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FINITE GROUPS WITH ABNORMAL MINIMAL NONNILPOTENT SUBGROUPS

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Abstract

We describe finite soluble nonnilpotent groups in which every minimal nonnilpotent subgroup is abnormal. We also show that if G is a nonsoluble finite group in which every minimal nonnilpotent subgroup is abnormal, then G is quasisimple and $Z(G)$ is cyclic of order $|Z(G)| \in \{1, 2, 3, 4\}$.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group; $G^{\mathfrak{N}}$ is the *nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with nilpotent quotient G/N; and $Z_\infty(G)$ is the hypercentre of G, that is, the largest normal subgroup of G such that $C_G(H/K) = G$ for every chief factor H/K of G below $Z_\infty(G)$. A nonnilpotent group G is called *minimal nonnilpotent* or a *Schmidt group* if every proper subgroup of G is nilpotent.

The structure of Schmidt groups is well known (see [10, III, Satz 5.2] and [2]) and such groups have deep applications in the theory of the classes of groups [3, 8]. Groups in which the condition of subnormality or generalised subnormality is satisfied for all or selected Schmidt subgroups are studied in [12, 17] and the recent papers [1, 9, 11, 13, 15, 19]. In this article, we consider, in a certain sense, the opposite situation.

A subgroup H of G is said to be *abnormal* in G if $x \in \langle H, H^x\rangle$ for all $x \in G$. From the results in [1, 9, 11, 13, 15, 19], it is natural to ask: *What is the structure of a group in which all Schmidt subgroups are abnormal?* We provide an answer to this question.

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We say that G is an *SA-group* if G is not nilpotent and every Schmidt subgroup of G is abnormal, and an *SSA-group* if G is a nonabelian simple SA-group and for every Schmidt subgroup H of G, we have $\pi(H) \cap \{2, 3\} \neq \emptyset$. The usefulness of the concept of an SSA-group is due to the fact that in any SA-group, any of its nonsoluble local subgroups is an SSA-group (see [6, page 444] and Theorem 1.2 below).

Our first result shows that the class of all soluble SA-groups is rather narrow.

THEOREM 1.1. *The group G is a soluble SA-group if and only if the following conditions hold.*

(i) $G = D \rtimes Q$, *where* $D = G^{\mathfrak{N}} \neq 1$ *is nilpotent,* $Q = \langle x \rangle$ *is a cyclic Sylow q-subgroup of G for some prime q dividing* $|G|$ *and* $F(G) = D\langle x^q \rangle$. *In particular,* $\langle x^q \rangle \leq Z(G)$.

(ii) $Z := Z_\infty(G) \cap D \leq \Phi(O_p(D))$ *for some prime p and, if* $Z \neq 1$, *then* $D = O_p(D)$ *is a Sylow p-subgroup of G.*

(iii) *For every prime r dividing* $|D|$ *and for the Sylow r-subgroup* D_r *of D:*

  (a) $R = (RQ)^{\mathfrak{N}}$ *for every normal r-subgroup R of G with* $Z < R$; *in particular,* $D_r = (D_rQ)^{\mathfrak{N}}$;

  (b) *if* H/K *is any chief factor of G between Z and* D_r, *then* $C_G(H/K) = F(G)$ *and* $|H/K| = r^n$, *where n is the smallest integer such that q divides* $r^n - 1$.

(iv) $ZQ = N_G(Q)$ *is a Carter subgroup of G and the set of all Carter subgroups of G coincides with the set of all its system normalisers. Moreover, a subgroup C of G is a Carter subgroup if and only if C is a maximal abnormal subgroup of a Schmidt subgroup of G.*

We do not know how wide the class of all nonsoluble SA-groups is (see Section 4). Nevertheless, using Theorem 1.1, we prove the following theorem which partially describes such groups.

