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Positive solutions for a class of quasilinear problems with critical growth in $\mathbb{R}^N$

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In this paper, we study the existence, multiplicity and concentration of positive solutions for a class of quasilinear problems

$$-\varepsilon^p\Delta_p u + V(x)|u|^{p-2}u = f(u) + |u|^{p^*-2}u, \quad x \in \mathbb{R}^N,$$
$$u \in W^{1,p}(\mathbb{R}^N), \quad u(x) > 0, \quad x \in \mathbb{R}^N,$$

where $-\Delta_p$ is the p-Laplacian operator for $2 \leqslant p < N$, $p^* = Np/(N-p)$, $\varepsilon > 0$ is a small parameter, $f(u)$ is a superlinear and subcritical nonlinearity that is continuous in u. Using a variational method, we first prove that for sufficiently small $\varepsilon > 0$ the system has a positive ground state solution u_ε with some concentration phenomena as $\varepsilon \to 0$. Then, by the minimax theorems and Ljusternik–Schnirelmann theory, we investigate the relation between the number of positive solutions and the topology of the set of the global minima of the potentials. Finally, we obtain some sufficient conditions for the non-existence of ground state solutions.

1. Introduction and main results

Consider the quasilinear problem

$$\left.\begin{aligned} -\varepsilon^p \operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = h(u),\\ u \in W^{1,p}(\mathbb{R}^N), \quad u(x) > 0, \quad x \in \mathbb{R}^N, \end{aligned}\right\} \tag{1.1}$$

where $2 \leqslant p < N$, $\varepsilon > 0$ is a small parameter, $h(u)$ is a superlinear term.

In recent years many mathematicians have studied (1.1). Especially when, for $p = 2$, it corresponds to the Schrödinger equation. Up to now, there has been a lot of work on existence and concentration phenomena of semi-classical states of nonlinear Schrödinger equations. For instance, see [13, 14, 18, 29, 30, 36] and the references therein. It is well known that the nonlinear Schrödinger equations arise in non-relativistic quantum mechanics. Consider the nonlinear Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + H(x)\psi - g(x,|\psi|)\psi, \tag{1.2}$$

where i is the imaginary unit, Δ is the Laplacian operator and $\hbar > 0$ is the Planck constant. Let $\psi(x,t)$ be a standing wave solution of (1.2) with the form

$$\psi(x,t) = u(x)\mathrm{e}^{-\mathrm{i}Et/\hbar}, \quad u(x) \in \mathbb{R}.$$

Then, $\psi(x,t)$ solves (1.2) if and only if $u(x)$ solves

$$-\frac{\hbar^2}{2m}\Delta u + A(x)u = h(x,u), \tag{1.3}$$

where $A(x) = H(x) - E$ is called the potential function and $h(x,u) = g(x,|u|)u$. If $h(x,u)$ is independent of x, then (1.3) is reduced to (1.1) with $\varepsilon = \hbar/\sqrt{2m}$ and $p = 2$.

For (1.3), many authors have focused on the case

$$\inf_{x\in\mathbb{R}^N} A(x) > 0. \tag{1.4}$$

In this case, and for $N = 1$ and $p = 4$, by the Lyapunov–Schmidt reduction arguments, Floer and Weinstein [18] first constructed semiclassical states, which concentrate near a non-degenerate critical point of A. Later, Oh [29, 30] generalized their results to the case of $N \geqslant 3$.

When the potential A has no non-degenerate critical point, under the assumption that

$$0 < \inf_{x\in\mathbb{R}^N} A(x) < \liminf_{|x|\to\infty} A(x), \tag{1.5}$$

Rabinowitz [31] obtained the existence result for (1.3) with $h = u^{p-1}$ ($2 < p < 2^* = 2N/(N-2)$ if $N \geqslant 3$, $p > 2$ if $N = 1, 2$) and $\varepsilon > 0$ being small. In [36] Wang improved Rabinowitz's result and obtained the concentration of the positive ground state solutions as $\varepsilon \to 0^+$ at global minimum points of A. For more information in the case of (1.4), we refer the reader to [13, 14] and the references therein. In the case of

$$\inf_{x\in\mathbb{R}^N} A(x) = 0, \tag{1.6}$$

the existence of semiclassical solutions for (1.3) was first proved in [6, 7] and then generalized in [8,9]. When $\varepsilon = 1$ and $p > 2$, (1.1) also arises in a lot of applications, such as image processing, non-Newtonian fluids and pseudo-plastic fluids, and some important results are obtained in [4, 10, 15].

Many previous results for (1.1) are obtained in the case of subcritical growth. However, in the presence of critical growth, the problem has also been widely studied.

Several papers have appeared recently about the semiclassical p-Laplacian problems involving critical growth (see [3,26] and the references therein). For convenience we write (1.1) with critical growth in the following form:

$$\left.\begin{aligned} -\varepsilon^p \operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u &= |u|^{p^*-2}u + f(u), \\ u \in W^{1,p}(\mathbb{R}^N), \quad u(x) > 0, \quad x &\in \mathbb{R}^N, \end{aligned}\right\} \tag{LP$_\varepsilon$}$$

where $p^* = Np/(N-p)$ and $f(u)$ is a subcritical term.

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Assume that $f \in C^1$ and that $V : \mathbb{R}^N \to \mathbb{R}$ is a function that is bounded from below away from 0 such that

$$\inf_{\partial\Omega} V > \inf_{\Omega} V,$$

where Ω is an open bounded subset of $\mathbb{R}^N$. By the local mountain pass theorem and truncation function technique, Marcos do Ó [26] obtained solutions of $(\mathrm{LP})_\varepsilon$ that concentrate around a local minima of V, that are not necessarily non-degenerate.

Later, using Ljusternik–Schnirelmann theory (see [39]) and minimax methods, under some assumption on f, the author [17] proved the existence of multiple positive solutions for $(\mathrm{LP})_\varepsilon$ that concentrate on the minima of $V(x)$ as $\varepsilon \to 0$. In [17], $f \in C^1$ and satisfies that

$$\left.\begin{aligned} &\frac{f(s)}{s^{p-1}} \text{ is increasing on } (0,\infty),\\ &0 < \mu F(s) = \mu \int_0^s f(t)\,\mathrm{d}t \leqslant s f(s), \quad \mu > p,\\ &f(s) \geqslant \lambda s^{q_1-1} \quad \text{for all } s > 0 \text{ with } \lambda > 0 \text{ and } q_1 > 0,\\ &\sigma \in (4,6), \quad C > 0,\\ &\lim_{|s|\to 0} \frac{|f(s)|}{|s|^{p-1}} = 0 \quad \text{and} \quad \lim_{|s|\to\infty} \frac{|f(s)|}{|s|^{q}} = 0, \quad q \in (p, p^*). \end{aligned}\right\} \tag{1.7}$$

Recently, Alves and Figueiredo [3] studied the quasilinear problem

$$\left.\begin{aligned} -\varepsilon^N \operatorname{div}(|\nabla u|^{N-2}\nabla u) + M(x)|u|^{N-2}u = f(u),\\ u \in W^{1,N}(\mathbb{R}^N), \quad u(x) > 0, \quad x \in \mathbb{R}^N, \end{aligned}\right\} \tag{B$_\varepsilon$}$$

where $\varepsilon > 0$ is a positive parameter, $N \geqslant 2$, $M : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function having critical exponential growth. By Lusternik–Schnirelmann category theory and minimax methods, the authors proved the existence, multiplicity and concentration of positive solutions for $(\mathrm{B})_\varepsilon$.

Since this phenomenon of concentration is very interesting for both mathematicians and physicists, motivated by [3, 17, 26], we continue to study $(\mathrm{LP})_\varepsilon$ when the potential V has a global minimum, and investigate the existence, multiplicity and concentration of positive solutions. More precisely, we focus on four points: a more general nonlinearity than in [3, 26], positive ground state solutions with some properties of concentration and exponential decay, the relation between the number of solutions and the topology of the set of the global minima of the potentials, and sufficient conditions for the non-existence of positive ground state solutions.

Before stating our theorems, we first give some assumptions. Assume that V satisfies one of the following two conditions.

($\mathcal{D}_0$) $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$.

($\mathcal{D}_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $0 < V^\infty = \limsup_{|x|\to\infty} V(x) \leqslant V(x)$ and $|\mathcal{K}| > 0$, where $\mathcal{K} = \{x \in \mathbb{R}^N,\ V(x) > V^\infty\}$.

The hypothesis $(\mathcal{D}_0)$ was first introduced by Rabinowitz [31] in the study of a nonlinear Schrödinger equation with subcritical growth. In this paper, without loss of generality, we also assume that $V_\infty < \infty$. This condition is made only for simplicity. Actually, it is even easier if the potential is large at ∞, since we have better embedding theorems in that case.

For the nonlinearity f we assume that the following hold.

(f_1) $f \in C(\mathbb{R}^N)$, $f(t) = o(t^{p-1})$ as $t \to 0$, $f(t)t > 0$ for all $t > 0$ and $f(t) = 0$ for $t \leqslant 0$.

(f_2) There exist $q, q_1 \in (p, p^*)$ and $c > 0$ such that

$$f(t) \geqslant c\lambda t^{q_1-1} \text{ for all } t > 0 \text{ with } \lambda > 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{t^q} = 0.$$

(f_3) $f(t)/t^{p-1}$ is strictly increasing on the interval $(0, +\infty)$.

Since we look for positive solutions, let $f(s) = 0$ for $s \leqslant 0$. Obviously, from conditions (f_1) and (f_2) it follows that

$$F(u) > 0, \quad pF(u) < f(u)u \quad \forall u \neq 0, \tag{1.8}$$

where $F(u) = \int_0^u f(s)\,\mathrm{d}s$. Set

$$\mathcal{V} := \{x \in \mathbb{R}^N \colon V(x) = V_0\}.$$

Without loss of generality, below we assume that $0 \in \mathcal{V}$, that is, $V(0) = V_0$. The limit problem associated with $(\mathrm{LP})_\varepsilon$ reads as

$$-\Delta_p u + V_0 u = f(u) + |u|^{p^*-2}u, \quad u \in W^{1,p}(\mathbb{R}^N). \tag{LP$_{V_0}$}$$

Let

$$\mathcal{Q}_\varepsilon(u) := \frac{1}{p}\int_{\mathbb{R}^N} (\varepsilon^p|\nabla u|^p + V(x)|u|^p) - \int_{\mathbb{R}^N} F(u) - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*},$$

which is called an energy function associated with $(\mathrm{LP})_\varepsilon$. Set

$$\ell_\varepsilon = \inf\{\mathcal{Q}_\varepsilon(u) \colon u \neq 0 \text{ is a solution of } (\mathrm{LP})_\varepsilon\}.$$

If $u^0 > 0$ and solves $(\mathrm{LP})_\varepsilon$, we say that u^0 is a positive solution. A positive solution u^0 with $\ell_\varepsilon = \mathcal{Q}_\varepsilon(u^0)$ is called a positive ground state solution. Denote by $\mathcal{L}'_\varepsilon$ the set of all positive ground state solutions of $(\mathrm{LP})_\varepsilon$. We recall that, if Y is a closed subset of a topological space X, the Ljusternik–Schnirelmann category $\mathrm{cat}_X(Y)$ is the least number of closed and contractible sets in X that cover Y.

Theorem 1.1. *Suppose that the assumptions $(\mathcal{D}_0)$ and (f_1)–(f_3) are satisfied. If one of the conditions*

(b_1) $N \geqslant p^2$,

(b_2) $p < N < p^2$, $p^* - p/(p+1) < q_1 < p^*$,

(b_3) $p < N < p^2$, $p^* - p/(p+1) \geqslant q_1$ *and large* λ

holds, then there exists $\varepsilon^* > 0$ *such that, for each* $\varepsilon \in (0, \varepsilon^*)$*, the following conclusions hold true.*

(i) *(LP)*$_\varepsilon$ *has one positive ground state solution* u_ε *in* $W^{1,p}(\mathbb{R}^N)$*.*

(ii) $\mathcal{L}'_\varepsilon$ *is compact in* $W^{1,p}(\mathbb{R}^N)$*.*

(iii) *There exists a maximum point* x_ε *of* u_ε *such that* $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{V}) = 0$*, and, for any sequences of such* x_ε*,* $h_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ *uniformly converges to a positive ground state solution of (LP)*$_{V_0}$*, as* $\varepsilon \to 0$*, where* $u_\varepsilon \in \mathcal{L}'_\varepsilon$*.*

(iv) $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$ *and* $u_\varepsilon \in C^{1,\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $\sigma \in (0, 1)$*. Furthermore, there exist constants* $C, c > 0$ *such that* $|u_\varepsilon(x)| \leqslant C\mathrm{e}^{-(c/\varepsilon)|x-x_\varepsilon|}$ *for all* $x \in \mathbb{R}^N$*.*

Theorem 1.2. *Let the assumptions* $(\mathcal{D}_0)$ *and* (f_1)*–*(f_3) *be satisfied. If* (b_1) *or* (b_2) *or* (b_3) *in theorem 1.1 holds, then, for each* $\delta > 0$*, there exist* $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$*, (LP)*$_\varepsilon$ *has at least* $\operatorname{cat}_{\mathcal{V}_\delta}(\mathcal{V})$ *positive solutions. Furthermore, if* u_ε *denotes one of these positive solutions and* $\sigma_\varepsilon \in \mathbb{R}^N$ *such that* $u_\varepsilon(\sigma_\varepsilon) = \max_{x\in\mathbb{R}^N} u_\varepsilon(x)$*, then one gets that*

(i) $\lim_{\varepsilon\to 0} V(\sigma_\varepsilon) = V_0$*,*

(ii) $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$ *and* $u_\varepsilon \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $\gamma \in (0, 1)$*. Furthermore, there exist constants* $C, c > 0$ *such that* $|u_\varepsilon(x)| \leqslant C\mathrm{e}^{-(c/\varepsilon)|x-\sigma_\varepsilon|}$ *for all* $x \in \mathbb{R}^N$*.*

Theorem 1.3. *If the assumptions* $(\mathcal{D}_1)$ *and* (f_1)*–*(f_3) *hold, then, for each* $\varepsilon > 0$*, (LP)*$_\varepsilon$ *has no positive ground state solution.*

Below, we compare our results with those in [17]. First, our nonlinearities are more general. In fact, in this paper f is only required to be a continuous function; moreover, we weaken the Ambrosetti–Rabinowitz condition (see (1.7)):

$$0 < \mu F(s) = \mu \int_0^s f(t)\,\mathrm{d}t \leqslant s f(s), \quad \mu > p.$$

Second, we have more information for the positive solutions, such as the relationship between the positive ground state solution of (LP)$_\varepsilon$ and (LP)$_{V_0}$, the exponential decay etc. Finally, we obtain some sufficient conditions for non-existence of positive ground state solutions.

