J. Austral. Math. Soc. (Series A) 35 (1983), 1–17

A SUBCLASS OF UNIVALENT FUNCTIONS

R. M. GOEL and BEANT SINGH MEHROK

(Received 23 December 1980, revised 19 June 1981)

Communicated by R. Vyborny

Abstract

Sharp results for the coefficient estimates, distortion theorems, radius of convexity, arc-length and area of the image curve are obtained for the class $R(A, B)$ of regular functions whose derivative is subordinate to $(1 + Az)/(1 + Bz)$, $-1 \leqslant B < A \leqslant 1$, in the unit disc $E = \{z: |z| < 1\}$. We also establish a convolution theorem for this class.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 30 A 32, 30 A 34, 30 A 42.

Keywords and phrases: quasi-subordination, subordination, majorization, subclass of univalent functions, convolution (Hadamard product).

1. Introduction

Let U denote the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \tag{1.1}$$

which are regular in $E = \{z: |z| < 1\}$ and satisfying there the conditions $w(0) = 0$ and $|w(z)| < 1$.

Let S denote the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

regular and univalent in E.

https://doi.org/10.1017/S1446788700024733 Published online by Cambridge University Press

Let $R(A, B)$ denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are regular in E and satisfying there

$$(1.3) \qquad f'(z) \prec \frac{1 + Az}{1 + Bz}, \qquad -1 \leqslant B < A \leqslant 1, z \in E.$$

Obviously $R(+1, -1)$ coincides with R, the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ regular in E and satisfying $\operatorname{Re} f'(z) > 0$, $z \in E$. Thus $R(A, B)$ is a subclass of $R(1, -1)$. To avoid repetition we lay down, once for all, that $-1 \leqslant B < A \leqslant 1$, and $z \in E$.

Let us set

$$(1.4) \qquad f'(z) = P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k.$$

Then by definition of subordination, $f \in R(A, B)$ if and only if $f'(z)$ has the representation

$$(1.5) \qquad f'(z) = P(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \qquad w \in U.$$

An easy computation shows that $f \in R(A, B)$ if and only if

$$(1.6) \qquad |f'(z) - 1| < |A - Bf'(z)|.$$

Alexander [1] and Wolff [17] made an early study of the class R. It follows from the Noshiro-Warschawski theorems [12, 16] that functions of the class R are equivalent in E. Hence $R(A, B)$ is a subclass of S.

MacGregor [9] investigated the properties of the class R, and subsequently, the same author [10] studied the subclass $R(1)$ of R of regular functions $f(z)$ satisfying the condition

$$(1.7) \qquad |f'(z) - 1| < 1.$$

The first author [5, 6] developed some properties of the subclass $S(\alpha)$ of R of regular functions $f(z)$ which satisfy the condition

$$(1.8) \qquad |f'(z) - \alpha| < \alpha, \qquad (\alpha > \tfrac{1}{2}).$$

Padmanabhan [13] investigated the subclass $R(\alpha)$ of R of regular functions $f(z)$ satisfying

$$(1.9) \qquad \left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \alpha, \qquad 0 < \alpha \leqslant 1.$$

Capling and Causey [4] also studied the class $R(\alpha)$ and improved some of the results due to Padmanabhan [13].

The following observations are obvious:

(i) $R(1, -1) \equiv R$,

(ii) $R(1, 0) \equiv R(1)$,

(iii) $R(1, 1/\alpha - 1) \equiv S(\alpha)$, $(\alpha > \frac{1}{2})$,
(iv) $R(\alpha, -\alpha) \equiv R(\alpha)$, $(0 < \alpha \leqslant 1)$.

Thus, $R(A, B)$ contains all the above mentioned classes, and therefore the view of Brickman [3, page 341], "idea of subordination has unified the geometric theory of functions" is strengthened.

In this paper, we obtain sharp result for coefficient estimates, distortion theorems, radius of convexity, arc-length and area of the image curve for the class $R(A, B)$. We also prove that if

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad h(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

belong to $R(A, B)$, then so does $F(z) = z + \frac{1}{2}\sum_{n=2}^{\infty} n a_n b_n z^n$.

Results due to MacGregor [9, 10], Padmanabhan [13], Capling and Causey [4] and the first author [5, 6] follow as special cases from our theorems.

2. Some preliminary lemmas

LEMMA 1. *If $g(z)$ and $G(z)$ are regular in $|z| < 1$ and $g(z)$ is subordinate to $G(z)$ $(g(z) \prec G(z))$ with $g(0) = G(0)$, then for $\lambda > 0$, $0 < r < 1$,*

$$\int_0^{2\pi} |g(re^{i\theta})|^{\lambda}\, d\theta \leqslant \int_0^{2\pi} |G(re^{i\theta})|^{\lambda}\, d\theta. \tag{2.1}$$

This lemma is due to Littlewood and its proof can be found in [7, page 484, Theorem 2; 8(1944), Theorem 210].