THEOREM 1.2. *If G is a nonsoluble SA-group, then the following conditions hold.*

(i) *G is quasisimple and* $Z(G)$ *is cyclic of order* $|Z(G)| \in \{1, 2, 3, 4\}$. *In particular,* $Z(G) \leq \Phi(H)$ *for every Schmidt subgroup H of G and* $U/Z(G)$ *is a Schmidt subgroup of* $G/Z(G)$ *if and only if U is a Schmidt subgroup of G.*

(ii) $G/Z(G)$ *is an SSA-group.*

(iii) *If* $N = N_G(P)$ *for some nonnormal p-subgroup P of G, then either N is a group of type (i) with* $|Z(G)| \in \{2, 3, 4\}$, *or N is nilpotent or* $|N/F(N)|$ *is a prime.*

2. Proof of Theorem 1.1

The first lemma is a corollary of the definition of abnormal subgroups.

LEMMA 2.1. *Let* $H \leq E$ *and* $N \trianglelefteq G$, *where H is abnormal in G. Then H is abnormal in E, E is abnormal in G and* HN/N *is abnormal in* G/N.

LEMMA 2.2. *Let* $N, H \leq G$, *where* $N \trianglelefteq G$ *and* $N \leq Z_\infty(G)$. *Then H is subnormal in HN and if H is abnormal in G, then* $N \leq H$.

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PROOF. First we show that H is subnormal in $E = HN$. Assume this is false and let G be a counterexample of minimal order. Then $H \neq G$. Since $N \leq Z_\infty(G) \cap E \leq Z_\infty(E)$, the hypothesis holds for (E, H, N). If $E < G$, then H is subnormal in E by the choice of G, and this contradicts the hypothesis. Therefore, $G = E = HN$. Let M be a maximal subgroup of G such that $H \leq M$. Then $M = H(M \cap N)$, where $M \cap N \leq Z_\infty(M)$, so the hypothesis holds for $(M, H, M \cap N)$ and hence H is subnormal in M and M is not normal in G. However, the hypothesis holds also for $(G/M_G, M/M_G, NM_G/M_G)$, so $M_G = 1$. Note also that $M \cap N < N_N(M \cap N)$ since $Z_\infty(G)$ is nilpotent and so $M \cap N$ is normal in G. Therefore, from $G = NM$ and $M_G = 1$, it follows that $M \cap N = 1$ and so N is a minimal normal subgroup of G contained in $Z_\infty(G)$. Hence $N \leq Z(G)$, so $G = MN \leq N_G(M)$ and this contradicts the hypothesis. Therefore, H is subnormal in $E = NH$. Finally, if H is abnormal in G, then H is abnormal in E by Lemma 2.1 and so $H = E$ by [5, I, Illustrations 6.19(b)]. The lemma is proved. □

The following lemma is well known (see, for example, [14, I, Lemma 4.1]).

LEMMA 2.3. *Let A be an abelian irreducible automorphism group of a p-group P of order $|P| = p^n$. Then A is a cyclic group and n is the smallest integer such that $|A|$ divides $p^n - 1$.*

PROOF OF THEOREM 1.1. First we show that if G is a soluble SA-group, then Conditions (i), (ii), (iii) and (iv) hold for G. Assume that this is false and let G be a counterexample of minimal order. Then G is not a Schmidt group since Conditions (i), (ii), (iii) and (iv) hold for every Schmidt group G by Proposition 1.9 in [8, Ch. 1] and the results in [2]. Let $D = G^{\mathfrak{N}}$ by the nilpotent residual of G. Then $D \neq 1$.

(1) *If $L \trianglelefteq T \leq G$, where T/L is nonnilpotent and either $L \neq 1$ or $T \neq G$, then Conditions (i), (ii), (iii) and (iv) hold for T/L.*

Let E/L be a Schmidt subgroup of T/L. Then E is not nilpotent, so it contains a Schmidt subgroup, A say, and A is abnormal in G by hypothesis. Then E is abnormal in T and so E/L is abnormal in T/L by Lemma 2.1. Therefore, the hypothesis holds for T/L, so we have (1) by the choice of G.

(2) *Every nonabnormal subgroup E of G is nilpotent.*

Since every nonnilpotent group possesses a Schmidt subgroup, this follows from Lemma 2.1 and the hypothesis.