The proof is based on the variational method. By comparing with the previous works, we may summarize as follows the main difficulties that one has to face in proving our theorems. On the one hand, as we see below, since the embeddings $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ (for all $t \in [p, p^*)$) and $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}_{\mathrm{loc}}(\mathbb{R}^N)$ are not compact, the lack of compactness prevents us from using the variational methods in a standard way. However, we make up the global compactness by the limit problem (LP)$_{V_0}$. To remedy the local compactness ($H^1(\mathbb{R}^N) \hookrightarrow L^{p^*}_{\mathrm{loc}}(\mathbb{R}^N)$), as in [17,26] we give some new estimates for the ground state level for the energy functional. On the other hand, in the previous papers [3,17,26], since f is a C^1-function, it follows that $\mathcal{Q}_\varepsilon \in C^2$ and $\mathcal{K}_\varepsilon \in C^1$, where $\mathcal{K}_\varepsilon$ is the Nehari manifold given by

$$\mathcal{K}_\varepsilon = \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \colon \mathcal{Q}'_\varepsilon(u)u = 0\}.$$

From these properties of $\mathcal{Q}_\varepsilon$ and $\mathcal{K}_\varepsilon$, one can easily deduce that critical points of $\mathcal{Q}_\varepsilon$ on $\mathcal{K}_\varepsilon$ are critical points of $\mathcal{Q}_\varepsilon$ on $W^{1,p}(\mathbb{R}^N)$. Furthermore, one can use the standard Ljusternik–Schnirelmann category theory on $\mathcal{K}_\varepsilon$ directly (see [11, 39]). However, in the present paper we cannot obtain these properties, since f is only continuous, and so $\mathcal{K}_\varepsilon$ is only a continuous sub-manifold in $W^{1,p}(\mathbb{R}^N)$. To overcome this difficulty, we should carefully study the elementary properties for $\mathcal{K}_\varepsilon$ as in [34]. By doing this we can reduce the variational problem for an indefinite functional to the minimax problem on a manifold and find positive solutions for $(\mathrm{LP})_\varepsilon$.

For the proof of our theorems, we consider an equivalent problem to $(\mathrm{LP})_\varepsilon$. For this purpose, making the change of variable $\varepsilon y = x$, we can rewrite $(\mathrm{LP})_\varepsilon$ as

$$-\Delta_p u + V(\varepsilon x)u = f(u) + |u|^{p^*-2}u, \quad u > 0,\ u \in W^{1,p}(\mathbb{R}^N). \qquad (\mathcal{P}_\varepsilon)$$

In the following we focus on this equivalent problem $(\mathcal{P}_\varepsilon)$.

2. Variational setting

In order to establish the variational setting for $(\mathcal{P}_\varepsilon)$, we first give some notation.

Let $L^p \equiv L^p(\mathbb{R}^N)$ be the usual Lebesgue space endowed with the norm

$$|u|_p^p = \int_{\mathbb{R}^N} |u|^p < \infty \quad \text{for } 1 \leqslant p < \infty, \quad |u|_\infty = \sup_{x\in\mathbb{R}^N} |u(x)|.$$

Let $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space endowed with the standard norm

$$\|u\|^p = \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p).$$

We denote by S_p the best Sobolev constant of the Sobolev embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, that is,

$$S_p = \inf\left\{\frac{|\nabla u|_p^p}{|u|_{p^*}^p} \colon u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}\right\},$$

where $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathcal{D}^{1,p}}^p = |\nabla u|_p^p$

Let $E = W^{1,p}(\mathbb{R}^N)$ and let $S = B_1(0) = \{u \in E \colon \|u\| = 1\}$.

The letters c, C, C_i are indiscriminately used to denote various positive constants whose exact values are irrelevant.

For any $\varepsilon > 0$, let $E_\varepsilon = \{u \in W^{1,p}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} V(\varepsilon x)u^2 < \infty\}$ denote the Sobolev space endowed with the norm

$$\|u\|_\varepsilon^p = \int_{\mathbb{R}^N} |\nabla u|^p + V(\varepsilon x)|u|^p \quad \text{for } u \in E_\varepsilon.$$

Clearly, $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ are equivalent norms for $\varepsilon > 0$ and $V_\infty < \infty$. Now, on E_ε we define the functional

$$\Psi_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p) - \int_{\mathbb{R}^N} F(u) - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} \quad \text{for } u \in E_\varepsilon.$$

Obviously, $\Psi_\varepsilon \in C^1(E_\varepsilon, \mathbb{R})$. A standard argument shows that critical points of Ψ_ε are solutions of $(\mathcal{P}_\varepsilon)$ (see [1, 3, 33]).

Let $\mathcal{N}_\varepsilon$ denote the Nehari manifold related to Ψ_ε, given by

$$\mathcal{N}_\varepsilon = \{u \in E_\varepsilon \setminus \{0\} \colon \Psi_\varepsilon'(u)u = 0\}.$$

Thus, for $u \in \mathcal{N}_\varepsilon$, it follows that

$$\int_{\mathbb{R}^N} (|\nabla u|^p + V_\varepsilon(x)|u|^p) = \int_{\mathbb{R}^N} f(u)u + \int_{\mathbb{R}^N} |u|^{p^*}, \tag{2.1}$$

where $V_\varepsilon(x) = V(\varepsilon x)$. This implies that, for $u \in \mathcal{N}_\varepsilon$,

$$\Psi_\varepsilon|_{\mathcal{N}_\varepsilon} = \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u)u - F(u)\right) + \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |u|^{p^*}. \tag{2.2}$$

Before proving some elementary properties for $\mathcal{N}_\varepsilon$, we first prove some properties for the functional Ψ_ε.

LEMMA 2.1. *Under the assumptions of* $(\mathcal{D}_0)$ *and* (f_1)*–*(f_3)*, we have that, for* $\varepsilon > 0$*, (i)* Ψ_ε *maps bounded sets in* E_ε *into bounded sets in* E_ε*, (ii)* Ψ_ε' *is weakly sequentially continuous in* E_ε*, (iii)* $\Psi_\varepsilon(t_n u_n) \to -\infty$ *as* $t_n \to \infty$*, where* $u_n \in \mathcal{E}$*, and* $\mathcal{E} \subset E_\varepsilon \setminus \{0\}$ *is a compact subset.*

Proof. (i) We follow the idea of [37]. From the conditions (f_1) and (f_3), we deduce that, for each $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(s)| \leqslant \epsilon |s|^{p-1} + C_\epsilon |s|^{q-1} \quad \text{and} \quad |F(s)| \leqslant \epsilon |s|^p + C_\epsilon |s|^q. \tag{2.3}$$

Let $\{u_n\}$ be a bounded sequence of E_ε. Then, for each $\varphi \in E_\varepsilon$, one deduces from $(\mathcal{D}_0)$ and (2.3) that

$$\begin{aligned} \Psi_\varepsilon'(u_n)\varphi &= \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n \nabla \varphi + V_\varepsilon(x)|u_n|^{p-1}\varphi) \\ &\quad + \int_{\mathbb{R}^N} f(u_n)\varphi + \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n \varphi \\ &\leqslant c\|u_n\|^{(p-1)/p} |\varphi|_p + c\|u_n\|^{(q-1)/q} |\varphi|_q + c\|u_n\|^{p^*-1} |\varphi|_{p^*} \\ &\leqslant c. \end{aligned}$$

(ii) To prove the conclusion (ii), one can refer to [3, 33]; we omit the details here.

(iii) Finally, we prove the conclusion (iii). Without loss of generality, we may assume that $\|u\|_\varepsilon = 1$ for each $u \in \mathcal{E}$. For $u_n \in \mathcal{E}$, after passing to a subsequence, we obtain that $u_n \to u \in S_\varepsilon := \{u \in E_\varepsilon \colon \|u\| = 1\}$. It follows from (1.8) that

$$\begin{aligned} \Psi_\varepsilon(t_n u_n) &= \frac{t_n^p}{p} \int_{\mathbb{R}^N} (|\nabla u_n|^p + V_\varepsilon(x)|u_n|^p) - \int_{\mathbb{R}^N} F(t_n u_n) + \frac{t_n^{p^*}}{p^*} \int_{\mathbb{R}^N} |u_n|^{p^*} \\ &\leqslant t_n^q \left(\frac{\int_{\mathbb{R}^N} (|\nabla u_n|^p + V_\varepsilon(x)|u_n|^p)}{t_n^{q-p}} - \frac{\int_{\mathbb{R}^N} F(t_n u_n)}{t_n^q} - t_n^{p^*-q} \int_{\mathbb{R}^N} |u_n|^{p^*} \right) \\ &\to -\infty \end{aligned}$$

as $n \to \infty$. □

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We are now ready to prove some elementary properties for $\mathcal{N}_\varepsilon$.

LEMMA 2.2. *Under the assumptions of lemma 2.1, for $\varepsilon > 0$ the following hold.*

(i) *For all $u \in S_\varepsilon$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon(u) = t_u u$ is the unique maximum of Ψ_ε on E_ε, where $S_\varepsilon = \{u \in E_\varepsilon \colon \|u\|_\varepsilon = 1\}$.*

(ii) *The set $\mathcal{N}_\varepsilon$ is bounded away from 0. Furthermore, $\mathcal{N}_\varepsilon$ is closed in E_ε.*

(iii) *There exists $\alpha > 0$ such that $t_u \geqslant \alpha$ for each $u \in S_\varepsilon$ and, for each compact subset $\mathcal{W} \subset S_\varepsilon$, there exists $C_\mathcal{W} > 0$ such that $t_u \leqslant C_\mathcal{W}$ for all $u \in \mathcal{W}$.*

(iv) *$\mathcal{N}_\varepsilon$ is a regular manifold diffeomorphic to the sphere of E_ε.*

(v) *$c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \Psi_\varepsilon \geqslant \rho > 0$ and Ψ_ε is bounded below on $\mathcal{N}_\varepsilon$, where $\rho > 0$ is independent of ε.*

Proof. We follow the idea of [37].

(i) For each $u \in S_\varepsilon$ and $t > 0$, we define $g(t) = \Psi_\varepsilon(tu)$. It is easy to verify that $g(0) = 0$, $g(t) < 0$ for $t > 0$ large. Moreover, we claim that $g(t) > 0$ for $t > 0$ small. Indeed, we derive, from the condition (2.3), that

$$\begin{aligned} g(t) &= \Psi_\varepsilon(tu) \\ &= \frac{t^p}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V(\varepsilon x)|u|^p) - \int_{\mathbb{R}^N} F(tu) - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}|u|^{p^*} \\ &\geqslant \frac{t^p}{p}\|u\|_\varepsilon^p - \epsilon t^p|u|_p^p - t^q cC_\epsilon|u|_q^q - ct^{p^*}|u|_{p^*}^{p^*} \\ &\geqslant \frac{t^p}{p}\|u\|_\varepsilon^2 - ct^p\epsilon\|u\|_\varepsilon^p - cC_\epsilon t^q\|u\|_\varepsilon^q - ct^{p^*}\|u\|_\varepsilon^{p^*}. \end{aligned}$$

Since we have $p < q < p^*$ and $\epsilon > 0$ small enough, we derive that $g(t) > 0$ for $t > 0$ small. Therefore, $\max_{t>0} g(t)$ is achieved at $t = t_u > 0$, so $g'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. Suppose that there exists $t'_u > t_u > 0$ such that $t'_u u, t_u u \in \mathcal{N}_\varepsilon$. It then follows from (2.1) that

$$\left.\begin{aligned} t_u^p\|u\|_\varepsilon^p &= \int_{\mathbb{R}^N} f(t_u u)t_u u + t_u^{p^*}\int_{\mathbb{R}^N}|u|^{p^*}, \\ (t'_u)^p\|u\|_\varepsilon^p &= \int_{\mathbb{R}^N} f(t'_u u)t'_u u + (t'_u)^{p^*}\int_{\mathbb{R}^N}|u|^{p^*}. \end{aligned}\right\} \tag{2.4}$$

We then see that

$$0 = \int_{\mathbb{R}^N}\left(\frac{f(t'_u u)}{(t'_u u)^{p-1}} - \frac{f(t_u u)}{(t_u u)^{p-1}}\right)u^p + ((t'_u)^{p^*-p} - t_u^{p^*-p})\int_{\mathbb{R}^N}|u|^{p^*},$$

which makes no sense in view of (f_2) and $t'_u > t_u > 0$. So the conclusion (i) follows.

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(ii) For $u \in \mathcal{N}_\varepsilon$, we infer from (2.1) and (2.3) that

$$\|u\|_\varepsilon^p \leqslant \epsilon|u|_p^p + C_\epsilon|u|_q^q + c|u|_{p^*}^{p^*} \leqslant c\epsilon\|u\|_\varepsilon^p + cC_\epsilon\|u\|_\varepsilon^q + c\|u\|^{p^*}.$$

So, for some $\kappa > 0$, we get that

$$\|u\|_\varepsilon \geqslant \kappa > 0. \tag{2.5}$$

Next, we prove that the set $\mathcal{N}_\varepsilon$ is closed in E_ε. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $u_n \to u$ in E_ε. In the following we prove that $u \in \mathcal{N}_\varepsilon$. By lemma 2.1, we have that $\Psi'_\varepsilon(u_n)$ is bounded; we then infer from

$$\Psi'_\varepsilon(u_n)u_n - \Psi'_\varepsilon(u)u = (\Psi'_\varepsilon(u_n) - \Psi'_\varepsilon(u))u - \Psi'_\varepsilon(u_n)(u_n - u) \to 0 \quad \text{as } n \to \infty,$$

that $\Psi'_\varepsilon(u)u = 0$. Moreover, it follows from (2.5) that $\|u\|_\varepsilon = \lim_{n\to\infty} \|u_n\|_\varepsilon \geqslant \kappa > 0$. So $u \in \mathcal{N}_\varepsilon$.

(iii) For $\{u_n\} \subset E_\varepsilon \setminus \{0\}$, there exist t_{u_n} such that $t_{u_n}u_n \in \mathcal{N}_\varepsilon$. By the conclusion (ii), one sees that $\|t_{u_n}u_n\|_\varepsilon = t_{u_n}\|u_n\|_\varepsilon \geqslant \kappa > 0$. It is impossible to have that $t_{u_n} \to 0$, as $n \to \infty$. To prove $t_u \leqslant C_{\mathcal{W}}$, for all $u \in \mathcal{W} \subset S_\varepsilon$, we argue by contradiction. Suppose that there exists $\{u_n\} \subset \mathcal{W} \subset S_\varepsilon$ such that $t_n = t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in E_ε and $u_n(x) \to u(x)$ almost everywhere (a.e.) on $\mathbb{R}^N$ after passing to a subsequence. Then, lemma 2.1 implies that $\Psi_\varepsilon(t_nu_n) \to -\infty$ as $n \to \infty$. However, from (2.2) we deduce that $\Psi_\varepsilon(t_nu_n) \geqslant 0$. This is a contradiction.

(iv) Define the mappings $\hat{m}_\varepsilon \colon E_\varepsilon \setminus \{0\} \to \mathcal{N}_\varepsilon$ and $m_\varepsilon \colon S_\varepsilon \to \mathcal{N}_\varepsilon$ by setting

$$\hat{m}_\varepsilon(u) = t_u u \quad \text{and} \quad m_\varepsilon = \hat{m}_\varepsilon|_{S_\varepsilon}. \tag{2.6}$$

By the conclusions (i)–(iii), we know that the conditions of [34, proposition 3.1] are satisfied. So, the mapping m_ε is a homeomorphism between S_ε and $\mathcal{N}_\varepsilon$, and the inverse of m_ε is given by

$$\check{m}_\varepsilon(u) = m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}. \tag{2.7}$$

Thus, $\mathcal{N}_\varepsilon$ is a regular manifold diffeomorphic to the sphere of E_ε.

