb{R}H^0(X, E \otimes L^d) \times T_x\mathbb{R}X \mid \dot\sigma(x) + \nabla_{|x}\sigma(\dot x) = 0\}.$$

As in the proof of Lemma 3.3.2, we deduce that the Jacobian determinant of the map

$$\dot\sigma \in H_x \mapsto \dot x = -(\nabla_{|x}\sigma^{\perp})^{-1}(\dot\sigma(x)) \in T_x\mathbb{R}X$$

equals $\sqrt{\pi^n d^n} |\det(a_j^i)_{1 \leqslant i,j \leqslant n}|(1 + o(1))$, so

$$|\det d_{|\sigma} ev^H_{(\sigma,x)}|^{-1} = \sqrt{\pi d^n} |\det(a_j^i)_{1 \leqslant i,j \leqslant n}|(1 + o(1)),$$

where the $o(1)$ term is uniformly bounded over $\mathbb{R}X$. From lemma 2.3.6 we deduce that $\mathbb{E}(\nu)$ becomes equivalent to

$$\left(\int_{M_n(\mathbb{R})} |\det(a_j^i)_{1 \leqslant i,j \leqslant n}| d\mu(a_j^i) \right) dvol_{h_L} = E_n(|\det|) dvol_{h_L}.$$

Formula (15.4.12) of [23] (see Remark 3.1.1) now gives

$$E_n(|\det|) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(1/2)} = \frac{1}{Vol_{FS}(\mathbb{R}P^n)}$$

(see Remark 2.14 of [14]), and hence the result. □

3.5. Equidistribution of critical points in the complex case

Let X be a smooth complex projective manifold of dimension n, (L, h_L) be a holomorphic Hermitian line bundle of positive curvature ω over X and (E, h_E) be a rank k holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. For every $d > 0$, we denote by L^d the dth tensor power of L and by h^d the induced Hermitian metric on L^d. We denote by $H^0(X, L^d)$ its complex vector space of global holomorphic sections and by N_d the dimension of $H^0(X, L^d)$. We denote then by $\langle ., . \rangle$ the L^2-Hermitian product on this vector space, defined by the relation

$$\forall \sigma, \tau \in H^0(X, L^d), \langle \sigma, \tau \rangle = \int_X h^d(\sigma, \tau) dx. \qquad (45)$$

The associated Gaussian measure is denoted by $\mu_{\mathbb{C}}$. It is defined, for every open subset U of $H^0(X, L^d)$, by

$$\mu_{\mathbb{C}}(U) = \frac{1}{\pi^{N_d}} \int_U e^{-\|\sigma\|^2} d\sigma, \qquad (46)$$

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where $d\sigma$ denotes the Lebesgue measure of $H^0(X, L^d)$. For every $d > 0$, we denote by Δ^d the discriminant hypersurface of $H^0(X, E \otimes L^d)$, that is the set of sections $\sigma \in H^0(X, E \otimes L^d)$ which do not vanish transversally. For every $\sigma \in H^0(X, E \otimes L^d) \setminus \{0\}$, we denote by C_σ the vanishing locus of σ in X. For every $\sigma \in H^0(X, E \otimes L^d) \setminus \Delta^d$, C_σ is then a smooth codimension k complex submanifold of X. We equip X with a Lefschetz pencil $p : X \dashrightarrow \mathbb{C}P^1$. We then denote, for every $d > 0$, by Δ^d_p the set of sections $\sigma \in H^0(X, E \otimes L^d))$ such that $\sigma \in \Delta^d$, or C_σ intersects the critical locus of p, or the restriction of p to C_σ is not a Lefschetz pencil. For d large enough, this extended discriminant locus is of measure 0 for the measure $\mu_{\mathbb{C}}$.

For every $\sigma \in H^0(X, E \otimes L^d) \setminus \Delta^d_p$, we denote by $Crit(p_{|C_\sigma})$ the set of critical points of the restriction of p to C_σ and set, for $1 \leqslant k \leqslant n-1$,

$$\nu(C_\sigma) = \frac{1}{d^n} \sum_{x \in Crit(p_{|C_\sigma})} \delta_x, \tag{47}$$

where δ_x denotes the Dirac measure of X at the point x. When $k = n$, $\nu(C_\sigma) = (1/d^n) \sum_{x \in C_\sigma} \delta_x$.

