Proceedings of the Edinburgh Mathematical Society (2018) **61**, 1055–1061
doi:10.1017/S0013091518000172

SOLVING THE EQUATION $\operatorname{div} v = F$ IN $\mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$

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(Received 23 August 2016; first published online 24 July 2018)

Abstract In the following note, we focus on the problem of *existence* of continuous solutions vanishing at infinity to the equation $\operatorname{div} v = f$ for $f \in L^n(\mathbb{R}^n)$ and satisfying an estimate of the type $\|v\|_\infty \leqslant C\|f\|_n$ for any $f \in L^n(\mathbb{R}^n)$, where $C > 0$ is related to the constant appearing in the Sobolev–Gagliardo–Nirenberg inequality for functions with bounded variation (BV functions).

Keywords: continuous solvability; divergence equation; Sobolev–Gagliardo–Nirenberg inequality

2010 *Mathematics subject classification:* Primary 46E40
Secondary 35A23; 35F05; 35D30

1. Introduction

It has been well known since Bourgain and Brezis' [**1**] that given a function $f \in L^n_\#(\mathbb{T}^n)$ (i.e. a function $f \in L^n(\mathbb{T}^n)$ satisfying $\int_{\mathbb{T}^n} f = 0$), there exists a bounded solution $v \in L^\infty(\mathbb{T}^n, \mathbb{R}^n)$ to the equation:

$$\operatorname{div} v = f \quad \text{in } \mathbb{T}^n, \tag{1}$$

satisfying the estimate

$$\|v\|_\infty \leqslant C\|f\|_n, \tag{2}$$

where $C > 0$ is independent of f. In fact, the authors even show that there exists a *continuous* solution to (1) satisfying estimate (2). They also show, though, that there is *no* bounded linear operator $S : L^n_\#(\mathbb{T}^n) \to L^\infty(\mathbb{T}^n, \mathbb{R}^n)$ yielding $\operatorname{div}(Sf) = f$ in the sense of distributions for any $f \in L^n_\#(\mathbb{T}^n)$; hence, (2) cannot be the consequence of any kind of bounded linear representation formula for the solutions of (1).

In the sequel, we denote by $\mathscr{C}_b(\mathbb{R}^n, \mathbb{R}^n)$ the spaces of bounded continuous vector fields, and by $\mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ the space of all continuous vector fields vanishing at infinity in $\mathbb{R}^n$.

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In the whole Euclidean space $\mathbb{R}^n$, it follows from De Pauw and Torres [3, Theorem 6.1] that, given $f \in L^n(\mathbb{R}^n)$, there exists a continuous vector field vanishing at infinity $v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} v = f$ in $\mathbb{R}^n$ in the sense of distributions. However, their result does not provide an estimate of type (2) in this case. One of the objectives of this short note is to show that, using the *existence* for each $f \in L^n(\mathbb{R}^n)$ of a continuous vector field $v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} v = f$, Bourgain and Brezis' proof of [1, Proposition 1] yields the existence for each $f \in L^n(\mathbb{R}^n)$ of a vector field $w \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} w = f$ in $\mathbb{R}^n$ as well as $\|w\|_\infty \leqslant C\|f\|_n$, where $C > 0$ is independent of f and related to the constant appearing in the Sobolev–Gagliardo–Nirenberg inequality for bounded variation (BV) functions. This is our Theorem 3.1.

2. Solvability in $\mathscr{C}_b(\mathbb{R}^n, \mathbb{R}^n)$ *vs* $\mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$

Let $n \geqslant 2$ and $1^* := n/(n-1)$ be the Sobolev exponent when $p = 1$. Consider $\mathrm{BV}_{1^*}(\mathbb{R}^n)$ the set of functions $g \in \mathrm{L}^{1^*}(\mathbb{R}^n)$ such that

$$\|Dg\| := \sup\left\{\int_{\mathbb{R}^n} g \operatorname{div} v : v \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^n), \|v\|_\infty \leqslant 1\right\} < \infty$$

and

$$BV_{1^*,c}(\mathbb{R}^n) := \{g \in \mathrm{BV}_{1^*}(\mathbb{R}^n) : \{g \neq 0\} \text{ is relatively compact in } \mathbb{R}^n\}.$$

Indeed, if $g \in \mathrm{BV}_{1^*}(\mathbb{R}^n)$, we may denote by $Dg : \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ the (finite) vector-valued Radon measure on $\mathbb{R}^n$ extending the distributional gradient of g.