(v) For $\varepsilon > 0$, $s > 0$ and $u \in E_\varepsilon \setminus \{0\}$, it follows from (2.3) that

$$\begin{aligned}\Psi_\varepsilon(su) &= \frac{s^p}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V_\varepsilon(x)|u|^p) - \int_{\mathbb{R}^N} F(su) - \frac{s^{p^*}}{p^*}\int_{\mathbb{R}^N}|u|^{p^*} \\ &\geqslant \frac{s^p}{p}\|u\|_\varepsilon^2 - s^pc\epsilon\|u\|_\varepsilon^p - s^qcC_\epsilon\|u\|_\varepsilon^q - cs^{p^*}\|u\|_\varepsilon^{p^*} \\ &= \frac{s^p}{p}(1 - cp\epsilon)\|u\|_\varepsilon^p - s^qcC_\epsilon\|u\|_\varepsilon^q - cs^{p^*}\|u\|_\varepsilon^{p^*}.\end{aligned}$$

So, there exists $\rho > 0$ such that $\Psi_\varepsilon(su) \geqslant \rho > 0$ for $s > 0$ small. On the other hand, we deduce from the conclusions (i)–(iii) that

$$c_\varepsilon = \inf_{\mathcal{N}_\varepsilon}\Psi_\varepsilon(u) = \inf_{w\in E_\varepsilon\setminus\{0\}}\max_{s>0}\Psi_\varepsilon(sw) = \inf_{w\in S_\varepsilon}\max_{s>0}\Psi_\varepsilon(sw). \tag{2.8}$$

So, we get that $c_\varepsilon \geqslant \rho > 0$ and $\Psi_\varepsilon|_{\mathcal{N}_\varepsilon} \geqslant \rho > 0$. □

We now consider the functionals $\hat{\Upsilon}_\varepsilon\colon E_\varepsilon \setminus \{0\} \to \mathbb{R}$ and $\Upsilon_\varepsilon\colon S_\varepsilon \to \mathbb{R}$ defined by

$$\hat{\Upsilon}_\varepsilon = \Psi_\varepsilon(\hat{m}_\varepsilon(u)) \quad \text{and} \quad \Upsilon_\varepsilon = \hat{\Upsilon}_\varepsilon|_{S_\varepsilon},$$

where $\hat{m}_\varepsilon(u) = t_u u$ is given in (2.6). As in [34], we have the following lemma.

Lemma 2.3 (Szulkin and Weth [34, corollary 3.3]). *Under the assumptions of lemma 2.1, we have, for $\varepsilon > 0$, that the following hold.*

(i) $\Upsilon_\varepsilon \in C^1(S_\varepsilon, \mathbb{R})$ *and*

$$\Upsilon'_\varepsilon(w)z = \|m_\varepsilon(w)\|_\varepsilon \Psi'_\varepsilon(m_\varepsilon(w))z \quad \text{for } z \in \mathcal{T}_w(S_\varepsilon).$$

(ii) *$\{w_n\}$ is a Palais–Smale sequence for Υ_ε if and only if $\{m_\varepsilon(w_n)\}$ is a Palais–Smale sequence for Ψ_ε. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a bounded Palais–Smale sequence for Ψ_ε, then $\check{m}_\varepsilon(u_n)$ is a Palais–Smale sequence for Υ_ε, where $\check{m}_\varepsilon(u)$ is given in (2.7).*

(iii) *We have*

$$\inf_{S_\varepsilon} \Upsilon_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \Psi_\varepsilon = c_\varepsilon.$$

Moreover, $z \in S_\varepsilon$ is a critical point of Υ_ε if and only if $m_\varepsilon(u)$ is a critical point of Ψ_ε, and the corresponding critical values coincide.

3. The periodic system

In this section we prove some properties of the ground state solution of the limit equation. Precisely, for each $\xi > 0$, we are concerned with the following equation:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \xi|u|^{p-u}u = f(u) + |u|^{p^*-2}u, \quad u > 0,\ u \in W^{1,p}(\mathbb{R}^N). \quad (\mathcal{P}_\xi)$$

For any $\xi > 0$, let $E_\xi = \{u \in W^{1,p}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} \xi u^p < \infty\}$ be a Banach space endowed the norm

$$\|u\|_\xi^p = \int_{\mathbb{R}^N} |\nabla u|^p + \xi|u|^p \quad \text{for } u \in E_\xi.$$

We then see that the energy functional corresponding to $(\mathcal{P}_\xi)$ is defined by

$$\Psi_\xi(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + \xi|u|^p) - \int_{\mathbb{R}^N} F(u) - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} \quad \text{for all } u \in E_\xi.$$

As in § 2, $\Psi_\varepsilon \in C^1(E_\xi, \mathbb{R})$ and a standard argument shows that critical points of Ψ_ε are solutions of $(\mathcal{P}_\xi)$. In order to find the critical points for the functional $(\mathcal{P}_\xi)$, we also use the Nehari manifold methods. The Nehari manifold corresponding to Ψ_ξ is defined by

$$\mathcal{N}_\xi = \{u \in E_\xi \setminus \{0\}\colon \Psi'_\xi(u)u = 0\}.$$

Thus, for $u \in \mathcal{N}_\xi$, one sees that

$$\int_{\mathbb{R}^N} (|\nabla u|^p + \xi|u|^p) = \int_{\mathbb{R}^N} f(u)u + \int_{\mathbb{R}^N} |u|^{p^*}. \quad (3.1)$$

This implies that, for $u \in \mathcal{N}_\xi$,

$$\Psi_\xi|_{\mathcal{N}_\xi} = \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u)u - F(u)\right) + \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |u|^{p^*}. \tag{3.2}$$

To prove some properties for the function Ψ_ξ, we need the following result.

LEMMA 3.1. *Let* $1 < r \leqslant \infty$, $1 \leqslant q < \infty$ *with* $q \neq Nr/(N-r)$ *if* $r < N$. *Assume that* ϕ_n *is bounded in* $L^q(\mathbb{R}^N)$, $|\nabla \phi_n|$ *is bounded in* $L^r(\mathbb{R}^N)$ *and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} |\phi_n|^q \to 0 \quad \textit{for some } R > 0.$$

Then, $\phi_n \to 0$ *in* $L^\sigma(\mathbb{R}^N)$ *for any* $\sigma \in (q, Nr/(N-r))$. *Moreover, if* ϕ_n *is bounded in* $L^p(\mathbb{R}^N)$, $|\nabla \phi_n|$ *is bounded in* $L^p(\mathbb{R}^N)$ *and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} |\phi_n|^{p^*} \to 0 \quad \textit{for some } R > 0.$$

Thus, $\phi_n \to 0$ *in* $L^k(\mathbb{R}^N)$ *for any* $k \in (q, Np/(N-p)]$.

Proof. For the proof of the first conclusion of this lemma, one can refer to [24,25,32]. We now prove the last conclusion. Clearly, it suffices to prove $\phi_n \to 0$ in $L^{p^*}(\mathbb{R}^N)$. It follows from the Hölder inequality that

$$\begin{aligned}\int_{B_r(y)} |\phi_n|^{p^*} &\leqslant \left(\int_{B_r(y)} |\phi_n|^{p^*}\,\mathrm{d}x\right)^{(p^*-p)/p^*} \left(\int_{B_r(y)} |\phi_n|^{p^*}\,\mathrm{d}x\right)^{p/p^*} \\ &\leqslant c\left(\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |\phi_n|^{p^*}\,\mathrm{d}x\right)^{(p^*-p)/p^*} \int_{\mathbb{R}^N} (|\nabla\phi_n|^p + |\phi_n|^p)\,\mathrm{d}x.\end{aligned}$$

Now, covering $\mathbb{R}^N$ by balls of radius r, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we find that

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi_n|^{p^*} &\leqslant c(N+1)\left(\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |\phi_n|^{p^*}\,\mathrm{d}x\right)^{(p^*-p)/p^*} \int_{\mathbb{R}^N} (|\nabla\phi_n|^p + |u_n|^p)\,\mathrm{d}x \\ &\leqslant c(N+1)\left(\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |\phi_n|^{p^*}\,\mathrm{d}x\right)^{(p^*-p)/p^*} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This completes the proof of the lemma. □

We are now ready to prove some elementary properties for $\mathcal{N}_\xi$.

LEMMA 3.2. *Under the assumptions of lemma 2.1, we have that, for* $\xi > 0$, *the following hold.*

(i) *For all* $u \in S_\xi := \{u \in E_\xi \colon \|u\|_\xi = 1\}$, *there exists a unique* $t_u > 0$ *such that* $t_u u \in \mathcal{N}_\xi$. *Moreover,* $m_\xi(u) = t_u u$ *is the unique maximum of* Ψ_ξ *on* E_ξ.

(ii) *The set* $\mathcal{N}_\xi$ *is bounded away from* 0. *Furthermore,* $\mathcal{N}_\xi$ *is closed in* E_ξ.

(iii) *There exists* $\delta > 0$ *such that* $t_u \geqslant \delta$ *for each* $u \in S_\xi$ *and, for each compact subset* $\mathcal{W} \subset S_\xi$*, there exists* $C_{\mathcal{W}} > 0$ *such that* $t_u \leqslant C_{\mathcal{W}}$ *for all* $u \in \mathcal{W}$*.*

(iv) $\mathcal{N}_\xi$ *is a regular manifold diffeomorphic to the sphere of* E_ξ*.*

(v) $c_\xi = \inf_{\mathcal{N}_\xi} \Psi_\xi > 0$ *and* $\Psi_\xi|_{\mathcal{N}_\xi}$ *is bounded below by some positive constant.*

Proof. Using the same arguments as those of lemma 2.2, one can easily prove the conclusions (i)–(v). We omit the details here. □

From lemma 3.2(i), we know that, for each $u \in E_\xi \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\xi$. So we define the mapping $\hat{m}_\xi \colon E_\xi \setminus \{0\} \to \mathcal{N}_\xi$ by $\hat{m}_\xi(u) = t_u u$. Clearly, $m_\xi = \hat{m}_\xi|_{S_\xi}$. Let

$$\hat{\Upsilon}_\xi \colon E_\xi \setminus \{0\} \to \mathbb{R}, \qquad \hat{\Upsilon}_\xi(w) := \Psi_\xi(\hat{m}_\xi(w)) \quad \text{and} \quad \Upsilon_\xi := \hat{\Upsilon}_\xi|_{S_\xi}.$$

If the inverse of the mapping m_ξ to S_ξ is given by

$$\check{m}_\xi = m_\xi^{-1} \colon \mathcal{N}_\xi \to S_\xi, \quad \check{m}_\xi = \frac{u}{\|u\|},$$

then we have the following lemma.

Lemma 3.3 (Szulkin and Weth [34, corollary 3.3]). *Under the assumptions of lemma 2.1, we have that, for* $\varepsilon > 0$*, the following hold.*

(i) $\Upsilon_\xi \in C^1(S_\xi, \mathbb{R})$ *and*

$$\Upsilon_\xi'(w)z = \|m_\xi(w)\|_\xi \Psi_\xi'(m_\xi(w))z \quad \text{for } z \in \mathcal{T}_w(S_\xi).$$

(ii) $\{w_n\}$ *is a Palais–Smale sequence for* Υ_ξ *if and only if* $\{m_\xi(w_n)\}$ *is a Palais–Smale sequence for* Ψ_ξ*. If* $\{u_n\} \subset \mathcal{N}_\xi$ *is a bounded Palais–Smale sequence for* Ψ_ξ*, then* $\check{m}_\xi(u_n)$ *is a Palais–Smale sequence for* Υ_ξ*, where* $\check{m}_\xi(u) = m_\xi^{-1}(u) = u/\|u\|_\xi$*.*

(iii) *We have*

$$\inf_{S_\xi} \Upsilon_\xi = \inf_{\mathcal{N}_\xi} \Psi_\xi = c_\xi.$$

Moreover, $z \in S_\xi$ *is a critical point of* Υ_ξ *if and only if* $m_\xi(u)$ *is a critical point of* Ψ_ξ*, and the corresponding critical values coincide.*

Remark 3.3. By lemma 3.1, we note that the infimum of Ψ_ξ over $\mathcal{N}_\xi$ has the following minimax characterization:

$$0 < c_\xi = \inf_{z \in \mathcal{N}_\xi} \Psi_\xi(z) = \inf_{w \in E_\xi \setminus \{0\}} \max_{s>0} \Psi_\xi(sw) = \inf_{w \in S_\xi} \max_{s>0} \Psi_\xi(sw). \tag{3.3}$$

Similarly to [26], one can easily prove the following mountain pass geometry of the functional $\Psi_\xi(u)$.

Lemma 3.4 (mountain pass geometry). *The functional* Ψ_ξ *satisfies the following conditions.*

(i) *There exist positive constants* β*,* α *such that* $\Psi_\xi(u) \geqslant \beta > 0$ *for* $\|u\|_\mu = \alpha$*.*

(ii) *There exists* $e \in E_\xi$ *with* $\|e\| > \alpha$ *such that* $\Psi_\xi(e) < 0$*.*

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From lemma 3.4, by using the Ambrosetti–Rabinowitz mountain pass theorem without the $(\mathrm{PS})_c$-condition (see [12, 27]), it follows that there exists a $(\mathrm{PS})_c$-sequence $\{u_n\} \subset E_\xi$ such that

$$\Psi_\xi(u_n) \to c'_\xi = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \Psi_\xi(\gamma(t)) \quad \text{and} \quad \Psi'_\xi(u_n) \to 0, \tag{3.4}$$

where $\Gamma = \{\gamma \in C(E_\xi, \mathbb{R})\colon \Psi_\xi(\gamma(0)) = 0,\ \Psi_\xi(\gamma(1)) < 0\}$. As in [31, proposition 3.11], we use the equivalent characterization of c'_ξ, which is more adequate for our purpose, given by

$$c'_\xi = \inf_{u\in E_\xi\setminus\{0\}} \max_{t>0} \Psi_\xi(tu) = c_\xi. \tag{3.5}$$

Here in the last equality we used (3.3). As in [17], we have the following estimates for c_μ.

Lemma 3.5. *If the conditions* $(\mathcal{D}_0)$ *and* (f_1)*–*(f_3) *hold, one gets that, for any* $0 < \xi \leqslant V_\infty$, *the number* c_ξ *satisfies*

$$0 < c_\xi < \frac{1}{N} S_p^{N/p},$$

where S_p *is the best Sobolev constant, namely,*

$$S_p = \inf\left\{ \frac{|\nabla u|_p^p}{|u|_{p^*}^p} \colon u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\} \right\}.$$

We are now ready to study the minimizing sequence for Ψ_ξ.

Lemma 3.6. *Let* $\{u_n\} \subset \mathcal{N}_\xi$ *be a minimizing sequence for* Ψ_ξ. *Then,* $\{u_n\}$ *is bounded. Moreover, there exist* $r, \delta > 0$ *and a sequence* $\{y_n\} \subset \mathbb{R}^N$ *such that*

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \delta > 0,$$

where $B_r(y_n) = \{y \in \mathbb{R}^N \colon |y - y_n| \leqslant r\}$ *for each* $n \in \mathbb{N}$.

Proof. We first prove that $\{u_n\}$ is bounded. Arguing by contradiction, suppose that there exists a sequence $\{u_n\} \subset \mathcal{N}_\xi$ such that $\|u_n\|_\mu \to \infty$ and $\Psi_\xi(u_n) \to c_\xi$. Let $z_n = u_n/\|u_n\|_\xi$. Then, $z_n \rightharpoonup z$ and $z_n(x) \to z_n(x)$ a.e. in $\mathbb{R}^N$ after passing to a subsequence. Moreover, we have that either $\{z_n\}$ is vanishing, i.e.