Robertson [14] introduced the concept of quasi-subordination. Let $g(z)$ and $G(z)$ be analytic in E. Let $\phi(z)$ be analytic and $|\phi(z)| \leqslant 1$ in E, such that $g(z)/\phi(z)$ is regular and subordinate to $G(z)$, for $z \in E$. Then $g(z)$ is said to be quasi-subordinate to $G(z)$, written as $g(z) \prec_q G(z)$, $z \in E$.

An equivalent condition for this is

$$g(z) = \phi(z)G(w(z)), \qquad |\phi(z)| \leqslant 1, w \in U, z \in E.$$

If $\phi(z) = 1$, then $g(z) = G(w(z))$ so that $g(z) \prec G(z)$ in E. If $w(z) = z$, then $g(z) = \phi(z)G(z)$, we say that $g(z)$ is majorized by $G(z)$ and we write it as $g(z) \ll G(z)$, $z \in E$.

LEMMA 2. *If $g(z) = \sum_{k=0}^{\infty} d_k z^k \prec_q G(z) = \sum_{k=0}^{\infty} D_k z^k$, then*

$$\sum_{k=0}^{n} |d_k|^2 \leqslant \sum_{k=0}^{n} |D_k|^2 \qquad (n = 0, 1, 2, \ldots). \tag{2.2}$$

This lemma is due to Robertson [14].

In particular (2.2) holds also when
(i) $g(z) \prec G(z)$,
(ii) $g(z) \ll G(z)$.
By the application of Lemma 2, we establish

LEMMA 3. *For $f \in R(A, B)$, if $f'(z) = P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, then*

$$|p_n| \leqslant (A - B), \qquad n \geqslant 1. \tag{2.3}$$

The bounds are sharp.

PROOF. From (1.5) we have

$$\sum_{k=1}^{\infty} p_k z^k = w(z)\left[(A - B) - B \sum_{k=1}^{\infty} p_k z^k\right].$$

By the application of Lemma 2(2.2), we get

$$\sum_{k=1}^{n} |p_k|^2 \leqslant (A - B)^2 + B^2 \sum_{k=1}^{n-1} |p_k|^2$$

or

$$|p_n|^2 \leqslant (A - B)^2 - (1 - B^2) \sum_{k=1}^{n-1} |p_k|^2 \leqslant (A - B)^2.$$

This yields (2.3).

Equality signs in (2.3) are attained for the functions $P_n(z)$ defined by

$$P_n(z) = \frac{1 + A\delta z^n}{1 + B\delta z^n}, \qquad |\delta| = 1.$$

In order to determine the radius of convexity, we need the following two lemmas.

LEMMA 4. *For $w \in U$ and $|z| = r$, we have*

$$|zw'(z) - w(z)| \leqslant \frac{r^2 - |w(z)|^2}{1 - r^2}. \tag{2.4}$$

This result is due to Singh and Goel proved in [15].

LEMMA 5. *Let*

$$p(z) = \frac{1 + Bw(z)}{1 + Aw(z)}, \qquad w \in U.$$

Then for $|z| = r < 1$,

$$\text{(2.5)}\quad \operatorname{Re}\left(Ap(z) + \frac{B}{p(z)}\right) + \frac{r^2|Ap(z) - B|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|}$$

$$\leqslant \begin{cases} \dfrac{AB(A + B)r^2 - 4ABr + (A + B)}{(1 - Ar)(1 - Br)}, & R_1 \leqslant R_0, \\ \dfrac{2}{(1 - r^2)}\left[(1 - ABr^2) - ((1 - A)(1 - B)(1 + Ar^2)(1 + Br^2))^{1/2}\right], & \\ & R_1 \geqslant R_0, A \neq 1, \end{cases}$$

where

$$\text{(2.6)}\qquad R_1 = \frac{1 - Br}{1 - Ar}, \qquad R_0^2 = \frac{(1 - B)(1 + Br^2)}{(1 - A)(1 + Ar^2)}.$$

The bounds are sharp.

