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Rayleigh Matroids

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Motivated by a property of linear resistive electrical networks, we introduce the class of Rayleigh matroids. These form a subclass of the balanced matroids defined by Feder and Mihail [9] in 1992. We prove a variety of results relating Rayleigh matroids to other well-known classes – in particular, we show that a binary matroid is Rayleigh if and only if it does not contain $\mathscr{S}_8$ as a minor. This has the consequence that a binary matroid is balanced if and only if it is Rayleigh, and provides the first complete proof in print that $\mathscr{S}_8$ is the only minor-minimal binary non-balanced matroid, as claimed in [9]. We also give an example of a balanced matroid which is not Rayleigh.

1. Introduction

For explanation of any undefined terms, we refer the reader to Oxley's book [18].

In 1992, Feder and Mihail [9] introduced the concept of a balanced matroid in relation to a conjecture of Mihail and Vazirani [17] regarding expansion properties of one-skeletons of $\{0,1\}$-polytopes. (Unfortunately, the term 'balanced' has also been used for matroids with at least three other meanings [3, 8, 12].) Let $\mathscr{M}$ be a matroid with ground set E. For our purposes we can assume that all matroids are loopless, and regard $\mathscr{M}$ as its set of bases. For disjoint subsets I, J of E, let

$$\mathscr{M}_I^J := \{B \smallsetminus I \ : \ B \in \mathscr{M} \text{ and } I \subseteq B \subseteq E \smallsetminus J\}$$

denote the minor of $\mathscr{M}$ obtained by contracting I and deleting J, and let M_I^J denote the the number of bases of $\mathscr{M}_I^J$. (This convention is slightly nonstandard since $\mathscr{M}_I^J = \varnothing$ if I is dependent – but this is convenient here.) Feder and Mihail say that $\mathscr{M}$ is *negatively*

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correlated provided that, for every $e, f \in E$ with e not a loop,

$$\frac{M_f}{M} \geqslant \frac{M_{ef}}{M_e},$$

and that $\mathscr{M}$ is *balanced* provided that every minor of $\mathscr{M}$ is negatively correlated. Since $M_e = M_{ef} + M_e^f$, $M_f = M_{ef} + M_f^e$, and $M = M_{ef} + M_e^f + M_f^e + M^{ef}$, the inequality above is equivalent to

$$\Delta M\{e,f\} := M_e^f M_f^e - M_{ef} M^{ef} \geqslant 0.$$

We briefly review the literature on balanced matroids in Section 2.

Stemming from a collaboration with James Oxley and Alan Sokal [7], we were motivated to consider the following similar condition on a matroid $\mathscr{M}$ with ground set E. Fix indeterminates $\mathbf{y} := \{y_e : \ e \in E\}$ indexed by E, and for disjoint subsets $I, J \subseteq E$ let $M_I^J(\mathbf{y}) := \sum_B \mathbf{y}^B$, with the sum over all bases B of $\mathscr{M}_I^J$ and with $\mathbf{y}^B := \prod_{e \in B} y_e$. We say that $\mathscr{M}$ is a *Rayleigh* matroid provided that whenever $y_c > 0$ for all $c \in E$, then for every pair of distinct $e, f \in E$,

$$\Delta M\{e,f\}(\mathbf{y}) := M_e^f(\mathbf{y}) M_f^e(\mathbf{y}) - M_{ef}(\mathbf{y}) M^{ef}(\mathbf{y}) \geqslant 0.$$

We call the polynomial $\Delta M\{e,f\}(\mathbf{y})$ the *Rayleigh difference of* $\{e,f\}$ *in* $\mathscr{M}$. This terminology is motivated by the Rayleigh monotonicity property of linear resistive electrical networks, as explained in Section 3. The main results of Section 3 are as follows:

- The class of Rayleigh matroids is closed by taking duals and minors.
- Every Rayleigh matroid is balanced.
- The class of Rayleigh matroids is closed by taking 2-sums.
- The class of balanced matroids is closed by taking 2-sums if and only if every balanced matroid is Rayleigh.
- A binary matroid is Rayleigh if and only if it does not contain $\mathscr{S}_8$ as a minor.
- A binary matroid is balanced if and only if it is Rayleigh.

These results were motivated by similar claims for balanced matroids for which complete published proofs are not available.

In Section 4 we discuss another class of matroids: the 'half-plane property' matroids, or HPP matroids for short. This class was, in part, the object of study in our collaboration with Oxley and Sokal [7]. We extend a theorem of Godsil [11] (itself a refinement of a theorem of Stanley [22]) from the class of regular matroids to the more general class of HPP matroids. The following consequence of this is the main result of Section 4:

- Every HPP matroid is a Rayleigh matroid.

In proving this we identify a spectrum of conditions between these two extremes.

In Section 5 we discuss some more specific examples. On the positive side:

- Every sixth-root of unity matroid is an HPP matroid.

(This is from [7].) In particular, all regular matroids (hence all graphs) are HPP matroids, and hence Rayleigh. Recent work of Choe [5, 6] shows that:

- All sixth-root of unity matroids are in fact 'strongly Rayleigh' in a sense distinct from the spectrum of conditions in Section 4.

Also:

- A binary matroid is strongly Rayleigh if and only if it is regular.
- Every matroid with a 2-transitive automorphism group is negatively correlated.
- Every matroid of rank at most three is Rayleigh (this is proved in [23]).

On the negative side:

- There is a rank 4 transversal matroid which is not balanced.

In particular, such matroids need not be HPP, which settles negatively a question left open in [7].

- Every finite projective geometry fails to be HPP.
- There is a balanced matroid which is not Rayleigh.

Combined with the results in Section 3, this shows that the class of balanced matroids is not closed by taking 2-sums.

We conclude in Section 6 with a few open problems.

2. Balanced matroids

Feder and Mihail [9] prove two main results about balanced matroids. First:

- Every regular matroid is balanced.

This establishes a large class of examples including, of course, all graphic or cographic matroids. (See Proposition 5.1 and Corollary 4.7 below.) Second:

- The basis-exchange graph of a balanced matroid has cutset expansion at least one.

To explain this, the *basis-exchange graph* of a matroid $\mathscr{M}$ is the simple graph with the set of bases of $\mathscr{M}$ as its vertex set, and with an edge $B_1 \sim B_2$ if and only if $|B_1 \triangle B_2| = 2$ (in which $\triangle$ denotes the symmetric difference of sets). A simple graph $G = (V, E)$ has *cutset expansion at least* ρ provided that, for every $\varnothing \neq S \subset V$,

$$\frac{|\{e \in E \,:\, e \cap S \neq \varnothing \text{ and } e \cap (V \smallsetminus S) \neq \varnothing\}|}{\min\{|S|, |V \smallsetminus S|\}} \geqslant \rho.$$

Such isoperimetric inequalities imply that a natural random walk on the graph converges rapidly to the uniform distribution on the vertices. This leads to an efficient algorithm for generating a random basis of a balanced matroid approximately uniformly. See [9] for details.

The matroid $\mathscr{S}_8$ is represented over $GF(2)$ by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & b \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

with $b = 0$, and the matroid $\mathscr{A}_8 = \mathscr{AG}(3, 2)$ is represented over $GF(2)$ by this matrix with $b = 1$. Feder and Mihail refer to unpublished work showing that $\mathscr{S}_8$ is the only minor-minimal binary non-balanced matroid. To our knowledge, the only argument in print for this claim is in Chapter 5 and Appendix D of Merino's thesis [16], but it contains an error. Specifically, the argument rests on five points.