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |z_n|^{p^*} = 0, \tag{3.6}$$

or non-vanishing, i.e. there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |z_n|^{p^*} \geqslant \delta > 0. \tag{3.7}$$

As in [21], we show that neither (3.6) nor (3.7) holds true, and this provides the desired contradiction.

If $\{z_n\}$ is vanishing, lemma 3.1 implies that $z_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, p^*]$. Therefore, from (2.3) we deduce that $\int_{\mathbb{R}^N} F(\ell z_n) \to 0$ as $n \to \infty$ for each $\ell \in \mathbb{R}$. So, we infer from lemma 3.2 that, for $\xi > 0$,

$$\begin{aligned} c_\xi + o(1) &\geqslant \Psi_\xi(u_n) \\ &\geqslant \Psi_\xi(\ell z_n) \\ &= \frac{\ell^p}{p} \int_{\mathbb{R}^N} (|\nabla z_n|^p + \xi |z_n|^p) - \int_{\mathbb{R}^N} F(\ell z_n) - \frac{\ell^{p^*}}{p^*} \int_{\mathbb{R}^N} |z_n|^{p^*} \\ &\geqslant \frac{\ell^p}{p} - \int_{\mathbb{R}^N} F(\ell z_n) - \frac{\ell^{p^*}}{p^*} \int_{\mathbb{R}^N} |z_n|^{p^*} \\ &\to \frac{\ell^p}{p} \end{aligned}$$

as $n \to \infty$. We now arrive at a contradiction if ℓ is large enough. Hence, non-vanishing must hold. It follows from (2.3) that

$$\int_{\mathbb{R}^N} F(u_n) \leqslant c\epsilon \|u_n\|_\xi^p + cC_\epsilon \|u_n\|_\xi^q. \tag{3.8}$$

So, from (3.7) and (3.8) we infer that, for n large,

$$0 \leqslant \frac{\Psi_\xi(u_n)}{\|u_n\|_\xi^{p^*}} = -\frac{1}{p^*} \int_{\mathbb{R}^N} |z_n|^{p^*} + o(1) \leqslant -\frac{1}{p^*} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |z_n|^{p^*} + o(1) < 0,$$

a contradiction.

Next, we prove the latter conclusion of this lemma. Since $\{u_n\}$ is bounded, if

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^p = 0,$$

we deduce from lemma 3.1 that $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for $t \in (p, p^*)$. We infer from (2.3) that $\int_{\mathbb{R}^N} F(u_n) \to 0$ as $n \to \infty$. Moreover, it follows from $\Psi'_\xi(u_n) u_n = 0$ that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^p + \xi |u_n|^p) = \int_{\mathbb{R}^N} u_n^{p^*} + o(1). \tag{3.9}$$

Assume that $\int_{\mathbb{R}^N} (|\nabla u_n|^p + \xi |u_n|^p) \to \gamma$. If $\gamma > 0$, it follows from $\Psi_\xi(u_n) \to c_\xi$ that

$$\frac{1}{p} \|u_n\|_\xi^p - \frac{1}{p^*} \int_{\mathbb{R}^N} |u_n|^{p^*} \to c_\xi.$$

Thus, we obtain that $c_\xi = \gamma / N$. On the other hand, we infer from $\gamma \geqslant S_p \gamma^{p/p^*}$ that

$$\gamma \geqslant S_p^{N/p}.$$

Therefore, we get that $c_\xi = (1/N)\gamma \geqslant (1/N) S_p^{N/p}$. This contradicts the conclusion of lemma 3.5. □

We now state the main results for the limit problem $(\mathcal{P}_\xi)$.

THEOREM 3.7. *Let the assumptions of theorem 1.1 be satisfied. Then, for each* $\xi > 0$*, the following conclusions hold.*

(i) *The problem* $(\mathcal{P}_\xi)$ *has at least one positive ground state solution* u_ξ *in* $E_\xi = W^{1,p}(\mathbb{R}^N)$.

(ii) $\lim_{|x|\to\infty} u_\xi(x) = 0$ *and* $u_\xi \in C^{1,\sigma}_{\text{loc}}$ *with* $\sigma \in (0,1)$*. Furthermore, there exist* $C, c > 0$ *such that* $u_\xi(x) \leqslant C\mathrm{e}^{-c|x|}$.

(iii) $\mathcal{L}_\xi$ *is compact in* E_ξ *for* $\xi > 0$*, where* $\mathcal{L}_\xi$ *denotes the set of all least energy solutions of* $(\mathcal{P}_\xi)$.

Proof. (i) From the conclusion of lemma 3.2(v) we know that $c_\xi > 0$ for each $\xi > 0$. Moreover, if $u_0 \in \mathcal{N}_\xi$ satisfies $\Psi_\xi(u_0) = c_\xi$, then $\check{m}_\xi(u_0)$ is a minimizer of Υ_ξ, and therefore a critical point of Υ_ξ, so u_0 is a critical point of Ψ_ξ by lemma 3.3. It remains to show that there exists a minimizer u of $\Psi_\xi|_{\mathcal{N}_\xi}$. By Ekeland's variational principle [39], there exists a sequence $\{\omega_n\} \subset S_\xi$ such that $\Upsilon_\xi(\omega_n) \to c_\xi$ and $\Upsilon'_\xi(\omega_n) \to 0$ as $n \to \infty$. Set $u_n = m_\xi(\omega_n) \in \mathcal{N}_\xi$ for all $n \in \mathbb{N}$. Then $\Psi_\xi(u_n) \to c_\xi$ and $\Psi'_\xi(u_n) \to 0$ as $n \to \infty$. Similarly to the proof of lemma 3.6, we know that $\{u_n\}$ is bounded and there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \delta > 0.$$

So, we can choose $r' > r > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\lim_{n\to\infty} \int_{B_{r'}(y_n)} |u_n|^p \geqslant \frac{\delta}{2} > 0. \tag{3.10}$$

Using that Ψ_ξ and $\mathcal{N}_\xi$ are invariant under translations, we may assume that $\{y_n\}$ is bounded in $\mathbb{R}^N$. So $u_n \rightharpoonup u \neq 0$ and $\Psi'_\xi(u) = 0$.

It remains to show that $\Psi_\xi(u) = c_\xi$. Since $\{u_n\}$ is bounded, by (1.8) and Fatou's lemma we get that

$$\begin{aligned} c_\xi &= \liminf_{n\to\infty} \left(\Psi_\xi(u_n) - \frac{1}{p}\Psi'_\xi(u_n)u_n\right) \\ &= \liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} \left(\frac{1}{p} f(u_n)u_n - F(u_n)\right)\right) \\ &\geqslant \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u)u - F(u)\right) \\ &= \Psi_\xi(u) - \frac{1}{p}\Psi'_\xi(u)u \\ &= \Psi_\xi(u). \end{aligned}$$

Hence, $\Psi_\xi(u) \leqslant c_\xi$. The reverse inequality follows from the definition of c_ξ since $u \in \mathcal{N}_\xi$. So, we prove that $\Psi_\xi(u) = c_\xi$. Finally, we need to find a positive ground state solution for $(\mathcal{P}_\xi)$. In fact, for each $u \in W^{1,p}(\mathbb{R}^N)$, there exists $t > 0$ such that $t|u| \in \mathcal{N}_\xi$. From the condition (f_1) and the form of Ψ_ξ, we deduce that $\Psi_\xi(t|u|) \leqslant \Psi_\xi(tu)$. Moreover, it follows from $u \in \mathcal{N}_\xi$ that $\Psi_\xi(tu) \leqslant \Psi_\xi(u)$. So, we prove that

$c_\xi = \Psi_\xi(u) \leqslant \Psi_\xi(t|u|) \leqslant \Psi_\xi(u)$. That is, $u_\xi = t|u|$ also attains the least energy on $\mathcal{N}_\xi$. In addition, from lemma 3.3 we infer that u_ξ is a non-negative ground state solution of Ψ_ξ. It follows from Harnack's inequality (see [19]) that $u_\xi > 0$ for all $x \in \mathbb{R}^N$. This finishes the proof of the conclusion (i).

(ii) Using the arguments of [20, 23, 26, 35], we have that $u \in L^t(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for $t \in [2, \infty]$ and $\alpha \in (0,1)$. Set

$$h(u) = f(u) - \xi|u|^{p-2}u + |u|^{p^*-2}u.$$

From (2.3), we infer that

$$|h(u)| \leqslant c(|u|^{p-1} + |u|^{q-1} + |u|^{p^*-1}).$$

It follows that

$$|h(u)|_{L^\tau(B_{2\rho})} \leqslant c(|u|^{p-1}_{L^{(p-1)\tau}(B_{2\rho})} + |u|^{q-1}_{L^{(q-1)\tau}(B_{2\rho})} + |u|^{p^*-1}_{L^{(p^*-1)\tau}(B_{2\rho})}), \tag{3.11}$$

where $\tau > N$ and $B_{2\rho} = \{x \in \mathbb{R}^N : |x - x_0| \leqslant 2\rho,\ x_0 \in \mathbb{R}^N\}$. Using $(\mathcal{P}_\xi)$ and the definition of the norm $\|\cdot\|_{W^{1,\tau}}$, we derive that

$$\|u\|_{W^{1,\tau}(B_{2\rho})} \leqslant c(|h(u)|_{L^\tau(B_{2\rho})} + |u|_{L^\tau(B_{2\rho})}) \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$\|u\|_{W^{1,\tau}(B_{2\rho})} \leqslant c(|u|^{p-1}_{L^{(p-1)\tau}(B_{2\rho})} + |u|^{q-1}_{L^{(q-1)\tau}(B_{2\rho})} + |u|_{L^\tau(B_{2\rho})} + |u|^{p^*-1}_{L^{(p^*-1)\tau}(B_{2\rho})}).$$

Since $\tau > N$, by Sobolev's embedding theorem (see [19]) one has that

$$\|u\|_{C^{0,\sigma}(\bar{B}_\rho)} \leqslant c(|u|^{p-1}_{L^{(p-1)\tau}(B_{2\rho})} + |u|^{q-1}_{L^{(q-1)\tau}(B_{2\rho})} + |u|_{L^\tau(B_{2\rho})} + |u|^{p^*-1}_{L^{(p^*-1)\tau}(B_{2\rho})}),$$

where $\sigma \in (0,1)$. Letting $|x_0| \to \infty$, we conclude that $\|u\|_{C^{0,\sigma}(\bar{B}_\rho)} \to 0$. Therefore, we get that $\lim_{|x|\to\infty} u(x) = 0$.

Next, we prove that $u(x) \leqslant C\mathrm{e}^{-c|x|}$. By (f_2) and the fact that the solutions u decay uniformly to 0 as $|x| \to \infty$, we can take $R_0 > 0$ such that

$$f(u(x))u^{1-p} + u^{p^*-p} \leqslant \frac{\xi}{2} \quad \text{for all } |x| \geqslant R_0.$$

Consequently,

$$-\Delta_p u + \frac{\xi}{2}u^{p-1} = f(u) + u^{p^*-1} - \frac{\xi}{2}u^{p-1} \leqslant 0 \quad \text{for all } |x| \geqslant R_0.$$

Let β and δ be positive constants such that $\xi/2 - (p-1)\beta^p > 0$ and $u \leqslant \delta\exp(-\beta R_0)$ for all $|x| = R_0$. Hence, the function $\eta(x) = \delta\exp(-\beta|x|)$ satisfies

$$-\Delta_p \eta + \frac{\xi}{2}\eta^{p-1} \geqslant \left(\frac{\xi}{2} - (p-1)\beta^p\right)\eta^{p-1} > 0 \quad \text{for all } x \neq 0.$$

Since $p \geqslant 2$, we have that the function $\chi\colon \mathbb{R}^N \to \mathbb{R},\ \chi(x) = |x|^p$ is convex (see [26, 40]); thus,

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geqslant C_p|x - y|^p \geqslant 0 \quad \text{for } p \geqslant 2,\ x, y \in \mathbb{R}^N. \tag{3.13}$$

We now take $\gamma = \max\{u - \eta, 0\} \in W_0^{1,p}(|x| > R_0)$ as a test function. Hence, combining these estimates,

$$
\begin{aligned}
0 &\geqslant \int_{\mathbb{R}^N} \left[(|\nabla u|^{p-2}\nabla u - |\nabla \eta|^{p-2}\nabla \eta)\nabla\gamma + \frac{\xi}{2}(u^{p-1} - \eta^{p-1})\gamma \right] \\
&\geqslant \frac{\xi}{2} \int_{x\in\mathbb{R}^N\colon\, u\geqslant\eta} (u^{p-1} - \eta^{p-1})(u - \eta) \\
&\geqslant 0 \quad \text{for } |x| \geqslant R_0.
\end{aligned}
$$

Therefore, the set $\{x \in \mathbb{R}^N : |x| \geqslant R_0 \text{ for } u(x) \geqslant \psi(x)\}$ is empty. From this we can easily conclude that

$$u(x) \leqslant C\mathrm{e}^{-c|x|}.$$

(iii) Let the bounded sequence $\{u_n\} \subset \mathcal{L}_\xi \cap \mathcal{N}_\xi$ such that $\Psi_\xi(u_n) = c_\xi$ and $\Psi'_\xi(u_n) = 0$. Without loss of generality we assume that $u_n \rightharpoonup u$ in E_ξ. As in the proof of the conclusion (i), one can easily prove that $\{u_n\}$ is non-vanishing, i.e.