PROOF. It is easy to see that the transformation

$$p(z) = \frac{1 + Bw(z)}{1 + Aw(z)}$$

maps $|w(z)| \leqslant r$ onto the circle $|p(z) - a| \leqslant d$, where

$$a = \frac{(1 - ABr^2)}{(1 - A^2r^2)} \quad \text{and} \quad d = \frac{(A - B)r}{(1 - A^2r^2)}.$$

Putting $p(z) = Re^{i\theta}$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ and denoting the left hand side of (2.5) by $T(R, \theta)$, we get

$$T(R, \theta) = (AR + B/R)\cos\theta + \frac{2(1 - ABr^2)\cos\theta}{(1 - r^2)} - \frac{(1 - A^2r^2)R}{(1 - r^2)} - \frac{(1 - B^2r^2)}{(1 - r^2)R}.$$

For extreme values of $T(R, \theta)$, $\partial T/\partial R = 0 = \partial T/\partial\theta$ which yield respectively

$$\text{(2.7)}\qquad \cos\theta = \frac{(1 - A^2r^2) - (1 - B^2r^2)/R^2}{(1 - r^2)(A - B/R^2)}, \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right),$$

and

$$\text{(2.8)}\qquad L(R)\sin\theta = 0,$$

where

$$L(R) = (AR + B/R) + \frac{2(1 - ABr^2)}{(1 - r^2)}. \tag{2.9}$$

Now we prove that $L(R)$ remains positive. If $B \geqslant 0$, $A > 0$, then $L(R) > 0$. We now consider the case when $B < 0$. The following cases arise.

Case I. $B < 0$, $A \geqslant 0$. Using the fact that $0 < \cos\theta \leqslant 1$, it follows from (2.7) that

$$\frac{1 - B^2r^2}{1 - A^2r^2} < R^2 \leqslant \frac{(1 - B)(1 + Br^2)}{(1 - A)(1 + Ar^2)}. \tag{2.10}$$

$L'(R) = (A - B/R^2) > 0$ and hence $L(R)$ attains its minimum value at

$$R = \left(\frac{1 - B^2r^2}{1 - A^2r^2}\right)^{1/2} = R_2, \quad \text{say}. \tag{2.11}$$

Now

$$L(R_2) = \left[(1 - r^2)(A + B) + 2(1 - A^2r^2)^{1/2}(1 - B^2r^2)^{1/2}\right] \times \frac{(1 - ABr^2)}{(1 - r^2)\left[(1 - A^2r^2)(1 - B^2r^2)\right]^{1/2}}$$

which is positive provided

$$(1 - r^2)(A + B) + 2\left((1 - A^2r^2)(1 - B^2r^2)\right)^{1/2} > 0. \tag{2.12}$$

If $(A + B) \geqslant 0$, there is nothing to prove, so we assume $(A + B) < 0$. (2.12) will hold if

$$4(1 - A^2r^2)(1 - B^2r^2) - (A + B)^2(1 - r^2)^2 > 0$$

or if

$$\left[(1 + B)(1 + Ar^2)(1 + A)(1 + Br^2)\right]\left[2(1 + ABr^2) - (A + B)(1 + r^2)\right] > 0$$

which is always true.

Case II. $B < 0$, $A < 0$. Consider the case when $L'(R) = (A - B/R^2) < 0$. Since $0 < \cos\theta \leqslant 1$, it follows from (2.7) that

$$\frac{(1 - B)(1 + Br^2)}{(1 - A)(1 + Ar^2)} \leqslant R^2 < \frac{1 - B^2r^2}{1 - A^2r^2}.$$

An easy computation would show that this does not hold.

Now, consider the case when $(A - B/R^2) \geqslant 0$. If $A - B/R^2 = 0$, then from (2.7), we have

$$R^2 = \frac{(1 - B^2r^2)}{(1 - A^2r^2)} = \frac{B}{A}$$

and it implies $(A - B)(1 + ABr^2) = 0$ which is evidently not possible. Thus the only case needed to be considered is when $(A - B/R^2) > 0$. Therefore, by (2.10),

$$\frac{1 - B^2r^2}{1 - A^2r^2} < R^2 < B/A.$$

The minimum value of $L(R)$ occurs at $R = R_2$ and $L(R_2) > 0$ if

$$(1 - r^2)(A + B) + 2\left[(1 - A^2r^2)(1 - B^2r^2)\right]^{1/2} > 0$$

which holds as proved in Case I (when $A + B$ is negative). For extreme values, from (2.8) and (2.7), we get

$$\theta = 0, \pi; \qquad R^2 = \frac{(1 - B)(1 + Br^2)}{(1 - A)(1 + Ar^2)} = R_0^2, \quad \text{say}.$$

It can be easily verified that $T(R, \theta)$ attains its maximum value at $(\theta = 0, R = R_0)$. So

$$T(R, \theta) \leqslant T(R_0, 0)$$
$$= \frac{2}{(1 - r^2)}\left[(1 - ABr^2) - ((1 - A)(1 - B)(1 + Ar^2)(1 + Br^2))^{1/2}\right].$$

It is easy to see that $R_0 \geqslant a - d = (1 + Br)/(1 + Ar)$. But R_0 is not always less than or equal to $a + d$. In case $R_0 \notin [a = d, a + d]$, the maximum of $T(R, 0)$ is attained at

$$R = R_1 = (a + d) = \frac{1 - Br}{1 - Ar}$$

and equals

$$T(R_1, 0) = \frac{AB(A + B)r^2 - 4ABr + (A + B)}{(1 - Ar)(1 - Br)}, \qquad R_1 \leqslant R_0.$$