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- The matroid $\mathscr{S}_8$ is not negatively correlated. This was observed by Seymour and Welsh [21] and is not hard to verify. (Labelling the ground set $\{1,\ldots,8\}$ corresponding to the columns of the above matrix, we have $(S_8)_1 = 28$, $(S_8)_8 = 20$, $(S_8)_{1,8} = 12$, and $S_8 = 48$, so that $\Delta S_8\{1,8\} = 28\cdot 20 - 12\cdot 48 = -16 < 0$.)
- The matroid $\mathscr{A}_8$ is a 'splitter' for the class of binary matroids which do not contain an $\mathscr{S}_8$ minor. More explicitly, if a connected binary matroid $\mathscr{M}$ with no $\mathscr{S}_8$ minor has $\mathscr{A}_8$ as a proper minor, then $\mathscr{M}$ can be expressed as a 2-sum with $\mathscr{A}_8$ as one of the factors. This is an unpublished result of Seymour and is explained in Appendix D of [16].
- Every binary matroid which does not contain $\mathscr{S}_8$ or $\mathscr{A}_8$ as a minor can be constructed from regular matroids, the Fano matroid $\mathscr{F}_7$, and its dual $\mathscr{F}_7^*$ by taking direct sums and 2-sums. This is due to Seymour [20].
- The matroids $\mathscr{A}_8$, $\mathscr{F}_7$, and $\mathscr{F}_7^*$ are balanced. This too is not difficult to verify and appears in Appendix D of [16].
- The class of balanced matroids is closed by taking 2-sums. This appears as Lemma 5.4.4 in [16], but the argument in support of it contains an error on the first part of p. 113. In fact, this claim is false (Theorem 5.12).

To explain the difficulty, consider a matroid $\mathscr{M}$ and distinct elements e, f, g of $E(\mathscr{M})$. Then, since $M = M_g + M^g$, *etc.*, a short calculation shows that

$$\Delta M\{e,f\} = \Delta M_g\{e,f\} + \Theta M\{e,f|g\} + \Delta M^g\{e,f\}, \tag{2.1}$$

in which

$$\Delta M_g\{e,f\} := M_{eg}^{f} M_{fg}^{e} - M_{efg} M_g^{ef},$$
$$\Delta M^g\{e,f\} := M_e^{fg} M_f^{eg} - M_{ef}^{g} M^{efg},$$

and the *central term for* $\{e,f\}$ *and* g *in* $\mathscr{M}$ is given by

$$\Theta M\{e,f|g\} := M_e^{fg} M_{fg}^{e} + M_f^{eg} M_{eg}^{f} - M_g^{ef} M_{ef}^{g} - M_{efg} M^{efg}.$$

Now let $\mathscr{Q}$ be another matroid, with $E(\mathscr{Q}) \cap E(\mathscr{M}) = \{g\}$, and consider the 2-sum $\mathscr{N} = \mathscr{M} \oplus_g \mathscr{Q}$ of $\mathscr{M}$ and $\mathscr{Q}$ along g. The set of bases of $\mathscr{N}$ is

$$\mathscr{N} := \{B_1 \cup B_2 : (B_1, B_2) \in (\mathscr{M}_g \times \mathscr{Q}^g) \cup (\mathscr{M}^g \times \mathscr{Q}_g)\}$$

by definition, so that $N = M_g Q^g + M^g Q_g$. Again, a short calculation shows that

$$\Delta N\{e,f\} = (Q^g)^2 \Delta M_g\{e,f\} + Q^g Q_g \Theta M\{e,f|g\} + (Q_g)^2 \Delta M^g\{e,f\}. \tag{2.2}$$

Now assume that $\mathscr{M}$ is balanced. If the class of balanced matroids is closed by taking 2-sums then $\Delta N\{e,f\} \geqslant 0$ for any balanced choice of $\mathscr{Q}$. That is, the quadratic polynomial

$$p(y) := y^2 \Delta M_g\{e,f\} + y\Theta M\{e,f|g\} + \Delta M^g\{e,f\}$$

is such that $p(\lambda) \geqslant 0$ for any rational number of the form $\lambda = Q^g/Q_g$ with $\mathscr{Q}$ balanced and $g \in E(\mathscr{Q})$.

For positive integers a and b, let $G(a,b)$ be the graph obtained from a path with b edges by replacing each edge by a parallel edges, then joining the end-vertices by a new 'root' edge. Label the root edge of $G(a,b)$ by g. The graphic (cycle) matroid $\mathscr{Q}(a,b)$ of $G(a,b)$

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is balanced by the result of Feder and Mihail. Now, since $Q(a,b)^g/Q(a,b)_g = a/b$, every positive rational number is of the form λ above.

Therefore, the polynomial $p(y)$ above must satisfy $p(\lambda) \geqslant 0$ for all $\lambda \geqslant 0$, and since both $\Delta M_g\{e,f\}$ and $\Delta M^g\{e,f\}$ are nonnegative, the zeros of $p(y)$ are either nonreal complex conjugates or are real and of the same sign. This implies that

$$\Theta M\{e,f|g\} \geqslant -2\sqrt{\Delta M_g\{e,f\}\Delta M^g\{e,f\}}.$$

This 'triple condition' on the balanced matroid $\mathscr{M}$ is necessary for all $\{e,f\}$ and g in $E(\mathscr{M})$ if the class of balanced matroids is closed by taking 2-sums. However, it is unclear whether or not this can be deduced from the hypothesis that $\mathscr{M}$ is balanced. The Rayleigh hypothesis, on the other hand, includes these triple conditions and can be carried through the 2-sum construction with ease, as we see in the next section.

3. Rayleigh matroids

The term 'Rayleigh matroid' is motivated by analogy with a property of electrical networks. Consider a (multi)graph $G = (V,E)$ together with a set $\mathbf{y} = \{y_e : e \in E\}$ of positive real numbers indexed by the edges of G. Thinking of each y_e as the electrical conductance of the edge $e \in E$, for any two vertices $a,b \in V$ we may ask for the value of the effective conductance $\mathscr{Y}_{ab}(G;\mathbf{y})$ of the graph as a whole, considered as a network joining the poles a and b. In 1847, Kirchhoff [14] proved that

$$\mathscr{Y}_{ab}(G;\mathbf{y}) = \frac{T(G;\mathbf{y})}{T(G/ab;\mathbf{y})},$$

in which $T(G;\mathbf{y}) := \sum_T \mathbf{y}^T$ with the sum over all spanning trees of G, and $T(G/ab;\mathbf{y})$ is defined similarly except that G/ab is the graph obtained from G by merging a and b into a single vertex.