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \frac{\delta}{2} > 0.$$

By the invariance of Ψ_ξ and $\mathcal{N}_\xi$ under translations of the form $u \mapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that $\{y_n\}$ is bounded in $\mathbb{Z}^N$. So $u_n \rightharpoonup u \neq 0$ and $\Psi'_\xi(u) = 0$. Moreover, repeating arguments as in the proof of the conclusion (i), one sees that $\Psi_\xi(u) = c_\xi$ and $\Psi'_\xi(u) = 0$. So, it follows from Fatou's lemma that

$$
\begin{aligned}
c_\xi &= \Psi_\xi(u) \\
&= \Psi_\xi(u) - \frac{1}{p}(\Psi'_\xi(u), u) \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u)u - F(u)\right) + \frac{(p^* - p)}{p^* p} \int_{\mathbb{R}^N} |u|^{p^*} \\
&\leqslant \liminf_{n\to\infty} \left[\int_{\mathbb{R}^N} \left(\frac{1}{p} f(u_n)u_n - F(u_n)\right) + \frac{(p^* - p)}{p^* p} \int_{\mathbb{R}^N} |u_n|^{p^*} \right] \\
&= \liminf_{n\to\infty} \left(\Psi_\xi(u_n) - \frac{1}{p}\Psi'_\xi(u_n)u_n \right) \\
&= c_\xi.
\end{aligned}
$$

Thus, we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n^{p^*} = \int_{\mathbb{R}^N} u^{p^*}.$$

By using the Brezis–Lieb lemma (see [39]), we obtain that $|u_n - u|_{L^{p^*}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Note that u_n satisfies

$$-\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + \xi|u_n|^{p-2}u_n = f(u_n) + |u_n|^{p^*-2}u_n. \tag{3.14}$$

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Using $u_n - u$ as a test function in (3.14), we conclude that, for each $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) + \xi|u_n|^{p-2}u_n(u_n - u)) \\ &= \xi \int_{\mathbb{R}^N} [f(u_n)(u_n - u) + |u_n|^{p^*-2}u_n(u_n - u)] \\ &\leqslant \beta\xi \int_{\mathbb{R}^N} |u_n||u_n - u| + cC_\beta \int_{\mathbb{R}^N} |u_n|^{q-1}|u_n - u| \\ &\leqslant c\beta + cC_\beta |u_n - u|_{L^{p^*}(\mathbb{R}^N)}. \end{aligned} \tag{3.15}$$

So it follows that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) + \xi|u_n|^{p-2}u_n(u_n - u)) = o(1) \quad \text{as } n \to \infty. \tag{3.16}$$

Similarly, since u satisfies the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \xi u = f(u) + |u|^{p^*-2}u, \tag{3.17}$$

we infer that

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla(u_n - u) + \xi|u|^{p-2}u(u_n - u)) = o(1). \tag{3.18}$$

From (3.13), (3.16) and (3.18), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla(u_n - u)|^p + \xi|u_n - u|^p) \leqslant \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u) \\ + \xi \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u|^{p-2}u, u_n - u) \to 0 \end{aligned} \tag{3.19}$$

as $n \to \infty$. So, we obtain that $\|u_n - u\|_{W^{1,p}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. □

REMARK 3.8. We point out that our arguments in this section can also be applied to the case of periodic potentials, or to the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = f(u) + |u|^{p^*-2}u, \quad u > 0,\ u \in W^{1,p}(\mathbb{R}^N), \tag{$\mathcal{P}_V$}$$

where $V(x)$ is a positive continuous periodic function in each variable. Using translation invariance of the problem, the proof is still valid. Thus, the conclusions of theorem 3.7 hold.

LEMMA 3.9. *Under the assumptions of lemma 2.1, we have that* $c_{\xi_1} > c_{\xi_2}$ *for* $\xi_1 > \xi_2$.

Proof. For $\xi_1, \xi_2 > 0$, one sees that $E_{\xi_1} = E_{\xi_2} = E$. Let $u_1 \in \mathcal{N}_{\xi_1}$ be such that

$$c_{\xi_1} = \Psi_{\xi_1}(u_1) = \max_{w \in E_{\xi_1}} \Psi_{\xi_1}(w).$$

On the other hand, let $u_2 \in E_{\xi_2}$ be such that

$$\Psi_{\xi_2}(u_2) = \max_{w \in E_{\xi_2}} \Psi_{\xi_2}(w).$$

Therefore, one sees that

$$\begin{aligned} c_{\xi_1} &\geqslant \Psi_{\xi_1}(u_2) \\ &= \Psi_{\xi_2}(u_2) + (\xi_1 - \xi_2)\int_{\mathbb{R}^N} u_2^p \\ &\geqslant c_{\xi_2} + (\xi_1 - \xi_2)\int_{\mathbb{R}^N} u_2^p \\ &> c_{\xi_2}. \end{aligned}$$

□

4. A compactness condition

In this section we prove some compactness results for the functional Ψ_ε. Precisely, we show that any minimizing sequence of Ψ_ε has a strongly convergent subsequence in E_ε. We begin with the following lemma.

LEMMA 4.1. *Under the assumptions of* ($\mathcal{D}_0$) *and* (f_1)–(f_3)*, we have that*

(i) $c_\varepsilon \geqslant c_{V_0}$ *for all* $\varepsilon > 0$,

(ii) $c_\varepsilon \to c_{V_0}$ *as* $\varepsilon \to 0$.

Proof. The idea of the proof comes from [16, 37].

(i) Since V is a bounded function, it is easy to check that, for all $\varepsilon > 0$ and $\xi > 0$, $E_\varepsilon = E_\xi = W^{1,p}(\mathbb{R}^N)$. To prove the first conclusion, we argue by contradiction and assume that $c_\varepsilon < c_{V_0}$ for some $\varepsilon > 0$. By the definition of c_ε, we can choose an $e \in E_\varepsilon \setminus \{0\}$ such that $\max_{s>0}\Psi_\varepsilon(se) < c_{V_0}$. Again by the definition of c_{V_0}, we know that $c_{V_0} \leqslant \max_{s>0}\Psi_{V_0}(se)$. Since $V_\varepsilon(x) \geqslant V_0$, $\Psi_\varepsilon(u) \geqslant \Psi_{V_0}(u)$ for all $u \in E_\varepsilon$, and we get

$$c_{V_0} > \max_{s>0}\Psi_\varepsilon(se) \geqslant \max_{s>0}\Psi_{V_0}(se) \geqslant c_{V_0},$$

a contradiction.

(ii) Set $V^0(x) = V(x) - V_0$ and $V_\varepsilon^0(x) = V^0(\varepsilon x)$. We then see that

$$\Psi_\varepsilon(u) = \Psi_{V_0}(u) + \int_{\mathbb{R}^N} V_\varepsilon^0(x)u^p.$$

Let $u \in \mathcal{N}_{V_0}$ be such that $c_{V_0} = \Psi_{V_0}(u) = \max_{w\in E_{V_0}\setminus\{0\}}\Psi_{V_0}(w)$. We take $v \in E_\varepsilon \setminus \{0\}$ such that

$$c_\varepsilon \leqslant \Psi_\varepsilon(v) = \max_{s>0}\Psi_\varepsilon(su) = \Psi_{V_0}(v) + \int_{\mathbb{R}^N} V_\varepsilon^0(x)v^p. \tag{4.1}$$

Obviously, for each $\epsilon > 0$ we can choose $R > 0$ such that

$$\int_{|x|>R} V_\varepsilon^0(x)|v|^p < c\epsilon. \tag{4.2}$$

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Moreover, since $0 \in \mathcal{V}$, one has that

$$\int_{|x|\leqslant R} V_\varepsilon^0(x)|v|^p \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we deduce that

$$\int_{\mathbb{R}^N} V_\varepsilon^0(x) v^p \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, we get that

$$\begin{aligned} c_\varepsilon &\leqslant \Psi_{V_0}(v) + o(1) \\ &\leqslant \max_{w \in E_{V_0}\setminus\{0\}} \Psi_{V_0}(w) + o(1) \\ &= \Psi_{V_0}(u) + o(1) \\ &= c_{V_0} + o(1). \end{aligned}$$

Furthermore, it follows from the conclusion (i) that

$$c_{V_0} \leqslant \lim_{\varepsilon\to 0} c_\varepsilon \leqslant \lim_{\varepsilon \to 0} \Psi_\varepsilon(v) = \Psi_{V_0}(v) \leqslant \Psi_{V_0}(u) = c_{V_0}.$$

Hence, we obtain $c_\varepsilon \to c_{V_0}$ as $\varepsilon \to 0$. □

From $(\mathcal{D}_0)$, we know that $V_0 < V_\infty$. So, we can choose $\ell > 0$ such that

$$V_0 < \ell < V_\infty.$$

As in [2, 17], we have the following lemmas.

Lemma 4.2. *Suppose that the assumptions of $(\mathcal{D}_0)$ and (f_1)–(f_3) hold. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $\Psi_\varepsilon(u_n) \to c$ with $c \leqslant c_\ell$ and $u_n \rightharpoonup 0$ in E_ε; then one of the following conclusions holds.*

(i) *$u_n \to 0$ in E_ε.*

(ii) *There exist a sequence $y_n \in \mathbb{R}^N$ and constants $r, \delta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_r(y_n)} u_n^p \geqslant \delta.$$

Lemma 4.3. *Let the assumptions of $(\mathcal{D}_0)$ and (f_1)–(f_3) be satisfied. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $\Psi_\varepsilon(u_n) \to c$ with $c \leqslant c_\ell$ and $u_n \rightharpoonup 0$ in E_ε, we have that $u_n \to 0$ in E_ε for $\varepsilon > 0$ small.*

Lemma 4.4. *Under the assumptions of $(\mathcal{D}_0)$ and (f_1)–(f_3), we have that if $\{v_n\} \subset S_\varepsilon$ such that $\Upsilon_{\varepsilon,\lambda}(v_n) \to c$ and $\Upsilon'_{\varepsilon,\lambda}(v_n) \to 0$ with $0 < c \leqslant c_\ell < c_{V_\infty}$, then $\{v_n\}$ has a convergent subsequence in E_ε.*

Proof. Let $u_n = m_\varepsilon(v_n)$. It follows from lemmas 2.2 and 2.3 and

$$\Psi_\varepsilon(u_n) \to c, \qquad \Psi'_\varepsilon(u_n) \to 0 \quad \text{and} \quad \Psi'_\varepsilon(u_n)u_n = 0.$$

By using similar arguments as in the proof of lemma 3.6, one can easily check that $\{u_n\}$ is bounded. So, there exists $u \in E_\varepsilon$ such that $u_n \rightharpoonup u$ in E_ε. Moreover, u is a critical point of Ψ'_ε. Set $w_n = u_n - u$. By the Brezis–Lieb lemma (see [39]), we have that

$$\int_{\mathbb{R}^N} |\nabla w_n|^p = \int_{\mathbb{R}^N} |\nabla u_n|^p - \int_{\mathbb{R}^N} |\nabla u|^p + o(1)$$

and

$$\int_{\mathbb{R}^N} |w_n|^p = \int_{\mathbb{R}^N} |u_n|^p - \int_{\mathbb{R}^N} |u|^p + o(1).$$

Moreover, as in [22], it follows that $\Psi_\varepsilon(w_n) = \Psi_\varepsilon(u_n) - \Psi_\varepsilon(u) + o(1)$ and $\Psi'_\varepsilon(w_n) \to 0$ as $n \to \infty$. It follows from $\Psi'_\varepsilon(u) = 0$ and (1.8) that

$$\Psi_\varepsilon(u) = \Psi_\varepsilon(u) - \frac{1}{p}\Psi'_\varepsilon(u)u = \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u)u - F(u)\right) \geqslant 0.$$

So, we deduce that $\Psi_\varepsilon(w_n) = \Psi_\varepsilon(u_n) - \Psi_\varepsilon(u) + o(1) \to c - y$ as $n \to \infty$, where $y = \Psi_\varepsilon(u) \geqslant 0$. Thus, it follows from $c_1 = c - y \leqslant c \leqslant c_\ell$ and lemma 4.3 that $w_n = u_n - u \to 0$ in E_ε. Obviously, $u \in \mathcal{N}_\varepsilon$. Since $u_n = t_n v_n$ and t_n is bounded, $t_n \to t \neq 0$ (if $t = 0$, one can deduce that $u = 0$). Moreover, from the boundedness of $\{v_n\}$, we infer that there exists v such that $v_n \rightharpoonup v$ in E. So, it follows from $t_n \to t$ and $u_n \to u$ that $v_n \to v$ and $u = tv$. □

We are now in a position to prove that $(\mathcal{P}_\varepsilon)$ has a positive ground state solution.

Lemma 4.5. *Under the assumptions of* $(\mathcal{D}_0)$ *and* (f_1)*–*(f_3)*, we have that* c_ε *is attained for all small* $\varepsilon > 0$.

Proof. It follows from lemma 2.2(v) that $c_\varepsilon \geqslant \rho > 0$ for each $\varepsilon > 0$. Moreover, if $u_\varepsilon \in \mathcal{N}_\varepsilon$ satisfies $\Psi_\varepsilon(u_\varepsilon) = c_\varepsilon$, then $\check{m}_\varepsilon(u_\varepsilon)$ is a minimizer of Υ_ε, and therefore a critical point of Υ_ε, so u_ε is a critical point of Ψ_ε by lemma 2.3. It remains to show that there exists a minimizer u_ε of $\Psi_\varepsilon|_{\mathcal{N}_\varepsilon}$. By Ekeland's variational principle [39], there exists a sequence $\{\nu_n\} \subset S_\varepsilon$ such that $\Upsilon_\varepsilon(\nu_n) \to c_\varepsilon$ and $\Upsilon'_\varepsilon(\nu_n) \to 0$ as $n \to \infty$. Set $w_n = m_\varepsilon(\nu_n) \in \mathcal{N}_\varepsilon$ for all $n \in \mathbb{N}$. Then, from lemma 2.3 again, we deduce that $\Psi_\varepsilon(w_n) \to c_\varepsilon$, $\Psi'_\varepsilon(w_n)w_n = 0$ and $\Psi'_\varepsilon(w_n) \to 0$ as $n \to \infty$. So, $\{w_n\}$ is a $(\mathrm{PS})_{c_\varepsilon}$-sequence for Ψ_ε. By lemmas 4.1 and 4.2, we know that $c_\varepsilon \leqslant c_\ell$ for $\varepsilon > 0$ small. Thus, from the proof of lemma 4.4, we infer that $u_n = w_n - w \to 0$ in E_ε. Therefore, we prove that $w \in \mathcal{N}_\varepsilon$ and $\Psi_\varepsilon(w) = c_\varepsilon$. □

Let $\mathcal{L}_\varepsilon$ denote the set of all positive ground state solutions of $(\mathcal{P}_\varepsilon)$. Similarly to theorem 3.7(iii), one has the following lemma.

Lemma 4.6. *Suppose that the assumptions of theorem 1.1 are satisfied. Then* $\mathcal{L}_\varepsilon$ *is compact in* $W^{1,p}(\mathbb{R}^N)$ *for all small* $\varepsilon > 0$.

Proof. Let the boundedness sequence $\{u_n\} \subset \mathcal{L}_\varepsilon \cap \mathcal{N}_\varepsilon$ such that $\Psi_\varepsilon(u_n) = c_\xi$ and $\Psi'_\varepsilon(u_n) = 0$. Without loss of generality we assume that $u_n \rightharpoonup u \in E_\varepsilon$. It then follows from the weak continuity of Ψ'_ε that $\Psi'_\varepsilon(u) = 0$. Set $w_n = u_n - u$. As in lemma 4.5, we can prove that $w_n \to 0$ in $W^{1,p}(\mathbb{R}^N)$. □

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5. Multiplicity and concentration of positive solutions

In this section, we are in a position to give the proof of the main results. We first prove the existence of multiple positive solutions to $(\mathcal{P}_\varepsilon)$. To do this, as in [2, 5, 17], we make good use of the ground state solution of $\mathcal{P}_{V^0}$. Precisely, let w be a ground state solution of $\mathcal{P}_{V^0}$ and let Φ be a smooth non-increasing function defined in $[0,\infty)$ such that $\Phi(s)=1$ if $0\leqslant s\leqslant \frac{1}{2}$ and $\Phi(s)=0$ if $s\geqslant 1$. For any $y\in\mathcal{V}$, we define

$$\psi_{\varepsilon,y}(x)=\Phi(|\varepsilon x-y|)w\biggl(\frac{\varepsilon x-y}{\varepsilon}\biggr). \tag{5.1}$$

There then exists $t_\varepsilon>0$ such that $\max_{t\geqslant 0}\Psi_\varepsilon(t\psi_{\varepsilon,y})=\Psi_\varepsilon(t_\varepsilon\Phi_{\varepsilon,y})$. We define $\rho_\varepsilon\colon\mathcal{V}\to\mathcal{N}_\varepsilon$ by $\rho_\varepsilon(y)=t_\varepsilon\psi_{\varepsilon,y}$. By the construction, $\rho_\varepsilon(y)$ has a compact support for any $y\in\mathcal{V}$. As in [2, 17], one can easily prove the following results.