If $R_1 \leqslant R_0$, equality sign in (2.5) holds for the function

$$p(z) = \frac{1 + Bz}{1 + Az}.$$

If $R_1 \geqslant R_0$, $A \neq 1$, equality sign in (2.5) holds for the function

$$p_0(z) = \frac{1 - (1 + B)z\cos\theta + Bz^2}{1 - (1 + A)z\cos\theta + Az^2}$$

https://doi.org/10.1017/S1446788700024733 Published online by Cambridge University Press

where

$$R_0 = \frac{1-(1+B)r\cos\theta + Br^2}{1-(1+A)r\cos\theta + Ar^2}. \tag{2.13}$$

Hence the lemma is established.

3. Coefficient estimates

THEOREM 3.1. *Let $f \in R(A, B)$ then*

$$|a_n| \leqslant \frac{(A-B)}{n}, \qquad n \geqslant 2. \tag{3.1}$$

The bounds are sharp for the functions $f_{(n-1)}(z)$ defined by

$$f_{(n-1)}(z) = \int_0^z \left(\frac{1+A\delta t^{n-1}}{1+B\delta t^{n-1}}\right) dt, \qquad |\delta| = 1. \tag{3.2}$$

PROOF. (3.1) follows on equating the coefficients of z^n in (1.4) and then using (2.3).

THEOREM 3.2. *If $f \in R(A, B)$ and if μ is a complex number, then*

$$|a_3 - \mu a_2^2| \leqslant \frac{(A-B)}{3} \max\left\{1, \left|B + \frac{3(A-B)}{4}\mu\right|\right\}. \tag{3.3}$$

The estimate is sharp.

PROOF. On equating the coefficients of z^2 and z^3 in (1.5), we get

$$c_1 = \frac{2a_2}{(A-B)}, \tag{3.4}$$

$$c_2 = \frac{3}{(A-B)}\left[a_3 + \frac{4B}{3(A-B)}a_2^2\right]. \tag{3.5}$$

Also

$$|c_2| \leqslant 1 - |c_1|^2.$$

Therefore, for every complex number v, we have

$$\begin{aligned} |c_2 - vc_1^2| &\leqslant |c_2| + |v||c_1|^2 \\ &\leqslant 1 + (|v| - 1)|c_1|^2 \\ &\leqslant \max\{1, |v|\}, \end{aligned} \tag{3.6}$$

since $|c_1| \leqslant 1$. The estimate (3.6) is sharp for $w(z) = z$ and $w(z) = z^2$ respectively for $|v| \leqslant 1$ and $|v| < 1$. From (3.4) and (3.5) we have

$$|a_3 - \mu a_2^2| = \frac{(A-B)}{3} |c_2 - v c_1^2| , \tag{3.7}$$

where

$$\mu = \frac{4}{3(A-B)}(v - B)$$

or

$$v = B + \frac{3(A-B)\mu}{4}. \tag{3.8}$$

(3.7) in conjunction with (3.6) and (3.7), yields (3.3). (3.3) is sharp, being attained for the function $f_1(z)$ and $f_2(z)$ defined, respectively, by

$$f_1'(z) = \frac{1+Az}{1+Bz} \quad \text{and} \quad f_2'(z) = \frac{1+Az^2}{1+Bz^2}.$$

4. Distortion theorems

THEOREM 4.1. *Let $f \in R(A, B)$, then for $|z| = r < 1$,*

$$|f'(z)| \leqslant \frac{1+Ar}{1+Br}; \tag{4.1}$$

$$\operatorname{Re} f'(z) \geqslant \frac{1-Ar}{1-Br}; \tag{4.2}$$

$$|f(z)| \leqslant \begin{cases} \dfrac{A}{B}\left[r + \left(\dfrac{1}{A} - \dfrac{1}{B}\right)\log(1+Br)\right], & B \neq 0, \\ r + \dfrac{A}{2}r^2, & B = 0, \end{cases} \tag{4.3}$$

$$|f(z)| \geqslant \begin{cases} \dfrac{A}{B}\left[r - \left(\dfrac{1}{A} - \dfrac{1}{B}\right)\log(1-Br)\right], & B \neq 0, \\ r - \dfrac{A}{2}r^2, & B = 0. \end{cases} \tag{4.4}$$

All the estimates are sharp.