(3) *$D < G$ and if $D \leq V < G$, where V is a maximal subgroup of G, then $V = F(G)$ is the largest normal nilpotent subgroup of G and G/D is a cyclic group of order q^r for some prime q.*

Since G is soluble, $D \neq G$. However, G/D is nilpotent, so each maximal subgroup V of G containing D is subnormal in G. Assume that V is not nilpotent. Then V is abnormal in G by (2), so V/D is abnormal in $G/D = Z_\infty(G/D)$ and hence $V/D = G/D$ by Lemma 2.2. This contradiction shows that V is nilpotent. If G/D

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has at least two distinct maximal subgroups V/D and W/D, then $G = \langle V, W\rangle$ is nilpotent since the subgroup generated by any two subnormal nilpotent subgroups of the group is nilpotent by [3, Theorem 6.3.3]. Therefore, G/D is a cyclic q-group for some prime q and $V = V^G$ is the largest normal nilpotent subgroup of G. Hence, we have (3).

(4) *Condition (i) holds for G.*

Let Q be a Sylow q-subgroup of G. Then $Q \cap D$ is a Sylow q-subgroup of D and D has a normal Hall q'-subgroup V since D is nilpotent by (3). The subgroup V is characteristic in D, so it is normal in G. Moreover,

$$G/V = DQ/V = QV/V \simeq Q/(Q \cap V) = Q/1$$

is nilpotent, so $D \le V$ and hence $Q \cap D = 1$. Therefore, $G = D \rtimes Q$, where $G/D \simeq Q = \langle x\rangle$ is a cyclic q-group and $F(G) = D\langle x^q\rangle$, again by (3). It follows that $\langle x^q\rangle \le Z(G)$.

(5) *Condition (ii) holds for G.*

Assume that $Z \neq 1$ and let L be a minimal normal subgroup of G contained in Z. Then $L \le Z(G)$. Let p be any prime dividing $|Z|$ and let Z_p be the Sylow p-subgroup of Z. We show that D is a Sylow p-subgroup of G. Assume that $D \neq D_p := O_p(D)$. Then for the p-complement V of D, we have $Z_p \le C_G(VQ)$ since $[Q, Z] = 1$ by [18, Appendixes, Theorem 6.2]. If VQ is not nilpotent and H is a Schmidt subgroup of VQ, then $Z_p \le H \le VQ$ by Lemma 2.2. However, $Z_p \cap VQ = 1$ and so $Z_p = 1$, and this contradicts the hypothesis. Hence, $G/D_p \simeq VQ$ is nilpotent, so $D_p \le D \le D_p$. Thus, $D = D_p$.

Now we show that $Z \le \Phi(D) = \Phi(D_p)$. Let $\Phi = \Phi(D)$. Then $\Phi \le \Phi(G)$ and so G/Φ is not nilpotent. First assume that $\Phi \neq 1$. Then Condition (ii) holds for G/Φ by (1). Hence,

$$\begin{aligned} Z\Phi/\Phi = (Z_\infty(G) \cap D)\Phi/\Phi \le Z_\infty(G/\Phi) \cap (D/\Phi) &= Z_\infty(G/\Phi) \cap (G^{\mathfrak{N}}/\Phi) \\ &= Z_\infty(G/\Phi) \cap (G/\Phi)^{\mathfrak{N}} \le \Phi(D/\Phi) = \Phi/\Phi, \end{aligned}$$

so $Z \le \Phi = \Phi(D)$. Finally, assume that $\Phi = 1$, that is, $D = D_p$ is an elementary abelian p-group. Then $D = N_1 \times \cdots \times N_t$, where $N_1, \ldots, N_t$ are minimal normal subgroups of G by Maschke's theorem. It is clear also that for some i, for $i = 1$ say, we have $N_1 = L \le Z(G)$. However, then $G/N_2 \times \cdots \times N_t \simeq N_1 Q$ is nilpotent and so $D \le N_2 \times \cdots \times N_t$. This contradiction completes the proof that $Z \le \Phi = \Phi(D)$. Therefore, (5) holds.