It is physically intuitive that if $y_c > 0$ for all $c \in E$ and y_e is increased, then $\mathscr{Y}_{ab}(G;\mathbf{y})$ does not decrease – this property is called *Rayleigh monotonicity*. (This will be proved below when we show that sixth-root of unity matroids – in particular, graphs – are Rayleigh matroids.) Nonnegativity of $\partial\mathscr{Y}_{ab}(G;\mathbf{y})/\partial y_e$ is equivalent to the inequality

$$\frac{\partial T(G;\mathbf{y})}{\partial y_e} T(G/ab;\mathbf{y}) \geqslant T(G;\mathbf{y})\frac{\partial T(G/ab;\mathbf{y})}{\partial y_e}.$$

Rephrasing this in terms of the graph H obtained from G by adjoining a new edge f with ends a and b, the inequality is

$$T_e^f(H;\mathbf{y})T_f(H;\mathbf{y}) \geqslant T^f(H;\mathbf{y})T_{ef}(H;\mathbf{y}),$$

in which $T_e^f(H;\mathbf{y})$ is the sum of $\mathbf{y}^T$ over all spanning trees T of the graph obtained by contracting e and deleting f from H, *etc.* A little cancellation shows that this is equivalent to the inequality

$$T_e^f(H;\mathbf{y})T_f^e(H;\mathbf{y}) - T_{ef}(H;\mathbf{y})T^{ef}(H;\mathbf{y}) \geqslant 0.$$

Replacing $T(H;\mathbf{y})$ by the basis-generating polynomial $M(\mathbf{y})$ of a more general matroid $\mathscr{M}$, we arrive at the condition $\Delta M\{e,f\}(\mathbf{y}) \geqslant 0$ defining Rayleigh matroids.

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To simplify notation, when calculating with Rayleigh matroids we will henceforth usually omit reference to the variables $\mathbf{y}$ – writing M_I^J instead of $M_I^J(\mathbf{y})$, *etc.* – unless a particular substitution of variables requires emphasis. We will also write '$\mathbf{y} > \mathbf{0}$' as shorthand for '$y_c > 0$ for all $c \in E$', and '$\mathbf{y} \equiv \mathbf{1}$' as shorthand for '$y_c = 1$ for all $c \in E$'.

Proposition 3.1. *A matroid $\mathcal{M}$ is Rayleigh if and only if the dual matroid $\mathcal{M}^*$ is Rayleigh.*

Proof. For disjoint subsets $I, J \subseteq E$ we have $M_I^{*J}(\mathbf{y}) = \mathbf{y}^E M_J^I(\mathbf{1}/\mathbf{y})$, in which $\mathbf{1}/\mathbf{y} := \{1/y_c : c \in E\}$. Therefore, the inequality $\Delta M^*\{e,f\}(\mathbf{y}) \geqslant 0$ is equivalent to the inequality $\Delta M\{e,f\}(\mathbf{1}/\mathbf{y}) \geqslant 0$. From this the result follows. □

Proposition 3.2. *If $\mathcal{M}$ is a Rayleigh matroid and $\mathcal{N}$ is a minor of $\mathcal{M}$, then $\mathcal{N}$ is a Rayleigh matroid.*

Proof. Since $\mathcal{M}$ is Rayleigh, for distinct $e, f, g \in E$ and $\mathbf{y} > \mathbf{0}$,

$$\Delta M\{e,f\} = y_g^2 \Delta M_g\{e,f\} + y_g \Theta M\{e,f|g\} + \Delta M^g\{e,f\} \tag{3.1}$$

is nonnegative. Take the limit of this as $y_g \to 0$ to see that $\Delta M^g\{e,f\} \geqslant 0$. Since $e, f \in E(\mathcal{M}^g)$ and $\mathbf{y} > \mathbf{0}$ are arbitrary, this shows that $\mathcal{M}^g$ is Rayleigh. Similarly, by considering the limit of $y_g^{-2}\Delta M\{e,f\}$ as $y_g \to \infty$ we see that $\mathcal{M}_g$ is Rayleigh. The case of a general minor is obtained by iteration of the above two cases. □

Corollary 3.3. *Every Rayleigh matroid $\mathcal{M}$ is balanced and satisfies the triple condition*

$$\Theta M\{e,f|g\} \geqslant -2\sqrt{\Delta M_g\{e,f\}\Delta M^g\{e,f\}}$$

for distinct $e, f, g \in E(\mathcal{M})$ when $\mathbf{y} > \mathbf{0}$.

Proof. If $\mathcal{M}$ is a Rayleigh matroid, then by setting $\mathbf{y} \equiv \mathbf{1}$ we see that $\mathcal{M}$ is negatively correlated. Since every minor of $\mathcal{M}$ is also Rayleigh, it follows that $\mathcal{M}$ is balanced.

For distinct $e, f, g \in E(\mathcal{M})$, when $y_c > 0$ for all $c \neq g$, the polynomial (in the variable y_g) displayed in (3.1) above is nonnegative for all $y_g > 0$. As in Section 2, this implies the desired inequality. □

Proposition 3.4. *Let $\mathcal{M}$ be a matroid with ground set E, and let I, J be disjoint subsets of E. If $\mathcal{M}$ is Rayleigh and $\mathbf{y} > \mathbf{0}$ then*

$$M_I M_J \geqslant M_{IJ} M.$$

Proof. The inequality is trivial if either I or J is dependent, so assume that both I and J are independent in $\mathcal{M}$.

We first prove the result for $I = \{e_1\}$ and $J = \{f_1, \ldots, f_k\}$. Notice that the Rayleigh difference of $\{e,f\}$ in $\mathcal{M}$ may also be expressed as $\Delta M\{e,f\} = M_e M_f - M_{ef} M$. Thus, the Rayleigh condition is that if $\mathbf{y} > \mathbf{0}$ then $M_e M_f \geqslant M_{ef} M$. Since every (contraction) minor of $\mathscr{M}$ is also Rayleigh, we see that if $\mathbf{y} > \mathbf{0}$ then

$$\frac{M_{e_1}}{M} \geqslant \frac{M_{e_1 f_1}}{M_{f_1}} \geqslant \frac{M_{e_1 f_1 f_2}}{M_{f_1 f_2}} \geqslant \cdots \geqslant \frac{M_{e_1 J}}{M_J}.$$

That is, $M_I M_J \geqslant M_{IJ} M$ in this case.

Viewed another way, we have shown that if $\mathscr{M}$ is Rayleigh and $\mathbf{y} > \mathbf{0}$ then for any non-loop $e_1 \in E$ and $J \subseteq E$, $M_J/M \geqslant M_{e_1 J}/M_{e_1}$. If now $I = \{e_1, e_2, \ldots, e_m\}$ is independent then, since each (contraction) minor of $\mathscr{M}$ is Rayleigh,

$$\frac{M_J}{M} \geqslant \frac{M_{e_1 J}}{M_{e_1}} \geqslant \frac{M_{e_1 e_2 J}}{M_{e_1 e_2}} \geqslant \cdots \geqslant \frac{M_{IJ}}{M_I}.$$

This implies the desired inequality. □

The probability space associated with $\mathscr{M}$ and $\mathbf{y} > \mathbf{0}$ assigns to each basis B of $\mathscr{M}$ the probability $\mathbf{y}^B/M(\mathbf{y})$. As in [9, 15, 19], Proposition 3.4 leads to the fact that any two increasing events with disjoint support in this space are negatively correlated, provided that $\mathscr{M}$ is Rayleigh.