Given a sequence $(g_k) \subseteq \mathrm{BV}_{1^*}(\mathbb{R}^n)$, we can write $g_k \twoheadrightarrow 0$ when $k \to \infty$ in cases where the following conditions hold:

(i) $g_k \rightharpoonup 0$, $k \to \infty$ weakly in $\mathrm{L}^{1^*}(\mathbb{R}^n)$;

(ii) $\sup_k \|Dg_k\| < +\infty$.

Next, we define *charges vanishing at infinity* according to De Pauw and Torres [3, Definition 3.1].

Definition. A *charge vanishing at infinity* is a linear functional $F : \mathrm{BV}_{1^*}(\mathbb{R}^n) \to \mathbb{R}$ having the property that $F(g_k) \to 0$, $k \to \infty$ for any sequence $(g_k) \subseteq \mathrm{BV}_{1^*}(\mathbb{R}^n)$ satisfying $g_k \twoheadrightarrow 0$, $k \to \infty$.

According to De Pauw and Torres [3, Theorem 6.1], charges vanishing at infinity are precisely the (extensions to $\mathrm{BV}_{1^*}(\mathbb{R}^n)$ of) distributions for which the equation $\operatorname{div} v = F$ has a continuous solution vanishing at infinity. We formulate their result as follows.

Theorem 2.1. *Assume that $F \in \mathscr{D}'(\mathbb{R}^n)$ is a distribution. The equation $\operatorname{div} v = F$ has a solution $v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ in the sense of distributions if and only if F extends to a charge vanishing at infinity.*

Remark 2.2. It is easy to observe that in the case where $F \in \mathscr{D}'(\mathbb{R}^n)$ extends to a charge vanishing at infinity, this extension is unique (and still denoted by F in the sequel).

https://doi.org/10.1017/S0013091518000172 Published online by Cambridge University Press

The following example shows that there exist distributions $F \in \mathscr{D}'(\mathbb{R}^n)$ having the property that $\operatorname{div} v = F$ is solvable in $\mathscr{C}_b(\mathbb{R}^n, \mathbb{R}^n)$ but *not* in $\mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$.

Example. Define for each $k \in \mathbb{N}$ a function $g_k \in \mathrm{BV}_{1^*}(\mathbb{R}^n)$ by $g_k := \chi_{[2k,2k+1]^n}$, where χ_E denotes the characteristic function of $E \subseteq \mathbb{R}^n$. It is clear that for $k \in \mathbb{N}$, we have:

$$\|Dg_k\| = \mathscr{H}^{n-1}(\partial[2k, 2k+1]^n) = 2n,$$

where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff outer measure in $\mathbb{R}^n$ (see, e.g. [4, § 2.1]). One also readily observes that $g_k \rightharpoonup 0$, $k \to \infty$ weakly in $\mathrm{L}^{1^*}(\mathbb{R}^n)$. Hence $g_k \twoheadrightarrow 0$, $k \to \infty$.

Now choose $\eta : \mathbb{R}^n \to \mathbb{R}$, a bounded continuous function satisfying the following properties:

(a) $0 \leqslant \eta \leqslant 1$ on $\mathbb{R}^n$;

(b) $\eta = 1$ on $[2k + 1/3, 2k + 2/3]^{n-1} \times \{2k + 1\}$ for each $k \in \mathbb{N}$;

(c) $\eta = 0$ outside $[2k + 1/4, 2k + 3/4]^{n-1} \times [2k + 1/2, 2k + 3/2]$.