(6) *Condition (iii) holds for G.*

Let $E = RQ$. If E is nilpotent, then $E < G$ and $G/C_G(R)$ is an r-group by (4), so $R \le Z = Z_\infty(G) \cap D$ by [18, Appendixes, Theorem 6.3] and this contradicts the hypothesis. Therefore, E is not nilpotent, so $E^{\mathfrak{N}} = R$ by (1). Finally, if $E = G$, then $R = D = E^{\mathfrak{N}}$ by (4). Hence, Condition (a) holds.

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Now, let H/K be any chief factor of G between Z and D_r. First we show that $C_G(H/K) = F(G) = D\langle x^q\rangle$. By [5, Ch. A, Theorem 13.8(b)], we have $F(G) \leq C_G(H/K)$. Assume that $F(G) < C_G(H/K)$. Then $C_G(H/K) = G$, so $Q \leq C_G(H/K)$. Let $E = HQ$. Then $H = E^{\mathfrak{N}}$ by (a), so E/K is not nilpotent. However, $Q \leq C_G(H/K)$, so $QK/K \leq C_{E/K}(H/K)$ and then $E/K = (H/K) \times (QK/K)$ is nilpotent, and this contradicts the hypothesis. Hence, $C_G(H/K) = F(G) = D\langle x^q\rangle$.

From $G = D \rtimes Q$, it follows that for every element $g \in G$, we have $g = dy$ for some $d \in D$ and $y \in Q$, where $d \in C_G(H/K)$, so $(hK)^g = (hK)^y$. Hence, Q acts irreducibly on H/K. Therefore, $Q/C_Q(H/K) = Q/\langle x^q\rangle$ is an abelian irreducible automorphim group for H/K. Hence, $|H/K| = r^n$, where n is the smallest integer such that q divides $r^n - 1$ by Lemma 2.3. Therefore, Condition (b) holds. Therefore, Condition (iii) holds for G.

(7) *Condition (iv) holds for G.*

Let $N = N_G(Q)$ and $D_0 = D \cap N$. Then $N = N \cap DQ = (N \cap D)Q = D_0 \times Q$ is nilpotent. However,

$$N_G(D_0 \times Q) = N_G(D_0) \cap N_G(Q) = D_0Q \cap N_G(D_0) = D_0(Q \cap N_G(D_0)) = D_0Q.$$

Hence, D_0Q is a Carter subgroup of G. In view of (4), $N = D_0Q$ is a system normaliser of G. Hence, N covers all central chief factors of G and N avoids all noncentral chief factors of G by [5, I, Theorem 5.6]. Therefore, $|N|$ is the product of the orders of all central factors of a chief series of G by [5, I, Theorem 5.7]. In view of (5) and (6), the product of the orders of all central factors of a chief series of G is $|Z||Q|$. However, $ZQ \leq N$, so $Z \times Q = D_0 \times Q$ and hence $Z = D_0$. Therefore, $ZQ = N_G(Q)$ is a Carter subgroup of G and the set of all Carter subgroups of G coincides with the set of all its system normalisers since in a soluble group, every two Carter subgroups and every two system normalisers are conjugate.

Now, let C be any Carter subgroup of G. Then $C = (ZQ)^a = ZQ^a$ for some $a \in G$ since any two Carter subgroups of a soluble group are conjugate. Let N/Z be a chief factor of G, where $N \leq D_r$. Then NQ^a is not nilpotent by (4). Hence, this subgroup contains a Schmidt subgroup H. Moreover, $Z \leq H$ by Lemma 2.2 since H is abnormal in G by hypothesis. Also we have $Q^b \leq H$ for some $b \in G$ since every subgroup of G not containing a conjugate of Q is nilpotent by (6). Therefore, H contains a Carter subgroup $ZQ^b = (ZQ)^b$ and so C is contained in some conjugate H^y of H. Hence, C is a maximal abnormal subgroup of H^y since H^y is not nilpotent but each of its maximal subgroups is nilpotent. Similarly, it can be proved that if H is a Schmidt subgroup of G, then each maximal abnormal subgroup of H is a Carter subgroup of G. Hence, we have (7).