LEMMA 5.1. *Under the assumptions of* $(\mathcal{D}_0)$ *and* (f_1)*–*(f_3)*, we have that the function* ρ_ε *such that* $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\rho_\varepsilon(y))=c_{V_0}$.

For each $\delta>0$, let $\varrho=\varrho(\delta)$ be such that $\mathcal{V}_\delta\subset B_\varrho(0)$. Let $\chi\colon\mathbb{R}^N\to\mathbb{R}^N$ be defined by $\chi(x)=x$ for $|x|\leqslant\varrho$ and by $\chi(x)=\varrho x/|x|$ for $|x|\geqslant\varrho$. Finally, we define $\beta_\varepsilon\colon\mathcal{N}_\varepsilon\to\mathbb{R}$ by

$$\beta_\varepsilon(u)=\frac{\int_{\mathbb{R}^N}\chi(\varepsilon x)u^2\,\mathrm{d}x}{\int_{\mathbb{R}^N}u^2\,\mathrm{d}x}.$$

As in the proof of lemma 5.1, it is easy to see that

$$\begin{aligned}\beta_\varepsilon(\rho_\varepsilon(y))&=\frac{\int_{\mathbb{R}^N}\chi(\varepsilon x)\rho_\varepsilon^2(y)\,\mathrm{d}x}{\int_{\mathbb{R}^N}\rho_\varepsilon(y)^2\,\mathrm{d}x}\\&=\frac{\int_{\mathbb{R}^N}\chi(\varepsilon x+y)|w(x)\Phi(|\varepsilon x|)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N}|w(x)\Phi(|\varepsilon x|)|^2\,\mathrm{d}x}\\&=y+\frac{\int_{\mathbb{R}^N}(\chi(\varepsilon x+y)-y)|w(x)\Phi(|\varepsilon x|)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N}|w(x)\Phi(|\varepsilon x|)|^2\,\mathrm{d}x}\\&=y+o(1)\end{aligned}$$

as $\varepsilon\to 0$, uniformly for $y\in\mathcal{N}_\varepsilon$. So we conclude that $\lim_{\varepsilon\to 0}\beta_\varepsilon(\rho_\varepsilon(y))=y$ uniformly for $y\in\mathcal{N}_\varepsilon$.

Next we prove some concentration phenomena for the positive ground state solutions of $(\mathcal{P}_\varepsilon)$. Before doing so, we start with the following preliminary lemma.

LEMMA 5.2. *Suppose that the assumptions of theorem 1.1 are satisfied. Let* $u_n\subset\mathcal{N}_{V_0}$ *be a sequence satisfying* $\Psi_{V_0}(u_n)\to c_{V_0}$*. Then, either* $\{u_n\}$ *has a subsequence strongly convergent in* $W^{1,p}(\mathbb{R}^N)$ *or there exists* $\{y_n\}\subset\mathbb{R}^N$ *such that the sequence* $w_n(x)=u_n(x+y_n)$ *converges strongly in* $W^{1,p}(\mathbb{R}^N)$*. In particular, there exists a minimizer of* c_{V_0}.

Proof. By lemma 3.2, we know that $\{u_n\}$ is a bounded sequence. Moreover, it follows that

$$\Psi_{V_0}(u_n)\to c_{V_0}\quad\text{and}\quad\Psi_{V_0}'(u_n)u_n=0. \tag{5.2}$$

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Hence, for some subsequence, still denoted by $\{u_n\}$, we may assume that there exists a $u \in W^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$.

(h$_1$) If $u \neq 0$, it follows that $u \in \mathcal{N}_{V_0}$. Thus, in the same way as in the proof of lemma 4.5, we can prove that $u_n \to u$ in E.

(h$_2$) If $u = 0$, as in lemma 3.2, we have that there exist $\{y_n\} \subset \mathbb{R}^N$, $r, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} u_n^2 \geqslant \delta. \tag{5.3}$$

We set $w_n(x) = u_n(x + y_n)$; then $\|w_n\|_{V_0} = \|u_n\|_{V_0}$, $\Psi_{V_0}(w_n) \to c_{V_0}$ and $\Psi'_{V_0}(w_n)w_n = 0$. It is clear that there exists $w \in W^{1,p}(\mathbb{R}^N)$ with $w \neq 0$ such that $w_n \rightharpoonup w$ in $W^{1,p}(\mathbb{R}^N)$. The proof then follows from the arguments used in the case of $u \neq 0$.

□

Lemma 5.3. *Let u_ε be the positive ground state solutions of ($\mathcal{P}_\varepsilon$) and let $0 \in \mathcal{V} = \{x \in \mathbb{R}^N : M(x) = V_0\}$. Under the assumptions of theorem 1.1, u_ε has a maximum point y_ε such that* $\mathrm{dist}(\varepsilon y_\varepsilon, \mathcal{V}) \to 0$. *Moreover, $v_\varepsilon(x) = u_\varepsilon(x + y_\varepsilon)$ converges in $W^{1,p}(\mathbb{R}^N)$ to a positive ground state solution of $\mathcal{P}_{V^0}$ as $\varepsilon \to 0$.*

Proof. Let $\varepsilon_j \to 0$, $u_j \in \mathcal{L}_{\varepsilon_j}$ such that $\Psi_{\varepsilon_j}(u_j) = c_{\varepsilon_j}$ and $\Psi'_{\varepsilon_j}(u_j) = 0$. Clearly, $\{u_j\} \subset \mathcal{N}_{\varepsilon_j}$. Using the same arguments as in lemma 4.4, one can easily check that $\{u_j\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. So we can assume that $u_j \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$. Moreover, since $\Psi_{\varepsilon_j}(u_j) = c_{\varepsilon_j} \to c_{V_0}$ as $j \to \infty$ according to lemma 4.1, then we have $c_{\varepsilon_j} \leqslant c_{V_\infty}$ for j large. Thus, similarly to the proof of lemma 4.4, we can prove that there exist $r, \delta > 0$ and a sequence $\{y'_j\} \subset \mathbb{R}^N$ such that

$$\liminf_{j\to\infty} \int_{B_r(y'_j)} u_j^p \geqslant \delta > 0. \tag{5.4}$$

Let $\{y_j\} \subset \mathbb{R}^N$ be such that

$$u_j(y_j) = \max_{y\in\mathbb{R}^N} u_j(y) \quad \forall j.$$

We claim that there exists $\kappa > 0$ (independent of j) such that

$$u_j(y_j) \geqslant \kappa > 0 \quad \text{uniformly for all } j \in \mathbb{N}. \tag{5.5}$$

Assume by contradiction that $u_j(y_j) \to 0$ as $j \to \infty$. We deduce from (5.4) that

$$0 < \delta \leqslant \int_{B_r(y'_j)} u_j^p \leqslant c u_j(y_j)^p \to 0 \quad \text{as } j \to \infty.$$

This is a contradiction. As in theorem 3.7, one can easily check that $u_j \in C^{1,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for each $j \in \mathbb{N}$. So it follows from (5.4) and (5.5) that there exist $R > r > 0$ and $\delta' > 0$ such that

$$\liminf_{j\to\infty} \int_{B_R(y_j)} |u_j|^p \geqslant \delta' > 0.$$

Set

$$v_j(x) = u_j(x+y_j) \quad \text{and} \quad \hat{V}_{\varepsilon_j}(x) = V(\varepsilon_j(x+y_j)).$$

Then, along a subsequence we have $v_j \rightharpoonup v \neq 0$ in $W^{1,p}(\mathbb{R}^N)$ and $v_j \to v$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$ (for all $t \in (p, Np/(N-p))$). We first claim that $v_j \to v \neq 0$ in $W^{1,p}(\mathbb{R}^N)$. In fact, according to lemma 3.2, we choose $t_j > 0$ such that $m_{V_0}(v_j) = t_j v_j \in \mathcal{N}_{V_0}$. Set $\tilde{v}_j = t_j v_j$. It follows from $(\mathcal{D}_0)$, $u_j \in \mathcal{N}_{\varepsilon_j}$ and lemma 4.1 that

$$\begin{aligned}
\Psi_{V_0}(\tilde{v}_j) &\leqslant \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla \tilde{v}_j|^p + \hat{V}_{\varepsilon_j}(x)|\tilde{v}_j|^p) - \int_{\mathbb{R}^N} F(\tilde{v}_j) - \frac{1}{p^*}\int_{\mathbb{R}^N}\tilde{v}_j^{p^*} \\
&= \Psi_{\varepsilon_j}(t_j u_j) \\
&\leqslant \Psi_{\varepsilon_j}(u_j) \\
&= c_{V_0} + o(1).
\end{aligned}$$

Note that $\Psi_{V_0}(\tilde{v}_j) \geqslant c_{V_0}$, and thus $\lim_{j\to\infty}\Psi_{V_0}(\tilde{v}_j) = c_{V_0}$. From lemma 3.2(vi), we infer that t_j is bounded. Without loss of generality we can assume that $t_j \to t \geqslant 0$. If $t = 0$, we have that $\tilde{v}_j = t_j v_j \to 0$ in view of the boundedness of v_j, and hence $\Psi_{V_0}(\tilde{v}_j) \to 0$ as $j \to \infty$, which contradicts $c_{V_0} > 0$. So, $t > 0$ and the weak limit of $\tilde{v}_j$ is different from 0. Let $\tilde{v}$ be the weak limit of $\tilde{v}_j$ in $W^{1,p}(\mathbb{R}^N)$. Since $t_n \to t > 0$ and $v_n \rightharpoonup v \neq 0$, we have, from the uniqueness of the weak limit, that $\tilde{v} = tv \neq 0$ and $\tilde{v} \in \mathcal{N}_{V_0}$. From lemma 5.2, $\tilde{v}_j \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N)$, and so $v_j \to v$ in $W^{1,p}(\mathbb{R}^N)$. This proves the claim for $v_j \to v \neq 0$ in $W^{1,p}(\mathbb{R}^N)$.

Obviously, v_j solves

$$-\operatorname{div}(|\nabla v_j|^{p-2}\nabla v_j) + \hat{M}_j(x)v_j = f(v_j) + |v_j|^{p^*-2}v_j \quad \text{in } \mathbb{R}^N. \qquad (\mathcal{P}^v_\varepsilon)$$

Correspondingly, the energy functional is defined as

$$P_{\varepsilon_j}(v_j) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v_j| + \hat{M}_j(x)v_j) - \int_{\mathbb{R}^N} F(v_j) + \int_{\mathbb{R}^N}|v_j|^{p^*} = \Psi_{\varepsilon_j}(u_j) = c_{\varepsilon_j}.$$

We next show that $\{\varepsilon_j y_j\}$ is bounded. To do this we borrow an idea of [16]. Assume by contradiction that $\varepsilon_j|y_j| \to \infty$. Without loss of generality assume that $V(\varepsilon_j y_j) \to \tilde{V}^\infty$. Clearly, $V_0 < \tilde{V}^\infty$ by $(\mathcal{D}_0)$. For each $\varphi \in W^{1,p}(\mathbb{R}^N)$, as in [3], one can easily derive that

$$\begin{aligned}
\lim_{j\to\infty}\int_{\mathbb{R}^N} f(v_j)\varphi &= \int_{\mathbb{R}^N} f(v)\varphi, \\
\lim_{j\to\infty}\int_{\mathbb{R}^N}\hat{V}_{\varepsilon_j}(x)|v_j|^{p-2}v_j\varphi &= \int_{\mathbb{R}^N}\tilde{V}^\infty|v|^{p-2}v\varphi, \\
\lim_{j\to\infty}\int_{\mathbb{R}^N}|v_j|^{p^*-2}v_j\varphi &= \int_{\mathbb{R}^N}|v|^{p^*-2}v\varphi.
\end{aligned}$$

Moreover, we claim that

$$\int_{\mathbb{R}^N}|\nabla v_j|^{p-2}\nabla v_j\nabla\varphi \to \int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla\varphi \quad \text{as } n \to \infty.$$

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Indeed, by the Hölder inequality, we deduce that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}|\nabla v_j|^{p-2}\nabla v_j\nabla\varphi-\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla\varphi\right| \\ &=\left|\int_{\mathbb{R}^N}(|\nabla v_j|^{p-2}-|\nabla v|^{p-2})\nabla v\nabla\varphi+\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla\varphi(\nabla v_j-\nabla v)\right| \\ &\leqslant\left(\int_{\mathbb{R}^N}(|\nabla v_j|^{p-2}-|\nabla v|^{p-2})^{p/(p-2)}\right)^{(p-2)/p}|\nabla v|_p|\nabla\varphi|_p \\ &\quad+|\nabla v_j-\nabla v|_p|\nabla\varphi|_p|\nabla v_j-\nabla v|_p^{p-2}. \qquad (5.6)\end{aligned}$$

Since $p/(p-2)>1$, we infer from the Brezis–Lieb lemma (see [39]) and $v_j\to v$ in $L^p(\mathbb{R}^N)$ that

$$\left(\int_{\mathbb{R}^N}(|\nabla v_j|^{p-2}-|\nabla v|^{p-2})^{p/(p-2)}\right)^{(p-2)/p}\to 0 \quad\text{and}$$
$$|\nabla v_j-\nabla v|_p|\nabla\varphi|_p|\nabla v_j-\nabla v|_p^{p-2}\to 0 \quad\text{as } j\to\infty. \qquad (5.7)$$

Combining (5.6) and (5.7) we derive that

$$\int_{\mathbb{R}^N}|\nabla v_j|^{p-2}\nabla v_j\nabla\varphi\to\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla\varphi \quad\text{as } n\to\infty.$$

So it follows that

$$\lim_{j\to\infty}P'_{\varepsilon_j}(v_j)\varphi=\int_{\mathbb{R}^N}(|\nabla v|^{p-2}\nabla v\nabla\varphi+\tilde V^\infty|v|^{p-2}v\varphi)-\int_{\mathbb{R}^N}f(v)\varphi=0.$$

Thus, v solves

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v)+\tilde V^\infty v=f(v)+|v|^{p^*-2}v \quad\text{in } \mathbb{R}^N. \qquad (\mathcal{P}_{\tilde V^\infty})$$

We denote the energy functional by

$$P_\infty(v)=\frac{1}{p}\int_{\mathbb{R}^N}|\nabla v|^p+\tilde V^\infty v^p-\int_{\mathbb{R}^N}F(v)\geqslant c_{\tilde V^\infty}.$$

Remark that, since $V_0<\tilde V^\infty$, one has $c_{\tilde V^\infty}>c_{V_0}$ by lemma 3.6. Moreover, since $P'_{\varepsilon_j}(v_j)v_j=\Psi'_{\varepsilon_j}(u_j)u_j=0$, it follows from Fatou's lemma and (1.8) that

$$\begin{aligned}\lim_{j\to\infty}c_{\varepsilon_j}&=\lim_{j\to\infty}P_{\varepsilon_j}(v_j)\\ &=\lim_{j\to\infty}\left(P_{\varepsilon_j}(v_j)(v_j)-\frac{1}{p}P_{\varepsilon_j}(v_j)'(v_j)v_j\right)\\ &=\liminf_{j\to\infty}\left[\int_{\mathbb{R}^N}\left(\frac{1}{p}f(v_j)v_j-F(v_j)\right)+\frac{p^*-p}{pp^*}\int_{\mathbb{R}^N}|v_j|^{p^*}\right]\\ &\geqslant\left[\int_{\mathbb{R}^N}\left(\frac{1}{p}f(v)v-F(v)\right)+\frac{p^*-p}{pp^*}\int_{\mathbb{R}^N}|v_j|^{p^*}\right]\\ &=P_\infty(v). \qquad (5.8)\end{aligned}$$

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Consequently, we infer from (5.8) that

$$c_{V_0} < c_{\tilde{V}\infty} \leqslant P_\infty(v) \leqslant \lim_{j\to\infty} c_{\varepsilon_j} = c_{V_0},$$

a contradiction. Thus, $\{\varepsilon_j y_j\}$ is bounded. Hence, we can assume that $x_j = \varepsilon_j y_j \to x_0$. Then v solves

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) + V(x_0)|v|^{p-2}v = f(v) + |v|^{p^*-2}v \quad \text{in } \mathbb{R}^N. \tag{$\mathcal{P}_{V^0}$}$$

It follows from $V(x_0) \geqslant V_0$ that

$$P_0(v) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x_0)|v|^p) - \int_{\mathbb{R}^N} F(v) - \frac{1}{p^*}\int_{\mathbb{R}^N}|v|^{p^*} \geqslant c_{V(x_0)} \geqslant c_{V_0}.$$

Similarly to (5.8), one gets

$$c_{V_0} = \lim_{j\to\infty} c_{\varepsilon_j} \geqslant P_0(v) \geqslant c_{V_0}.$$

This implies that $P_0(v) = c_{V_0}$, and hence $V(x_0) = V_0$. So, by lemma 4.1, $x_0 \in \mathcal{V}$. □

We now study the exponential decay for the ground state solution.