Let $e_n := (0, \ldots, 0, 1) \in \mathbb{R}^n$, define $w \in \mathscr{C}_b(\mathbb{R}^n, \mathbb{R}^n)$ by $w = \eta e_n$ and let the distribution $F \in \mathscr{D}'(\mathbb{R}^n)$ be defined by:

$$F(\varphi) := -\int_{\mathbb{R}^n} w \cdot \nabla\varphi \,\mathrm{d}x,$$

for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

Since F is the distributional divergence of w, it is clear that on the one hand the equation $\operatorname{div} v = F$ has w as a continuous solution. Yet, on the other hand, *if* the equation $\operatorname{div} v = F$ admitted a solution $w_0 \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$, one would be able to extend the distribution F to a charge vanishing at infinity $\bar{F} : \mathrm{BV}_{1^*}(\mathbb{R}^n) \to \mathbb{R}$ according to Theorem 2.1. However, in this case, a routine approximation argument shows that for any $g \in \mathrm{BV}_{1^*,c}(\mathbb{R}^n)$, we would have:

$$\bar{F}(g) = -\int_{\mathbb{R}^n} w \cdot \mathrm{d}[Dg].$$

This would yield, for any $k \in \mathbb{N}$:

$$\bar{F}(g_k) \geqslant \tfrac{2}{3},$$

and hence $\bar{F}(g_k) \not\twoheadrightarrow 0$, $k \to \infty$ while $g_k \twoheadrightarrow 0$, $k \to \infty$, which would be in contradiction to the fact that $\bar{F}$ is a charge vanishing at infinity.

We have thus shown that:

(I) $\operatorname{div} v = F$ is solvable in $\mathscr{C}_b(\mathbb{R}^n, \mathbb{R}^n)$;

(II) $\operatorname{div} v = F$ is not solvable in $\mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$.

3. A Bourgain–Brezis-like estimate for solving $\operatorname{div} v = F$ in $\mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$

In the present section, we would like to make the simple observation that in the case where $F \in L^n(\mathbb{R}^n)$, De Pauw and Torres' existence theorem (Theorem 2.1) in fact implies a more precise result along the lines of Bourgain and Brezis' paper [**1**, Proposition 1]. Before we state our result, recall that the Sobolev–Gagliardo–Nirenberg inequality for BV functions ([**4**, Theorem 1, p. 189]) ensures the existence of a constant $\kappa = \kappa(n) > 0$ (depending only on the dimension n) such that:

$$\|g\|_{1^*} \leqslant \kappa(n)\|Dg\| \quad \text{for all } g \in \mathrm{BV}_{1^*}(\mathbb{R}^n). \tag{3}$$

Theorem 3.1. *Given $F \in L^n(\mathbb{R}^n)$, there exists $v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ satisfying* $\operatorname{div} v = F$ *in $\mathbb{R}^n$, together with the estimate:*

$$\|v\|_\infty \leqslant 2\kappa(n)\|F\|_n, \tag{4}$$

where $\kappa(n) > 0$ is the constant appearing in the Sobolev–Gagliardo–Nirenberg inequality for BV functions stated above (3).

Remark 3.2. Given $F \in L^n(\mathbb{R}^n)$, De Pauw and Torres observed that the map $\mathrm{BV}_{1^*} \to \mathbb{R}$ given by $g \mapsto \int_{\mathbb{R}^n} Fg$ defines a charge vanishing at infinity ([**3**, Proposition 3.4]). It follows that there exists $v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} v = F$ according to Theorem 2.1. We improve this by showing that v can be chosen to verify (4).

Proof of Theorem 3.1. As in [**1**, Proof of Proposition 1], let

$$X := \{v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n) : \operatorname{div} v \in \mathrm{L}^n(\mathbb{R}^n)\}, \qquad Y := \mathrm{L}^n(\mathbb{R}^n),$$

endow X with the usual norm $\|v\|_\infty := \sup\{|v(x)| : x \in \mathbb{R}^n\}$, Y with the Lebesgue norm $\|F\|_n$, and define an (unbounded) operator with domain $D(A) := X \subseteq \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ by:

$$A : D(A) \subseteq \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n) \to Y, \qquad v \mapsto A(v) := \operatorname{div} v.$$

Recall also that we denote by $G(A)$, $N(A)$ and $R(A)$, the graph, the null-set and the range of A, respectively. Given $Z \subseteq X$, we also let $Z^\perp := \{\mu \in X^* : \langle \mu, v\rangle = 0 \text{ for all } v \in Z\}$.