From (3)–(7), it follows that Conditions (i), (ii), (iii) and (iv) hold for G, contrary to the choice of G. This contradiction completes the proof of the necessity of the condition of the theorem.

Conversely, assume that Conditions (i), (ii), (iii) and (iv) hold for G. Then G is a nonnilpotent soluble group. Let H be any Schmidt subgroup of G. Then for some

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Carter subgroup C of G, we have $C \leq H$ by Condition (iv), so H is abnormal in G by Lemma 2.1 since every Carter subgroup of G is abnormal by [10, VI, Satz 12.2(c)]. Therefore, every Schmidt subgroup of G is abnormal in G.

The theorem is proved. □

3. Proof of Theorem 1.2

The following lemma can be proved similarly to Lemma 6.3 in [10, VI].

LEMMA 3.1. *Let p be a prime and $K \leq H$ normal subgroups of G, where $K \leq \Phi(G)$. If H/K is p-closed, then H is p-closed.*

PROOF OF THEOREM 1.2. Assume that this theorem is false and let G be a counterexample of minimal order.

(1) *If $L \trianglelefteq T \leq G$, where T/L is nonsoluble and either $L \neq 1$ or $T \neq G$, then Conditions (i) and (ii) hold for T/L.*

Since every Schmidt subgroup of T/L is abnormal in T/L (see (1) in the proof of Theorem 1.1), this follows from the choice of G.

(2) *If H/K is a nonabelian chief factor of G such that K is soluble, then $H/K = G/K$ is a nonabelian simple group and K is the soluble radical of G (that is, every normal soluble subgroup of G is contained in K). Hence, $G' = G$ and a Sylow 2-subgroup G_2 of G is not cyclic.*

Let L/K be a minimal normal subgroup of H/K. Then L/K is a nonabelian simple group. Let A be a Schmidt subgroup of L. Then A is abnormal in G, so $L = H = G$. Hence, G/K is a nonabelian simple group. Assume that $G' < G$. Then $G'K = G$, hence $G'/(G' \cap K) \simeq G/K$ is a nonabelian chief factor of G such that $G' \cap K$ is soluble and so $G' = G$, and this contradicts the hypothesis. Hence, $G' = G$, so G_2 is not cyclic by [10, IV, Satz 2.8].

(3) *K is nilpotent.*

Assume that this is false and let R be a minimal normal subgroup of G contained in K. Then $R \leq O_p(G)$ for some prime p since K is soluble. Moreover, G/R is quasisimple by (1), where $(G/R)/(K/R) \simeq G/K$, so $K/R \leq Z(G/R)$ and hence K/R is nilpotent. If G has a minimal normal subgroup $N \neq R$, then $K/1 = K/(R \cap N)$ is nilpotent. Hence, R is the unique minimal normal subgroup of G and, by Lemma 3.1, $R \nleq \Phi(G)$ since K is not nilpotent. Let M be a maximal subgroup of G such that $G = RM$. Then M is not nilpotent since $G' = G$ and $R \cap M = 1 = C_G(R) \cap M$ since both these intersections are normal in G, so $C_G(R) = R(C_G(R) \cap M) = R$ and so $|O_p(G/R)| = 1 = |O_p(M)|$ by [5, Ch. A, Lemma 13.6(b)]. It follows that for some prime $q \neq p$, the group M is not q-nilpotent and hence M possesses a q-closed Schmidt subgroup A of the form $A = A_q \rtimes A_r$ for some prime $r \neq q$ by [10, IV, Satz 5.4]. Let $E = RA$. Then E is a soluble nonnilpotent group with abnormal Schmidt subgroups by Lemma 2.1. Therefore, from

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Theorem 1.1, it follows that $E = D \rtimes V$, where $D = E^{\mathfrak{N}}$ is a nilpotent Hall subgroup of E and V is a cyclic Sylow t-subgroup of E for some prime $t \in \{p, q, r\}$. However, E/RA_q is nilpotent, so $D = RA_q$ and $V \simeq A_r$. Therefore, $A_q \le C_G(R) = R$. This contradiction completes the proof of (3).