Lemma 5.4. *Suppose that u_ε is a positive ground state solution of ($\mathcal{P}_\varepsilon$) for sufficiently small $\varepsilon > 0$. Then, under the assumptions of theorem 1.1, we have that $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$ and $u_\varepsilon \in C^{1,\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$ for $\sigma \in (0,1)$. Furthermore, there exist $C, c > 0$ such that $u_\varepsilon(x) \leqslant C\mathrm{e}^{-c|x-y_\varepsilon|}$, where $u_\varepsilon(y_\varepsilon) = \max_{x\in\mathbb{R}^N} u_\varepsilon(x)$.*

Proof. As in the proof of theorem 3.7(ii), we know that, for each $\varepsilon > 0$ small, $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$ and $u_\varepsilon \in C^{1,\sigma}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for $\sigma \in (0,1)$. In the following, we prove the exponential decay for the positive solution of u_ε. Let $\varepsilon_j \to 0$ and let $u_j \in \mathcal{L}_{\varepsilon_j}$ such that $\Psi_{\varepsilon_j}(u_j) = c_{\varepsilon_j}$ and $\Psi'_{\varepsilon_j}(u_j) = 0$. As in the proof of lemma 5.3, we have that $v_j = u_j(x + y_j)$ such that

$$-\operatorname{div}(|\nabla v_j|^2\nabla v_j) + \hat{V}_{\varepsilon_j}(x)|v_j|^{p-2}v_j = f(v_j) + |v_j|^{p^*-2}v_j \quad \text{in } \mathbb{R}^N \tag{$\mathcal{P}^v_\varepsilon$}$$

and $v_j \to v \neq 0$ in $W^{1,p}(\mathbb{R}^N)$, where $u_j(y_j) = \max_{y\in\mathbb{R}^N} u_j(y)$.

Next we use the Moser iterative method (see [17, 26, 28]) to prove the regularity of the solution of ($\mathcal{P}^v_\varepsilon$). Set $\beta_n = p\rho^n$ and $\rho = N/(N-p)$. From above we know that $v_j \in L^{\beta_1}(\mathbb{R}^N)$. For the function $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$, we use the function $\psi = \eta^p v_j v_{l,j}^{k_n}$ as the test function in ($\mathcal{P}^v_\varepsilon$), where $k_n = p(\rho^n - 1)$ and $v_{l,j} = \min\{l, v_j\}$. Thus, it follows from (f_1) and (f_3) that, for each $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N}[|\nabla v_j|^{p-2}\nabla v_j\nabla\psi + \hat{V}_{\varepsilon_j}(x)v_j^{p-1}\psi] &= \int_{\mathbb{R}^N} f(v_j)\psi + |v_j|^{p^*-2}v_j\psi \\ &\leqslant \int_{\mathbb{R}^N}[\epsilon v_j^{p-1} + C_\epsilon v_j^{p^*-1}]\psi,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N}|\nabla v_j|^{p-2}\nabla v_j\nabla\psi \leqslant C_\epsilon\int_{\mathbb{R}^N} v_j^{p^*-1}\psi.$$

A direct computation shows that

$$\int_{\mathbb{R}^N} |\nabla v_j|^p \eta^p v_{l,j}^{k_n} + k_n \int_{\mathbb{R}^N} \eta^p v_j v_{l,j}^{k_n-1} |\nabla v_j|^{p-2} \nabla v_j \nabla v_{l,j}$$
$$\leqslant -p \int_{\mathbb{R}^N} \eta^{p-1} v_j v_{l,j}^{k_n} |\nabla v_j|^{p-2} \nabla v_j \nabla \eta + C_\epsilon \int_{\mathbb{R}^N} v_j^{p^*} v_{l,j}^{k_n} \eta^p. \quad (5.9)$$

We deduce from Young's inequality that

$$\bigg| \int_{\mathbb{R}^N} \eta^{p-1} v_j v_{l,j}^{k_n} |\nabla v_j|^{p-2} \nabla v_j \nabla \eta \bigg|$$
$$\leqslant \frac{(p-1)\epsilon^{p/(p-1)}}{p} \int_{\mathbb{R}^N} \eta^p v_{l,j}^{k_n} |\nabla v_j|^p + \frac{1}{p\epsilon^p} \int_{\mathbb{R}^N} \nabla v_j^p v_{l,j}^{k_n} |\nabla \eta|^p. \quad (5.10)$$

On the other hand, we infer from the Gagliardo–Nirenberg–Sobolev inequality that

$$\begin{aligned} |\eta v_j v_{l,j}^{k_n/p}|_{L^{p^*}}^p &= \bigg(\int_{\mathbb{R}^N} (\eta v_j v_{l,j}^{k_n/p})^{p^*} \bigg)^{p/p^*} \\ &\leqslant M \bigg(\int_{\mathbb{R}^N} |\nabla \eta|^p v_j^p v_{l,j}^{k_n} + \int_{\mathbb{R}^N} \eta^p v_{l,j}^{k_n} |\nabla v_j|^p \bigg) \\ &\quad + \bigg(\frac{k_n}{p} \bigg)^p \int_{\mathbb{R}^N} \eta^p v_j^p v_{l,j}^{k_n-p} |\nabla v_{l,j}|^p, \end{aligned} \quad (5.11)$$

where the constant $M = M(N, p, \epsilon)$. Moreover, since

$$\int_{\mathbb{R}^N} \eta^p v_j^p v_{l,j}^{k_n-p} |\nabla v_{l,j}|^p \leqslant \int_{\mathbb{R}^N} \eta^p v_j v_{l,j}^{k_n-1} |\nabla v_j|^{p-2} \nabla v_j \nabla v_{l,j},$$

it follows from (5.9)–(5.11) that

$$|\eta v_j v_{l,j}^{k_n/p}|_{L^{p^*}}^p \leqslant M \rho^{p(n-1)} \bigg(\int_{\mathbb{R}^N} |\nabla \eta|^p v_j^p v_{l,j}^{k_n} + \int_{\mathbb{R}^N} v_j^{p^*} v_{l,j}^{k_n} \eta^p \bigg).$$

To obtain the estimate for $|v_j|_{L^{\beta_{n+1}}(|x| \geqslant R)}$ for some large $R > 0$, we define the function $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\eta = 1$ if $|x| \geqslant R$, $\eta = 0$ if $|x| \leqslant R - r$ and $|\nabla \eta| \leqslant 1$. So, it follows from the Hölder inequality that

$$\int_{\mathbb{R}^N} \eta^p v_j^{p^*} v_{l,j}^{k_n} \leqslant |\eta v_j v_{l,j}^{k_n/p}|_{L^{p^*}}^p |v_j|_{L^{p^*}(|x| \geqslant R/2)}^{p^*-p}.$$

Therefore, we obtain that

$$|\eta v_j v_{l,j}^{k_n/p}|_{L^{p^*}}^p \leqslant M \rho^{p(n-1)} (||\nabla \eta| v_j v_{l,j}^{k_n/p}|_{L^p}^p + |\eta v_j v_{l,j}^{k_n/p}|_{L^{p^*}}^p |v_j|_{L^{p^*}(|x| \geqslant R/2)}^{p^*-p}).$$

Since $v_j \to v$ in $W^{1,p}(\mathbb{R}^N)$, we can take R large enough such that

$$M \rho^{p(n-1)} |v_j|_{L^{p^*}(|x| \geqslant R/2)}^{p^*-p} \leqslant 1 \quad \text{for all } j.$$

Thus, we get that

$$\begin{aligned}|\eta v_j v_{l,j}^{k_n/p}|_{L^{p^*}(|x|\geqslant R)}^p &\leqslant |\eta v_j v_{l,j}^{k_n/p}|_{L^{p^*}}^p \\ &\leqslant M\rho^{p(n-1)}||\nabla\eta| v_j v_{l,j}^{k_n/p}|_{L^p}^p \\ &= M\rho^{p(n-1)}\int_{\mathbb{R}^N}|\nabla\eta|^p v_j^p v_{l,j}^{k_n} \\ &\leqslant M\rho^{p(n-1)}\int_{|x|\geqslant R/2} v_j^{\beta_n},\end{aligned}$$

where $M = M(N, p, \epsilon, R)$. Therefore, letting $l \to \infty$, by the dominated convergence theorem, one has that

$$|v_j|_{L^{\beta_{n+1}}(|x|\geqslant R)} \leqslant M^{1/\beta_n}\rho^{p(n-1)/\beta_n}|v_j|_{L^{\beta_n}(|x|\geqslant R/2)} \quad \forall j.$$

Interaction yields that

$$|v_j|_{L^{\beta_{n+1}}(|x|\geqslant R)} \leqslant M^{\sum 1/\beta_n}\rho^{\sum p(n-1)/\beta_n}|v_j|_{L^{\beta_1}(|x|\geqslant R/2)} \quad \forall j.$$

By the convergence of $\{v_j\}$ to v in $W^{1,p}(\mathbb{R}^N)$, we know that, for each $\tau > 0$, there exists $R > 0$ such that

$$|v_j|_{L^\infty(|x|\geqslant R)} < \tau.$$

Thus, we prove that

$$\lim_{|x|\to\infty} v_j(x) = 0 \quad \text{uniformly for all } j \in \mathbb{N}.$$

From this we deduce that there exists $\varepsilon_0 > 0$ such that

$$\lim_{|x|\to\infty} v_\varepsilon(x) = 0 \quad \text{uniformly for all } \varepsilon \in (0, \varepsilon_0].$$

So, by using the same arguments as in the proof of theorem 3.7(ii), we know that there exist $C, \delta > 0$ (independent of ε) such that

$$v_\varepsilon(x) \leqslant C\mathrm{e}^{-\delta|x|},$$

where $v_\varepsilon = u_\varepsilon(x + y_\varepsilon)$ and $u_\varepsilon(y_\varepsilon) = \max_{y\in\mathbb{R}^N} u_\varepsilon$. Thus, the conclusions of this lemma hold. □

To prove the concentration phenomenon for the positive solutions of $(\mathcal{P}_\varepsilon)$, we need the following results, which are due to [2, 17].

Lemma 5.5. *Under the assumptions of theorem 1.1 or theorem 1.2, if $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ such that $\Psi_{\varepsilon_n}(u_n) \to c_{V_0}$, then there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\tilde{y}_n = \varepsilon_n y_n \to y \in \mathcal{V}$.*

Let $\alpha(\varepsilon)$ be any positive function tending to 0 as $\varepsilon \to 0$, and let

$$\mathscr{D}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : \Psi_\varepsilon(u) \leqslant c_{V_0} + \alpha(\varepsilon)\}.$$

For any $y \in \mathcal{V}$, we deduce from lemma 5.1 that $\alpha(\varepsilon) = |\Psi_\varepsilon(\rho_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$. Thus, $\rho_\varepsilon(y) \in \mathscr{D}_\varepsilon$ and $\mathscr{D}_\varepsilon \neq \emptyset$ for $\varepsilon > 0$. By the same argument as in [2, lemma 4.4], we can obtain the following property on $\mathscr{D}_\varepsilon$.

LEMMA 5.6. *Suppose that the assumptions of theorem 1.1 or theorem 1.2 are satisfied. Then, for any* $\delta > 0$, *there holds that* $\lim_{\varepsilon\to 0}\sup_{u\in\mathscr{D}_\varepsilon}\operatorname{dist}(\beta_\varepsilon(u),\mathcal{V}_\delta) = 0$.

LEMMA 5.7. *Suppose that the assumptions of theorem 1.1 or theorem 1.2 are satisfied. Assume that* u_n *satisfies* $\Psi_{\varepsilon_n}(u_n) \to c_{V_0}$ *and there exist* $r, \delta > 0$ *and a sequence* $\{y_n\} \subset \mathbb{R}^N$ *such that* $\liminf_{n\to\infty}\int_{B_r(y_n)} u_n^p \geqslant \delta > 0$; *moreover, assume that* $v_n(x) = u_n(x + y_n)$ *satisfies the problem*

$$-\operatorname{div}(|\nabla v_n|^2\nabla v_n) + \hat{V}_{\varepsilon_n}(x)|v_n|^{p-2}v_n = f(v_n) + |v_n|^{p^*-2}v_n \quad \text{in } \mathbb{R}^N, \qquad (\mathcal{P}_\varepsilon^*)$$

where $\hat{V}_{\varepsilon_n}(x) = V(\varepsilon_n x + \varepsilon_n y_n)$ *and* y_n *is given in lemma 5.3. We then have that* $v_n \to v$ *in* $W^{1,p}(\mathbb{R}^N)$ *with* $v \neq 0$, $v_n \in L^\infty(\mathbb{R}^N)$ *and* $\|v_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C$ *for all* $n \in \mathbb{N}$. *Furthermore,* $\lim_{|x|\to\infty} v_n(x) = 0$ *uniformly for* $n \in \mathbb{N}$ *and* $v_n(x) \leqslant c\mathrm{e}^{-c|x-y_n|}$.

Proof. Since v_n satisfies $(\mathcal{P}_\varepsilon^*)$, we know that $\Psi'_{\varepsilon_n}(v_n) = 0$. Moreover, $\Psi_{\varepsilon_n}(u_n) \to c_{V_0}$. Using the same arguments as in lemma 5.4, one can obtain the conclusion of this lemma. We omit the details here. □

Proof of theorem 1.1. Go back to $(\mathrm{LP})_\varepsilon$ with the variable substitution $x \mapsto x/\varepsilon$. Lemma 4.5 implies that $(\mathrm{LP})_\varepsilon$ has at least one positive ground state solution $u_\varepsilon \in W^{1,p}(\mathbb{R}^N)$ for all $\varepsilon > 0$ small. The conclusions (ii) and (iii) follow from lemmas 4.6 and 5.3, respectively. Finally, it follows from lemma 5.4 that the conclusion (iv) of theorem 1.1 holds. □

LEMMA 5.8. *Under the assumptions of theorem 1.2,* $(\mathcal{P}_\varepsilon)$ *has at least* $\operatorname{cat}_{\mathcal{V}_\delta}(\mathcal{V})$ *positive solutions for sufficiently small* $\varepsilon > 0$.