Claim 3.3. *The operator A is densely defined and closed.*

To show that A is densely defined, it suffices to observe that one has $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^n) \subseteq D(A)$. To show that A is closed, assume that $((v_k, F_k)) \subseteq G(A)$ converges in $\mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n) \times Y$ to some $(v, F) \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n) \times Y$. We hence have $\operatorname{div} v_k = F_k$ in the sense of distributions, for each $k \in \mathbb{N}$, from which it readily follows that one also has $\operatorname{div} v = F$ in the sense of distributions. From the fact that $F \in \mathrm{L}^n(\mathbb{R}^n)$, we infer that $v \in X = D(A)$ and $A(v) = F$; it follows that $(v, F) \in G(A)$, and A has a closed graph.

Claim 3.4. *The operator A is onto, i.e. $R(A) = Y$.*

Observe indeed that Remark 3.2 implies that any $F \in Y = \mathrm{L}^n(\mathbb{R}^n)$ defines a charge vanishing at infinity, from which Theorem 2.1 implies the existence of $v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$ with $A(v) = \operatorname{div} v = F$; this proves the claim.

Claim 3.5. *The domain* $D(A^*)$ *of the adjoint operator* A^* *satisfies* $D(A^*) \subseteq \mathrm{BV}_{1^*}$.

Assuming that $g \in \mathrm{L}^{1^*}(\mathbb{R}^n) \simeq \mathrm{L}^n(\mathbb{R}^n)^*$ satisfies $g \in D(A^*)$, we have a constant $C > 0$ such that:

$$\left| \int_{\mathbb{R}^n} g \operatorname{div} v \, \mathrm{d}x \right| = |\langle A(v), g\rangle| \leqslant C\|v\|_\infty \quad \text{for all } v \in D(A).$$

Using the inclusion $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^n) \subseteq D(A)$, this yields, in particular:

$$\|Dg\| := \sup \left\{ \int_{\mathbb{R}^n} g \operatorname{div} v \, \mathrm{d}x : v \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^n), \|v\|_\infty \leqslant 1 \right\} \leqslant C,$$

which implies that $g \in \mathrm{BV}_{1^*}(\mathbb{R}^n)$.

In order to complete the proof of Theorem 3.1, let us proceed towards a contradiction. Fix $F \in Y$ with $\|F\|_n = 1$ and assume that the two following convex subsets of X,

$$K := \{v \in X : \operatorname{div} v = F\} \quad \text{and} \quad L := \{v \in X : \|v\|_\infty \leqslant 2\kappa(n)\},$$

satisfy $K \cap L = \emptyset$, where $\kappa(n) > 0$ is any constant such that inequality (3) holds for any $g \in \mathrm{BV}_{1^*}(\mathbb{R}^n)$.

It then follows from the geometric form of the Hahn–Banach theorem [**2**, Theorem 1.6] that there exists a non-zero $\mu \in X^*$ together with a real number $\alpha \in \mathbb{R}$ such that one has:

$$\langle \mu, v\rangle \geqslant \alpha \text{ for each } v \in K \quad \text{and} \quad \langle \mu, v\rangle \leqslant \alpha \text{ for each } v \in L.$$

Claim 3.6. *One has* $\mu \in N(A)^\perp$.

To prove this claim, fix $v \in N(A)$. Choose $v_0 \in K$ (this is possible since $K \neq \emptyset$ according to Claim 3.4) and observe that for any $\lambda \in \mathbb{R}$, one has $v_0 + \lambda v \in K$. Hence, for any $\lambda \in \mathbb{R}$:

$$\alpha \leqslant \langle \mu, v_0 + \lambda v\rangle = \langle \mu, v_0\rangle + \lambda\langle \mu, v\rangle,$$

which is impossible unless $\langle \mu, v\rangle = 0$.