PROOF. From (1.5), it is easy to establish (4.1) and (4.2). Using (4.1),

$$|f(z)| \leqslant \int_0^r |f'(te^{i\theta})|\, dt \leqslant \int_0^r \frac{1+At}{1+Bt}\, dt$$

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which yields (4.3). Again using (4.2),

$$|f(z)| \geqslant \int_0^r \operatorname{Re} f'(te^{i\beta})\,dt \geqslant \int_0^r \frac{1-At}{1-Bt}\,dt$$

which gives (4.4). (4.1) and (4.3) are sharp, being attained for the function

$$f_1(z) = \begin{cases} \dfrac{A}{B}\left[z + \left(\dfrac{1}{A} - \dfrac{1}{B}\right)\log(1+Bz)\right], & B \neq 0, \\ z + \dfrac{A}{2}z^2, & B = 0. \end{cases}$$

(4.2) and (4.4) are sharp for the function

$$f_{(1)}(z) = \begin{cases} \dfrac{A}{B}\left[z - \left(\dfrac{1}{A} - \dfrac{1}{B}\right)\log(1-Bz)\right], & B \neq 0, \\ z - \dfrac{A}{2}z^2, & B = 0. \end{cases}$$

Let W be any complex number such that

$$|W| < \begin{cases} \dfrac{A}{B}\left[r - \left(\dfrac{1}{A} - \dfrac{1}{B}\right)\log(1-Br)\right], & B \neq 0, \\ r - \dfrac{A}{2}r^2, & B = 0. \end{cases}$$

By Rouche's Theorem it follows that $f(z)$ and $f(z) - W$ have the same number of zeros in $|z| < r$, that is, precisely one. Hence we have the following:

COROLLARY. *Every function $f(z)$ in $R(A, B)$ maps E onto a domain which covers the disc*

$$|W| < \begin{cases} \dfrac{A}{B}\left[1 - \left(\dfrac{1}{A} - \dfrac{1}{B}\right)\log(1-B)\right], & B \neq 0, \\ 1 - \dfrac{A}{2}, & B = 0. \end{cases}$$

5. Argument of $f'(z)$

THEOREM 5.1. *If $f \in R(A, B)$ then*

$$(5.1) \qquad |\arg f'(z)| \leqslant \sin^{-1}\frac{(A-B)r}{1-ABr^2}, \qquad |z| = r.$$

The result is sharp.

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PROOF. It is easy to show that $f'(z) = (1 + Aw(z))/(1 + Bw(z))$ maps $|w(z)| \leq r$ onto the circle

$$\left| f'(z) - \frac{(1 - ABr^2)}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{(1 - B^2r^2)}. \tag{5.2}$$

(5.1) is an immediate consequence of (5.2). The result is sharp, being attained for the function $f_0(z)$ defined by

$$f_0'(z) = \frac{1 + A\delta z}{1 + B\delta z}, \tag{5.3}$$

where

$$\delta = \frac{z}{r}\left[\frac{-(A + B)r + i\left((1 - A^2r^2)(1 - B^2r^2)\right)^{1/2}}{1 + ABr^2}\right].$$

6. Convex set of functions

THEOREM 6.1. *If f and $h \in R(A, B)$, then*

$$\lambda f + (1 - \lambda)h \in R(A, B), \qquad (0 \leq \lambda \leq 1).$$

PROOF. By definition,

$$f'(z) \prec \frac{1 + Az}{1 + Bz}, \tag{6.1}$$

$$h'(z) \prec \frac{1 + Az}{1 + Bz}. \tag{6.2}$$

Since $(1 + Az)/(1 + Bz)$ is convex univalent in E, it follows by a result due to Bernardi [2, page 57, Example 2] that

$$\lambda f'(z) + (1 - \lambda)h'(z) \prec \frac{1 + Az}{1 + Bz}.$$

Hence

$$\lambda f' + (1 - \lambda)h' \in R(A, B).$$

7. Radius of convexity

THEOREM 7.1. *Let $f \in R(A, B)$, then*

(i) *for $A_0 \leq A \leq 1$, $f(z)$ is convex in $|z| < r_0$,*
where r_0 is the smallest positive root of

$$ABr^2 - 2Ar + 1 = 0; \tag{7.1}$$

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(ii) *for* $-1 < A \leqslant A_0$, $f(z)$ *is convex in* $|z| < r_1$, *where* r_1 *is the smallest positive root of*

(7.2)

$$A(1-r^2) - \left[(1-ABr^2) - ((1-A)(1-B)(1+Ar^2)(1+Br^2))^{1/2}\right] = 0;$$

$$A_0 = \frac{(2+B-2B^2) + (20-36B+21B^2-4B^3)^{1/2}}{2(4-2B-B^2)}.$$

The results are sharp.