Theorem 3.5. *Let $\mathscr{M}$ and $\mathscr{Q}$ be matroids with $E(\mathscr{M}) \cap E(\mathscr{Q}) = \{g\}$, and let $\mathscr{N} = \mathscr{M} \oplus_g \mathscr{Q}$ be the 2-sum of $\mathscr{M}$ and $\mathscr{Q}$ along g. If $\mathscr{M}$ and $\mathscr{Q}$ are Rayleigh matroids then $\mathscr{N}$ is a Rayleigh matroid.*

Proof. Fix $y_c > 0$ for all $c \in E(\mathscr{N})$, and consider any $e, f \in E(\mathscr{N})$. We must show that $\Delta N\{e,f\} \geqslant 0$. Up to symmetry of the hypotheses there are essentially two cases:

(i) $e \in E(\mathscr{M}) \smallsetminus \{g\}$ and $f \in E(\mathscr{Q}) \smallsetminus \{g\}$,
(ii) $\{e,f\} \subseteq E(\mathscr{M}) \smallsetminus \{g\}$.

For case (i) a short calculation using $N = M_g Q^g + M^g Q_g$, *etc.*, shows that

$$\Delta N\{e,f\} = \Delta M\{e,g\} \Delta Q\{f,g\}.$$

Since $\mathscr{M}$ and $\mathscr{Q}$ are Rayleigh and $\mathbf{y} > \mathbf{0}$, both factors on the right are nonnegative, so that $\Delta N\{e,f\} \geqslant 0$ as well.

For case (ii) we use the formula (2.2) above, which now refers to polynomials in $\mathbf{y}$. If $Q^g(\mathbf{y}) = 0$ or $Q_g(\mathbf{y}) = 0$ then $\Delta N\{e,f\} \geqslant 0$ because both $\mathscr{M}^g$ and $\mathscr{M}_g$ are Rayleigh. Otherwise, by defining $w_c := y_c$ for all $c \in E(\mathscr{M}) \smallsetminus \{g\}$ and $w_g := Q^g(\mathbf{y})/Q_g(\mathbf{y})$, we see that

$$\Delta N\{e,f\}(\mathbf{y}) = Q_g(\mathbf{y})^2 \Delta M\{e,f\}(\mathbf{w}) \geqslant 0,$$

since $\mathbf{w} > \mathbf{0}$ and $\mathscr{M}$ is Rayleigh.

This proves that $\mathscr{N} = \mathscr{M} \oplus_g \mathscr{Q}$ is Rayleigh. □

For a matroid $\mathscr{M}$ and a set $\mathbf{m} := \{m_e : \ e \in E(\mathscr{M})\}$ of positive integers indexed by $E(\mathscr{M})$, let $\mathscr{M}[\mathbf{m}]$ be the matroid obtained from $\mathscr{M}$ by replacing each element $e \in E(\mathscr{M})$ by a parallel class of m_e elements. Equivalently, $\mathscr{M}[\mathbf{m}]$ is obtained from $\mathscr{M}$ by attaching the uniform matroid $\mathscr{U}_{1,1+m_e}$ to $\mathscr{M}$ by a 2-sum along e, for each $e \in E(\mathscr{M})$.

Theorem 3.6. *For a matroid $\mathscr{M}$, the following conditions are equivalent:*

(a) *the matroid $\mathscr{M}$ is Rayleigh,*
(b) *every matroid of the form $\mathscr{M}[\mathbf{m}]$ is Rayleigh,*
(c) *every matroid of the form $\mathscr{M}[\mathbf{m}]$ is balanced,*
(d) *every matroid of the form $\mathscr{M}[\mathbf{m}]$ is negatively correlated.*

Proof. To see that (a) implies (b), we note that a matroid $\mathscr{Q}$ of rank one is Rayleigh since for any $e, f \in E(\mathscr{Q})$ we have $\mathscr{Q}_{ef} \equiv 0$. Thus, since $\mathscr{M}[\mathbf{m}]$ is expressed as a 2-sum of Rayleigh matroids, it is also Rayleigh, by Theorem 3.5.

That (b) implies (c) is immediate from Corollary 3.3, and that (c) implies (d) is immediate from the definitions.

To see that (d) implies (a) assume that $\mathscr{M}$ is not Rayleigh. Thus, there exist distinct $e, f \in E(\mathscr{M})$ and positive real numbers $\mathbf{y} > \mathbf{0}$ such that $\Delta M\{e,f\}(\mathbf{y}) < 0$. Since the rational numbers are dense in the real numbers, there are positive rationals $\mathbf{q} = \{q_c : c \in E\}$ such that $\Delta M\{e,f\}(\mathbf{q}) < 0$. Let D be the smallest positive common denominator of all the numbers $\{q_c : c \in \smallsetminus\{e,f\}\}$, and for $c \in E \smallsetminus \{e,f\}$ let $m_c := Dq_c$, a positive integer. Since $\Delta M\{e,f\}(\mathbf{y})$ is independent of y_e and y_f we may put $m_e := m_f := 1$. Since $\Delta M\{e,f\}(\mathbf{y})$ is homogeneous of degree $2r-2$ (where r is the rank of $\mathscr{M}$) we have

$$\Delta M\{e,f\}(\mathbf{m}) = D^{2r-2}\Delta M\{e,f\}(\mathbf{q}) < 0.$$

However, we also have $\Delta M[\mathbf{m}]\{e,f\}(\mathbf{1}) = \Delta M\{e,f\}(\mathbf{m}) < 0$, so that $\mathscr{M}[\mathbf{m}]$ is not negatively correlated. □

Corollary 3.7. *The following statements are equivalent:*

(a) *every balanced matroid is Rayleigh,*
(b) *the class of balanced matroids is closed by taking 2-sums.*

Proof. To show that (a) implies (b), let $\mathscr{M}$ and $\mathscr{Q}$ be balanced matroids such that $E(\mathscr{M}) \cap E(\mathscr{Q}) = \{g\}$. By (a) both $\mathscr{M}$ and $\mathscr{Q}$ are Rayleigh, so that $\mathscr{M} \oplus_g \mathscr{Q}$ is Rayleigh by Theorem 3.5, and hence balanced by Corollary 3.3.

To show that (b) implies (a), consider a balanced matroid $\mathscr{M}$. Since uniform matroids of rank one are balanced, the hypothesis (b) implies that every matroid of the form $\mathscr{M}[\mathbf{m}]$ is balanced. By Theorem 3.6, it follows that $\mathscr{M}$ is Rayleigh. □

In Theorem 5.12 we will see that the two statements of Corollary 3.7 are in fact false.

Theorem 3.8. *A binary matroid is Rayleigh if and only if it does not contain $\mathscr{S}_8$ as a minor.*

Proof. The outline of the argument has been sketched in Section 2 (for balanced matroids in place of Rayleigh matroids). For the first point, since $\mathscr{S}_8$ is not negatively correlated it is not balanced, hence not Rayleigh. The second and third points need no revision, and the fifth point is substantiated for Rayleigh matroids by Theorem 3.5.