Claim 3.7. *One has* $\|\mu\| \leqslant (\alpha/(2\kappa(n)))$ *(yielding in particular that* $\alpha > 0$*).*

To show this, fix $v \in X$ satisfying $\|v\|_\infty \leqslant 1$, observe that we have $2\kappa(n)v \in L$ and compute:

$$2\kappa(n)\langle \mu, v\rangle = \langle \mu, 2\kappa(n)v\rangle \leqslant \alpha.$$

The desired inequality follows, as v is arbitrary.

Claim 3.8. *There exists* $g \in \mathrm{BV}_{1^*}(\mathbb{R}^n)$ *such that* $\mu = -Dg$ *on* X *and* $\|Dg\| = \|\mu\|$*; in particular* μ *is (or rather extends to) a vector-valued Radon measure.*

Indeed, we have already shown (Claim 3.4) that $R(A) = Y$; in particular, A has closed range in Y. It follows from the Closed Range theorem [**2**, Theorem 2.19] that

$R(A^*) = N(A)^\perp$. Claim 3.6 thus ensures the existence of $g \in D(A^*) \subseteq \mathrm{BV}_{1^*}(\mathbb{R}^n)$, for which $\mu = -A^*(g)$. Yet, we can compute for $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^n) \subseteq X$:

$$\langle \mu, v\rangle = \langle A^*(g), v\rangle = \langle g, A(v)\rangle = \int_{\mathbb{R}^n} g \operatorname{div} v \,\mathrm{d}x = -\int_{\mathbb{R}^n} v \cdot d[Dg] = \langle -Dg, v\rangle.$$

The claim follows since it is clear that $\|Dg\| = \sup\{\langle Dg, v\rangle : v \in X, \|v\|_\infty \leqslant 1\}$.

Coming back to the proof of Theorem 3.1, observe now that it follows from the Sobolev–Gagliardo–Nirenberg inequality for BV functions (3) that:

$$\|g\|_{1^*} \leqslant \kappa(n)\|Dg\| = \kappa(n)\|\mu\| \leqslant \frac{\alpha}{2}.$$

Now, we compute for any $v \in K$, using the equality $\operatorname{div} v = F$ and a standard smoothing argument:

$$\alpha \leqslant \langle \mu, v\rangle = -\langle Dg, v\rangle = \int_{\mathbb{R}^n} gF \,\mathrm{d}x \leqslant \|F\|_n \|g\|_{1^*} \leqslant \frac{\alpha}{2},$$

which is impossible since α is positive. This concludes our proof. □

Remark 3.9. If one is solely interested in the existence of *some* constant $C > 0$ providing the inequality (4) in the statement of Theorem 3.1, the latter result becomes a simple consequence of the open mapping theorem. Indeed, endowing the space X defined above with the norm $\|v\|' := \|v\|_\infty + \|\operatorname{div} v\|_n$, it is easy to see that $(X, \|\cdot\|')$ becomes a complete space. The bounded linear operator $A : (X, \|\cdot\|') \to L^n(\mathbb{R}^n)$ defined by $A(v) := \operatorname{div} v$ is then surjective according to De Pauw and Torres' existence theorem (see Theorem 2.1 above) and [**3**, Proposition 3.4]; the open mapping theorem [**2**, Theorem 2.6, p. 35] then ensures the existence of a constant $c > 0$ such that $A(B_X(0, 1)) \supseteq B_Y(0, c)$. We now let $C := (1/c)$.

Fix a non-zero $F \in L^n(\mathbb{R}^n)$ and let $G := cF/\|F\|_n$ so that $\|G\|_n = c$. Hence, there exists $w \in X$ satisfying $\|w\|' \leqslant 1$ and $G = A(w) = \operatorname{div} w$. Letting $v := \|F\|_n w/c$, we get $v \in \mathscr{C}_0(\mathbb{R}^n, \mathbb{R}^n)$, $\operatorname{div} v = F$ and:

$$\|v\|_\infty \leqslant \|v\|' = \frac{1}{c} \cdot \|F\|_n \|w\|' \leqslant C\|F\|_n,$$

thus yielding an estimate of type (4).

Acknowledgements. We thank Professor Thierry De Pauw (Institut de Mathématiques de Jussieu) for helpful discussions concerning this work. We also thank the referee for her/his helpful comments, and in particular for suggesting the argument contained in Remark 3.9.

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