Theorem 3.5.1. *Let X be a smooth complex projective manifold of dimension n, (L, h_L) be a holomorphic Hermitian line bundle of positive curvature ω over X and (E, h_E) be a rank k holomorphic Hermitian vector bundle, with $1 \leqslant k \leqslant n$. Let $p : X \dashrightarrow \mathbb{C}P^1$ be a Lefschetz pencil. Then, the measure $\mathbb{E}(\nu)$ defined by (47) weakly converges to $\binom{n-1}{k-1}\omega^n$ as d grows to infinity.*

When $k = 1$, Theorem 3.5.1 reduces to Theorem 3 of [15]; see also Theorem 1.3 of [14].

Proof. The proof goes along the same lines as that of Theorem 3.1.2, so we only give a sketch of it. Firstly, the analogue of Proposition 3.2.1 provides

$$\mathbb{E}(\nu) = \frac{1}{d^n} (\pi_2)_* (\pi_1^* d\mu_{\mathbb{C}}),$$

and at every point $x \in X \setminus (Crit(p) \cup \mathrm{Base}(p))$, where $\mathrm{Base}(p)$ denotes the base locus of p,

$$(\pi_2)_* (\pi_1^* d\mu_{\mathbb{C}})_{|x} = \frac{1}{\pi^n} \int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} d\mu_{\mathbb{R}}(\sigma) \frac{\omega^n}{n!};$$

see Proposition 2.10 of [14]. Choosing complex coefficients in decomposition (41), Lemmas 3.3.1 and 3.3.2 remain valid in the complex setting; see Remark 2.3.8. We deduce that

$$\begin{aligned} \mathbb{E}(\nu) &= \frac{1}{\pi^n d^n} \left(\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} d\mu_{\mathbb{C}}(\sigma) \right) \frac{\omega^n}{n!} \\ &\underset{d\to\infty}{\sim} \frac{1}{\pi^n d^n} \left(\int_{\mathrm{Gr}_{\mathbb{C}}(n-k, \ker d_{|x} p) \times (E \otimes L^d)^*_{|x} \setminus \{0\}} e^{-|a_1^1|^2} \frac{dK \wedge d\lambda}{\pi^{(n-k)(k-1)+k}} \cdots \right. \\ &\qquad \left. \cdots \int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} |\det d_{|\sigma} \rho_x^{\perp}|^{-2} d\mu_{\mathbb{C}}(\sigma) \right) \frac{\omega^n}{n!}, \end{aligned}$$

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with $|a_1^1|$ given by (40); see Lemma 2.3.6 as before. Here, $\mathrm{Gr}_{\mathbb{C}}(n-k, \ker d_{|x} p)$ denotes the Grassmann manifold of $n-k$-dimensional complex linear subspaces of $\ker d_{|x} p$. From the complex versions of Lemma 2.3.5 and 2.3.6 (see Remark 2.3.8 and the relation (40)), we deduce that for every $x \in X \setminus (Crit(p) \cup \mathrm{Base}(p))$ and every $(K, \lambda) \in \mathrm{Gr}(n-k, \ker d_{|x} p) \times (E \otimes L^d)^*_{|x} \setminus \{0\}$,

$$\int_{\rho_x^{-1}(K,\lambda)} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} |\det d_{|\sigma} \rho_x^{\perp}|^{-2} d\mu_{\mathbb{C}}(\sigma)$$
$$\underset{d\to\infty}{\sim} \frac{|a_1^1|^4 \pi^n d^n}{\|\lambda\|^{2k}} \int_{M_{k-1}(\mathbb{C})} |\det(a^i_j)_{2\leqslant i,j\leqslant k}|^{2(n-k+2)} d\mu(a^i_j) \dots$$
$$\dots \int_{Sym_{\mathbb{C}}(n-k)} |\det(\tilde{a}^1_{lm})_{k+1\leqslant l,m\leqslant n}|^2 d\mu(\tilde{a}^1_{lm}).$$

We deduce that $\mathbb{E}(\nu)$ is equivalent to

$$\frac{\|d_{|x} p\|^4}{(\pi \delta_L d^{n+1})^2} \frac{1}{\pi^{(n-k)(k-1)+k}} Vol(\mathrm{Gr}_{\mathbb{C}}(k-1, n-1)) \dots$$
$$\dots \; E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) e_{\mathbb{C}}(n-k) \left(\int_{(E\otimes L^d)^*_{|x}\setminus\{0\}} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^{2(k+2)}} d\lambda \right) \frac{\omega^n}{n!},$$

where $e_{\mathbb{C}}(n-k) = \int_{Sym_{\mathbb{C}}(n-k)} |\det A|^2 d\mu_{\mathbb{C}}(A)$ and

$$E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) = \int_{M_{k-1}(\mathbb{C})} |\det A|^{2(n-k+2)} d\mu_{\mathbb{C}}(A).$$