It remains to show that the matroids $\mathscr{A}_8$, $\mathscr{F}_7$, and $\mathscr{F}_7^*$ are Rayleigh. Since $\mathscr{F}_7$ is obtained from $\mathscr{A}_8$ by contracting any element, Propositions 3.1 and 3.2 imply that it is enough to

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show that $\mathscr{A}_8$ is Rayleigh. Let the ground set of $\mathscr{A}_8$ be $E = \{1, \ldots, 8\}$ corresponding to the columns of the representing matrix in Section 2. The automorphism group of $\mathscr{A}_8$ is 2-transitive on E, so in order to check that this matroid is Rayleigh it suffices to show that $\Delta A_8\{7,8\} \geqslant 0$ when $\mathbf{y} > \mathbf{0}$. A direct computation with the aid of Maple 6.01 shows that $\Delta A_8\{7,8\}$ is a positive sum of monomials and squares of binomials such as

$$(y_1 y_5 y_6 - y_3 y_4 y_5)^2.$$

Since this is clearly nonnegative for $\mathbf{y} > \mathbf{0}$ we see that $\mathscr{A}_8$ is Rayleigh. This completes the proof. □

Corollary 3.9. *A binary matroid is balanced if and only if it is Rayleigh.*

Proof. By Corollary 3.3, every Rayleigh matroid is balanced. If $\mathscr{M}$ is a balanced matroid then $\mathscr{M}$ does not contain $\mathscr{S}_8$ as a minor, since $\mathscr{S}_8$ is not negatively correlated. If $\mathscr{M}$ is also binary then $\mathscr{M}$ is Rayleigh, by Theorem 3.8. □

4. Half-plane property matroids

A polynomial $P(\mathbf{y}) = \sum_\alpha c_\alpha \mathbf{y}^\alpha$ in several complex variables $\mathbf{y} = \{y_e : e \in E\}$ has the *half-plane property* provided that, whenever $\mathrm{Re}(y_e) > 0$ for all $e \in E$, then $P(\mathbf{y}) \neq 0$. We say that a matroid $\mathscr{M}$ is a *half-plane property matroid* (HPP matroid, for short) if its basis-generating polynomial $M(\mathbf{y})$ has the half-plane property. This class of polynomials is investigated thoroughly in [7], from which we take the following facts without proof.

Lemma 4.1. ([10], Theorem 18, or [7], Proposition 3.4.) *Let $P(\mathbf{y})$ be a polynomial in the variables $\mathbf{y} = \{y_e : e \in E\}$, fix $e \in E$, and let $P(\mathbf{y}) = \sum_{j=0}^{n} P_j(y_c : c \neq e\})y_e^j$. If P has the half-plane property then each P_j has the half-plane property.*

Lemma 4.2. ([7], Proposition 5.2) *Let $P(\mathbf{y})$ be a homogeneous polynomial in the variables $\mathbf{y} = \{y_e : e \in E\}$. For nonnegative real numbers $\mathbf{a} = \{a_e : e \in E\}$ and $\mathbf{b} = \{b_e : e \in E\}$, let $P(\mathbf{a}x + \mathbf{b})$ be the polynomial obtained by substituting $y_e = a_e x + b_e$ for each $e \in E$. The following are equivalent:*

(a) *$P(\mathbf{y})$ has the half-plane property,*
(b) *for all sets of nonnegative real numbers $\mathbf{a}$ and $\mathbf{b}$, $P(\mathbf{a}x + \mathbf{b})$ has only real zeros.*

Proposition 4.3. ([7], Propositions 3.1, 4.1, 4.2, and Corollary 4.9) *The class of HPP matroids is closed by taking duals, minors, and 2-sums.*

Theorem 4.4 was proved for regular matroids and $\mathbf{y} \equiv \mathbf{1}$ by Godsil [11].

Theorem 4.4. *Let $\mathscr{M}$ be a matroid on a set E. Let $(S, T, C_1, \ldots, C_k)$ be an ordered partition of E into pairwise disjoint nonempty subsets, and fix nonnegative integers $c_1, \ldots, c_k$. For each $0 \leqslant j \leqslant |S|$, let $M_j(\mathbf{y}) := \sum_B \mathbf{y}^B$, with the sum over all bases B of $\mathscr{M}$ such that $|B \cap S| = j$*

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and $|B \cap C_i| = c_i$ for all $1 \leqslant i \leqslant k$. If $\mathscr{M}$ is an HPP matroid and $\mathbf{y} > \mathbf{0}$, then the polynomial $\sum_{j=0}^{|S|} M_j(\mathbf{y})x^j$ in the variable x has only real zeros.

Proof. Let $\mathscr{M}$ be an HPP matroid and fix $\mathbf{y} > \mathbf{0}$. Let s, t, and $z_1, \ldots, z_k$ be indeterminates, and for $e \in E$ put

$$u_e := \begin{cases} y_e s & \text{if } e \in S, \\ y_e t & \text{if } e \in T, \\ y_e z_i & \text{if } e \in C_i. \end{cases}$$

Then $M(\mathbf{u})$ is a homogeneous polynomial with the half-plane property in the variables $s, t, z_1, \ldots, z_k$. By repeated application of Lemma 4.1, the coefficient $M_{\mathbf{c}}(s,t)$ of $z_1^{c_1} \cdots z_k^{c_k}$ in $M(\mathbf{u})$ also has the half-plane property, and is homogeneous. In fact,

$$M_{\mathbf{c}}(s,t) = \sum_{j=0}^{|S|} M_j(\mathbf{y}) s^j t^{d-j},$$

in which $d = \mathrm{rank}(\mathscr{M}) - (c_1 + \cdots + c_k)$. Upon substituting $s = x$ and $t = 1$ in $M_{\mathbf{c}}(s,t)$, Lemma 4.2 implies that $\sum_{j=0}^{|S|} M_j(\mathbf{y})x^j$ has only real zeros, as claimed. □

Newton's inequalities (item (51) of [13]) state that if a polynomial $\sum_{j=0}^{n} a_j x^j$ with real coefficients has only real zeros then $\binom{n}{j}^{-2} a_j^2 \geqslant \binom{n}{j-1}^{-1} \binom{n}{j+1}^{-1} a_{j-1} a_{j+1}$ for all $1 \leqslant j \leqslant n-1$. That is, the sequence $\{\binom{n}{j}^{-1} a_j\}$ is *logarithmically concave*. Thus, Theorem 4.4 implies the following corollary, first proved for regular matroids and $\mathbf{y} \equiv \mathbf{1}$ by Stanley [22].

Corollary 4.5. *With the hypothesis and notation of Theorem 4.4, for each $1 \leqslant j \leqslant |S| - 1$,*

$$\frac{M_j(\mathbf{y})^2}{\binom{|S|}{j}^2} \geqslant \frac{M_{j-1}(\mathbf{y})}{\binom{|S|}{j-1}} \cdot \frac{M_{j+1}(\mathbf{y})}{\binom{|S|}{j+1}}.$$

Corollary 4.5 can be viewed as a quantitative strengthening of the basis exchange axiom for HPP matroids, as requested in Question 13.9 of [7].

For a subset $S \subseteq E(\mathscr{M})$ and natural number j, let $M(S, j; \mathbf{y}) = \sum_B \mathbf{y}^B$, with the sum over all bases B of $\mathscr{M}$ such that $|B \cap S| = j$. For each positive integer m, consider the following conditions on a matroid $\mathscr{M}$.

RZ[*m*]. If $\mathbf{y} > \mathbf{0}$, then for all $S \subseteq E$ with $|S| \leqslant m$ the polynomial $\sum_{j=0}^{|S|} M(S,j;\mathbf{y})x^j$ has only real zeros.