PROOF. Differentiating logarithmically, (1.5) yields

$$1 + \frac{zf''(z)}{f'(z)} = 1 + (A-B)\frac{zw'(z)}{(1+Aw(z))(1+Bw(z))}. \tag{7.3}$$

(7.3) together with Lemma 4 gives

(7.4)

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant 1 + (A-B)$$
$$\times\left[\operatorname{Re}\frac{w(z)}{(1+Aw(z))(1+Bw(z))} - \frac{r^2 - |w(z)|^2}{(1-r^2)\,|(1+Aw(z))(1+Bw(z))|}\right].$$

Putting $p(z) = (1+Bw(z))/(1+Aw(z))$, and using Lemma 5,

(7.5)
$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right)$$
$$\geqslant \begin{cases} \dfrac{2A}{A-B} - \dfrac{AB(A+B)r^2 - 4ABr + (A+B)}{(A-B)(1-Ar)(1-Br)}, & R_1 \leqslant R_0, \\[2ex] \dfrac{2A}{A-B} - 2\left[\dfrac{(1-ABr^2) - ((1-A)(1-B)(1+Ar^2)(1+Br^2))^{1/2}}{(1-r^2)(A-B)}\right], & \\ & R_1 \geqslant R_0,\ A \neq 1. \end{cases}$$

(7.1) and (7.2) follow by equating the right hand sides of (7.5) to zero.

The equation $R_0 = R_1$ yields

$$ABr^4 - 2ABr^3 + [2(A+B) - AB - 1]r^2 - 2r + 1 = 0. \tag{7.6}$$

Elimination of r between (7.1) and (7.6) leads to

$$(4-2B-B^2)A^2 - (2+B-2B^2)A - (1-B)^2 = 0. \tag{7.7}$$

(7.7), on verification of the signs, yields

$$A = \frac{(2 + B - 2B^2) + (20 - 36B + 21B^2 - 4B^3)^{1/2}}{2(4 - 2B - B^2)} = A_0, \quad \text{say}.$$

The results are sharp, being attained respectively, for the functions $f_1(z)$ and $f_\theta(z)$ defined by

$$f_1'(z) = \frac{1 + Az}{1 + Bz}, \qquad f_\theta'(z) = \frac{1 - (1 + A)z\cos\theta + Az^2}{1 - (1 + B)z\cos\theta + Bz^2},$$

where θ is defined by (2.13).

REMARK 1. Radii of convexity for the classes R, $R(1)$ and $S(\alpha)$, at once, follow from (7.1).

REMARK 2. On taking $A = \alpha$, $B = -\alpha$ $(0 < \alpha \leqslant 1)$ in (7.7), we get $\alpha^4 - 4\alpha^3 - 4\alpha^2 + 4\alpha + 1 = 0$ which gives

$$\alpha = \frac{(2^{1/2} - 1)(3^{1/2} + 1)}{2^{1/2}} = \alpha_0, \qquad \text{say}.$$

Hence

(i) for $\alpha_0 \leqslant \alpha \leqslant 1$, $f(z)$ maps $|z| < (2^{1/2} - 1)/\alpha$ onto a convex domain;

(ii) for $0 < \alpha \leqslant \alpha_0$, $f(z)$ maps

$$|z| < \left[\frac{(\alpha^2 - 1) + ((1 - \alpha^2)(1 + 4\alpha - \alpha^2))^{1/2}}{2\alpha(1 + \alpha)}\right]^{1/2}$$

onto a convex domain. This result was established by Padmanabhan in [13] and also by Capling and Causey in [4].

8. Arc-length and area of the image curve

THEOREM 8.1. *Let $f \in R(A, B)$ and $L_r(f)$ denotes the length of the image of $|z| = r$ under $f(z)$, $0 < r < 1$, then*

(8.1)

$$L_r(f) \leqslant \begin{cases} \pi r\left[\left|\dfrac{A + B}{B}\right| - \dfrac{(A - B)}{|B|} - 2\dfrac{(A - B)}{\pi B}\log\left(\dfrac{1 - Br}{1 + Br}\right)\right], & B \neq 0, \\ r\displaystyle\int_0^{2\pi} |1 + Are^{i\theta}|\, d\theta, & B = 0. \end{cases}$$

The results are sharp.

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PROOF. In Lemma 1, set $g(z) = f'(z)$, $G(z) = (1 + Az)/(1 + Bz)$ and $\lambda = 1$. Then

$$\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \leqslant \int_0^{2\pi} \left| \frac{1 + Are^{i\theta}}{1 + Bre^{i\theta}} \right| d\theta. \tag{8.2}$$

Now

$$\begin{aligned} L_r(f) &= \int_{|z|=r} |f'(z)||dz| \\ &= \int_0^{2\pi} |f'(re^{i\theta})|\, r\, d\theta. \end{aligned}$$