(4) *G has a p-closed Schmidt subgroup $A = A_p \rtimes A_q$, where $p \in \pi(A)$, for every prime p dividing $|G/K|$.*

From (2), it follows that G/K is not p-nilpotent, so some subgroup E/K of G/K is a p-closed Schmidt group with $\pi(E/K) = \{p, q\}$. Let U be a minimal supplement to K in E. Then $U \cap K \le \Phi(U)$, so U is a p-closed nonnilpotent group by Lemma 3.1 with $\pi(U) = \{p, q\}$. Then U has a p-closed Schmidt subgroup A with $p \in \pi(A)$.

(5) $K \le Z_\infty(G)$.

Assume that $K \nleq Z_\infty(G)$ and let $C = C_G(K)$. Then $C \ne G$. If $C \nleq K$, then $G = KC$ by (2) and so from the isomorphism $G/K \simeq C/(C \cap K)$ and (2), it follows that $C = G$ and this contradicts the hypothesis. Hence, $C \le K$.

Let V be the Hall $2'$-subgroup of K. The subgroup V is characteristic in K, so it is normal in G. Assume that $V \nleq Z_\infty(G)$. Then $V \ne 1$, so $K/V \le Z(G/V)$ by (1) and (2). If G_2V is nilpotent, then $G_2 \le C_G(V)$. Since G/K is a nonabelian simple group, $G_2 \nleq K$ by the Feit–Thompson theorem. Hence, $C_G(V) \nleq K$, which implies that $G = C_G(V)K$ and so $G = C_G(V)$ by (2). Therefore, $V \le Z(G)$ and this contradicts the hypothesis. Hence, G_2V is a soluble nonnilpotent group and every Schmidt subgroup of G_2V is abnormal in G_2V, so G_2 is cyclic by Theorem 1.1, contrary to (2). Therefore, $V \le Z_\infty(G)$. Since also we have $K/V \le Z(G/V)$, it follows that $K \le Z_\infty(G)$ by the Jordan–Hölder theorem for the chief series, contrary to our assumption on K. Hence, $K \le G_2$.

Finally, G has a p-closed Schmidt subgroup $A = A_p \rtimes A_q$, where $p \in \pi(A)$, for every prime $p \ne 2$ dividing $|G/K|$ by (4). Then $(KA)^{\mathfrak{N}} = KA_p = K \times A_p$ is nilpotent by Theorem 1.1. Therefore, $A_p \le C_G(K) = K$. This contradiction completes the proof of (5).

(6) *G is quasisimple. Hence, $K = Z(G) \le \Phi(G)$.*

Since $G/C_G(K)$ is nilpotent by [5, IV, Theorem 6.10] and (5), $K = Z(G) \le \Phi(G)$ by (2). Hence, we have (6).

(7) *$K \le \Phi(H)$ for every Schmidt subgroup H of G. Hence, K is a cyclic p-group for some prime p.*

Let H be a Schmidt subgroup of G. Then $K \le H$ by (5) and Lemma 2.2. Moreover, if V is a maximal subgroup of H, then V is nilpotent and so, in fact, $K \le V$. Hence, $K \le \Phi(H)$.

Now observe that $\pi(K) \subseteq \{2, p\}$ for some prime $p \ne 2$ since G has a Schmidt subgroup A with $2 \in \pi(A)$ by (2) and (4). From (2) and Burnside's p^aq^b-theorem, it follows that for some prime q dividing $|G/K|$, we have $2 \ne q \ne p$. However, G has a

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q-closed Schmidt subgroup $A = A_q \rtimes A_r$ by (4) and we also have $K \leq A$. Hence, $K \leq A_r$ is a cyclic r-group and so we have (7).

(8) *Condition (i) holds for G.*

From (6) and (7), we have $K = Z(G) \leq H$ for every Schmidt subgroup H of G. Now we show that $|K| \in \{1, 2, 3, 4\}$. Assume that $K \neq 1$. From (6) and (7), it follows that K is cyclic and $|K|$ divides the order of the Schur multiplier $M(G/K)$ of G/K. Hence, $|K| \in \{2, 3, 4\}$ (see Section 4.15(A) in [7, Ch. 4]).