LC[*m*]. If $\mathbf{y} > \mathbf{0}$, then for all $S \subseteq E$ with $|S| \leqslant m$ the sequence $\{\binom{|S|}{j}^{-1} M(S,j;\mathbf{y})\}$ is logarithmically concave.

The $k = 0$ case of Theorem 4.4 implies that an HPP matroid is RZ[m] for all m, and Newton's Inequalities show that RZ[m] implies LC[m] for every m. The implications

RZ[m] $\Longrightarrow$ RZ[$m-1$] and LC[m] $\Longrightarrow$ LC[$m-1$] are trivial, as are the conditions RZ[1] and LC[1]. Thus, the weakest nontrivial condition among these is LC[2].

Theorem 4.6. *The following conditions are equivalent:*

(a) *the matroid* $\mathscr{M}$ *is* LC[2],
(b) *the matroid* $\mathscr{M}$ *is* RZ[2],
(c) *the matroid* $\mathscr{M}$ *is Rayleigh,*
(d) *the matroid* $\mathscr{M}$ *is* LC[3].

Proof. Conditions (a) and (b) are equivalent because a quadratic polynomial has only real zeros if and only if its discriminant is nonnegative.

To show that (a) implies (c) assume that $\mathscr{M}$ is LC[2], and choose distinct $e, f \in E$. Since $\mathscr{M}$ is LC[2], if $w_c > 0$ for all $c \in E$ then

$$\left(w_e M_e^f(\mathbf{w}) + w_f M_f^e(\mathbf{w})\right)^2 \geqslant 4 w_e w_f M_{ef}(\mathbf{w}) M^{ef}(\mathbf{w}).$$

In particular, if $\mathbf{y} > \mathbf{0}$ then let

$$w_c := \begin{cases} y_c & \text{if } \ c \notin \{e, f\}, \\ M_f^e(\mathbf{y}) & \text{if } \ c = e, \\ M_e^f(\mathbf{y}) & \text{if } \ c = f. \end{cases}$$

The inequality above becomes

$$\left(2 M_e^f(\mathbf{y}) M_f^e(\mathbf{y})\right)^2 \geqslant 4 M_e^f(\mathbf{y}) M_f^e(\mathbf{y}) M_{ef}(\mathbf{y}) M^{ef}(\mathbf{y}).$$

After some cancellation, this shows that

$$M_e^f(\mathbf{y}) M_f^e(\mathbf{y}) \geqslant M_{ef}(\mathbf{y}) M^{ef}(\mathbf{y}).$$

Hence, $\mathscr{M}$ is Rayleigh.

To show that (c) implies (d) assume that $\mathscr{M}$ is Rayleigh, and let $\mathbf{y} > \mathbf{0}$. We use the elementary inequality, for $m \geqslant 2$ real numbers $R_1, \ldots, R_m$:

$$(R_1 + \cdots + R_m)^2 \geqslant \frac{2m}{m-1} \sum_{\{i,j\} \subseteq \{1,\ldots,m\}} R_i R_j.$$

Apply this inequality when $S = \{e_1, \ldots, e_m\} \subseteq E(\mathscr{M})$ and $R_i := y_{e_i} M_{e_i}^{S \smallsetminus e_i}(\mathbf{y})$ for $1 \leqslant i \leqslant m$, with the result that

$$\begin{aligned} M(S, 1; \mathbf{y})^2 &\geqslant \frac{2|S|}{|S|-1} \sum_{\{e,f\} \subseteq S} y_e y_f M_e^{S \smallsetminus e}(\mathbf{y}) M_f^{S \smallsetminus f}(\mathbf{y}) \\ &\geqslant \frac{2|S|}{|S|-1} \sum_{\{e,f\} \subseteq S} y_e y_f M_{ef}^{S \smallsetminus ef}(\mathbf{y}) M^S(\mathbf{y}) \\ &= \frac{2|S|}{|S|-1} M(S, 0; \mathbf{y}) M(S, 2; \mathbf{y}). \end{aligned} \tag{4.1}$$

The second inequality uses the fact that each of the deletion minors $\mathscr{M}^{S \smallsetminus ef}$ of $\mathscr{M}$ is Rayleigh, by Proposition 4.3.

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This implies that $\mathcal{M}$ is LC[2] and verifies one of the inequalities of the condition LC[3] when $|S| = 3$. It remains to show that if $|S| = 3$ then $M(S,2;\mathbf{y})^2 \geqslant 3M(S,1;\mathbf{y})M(S,3;\mathbf{y})$. To do this we apply the above inequality to the dual matroid $\mathcal{M}^*$, which is also Rayleigh. Since

$$M^*(S,j;\mathbf{y}) = \mathbf{y}^E M(S,3-j;\mathbf{1}/\mathbf{y})$$

for $0 \leqslant j \leqslant 3$, the inequalities (4.1) imply the required result, showing that $\mathcal{M}$ is LC[3].

That (d) implies (a) is trivial. This completes the proof. □

The ideas in the proof of Theorem 4.6 are carried further in Section 4 of [24].

Corollary 4.7. *Every HPP matroid is a Rayleigh matroid.*

5. Examples

A matrix A of complex numbers is a *sixth-root of unity* matrix provided that every nonzero minor of A is a sixth-root of unity. A matroid $\mathcal{M}$ is a *sixth-root of unity matroid* provided that it can be represented over the complex numbers by a sixth-root of unity matrix. For example, every regular matroid is a sixth-root of unity matroid. Whittle [25] has shown that a matroid is a sixth-root of unity matroid if and only if it is representable over both $GF(3)$ and $GF(4)$. For graphs, Proposition 5.1 is part of the 'folklore' of electrical engineering. We take it from Corollary 8.2(a) and Theorem 8.9 of [7], but repeat the short and interesting proof for completeness.

Proposition 5.1. *Every sixth-root of unity matroid is an HPP matroid.*

Proof. Let A be a sixth-root of unity matrix of full row-rank r, representing the matroid $\mathcal{M}$, and let A^* denote the conjugate transpose of A. Index the columns of A by the set E, and let $Y := \mathrm{diag}(y_e : e \in E)$ be a diagonal matrix of indeterminates. For an r-element subset $S \subseteq E$, let $A[S]$ denote the square submatrix of A supported on the set S of columns. By the Binet–Cauchy formula,

$$\det(AYA^*) = \sum_{S \subseteq E:\ |S|=r} |\det A[S]|^2 \mathbf{y}^S = M(\mathbf{y})$$

is the basis-generating polynomial of $\mathcal{M}$, since $|\det A[S]|^2$ is 1 or 0 according to whether or not S is a basis of $\mathcal{M}$.

Now we claim that if $\mathrm{Re}(y_e) > 0$ for all $e \in E$, then AYA^* is nonsingular. This suffices to prove the result. Consider any nonzero vector $\mathbf{v} \in \mathbb{C}^r$. Then $A^*\mathbf{v} \neq \mathbf{0}$ since the columns of A^* are linearly independent. Therefore

$$\mathbf{v}^* AYA^* \mathbf{v} = \sum_{e \in E} y_e |(A^*\mathbf{v})_e|^2$$

has strictly positive real part, since for all $e \in E$ the numbers $|(A^*\mathbf{v})_e|^2$ are nonnegative reals and at least one of these is positive. In particular, for any nonzero $\mathbf{v} \in \mathbb{C}^r$, the vector $AYA^*\mathbf{v}$ is nonzero. It follows that AYA^* is nonsingular, completing the proof. □

Proposition 5.1 and Corollary 4.7 show that every sixth-root of unity matroid is Rayleigh. This implies the result of Feder and Mihail [9] that every regular matroid is balanced. In fact, even more is true. Enhancing Feder and Mihail's proof, Choe [5, 6] has recently shown the following.