By (8.2),

$$\begin{aligned} L_r(f) &\leqslant r\int_0^{2\pi}\left|\frac{1 + Are^{i\theta}}{1 + Bre^{i\theta}}\right| d\theta \\ &= r\int_0^{2\pi}\left|\left(\frac{A+B}{2B} - \frac{(A-B)(1-B^2r^2)}{2B(1 + 2Br\cos\theta + B^2r^2)}\right) + i\frac{(A-B)r\sin\theta}{1 + 2Br\cos\theta + B^2r^2}\right| d\theta \\ &\leqslant \pi r\left|\frac{A+B}{B}\right| + \frac{(A-B)r}{2|B|}\int_0^{2\pi}\frac{(1-B^2r^2)}{1 + 2Br\cos\theta + B^2r^2}\, d\theta \\ &\quad + (A-B)r\int_0^{2\pi}\frac{r|\sin\theta|}{1 + 2Br\cos\theta + B^2r^2}\, d\theta \\ &= \pi r\left|\frac{A+B}{B}\right| + \frac{\pi(A-B)r}{|B|} - \frac{(A-B)r}{B}\int_0^{\pi}\frac{-2Br\sin\theta}{1 + 2Br\cos\theta + B^2r^2}\, d\theta \\ &= \pi r\left[\left|\frac{A+B}{B}\right| + \frac{(A-B)}{|B|} - 2\frac{(A-B)}{\pi B}\log\left(\frac{1-Br}{1+Br}\right)\right]. \end{aligned}$$

For $B = 0$, result is trivial.

The extremal function $f_0(z)$ is given by

$$f_0'(z) = \frac{1 + A\delta z}{1 + B\delta z}, \qquad |\delta| = 1. \tag{8.3}$$

COROLLARY. *For the class* $R(\alpha)$, *we deduce, from* (8.1),

$$L_r(f) \leqslant 2\pi r + 4r\log\left(\frac{1+\alpha r}{1-\alpha r}\right).$$

This is a result established by Capling and Causey [4].

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THEOREM 8.2. *If $f \in R(A, B)$, and if $A_r(f)$ denotes the area of image of $|z| = r$ under $f(z)$, $0 < r < 1$, then*

(8.4)

$$A_r(f) \leqslant \begin{cases} \pi r^2\left[\left(1 - \frac{(A-B)^2}{B^2}\right) - \frac{(A-B)^2}{B^4 r^2}\log(1 - B^2 r^2)\right], & B \neq 0, \\ \pi r^2\left[1 + \frac{A^2 r^2}{2}\right], & B = 0. \end{cases}$$

The inequalities are sharp.

(8.4) are direct consequences of Lemma 1 ($\lambda = 2$) and interior area theorem. Equality sign is attained for the function $f_0(z)$ defined by (8.3).

COROLLARY. *For the class $R(\alpha)$, we have from* (8.4),

$$A_r(f) \leqslant \pi r^2\left[-3 - \frac{4}{\alpha^2 r^2}\log(1 - \alpha^2 r^2)\right].$$

This is a result due to Capling and Causey [4].

9. Convolution

THEOREM 9.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to the class $R(A, B)$, then so does*

$$F(z) = z + \frac{1}{2}\sum_{n=2}^{\infty} n a_n b_n z^n.$$

PROOF. Since $f \in R(A, B)$, it follows by (1.6) that $|f'(z) - 1| < |A - Bf'(z)|$. It is equivalent to

$$|f'(z) - b| \leqslant C \tag{9.1}$$

where $b = (1 - AB)/(1 - B^2)$, $C = (A - B)/(1 - B^2)$. It is easy to see that $1 - b < C \leqslant b$. We know that if $H(z) = \sum_{n=0}^{\infty} h_n z^n$ is regular for $|z| < 1$ and $|H(z)| \leqslant M$, then, by [11, page 101],

$$\sum_{n=0}^{\infty} |h_n|^2 \leqslant M^2. \tag{9.2}$$

Applying (9.2) to (9.1), we get $(1 - b)^2 + \sum_{n=2}^{\infty} n^2 |a_n|^2 < C^2$ or

$$\sum_{n=2}^{\infty} n^2 |a_n|^2 < \frac{(A-B)^2}{(1-B^2)^2}. \tag{9.3}$$

Similarly

$$\sum_{n=2}^{\infty} n^2 |b_n|^2 < \frac{(A-B)^2}{(1-B^2)^2}. \tag{9.4}$$

Now

$$\begin{aligned}
|F'(z) - b|^2 &= \left|(1-b) + \frac{1}{2}\sum_{n=2}^{\infty} n^2 a_n b_n z^{n-1}\right|^2 \\
&\leqslant (1-b)^2 + (1-b)\sum_{n=2}^{\infty} n^2 |a_n|\,|b_n| r^{n-1} + \frac{1}{4}\left|\sum_{n=2}^{\infty} n^2 a_n b_n z^{n-1}\right|^2 \\
&\leqslant (1-b)^2 + (1-b)\left(\sum_{n=2}^{\infty} n^2 |a_n|^2 r^{n-1}\right)^{1/2}\left(\sum_{n=2}^{\infty} n^2 |b_n|^2 r^{n-1}\right)^{1/2} \\
&\quad + \frac{1}{4}\left(\sum_{n=2}^{\infty} n^2 |a_n|^2 r^{n-1}\right)\left(\sum_{n=2}^{\infty} n^2 |b_n|^2 r^{n-1}\right)
\end{aligned}$$