Next assume that H/K is a Schmidt subgroup of G/K. Then H is not nilpotent, so it has a Schmidt subgroup U and we have $K \leq U$. Moreover, U/K is not nilpotent since $K \leq Z(U)$ and so $U = H$ since every proper subgroup of H/K is nilpotent. Similarly, it can be proved that if H is a Schmidt subgroup of G, then $K < H$ and H/K is a Schmidt subgroup of G/K. Therefore, (8) holds.

(9) *Condition (ii) holds for G.*

This follows from Condition (i).

(10) *Condition (iii) holds for G.*

If N is soluble and N is not nilpotent, then $|N/F(N)|$ is a prime by Theorem 1.1. Finally, suppose that $N = N_G(P)$ is not soluble. Then N is a group of type (i) with $|Z(G)| \in \{2, 3, 4\}$. Indeed, this follows from (1), if $N < G$ and from (8), in the case when $N = G$.

The theorem is proved. □

4. Final remarks, examples and open questions

EXAMPLE 4.1.

(1) Let E be an extraspecial group of order 3^7 and exponent 3. Then $\mathrm{Aut}(E)$ contains an element α of order 7 which operates irreducibly on E/Z_E and centralises $Z(E)$ by Lemma 20.13 in [5, Ch. A]. Let E_1 and E_2 be two copies of the group E and let $P = E_1 \curlyvee E_2 := (E_1 \times E_2)/D$, where $D = \{(a, a^{-1}) \mid a \in Z(E)\}$ is the direct product of the groups E_1 and E_2 with joint centre (see [10, page 49]). Then α induces an automorphism of order 7 on P and for the group $G_1 = P \rtimes \langle\alpha\rangle$, all Conditions (i), (ii), (iii) and (iv) are fulfilled for G_1 with $Z = Z(E)$.

Now let $G_2 = C_{57} \rtimes \langle\alpha\rangle$, where α is an element of order 7 in $\mathrm{Aut}(C_{57})$. Let $\phi_i : G_i \to \langle\alpha\rangle$ be an epimorphism of G_i onto $\langle\alpha\rangle$ and let

$$G = G_1 \curlywedge G_2 = \{(g_1, g_2) \mid g_i \in G_i, \phi_1(g_1) = \phi_2(g_2)\}$$

be the direct product of the groups G_1 and G_2 with joint factor group $\langle\alpha\rangle$ (see [10, page 50]). Then Conditions (i), (iii) and (iv) are fulfilled for G.

(2) The alternating group A_5 of degree 5 is an *SA*-group and an *SSA*-group.

(3) It is well known that the alternating group A_{13} possesses a Frobenius subgroup $C_{13} \rtimes C_6 = (C_{13} \rtimes C_3) \times C_2$ (see [4, page 104]), where $C_{13} \rtimes C_3$ is a Schmidt subgroup of A_{13}. Hence, A_{13} is neither an *SA*-group nor an *SSA*-group.

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REMARK 4.2

(1) If G is a soluble SA-group and D_r a Sylow r-subgroup of G for some prime r dividing $G^{\mathfrak{N}}$, then (using Theorem 1.1) it can be proved by direct verification that all chief factors of G between Z and D_r are G-isomorphic.
(2) In fact, Theorem 1.2 reduces the problem of classification of all nonsoluble SA-groups to the classification of all nonabelian simple SA-groups.

Remark 4.2(2) is a motivation for the following natural questions.

QUESTION 4.3. Classify all nonabelian simple SA-groups.

QUESTION 4.4. Classify all nonabelian simple groups in which every nonsoluble local subgroup is an SSA-group.

QUESTION 4.5. Classify all nonabelian simple groups in which every Schmidt subgroup is self-normalising.

In Ref. [16], Thompson classified nonsoluble groups all of whose local subgroups are soluble. This classical result makes it natural to ask: *What is the structure of a nonsoluble group in which every nonsoluble local subgroup is quasisimple?*

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