Theorem 5.2. (Choe [5, 6]) *Let $\mathscr{M}$ be a sixth-root of unity matroid, and let $e, f \in E(\mathscr{M})$ be distinct. There are sixth-roots of unity $C_{ef}(S)$ for each $S \subset E \smallsetminus \{e, f\}$ such that both $S \cup \{e\}$ and $S \cup \{f\}$ are bases of $\mathscr{M}$, such that*

$$\Delta M\{e, f\}(\mathbf{y}) = \left(\sum_S C_{ef}(S)\mathbf{y}^S\right)\left(\sum_S \overline{C_{ef}(S)}\mathbf{y}^S\right).$$

Since the factors on the right-hand side are complex conjugates when all the y_e are real, Theorem 5.2 shows that for a sixth-root of unity matroid $\mathscr{M}$ and distinct $e, f \in E(\mathscr{M})$, the Rayleigh difference $\Delta M\{e, f\}(\mathbf{y})$ is nonnegative for *any real values* of the variables $\mathbf{y}$: positive, negative, or zero. We shall call such matroids *strongly Rayleigh.*

Proposition 5.3. *Let $\mathscr{M}$ be a strongly Rayleigh matroid on the set E. Then, for all distinct $e, f, g \in E$ and $\mathbf{y} \in \mathbb{R}^E$,*

$$|\Theta M\{e, f|g\}| \leqslant 2\sqrt{\Delta M_g\{e, f\}\Delta M^g\{e, f\}}.$$

Proof. For a strongly Rayleigh matroid $\mathscr{M}$ and real numbers $\mathbf{y} \in \mathbb{R}^E$ we have $\Delta M\{e, f\} \geqslant 0$. Considered as a quadratic polynomial in y_g, this does not change sign for $y_g \in \mathbb{R}$, and therefore it has a nonpositive discriminant. This gives the stated inequality. □

Arguments directly analogous to those in Section 3 suffice to prove the following, and the details are omitted.

Proposition 5.4. *The class of strongly Rayleigh matroids is closed by taking duals, minors, and 2-sums.*

Theorem 5.5. *A binary matroid is strongly Rayleigh if and only if it is regular.*

Proof. It is a theorem of Tutte that a binary matroid is regular if and only if it does not contain $\mathscr{F}_7$ or $\mathscr{F}_7^*$ as a minor (Theorems 13.1.1 and 13.1.2 of Oxley [18], for example). Regular matroids are strongly Rayleigh by Theorem 5.2. By Proposition 5.4, to prove the converse it suffices to show that $\mathscr{F}_7$ is not strongly Rayleigh. Label the elements of $E(\mathscr{F}_7)$ by $\{1, \ldots, 7\}$ corresponding to the columns of the representing matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

over $GF(2)$. To simplify notation we will write F_1^{26} instead of $(F_7)_1^{2,6}$, *etc.* With the substitutions $y_3 = y_5 = 2$ and $y_4 = y_7 = -1$, we have $F_{126} = 0$, $F_{12}^6 = F_{16}^2 = F_{26}^1 = 2$, $F_1^{26} = -8$,

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$F_2^{16} = F_6^{12} = 1$, and $F^{126} = -4$. Therefore, with $y_6 = t$ we have

$$\Delta F\{1,2\} = F_1^2 F_2^1 - F_{12}F^{12}$$
$$= (2t-8)(2t+1) - (2)(t-4) = 4t(t-4).$$

For any $0 < t < 4$ we have $\Delta F\{1,2\} < 0$, so that $\mathscr{F}_7$ is not strongly Rayleigh. □

In the case of graphs, Theorem 5.2 specializes to the following combinatorial identity: see also equation (2.34) of Brooks, Smith, Stone, and Tutte [2], Theorem 2.1 of Feder and Mihail [9], and several of the identities in Section 3.8 of Balabanian and Bickart [1].

Theorem 5.6. *Let $G = (V, E)$ be a connected (multi)graph, and let $\mathscr{G}$ be the graphic matroid of G. For distinct $e, f \in E$, fix arbitrary orientations of e and f, and for each $S \subset E \smallsetminus \{e, f\}$ such that both $S \cup \{e\}$ and $S \cup \{f\}$ are spanning trees of G, let $C_{ef}(S) := \pm 1$ according to whether or not e and f are directed consistently around the unique cycle of $S \cup \{e\} \cup \{f\}$. Then*

$$G_e^f(\mathbf{y})G_f^e(\mathbf{y}) - G_{ef}(\mathbf{y})G^{ef}(\mathbf{y}) = \left(\sum_S C_{ef}(S)\mathbf{y}^S\right)^2.$$

A combinatorial proof of this fact is greatly to be desired.

Chavez [4] has shown that every finite projective geometry is negatively correlated. More generally, we have the following proposition.

Proposition 5.7. *If a matroid admits a 2-transitive group of automorphisms then it is negatively correlated.*

Proof. Let $\mathscr{M}$ be a matroid of rank r on $m \geqslant 2$ elements which has a 2-transitive automorphism group, and let $M = M(\mathbf{1})$, *etc.* Let $e, f \in E$ be distinct. By transitivity of the automorphism group, $mM_e = mM_f = rM$. By 2-transitivity of the automorphism group, $m(m-1)M_{ef} = r(r-1)M$. Thus

$$\Delta M\{e, f\} = M_e M_f - M_{ef}M = \frac{M^2 r(m-r)}{m^2(m-1)} \geqslant 0$$

since $r \leqslant m$. □

The next result goes in the other direction.

Proposition 5.8. *Every finite projective geometry is not an HPP matroid.*

Proof. Every finite projective geometry contains a finite projective plane as a minor, so it suffices to prove that finite projective planes are not HPP matroids. In fact, a projective plane of order q fails the condition RZ$[q+1]$, as can be seen by taking $S \subseteq E$ to be a line of the plane and $\mathbf{y} \equiv \mathbf{1}$. Then the relevant polynomial is $Ax^2 + Bx + C$ with

$$A = q^2 \binom{q+1}{2} = \frac{(q+1)q^3}{2},$$
$$B = (q+1)\left[\binom{q^2}{2} - q\binom{q}{2}\right] = \frac{(q+1)q^3(q-1)}{2},$$
$$C = \binom{q^2}{3} - (q+1)q\binom{q}{3} = \frac{(q+1)q^3(q-1)^2}{6},$$

which has discriminant $-(q+1)^2 q^6 (q-1)^2/12$, and thus has non-real zeros. Theorem 4.4 thus implies that a projective plane of order q is not an HPP matroid. □

In Section 10.5 of [7], the question is raised whether or not every transversal matroid is an HPP matroid. Numerical experiments support this idea for transversal matroids of rank three, but we can no longer hope for much more than this, as we now see.