(by Cauchy-Schwarz inequality)

$$\begin{aligned}
&\leqslant (1-b)^2 + (1-b)\left(\sum_{n=2}^{\infty} n^2 |a_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty} n^2 |b_n|^2\right)^{1/2} \\
&\quad + \frac{1}{4}\left(\sum_{n=2}^{\infty} n^2 |a_n|^2\right)\left(\sum_{n=2}^{\infty} n^2 |b_n|^2\right) \\
&\leqslant (1-b)^2 + (1-b)\frac{(A-B)^2}{(1-B^2)^3} + \frac{1}{4}\frac{(A-B)^4}{(1-B^2)^4}
\end{aligned}$$

[using (9.3) and (9.4)]

$$= \frac{B^2(A-B)^2}{(1-B^2)^2} + \frac{B(A-B)^3}{(1-B^2)^3} + \frac{1}{4}\frac{(A-B)^4}{(1-B^2)^4}.$$

$F \in R(A, B)$ if

$$\frac{B^2(A-B)^2}{(1-B^2)^2} + \frac{B(A-B)^3}{(1-B^2)^3} + \frac{1}{4}\frac{(A-B)^4}{(1-B^2)^4} < \frac{(A-B)^2}{(1-B^2)^2}.$$

This gives on simplification, $(A+B) < 2$ which is true. Hence $F \in R(A, B)$.

REMARK. The first author [6] proved this theorem for the class $S(\alpha)$.

Acknowledgement

The authors would like to thank the referee for a number of suggestions.

References

[1] J. W. Alexander, 'Functions which map the interior of the unit circle upon simple regions', *Ann. of Math.* **17** (1915), 12–22.

[2] S. D. Bernardi, 'Special classes of subordinate functions', *Duke Math. J.* **33** (1966), 55–67.

[3] L. Brickman, 'Subordinate families of analytic functions', *Illinois J. Math.* **15** (1971), 241–248.

[4] T. R. Capling and W. M. Causey, 'A class of univalent functions', *Proc. Amer. Math. Soc.* **39** (1973), 357–361.

[5] R. M. Goel, 'A class of univalent functions whose derivatives have positive real part in the unit disc', *Nieuw Arch. Wisk.* (3), **15** (1967), 55–63.

[6] R. M. Goel, 'A class of analytic functions whose derivatives have positive real part in the unit disc', *Indian J. Math.* **3** (1971), 141–145.

[7] J. E. Littlewood, 'On inequalities in the theory of functions', *Proc. London Math. Soc.* (2), **23** (1925), 481–519.

[8] J. E. Littlewood, *Lectures on the theory of functions*, pp. 163–185 (Oxford Univ. Press, London, 1944).

[9] T. H. MacGregor, 'Functions whose derivative has a positive real part', *Trans. Amer. Math. Soc.* **104** (1962), 532–537.

[10] T. M. MacGregor, 'A class of univalent functions', *Proc. Amer. Math. Soc.* **15** (1964), 311–317.

[11] Z. Nehari, *Conformal mapping* (McGraw Hill, New York, 1952).

[12] K. Noshiro, 'On the theory of schlicht functions', *J. Fac. Sci. Hokkaido Univ.* (1), **2** (1934-35), 129–155.

[13] K. S. Padmanabhan, 'On a certain class of functions whose derivatives have a positive real part in the unit disc', *Ann. Polon. Math.* **23** (1970), 73–81.

[14] M. S. Robertson, 'Quasi-subordinate functions', *Mathematical essays dedicated to A. J. MacIntyre*, pp. 311–330 (Ohio Univ. Press, Athens, Ohio, 1967).

[15] V. Singh and R. M. Goel, 'On radii of convexity and starlikeness of some classes of functions', *J. Math. Soc. Japan* **23** (1971), 323–339.

[16] S. S. Warschawaki, 'On the higher derivatives at the boundary in conformal mappings', *Trans. Amer. Math. Soc.* **8** (1935), 310–340.

[17] J. Wolff, 'L'intégrale d'une fonction holomorphe et à partie réelle positive dans un demiplan est univalente', *C. R. Acad. Sci. Paris Sér A-B* **198** (1934), 1209–1210.

Department of Mathematics
Panjabi University
Patiala-147002
(Panjab State)
India