Now,

$$\frac{\|d_{|x} p\|^4}{(\pi \delta_L d^{n+1})^2} \int_{(E\otimes L^d)^*_{|x}\setminus\{0\}} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^{2k+4}} d\lambda = Vol(S^{2k-1}) \frac{\|d_{|x} p\|^4}{(\pi \delta_L d^{n+1})^2} \int_0^{+\infty} \frac{e^{-|a_1^1|^2}}{\|\lambda\|^5} d\|\lambda\|$$
$$= Vol(S^{2k-1}) \int_0^{+\infty} e^{-r^2} r^3 dr = \frac{1}{2} Vol(S^{2k-1}).$$

Hence, $\mathbb{E}(\nu)$ is equivalent to

$$\frac{1}{2\pi^{(n-k)(k-1)+k}} Vol(\mathrm{Gr}_{\mathbb{C}}(k-1, n-1)) Vol(S^{2k-1}) \; E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) e_{\mathbb{C}}(n-k) \frac{\omega^n}{n!},$$

where $e_{\mathbb{C}}(n-k) = (n-k+1)!$ by Proposition 3.8 of [14], $Vol(S^{2k-1}) = 2\pi^k/(k-1)!$,

$$E^{\mathbb{C}}_{k-1}(|\det|^{2(n-k+2)}) = \frac{\prod_{j=1}^{k-1} \Gamma((n-k+2)+j)}{\prod_{j=1}^{k-1} \Gamma(j)} = \frac{\prod_{j=n-k+3}^{n+1} \Gamma(j)}{\prod_{j=1}^{k-1} \Gamma(j)}$$

by formula 15.4.12 of [23] and

$$Vol(\mathrm{Gr}_{\mathbb{C}}(k-1, n-1)) = \frac{\prod_{j=1}^{k-1} \Gamma(j)}{\prod_{j=n-k+1}^{n-1} \Gamma(j)} \pi^{(k-1)(n-k)}$$

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by a computation analogous to the one given for the real case by Remark 3.1.1. We conclude that $\mathbb{E}(\nu)$ weakly converges to $\binom{k-1}{n-1}\omega^n$, where the convergence is dominated by a function in $L^1(X, (\omega^n/n!))$, for it has poles of order at most $2(n-k)$ near the critical points of p and at most 2 near the base points; see [15]. □

Corollary 3.5.2. *Under the hypotheses of Theorem 3.5.1, for every generic* $\sigma \in \mathbb{R}H^0(X, E \otimes L^d)$*, let* $|Crit p_{|C_\sigma}|$ *be the number of critical points of* $p_{|C_\sigma}$*. Then,*

$$\frac{1}{d^n}\mathbb{E}(|Crit p_{|C_\sigma}|) \underset{d\to\infty}{\sim} \binom{k-1}{n-1}\int_X c_1(L)^n.$$

Proof. Corollary 3.5.2 follows from Theorem 3.5.1 by integration of 1 over X. A direct proof can be given though. The modulus of p is a Morse function on $C_\sigma \setminus (\text{Base}\,(p) \cup F_0 \cup F_\infty)$, where F_0 (resp. F_∞) is the fiber of 0 (resp. of ∞) of $p : X \dashrightarrow \mathbb{C}P^1$. Moreover, the index of every critical point of $|p|$ is $n-k$. As in the proofs of Propositions 1 and 2 in [15], we deduce that $\mathbb{E}(|Crit p_{|C_\sigma}|)$ is equivalent to $|\chi(C_\sigma)|$ as d grows to infinity. Now,

$$\chi(C_\sigma) = \int_{C_\sigma} c_{n-k}(C_\sigma) = \int_X c_{n-k}(C_\sigma) \wedge c_k(E \otimes L^d),$$

while from the adjunction formula, $c(C_\sigma) \wedge c(E \otimes L^d)_{|C_\sigma} = c(X)$. Moreover, for $0 \leqslant i \leqslant k$, $c_i(E \otimes L^d) = \binom{k}{i} d^i c_1(L)^i + o(d^i)$, so

$$c(E \otimes L^d) = (1 + dc_1(L))^k + o((1 + dc_1(L))^k).$$

From the formula $(1+x)^{-k} = \sum_{j=0}^{\infty}(-1)^j((k-1+j)!/j!(k-1)!)x^j$, we then deduce that $c_{n-k}(C_\sigma) = (-1)^{n-k}\binom{n-1}{k-1}d^{n-k}c_1(L)^{n-k} + o(d^{n-k})$ and finally that

$$\chi(C_\sigma) = (-1)^{n-k}\binom{n-1}{k-1}d^n\int_X c_1(L)^n + o(d^n).$$

Hence the result. □

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