Proposition 5.9. *There is a transversal matroid of rank* 4 *which is not balanced.*

Proof. Let $\mathscr{L}$ be the matroid on the set $E = \{1, 2, \ldots, 10, e, f\}$ for which the bases are the transversals to the four sets $\{1, 2, 3, 4, f\}$, $\{5, 6, 7, f\}$, $\{8, 9, 10, f\}$, and $\{1, 2, 3, 5, 6, 8, 9, e, f\}$. A direct computation shows that $L_e = 80$, $L_f = 168$, $L_{ef} = 33$, and $L = 436$, so that $\Delta L\{e, f\} = -948 < 0$. □

Theorem 5.10 is proved in [23].

Theorem 5.10. *Every matroid of rank* (*or corank*) *at most* 3 *is Rayleigh.*

Corollary 5.11. *Every matroid with at most* 7 *elements is Rayleigh.*

Alternatively, Corollary 5.11 can be proved by directly checking the Rayleigh property for the nine matroids with rank three and seven elements that are not known to be HPP (Table 2 and Appendix A.2 of [7]). This was the original proof.

Theorem 5.12. *The class of balanced matroids is not closed by taking* 2*-sums.*

Proof. By Corollary 3.7 it suffices to give an example of a matroid which is balanced but not Rayleigh. The matroid $\mathscr{J}'$ represented over $\mathbb{R}$ by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 3 \end{bmatrix}$$

is such an example. Let $E(\mathscr{J}') = \{1, \ldots, 8\}$ corresponding to the columns of the above matrix. By Corollary 5.11, every proper minor of $\mathscr{J}'$ is Rayleigh, so it suffices to show that

$\mathcal{J}'$ is negatively correlated but not Rayleigh. Straightforward Maple-aided calculations show that $\mathcal{J}'$ is negatively correlated. However, if the elements are assigned weights $y_2 = y_3 = y_4 = t$ and $y_5 = y_6 = y_7 = 1$, then

$$\Delta J'\{1,8\}(\mathbf{y}) = (t+1)^3(t-1)(t^2+t-1)$$

and therefore $\Delta J'\{1,8\} < 0$ if $(\sqrt{5}-1)/2 < t < 1$. Therefore, $\mathcal{J}'$ is not Rayleigh. □

(The matroid $\mathcal{J}'$ in the proof of Theorem 5.12 is similar in structure to the sixth-root of unity matroid called $\mathcal{J}$ by Oxley [18].)

6. Open problems

The class of Rayleigh matroids is naturally motivated by generalization of a physically intuitive property, and it has some useful structure and relevance to other interesting classes of matroids. There are still many unsolved problems concerning these ideas, among them the following.

With regard to finding more examples of Rayleigh matroids, Theorems 3.8, 5.12, and Proposition 5.9 show that we can not hope for all matroids of rank 4 to be Rayleigh:

- Characterize the class of rank 4 Rayleigh matroids by means of excluded (deletion) minors.

With Theorem 3.8 in mind:

- Characterize the class of ternary Rayleigh matroids by means of excluded minors.
- Characterize the class of $GF(4)$-representable Rayleigh matroids by means of excluded minors.

Proposition 5.1 provides a starting point for these problems, from which the method of proof of Theorem 3.8 could be launched. Completing either of these projects will require a substantial amount of work, but should be well worth it.

Concerning the spectrum of conditions between the HPP and Rayleigh property:

- Is there a Rayleigh matroid which is not LC[4]?

Regarding Theorem 5.2:

- Are there strongly Rayleigh matroids which are not HPP, or not sixth-root of unity?
- Is every HPP matroid strongly Rayleigh?

Finally, in order to better understand the enumerative combinatorics of graphs:

- Find a combinatorial (bijective) proof of Theorem 5.6.

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References

[1] Balabanian, N. and Bickart, T. A. (1969) *Electrical Network Theory*, Wiley, New York.
[2] Brooks, R. L., Smith, C. A. B., Stone, A. H. and Tutte, W. T. (1940) The dissection of rectangles into squares. *Duke Math. J.* **7** 312–340.
[3] Bruter, C. P. (1971) Déformations des matroïdes. *CR Acad. Sci. Paris Sér. A–B* **273** A9–A10.
[4] Chavez Lomelí, L. E. (1994) The basis problem for matroids. MSc thesis, University of Oxford.
[5] Choe, Y.-B. (2003) Polynomials with the half-plane property and Rayleigh monotonicity. PhD thesis, University of Waterloo.
[6] Choe, Y.-B. Rayleigh monotonicity of sixth-root of unity matroids. In preparation.
[7] Choe, Y.-B., Oxley, J. G., Sokal, A. D. and Wagner, D. G. (2004) Homogeneous polynomials with the half-plane property. *Adv. Appl. Math.* **32** 88–187.
[8] Corp, J. and McNulty, J. (2001) On a characterization of balanced matroids. *Ars Combin.* **58** 111–112.
[9] Feder, T. and Mihail, M. (1992) Balanced matroids. In *Proc. 24th Annual ACM* (STOC), Victoria BC, ACM Press, New York.
[10] Fettweis, A. and Basu, S. (1987) New results on stable multidimensional polynomials, Part I: Continuous case. *IEEE Trans. Circuits Systems* **34** 1221–1232.
[11] Godsil, C. D. (1984) Real graph polynomials. In *Progress in Graph Theory* (Waterloo, Ont., 1982), Academic Press, Toronto, pp. 281–293.
[12] Harary, F. and Lindström, B. (1981) On balance in signed matroids. *J. Combin. Inform. System Sci.* **6** 123–128.
[13] Hardy, G. H., Littlewood, J. E. and Pólya, G. (1988) *Inequalities*, reprint of the 1952 edition, Cambridge University Press, Cambridge.
[14] Kirchhoff, G. (1847) Über die Auflösung der Gleichungen, auf welche man bei der Untersuchungen der linearen Vertheilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.* **72** 497–508.
[15] Lyons, R. D. (2003) Determinantal probability measures. *Publ. Math. Inst. Hautes Études Sci.* **98** 167–212.
[16] Merino, C. (1999) Matroids, the Tutte polynomial and the chip firing game. PhD thesis, Somerville College, University of Oxford.
[17] Mihail, M. and Vazirani, U. (1989) On the expansion of 0–1 polytopes. Technical Report 05–89, Harvard University.
[18] Oxley, J. G. (1992) *Matroid Theory*, Oxford University Press, New York.
[19] Pemantle, R. (2000) Toward a theory of negative dependence. *J. Math. Phys.* **41** 1371–1390.
[20] Seymour, P. D. (1980) Matroids and multicommodity flows. *Europ. J. Combin.* **2** 257–290.
[21] Seymour, P. D. and Welsh, D. J. A. (1975) Combinatorial applications of an inequality from statistical mechanics. *Math. Proc. Camb. Phil. Soc.* **75** 495–495.
[22] Stanley, R. P. (1981) Two combinatorial applications of the Aleksandrov–Fenchel inequalities. *J. Combin. Theory Ser. A* **31** 56–65.
[23] Wagner, D. G. (2005) Rank three matroids are Rayleigh. *Electronic J. Combin.* **12** N8 1–11.
[24] Wagner, D. G. (2005) Matroid inequalities from electrical network theory. *Electronic J. Combin.* **11** A1 1–17.
[25] Whittle, G. (1997) On matroids representable over $GF(3)$ and other fields. *Trans. Amer. Math. Soc.* **349** 579–603.

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