Proof. To prove $(\mathcal{P}_\varepsilon)$ has at least $\operatorname{cat}_{\mathcal{V}_\delta}(\mathcal{V})$ positive solutions, since $\mathcal{N}_\varepsilon$ is not a C^1-submanifold of E_ε, we cannot apply the category theorem directly. Fortunately, from lemma 2.2, we know that the mapping m_ε is a homeomorphism between $\mathcal{N}_\varepsilon$ and S_ε, and S_ε is a C^1-submanifold of E_ε. So we can apply this theorem to $\Upsilon_\varepsilon(w) = \Psi_\varepsilon(\hat{m}_\varepsilon(w))|_{S_\varepsilon} = \Psi_\varepsilon(m_\varepsilon(w))$, where Υ_ε is given in lemma 2.3.

Define

$$\mu_{\varepsilon,1}(y) = m_\varepsilon^{-1}(t_\varepsilon\psi_{\varepsilon,y}) = m_\varepsilon^{-1}(\rho_\varepsilon(y)) = \frac{t_\varepsilon\psi_{\varepsilon,y}}{\|t_\varepsilon\psi_{\varepsilon,y}\|} = \frac{\psi_{\varepsilon,y}}{\|\psi_{\varepsilon,y}\|}$$

for $y \in \mathcal{V}$. It follows from lemma 5.1 that

$$\lim_{\varepsilon\to 0}\Upsilon_\varepsilon(\mu_{\varepsilon,1}(y)) = \lim_{\varepsilon\to 0}\Psi_\varepsilon(\rho_\varepsilon(y)) = c_{V_0}. \qquad (5.12)$$

Furthermore, we set

$$\mathscr{D}_{\varepsilon,1} := \{w \in S_\varepsilon : \Upsilon_\varepsilon(w) \leqslant c_{V_0} + \alpha(\varepsilon)\}, \qquad (5.13)$$

where $\alpha(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. It follows from (5.12) that $\alpha(\varepsilon) = |\Upsilon_\varepsilon(\mu_{\varepsilon,1}(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$. Thus, $\mu_{\varepsilon,1}(y) \in \mathscr{D}_{\varepsilon,1}$ and $\mathscr{D}_{\varepsilon,1} \neq \emptyset$ for any $\varepsilon > 0$. Recall that $\mathscr{D}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : \Psi_\varepsilon(u) \leqslant c_{V_0} + \alpha(\varepsilon)\}$. From lemmas 2.2, 2.3, 5.1 and 5.6, we know that, for any $\varepsilon > 0$ sufficiently small, the diagram

$$\mathcal{V} \xrightarrow{\rho_\varepsilon} \mathscr{D}_\varepsilon \xrightarrow{m_\varepsilon^{-1}} \mathscr{D}_{\varepsilon,1} \xrightarrow{m_\varepsilon} \mathscr{D}_\varepsilon \xrightarrow{\beta_\varepsilon} \mathcal{V}_\delta \qquad (5.14)$$

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is well defined. By the arguments in the paragraph just before lemma 5.2, we see that

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\rho_\varepsilon(y)) = y \quad \text{uniformly in } y \in \mathcal{V}. \tag{5.15}$$

For $\varepsilon > 0$ small enough, we define $\beta_\varepsilon(\rho_\varepsilon(y)) = y+\lambda(y)$ for $y \in \mathcal{V}$, where $|\lambda(y)| < \delta/2$ uniformly for $y \in \mathcal{V}$. Define $H(t, y) = y + (1 - t)\lambda(y)$. Then, $H\colon [0, 1] \times \mathcal{V} \to \mathcal{V}_\delta$ is continuous. Obviously, $H(0, y) = \beta_\varepsilon(\rho_\varepsilon(y))$, $H(1, y) = y$ for all $y \in \mathcal{V}$. Let $\xi_{\varepsilon,1} = m_\varepsilon^{-1} \circ \rho_\varepsilon$ and $\beta_{\varepsilon,1} = \beta_\varepsilon \circ m_\varepsilon$. Thus, we obtain that the composite mapping $\beta_{\varepsilon,1} \circ \xi_{\varepsilon,1} = \beta_\varepsilon \circ \rho_\varepsilon$ is homotopic to the inclusion mapping $\mathrm{id}\colon \mathcal{V} \to \mathcal{V}_\delta$. So it follows from [11, lemma 2.2] that

$$\operatorname{cat}_{\mathscr{D}_{\varepsilon,1}}(\mathscr{D}_{\varepsilon,1}) \geqslant \operatorname{cat}_{\mathcal{V}_\delta}(\mathcal{V}). \tag{5.16}$$

On the other hand, let us choose a function $\alpha(\varepsilon) > 0$ such that $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $(c_{V_0} + \alpha(\varepsilon))$ is not a critical level for $\varUpsilon_\varepsilon$. For $\varepsilon > 0$ small enough, we deduce from lemma 4.5 that $\varUpsilon_\varepsilon$ satisfies the Palais–Smale condition in $\mathscr{D}_{\varepsilon,1}$. So, it follows from [11, theorem 2.1] that $\varUpsilon_\varepsilon$ has at least $\operatorname{cat}_{\mathscr{D}_{\varepsilon,1}}(\mathscr{D}_{\varepsilon,1})$ critical points on $\mathscr{D}_{\varepsilon,1}$. By lemma 2.3(iii), we conclude that $\varPsi_\varepsilon$ has at least $\operatorname{cat}_{\mathcal{V}_\delta}(\mathcal{V})$ critical points. □

Proof of theorem 1.2. From the above arguments we know that $(\mathcal{P}_\varepsilon)$ has at least $\operatorname{cat}_{\mathcal{V}_\delta}(\mathcal{V})$ positive solutions. Go back to $(\mathrm{LP})_\varepsilon$ with the variable substitution $x \mapsto x/\varepsilon$. We obtain that $(\mathrm{LP})_\varepsilon$ has at least $\operatorname{cat}_{\mathcal{V}_\delta}(\mathcal{V})$ positive solutions. In the following we prove the concentration phenomena for positive solutions. Let u_{ε_n} denote a positive solution of $(LP)_\varepsilon$. Then, $v_n(x) = u_n(x + y_n)$ is a solution of the problem

$$-\operatorname{div}(|\nabla v_n|^{p-2}\nabla v_n) + \hat{V}_{\varepsilon_n}(x)|v_n|^{p-2}v_n = f(v_n) + |v_n|^{p^*-2}v_n \quad \text{in } \mathbb{R}^N,$$

where $\hat{V}_{\varepsilon_n}(x) = V(\varepsilon_n x + \varepsilon_n y_n)$ and y_n is given in lemma 5.5. Furthermore, up to a subsequence, it follows from lemma 5.5 that $v_n \to v$ and $\tilde{y}_n = \varepsilon_n y_n \to y \in \mathcal{V}$. As in [17, lemma 4.5], we have that there exists a $\delta > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \geqslant \delta > 0$. Let ν_n be the global maximum of v_n; we infer from lemma 5.7 and the claim above that $\{\nu_n\} \subset B_R(0)$ for some $R > 0$. Thus, the global maximum of u_{ε_n} given by $z_n = y_n + \nu_n$ satisfies $\varepsilon_n z_n = \tilde{y}_n + \varepsilon_n\nu_n$. Since $\{\nu_n\}$ is bounded, it follows that $\varepsilon_n z_n \to y \in \mathcal{V}$. Moreover, since the function $h_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a positive solution of $(\mathrm{LP})_\varepsilon$, the maximum points σ_ε and z_ε of h_ε and u_ε, respectively, satisfy the equality $\sigma_\varepsilon = \varepsilon z_\varepsilon$. So, we have that

$$\lim_{\varepsilon\to 0} V(\sigma_\varepsilon) = \lim_{n\to\infty} V(\varepsilon_n z_n) = V_0.$$

Finally, from the above arguments and lemma 5.7, it follows from the boundedness of $\{\nu_n\}$ that $u_n(x) \leqslant c\mathrm{e}^{-c|x-z_n+\nu_n|} \leqslant c\mathrm{e}^{-c|x-z_n|}$. So, we conclude that u_ε satisfies theorem 1.2(ii). □

Proof of theorem 1.3. We use the idea of [37, 38] to prove this conclusion, since, for each $\varepsilon > 0$, we have $E = W^{1,p}(\mathbb{R}^N) = E_\varepsilon$. Therefore, to prove the conclusion, we first claim that $c_\varepsilon = c_{V^\infty}$ for each $\varepsilon > 0$. In fact, as in lemma 4.2, since $V(x) \leqslant V^\infty$, one can easily check that $c_\varepsilon \geqslant c_{V^\infty}$. So, in order to prove $c_{V^\infty} = c_\varepsilon$, it suffices to show that

$$c_{V^\infty} \leqslant c_\varepsilon. \tag{5.17}$$

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By theorem 3.7, we know that there exist $e \in S_{V^\infty} = \{u \in W^{1,p}(\mathbb{R}^N) \colon \|u\|_{V^\infty} = 1\}$ and $s_0 > 0$ such that $u_0 = m_{V^\infty}(e) = s_0 e$ is a positive ground state solution of $(\mathcal{P}_{V^\infty})$. Moreover, $m_{V^\infty}(e)$ is the unique global maximum of Ψ_{V^∞} on E. Set $w_n = e(\cdot - y_n)$, where $y_n \in \mathbb{R}^N$ and $|y_n| \to \infty$ as $n \to \infty$. Then, by lemma 2.2, it follows that, for each n, $m_\varepsilon(w_n) = \hat{m}_\varepsilon(w_n) \in \mathcal{N}_\varepsilon$ is the unique global maximum of Ψ_ε on E. Therefore, we get

$$\begin{aligned}
c_\varepsilon &\leqslant \Psi_\varepsilon(m_\varepsilon(w_n)) \\
&= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla m_\varepsilon(w_n)|^p + V_\varepsilon(x)|m_\varepsilon(w_n)|^p) \\
&\qquad - \int_{\mathbb{R}^N} F(m_\varepsilon(w_n)) - \frac{1}{p^*}\int_{\mathbb{R}^N} |m_\varepsilon(w_n)|^{p^*} \\
&= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla m_\varepsilon(e)|^p + V(\varepsilon x + \varepsilon y_n)|m_\varepsilon(e)|^p) \\
&\qquad - \int_{\mathbb{R}^N} F(m_\varepsilon(e)) - \frac{1}{p^*}\int_{\mathbb{R}^N} |m_\varepsilon(e)|^{p^*} \\
&= \Psi_{V^\infty}(m_\varepsilon(e)) + \int_{\mathbb{R}^N} (V(\varepsilon x + \varepsilon y_n) - V^\infty) m_\varepsilon^p(e) \\
&\leqslant c_{V^\infty} + \int_{\mathbb{R}^N} (V(\varepsilon x + \varepsilon y_n) - V^\infty) m_\varepsilon^p(e).
\end{aligned} \tag{5.18}$$

It is clear that, for each $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{|x| \geqslant R} (V(\varepsilon x + \varepsilon y_n) - V^\infty)(m_\varepsilon^p(e)) \leqslant c\epsilon. \tag{5.19}$$

Moreover, we conclude from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\lim_{n\to\infty} \int_{|x|<R} &(V(\varepsilon x + \varepsilon y_n) - V^\infty) m_\varepsilon^p(e) \\
&= \int_{|x|<R} \Big(\lim_{n\to\infty} V(\varepsilon x + \varepsilon y_n) - V^\infty\Big) m_\varepsilon^p(e) \\
&\leqslant \int_{|x|<R} \Big(\limsup_{n\to\infty} V(\varepsilon x + \varepsilon y_n) - V^\infty\Big) m_\varepsilon^p(e) \\
&= 0.
\end{aligned} \tag{5.20}$$

So it follows from (5.19) and (5.20) that

$$\int_{x \in \mathbb{R}^N} (V(\varepsilon x + \varepsilon y_n) - V^\infty) m_\varepsilon^p(e) = o(1), \tag{5.21}$$

where $o(1) \to 0$ as $n \to \infty$. So, it follows that $c_{V^\infty} = c_\varepsilon$ for $\varepsilon > 0$.

Finally, assume, seeking a contradiction, for some $\varepsilon_0 > 0$, that there exists $0 < \hat{u} \in \mathcal{N}_{\varepsilon_0}$ such that $c_{\varepsilon_0} = \Psi_{\varepsilon_0}(\hat{u})$. From lemma 2.2(iv), we deduce that there exists $\hat{e} \in S_{\varepsilon_0}$ such that $\hat{u} = m_{\varepsilon_0}(\hat{e}) = s_1 \hat{e}$, where $s_1 > 0$. From lemma 2.2 again, we infer that $m_{\varepsilon_0}(\hat{e}) = \hat{m}_{\varepsilon_0}(\hat{e})$ is the unique global maximum of Ψ_{ε_0} on E. We first have that $c_{V^\infty} \leqslant \Psi_{V^\infty}(m_{V^\infty}(\hat{e})) = \max_{u \in E} \Psi_{V^\infty}(u)$. On the other hand, by $(\mathcal{D}_1)$,

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it follows that $V(x) \geqslant V^\infty$ for all $x \in \mathbb{R}^N$ and $\Psi_{V^\infty}(u) \leqslant \Psi_{\varepsilon_0}(u)$ for each $u \in E$. Thus,

$$c_{V^\infty} \leqslant \Psi_{V^\infty}(m_{V^\infty}(\hat{e})) \leqslant \Psi_{\varepsilon_0}(m_{V^\infty}(\hat{e})) \leqslant \Psi_{\varepsilon_0}(m_{\varepsilon_0}(\hat{e})) = c_{\varepsilon_0} = c_{V^\infty}.$$

This implies that $c_{V^\infty} = \Psi_{V^\infty}(m_{V^\infty}(\hat{e})) = \Psi_{\varepsilon_0}(m_{V^\infty}(\hat{e}))$. Moreover, $u^\infty = m_{V^\infty}(\hat{e})$ satisfies

$$-\operatorname{div}(|\nabla u^\infty|^{p-2}\nabla u^\infty) + V^\infty |u^\infty|^{p-2}u^\infty = f(u^\infty) + |u^\infty|^{p^*-2}u^\infty \quad \text{in } \mathbb{R}^N. \quad (\mathcal{P}_{V^\infty})$$

As in the proof of theorem 3.7(i), one can easily check that $u^\infty(x) > 0$ in $\mathbb{R}^N$. However, one has that

$$\Psi_{V^\infty}(u^\infty) = \Psi_{\varepsilon_0}(u^\infty) + \int_{\mathbb{R}^N} (V^\infty - V(\varepsilon_0 x))(u^\infty)^p. \tag{5.22}$$

Furthermore, we deduce from $(\mathcal{D}_1)$ that

$$\int_{\mathbb{R}^N} (V^\infty - V(\varepsilon_0 x))(u^\infty)^p < 0. \tag{5.23}$$

Thus, $\Psi_{V^\infty}(u^\infty) < \Psi_{\varepsilon_0}(u^\infty)$. This is a contradiction. □

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