Proceedings of the Edinburgh Mathematical Society (2017) **60**, 973–1001
DOI:10.1017/S0013091516000456

ON THE UNIVERSAL SL_2-REPRESENTATION RINGS OF FREE GROUPS

TAKAO SATOH

Department of Mathematics, Faculty of Science Division II, Tokyo University of Science, 1–3 Kagurazaka, Shinjuku, Tokyo 162-8601, Japan (takao@rs.tus.ac.jp)

(Received 2 October 2014)

Abstract In this paper, we give an explicit realization of the universal SL_2-representation rings of free groups by using 'the ring of component functions' of $\mathrm{SL}(2,\mathbb{C})$-representations of free groups. We introduce a descending filtration of the ring, and determine the structure of its graded quotients. Then we study the natural action of the automorphism group of a free group on the graded quotients, and introduce a generalized Johnson homomorphism. In the latter part of this paper, we investigate some properties of these homomorphisms from a viewpoint of twisted cohomologies of the automorphism group of a free group.

Keywords: universal SL_2-representation; Fricke characters; character variety; Johnson homomorphism

2010 *Mathematics subject classification:* Primary 20F28
Secondary 20G05; 13A15; 13C05

1. Introduction

Let F_n be a free group of rank n generated by $x_1, \dots, x_n$. We denote by $R(F_n)$ the set $\mathrm{Hom}(F_n, \mathrm{SL}(2,\mathbb{C}))$ of all $\mathrm{SL}(2,\mathbb{C})$-representations of F_n. Let $\mathcal{F}(R(F_n),\mathbb{C})$ be the set $\{\chi\colon R(F_n) \to \mathbb{C}\}$ of all complex-valued functions on $R(F_n)$. Then we can regard $\mathcal{F}(R(F_n),\mathbb{C})$ as a $\mathbb{C}$-algebra in a natural way from the pointwise product. (See § 4 for details.) For any $x \in F_n$ and any $1 \leqslant i, j \leqslant 2$, we define the element $a_{ij}(x)$ of $\mathcal{F}(R(F_n),\mathbb{C})$ to be

$$(a_{ij}(x))(\rho) := \text{the } (i,j)\text{-component of } \rho(x)$$

for any $\rho \in R(F_n)$. We call the map $a_{ij}(x)$ the (i,j)-component function of x, or simply a component function of x. Let $\mathfrak{R}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-subalgebra of $\mathcal{F}(R(F_n),\mathbb{C})$ generated by all $a_{ij}(x)$ for $x \in F_n$ and $1 \leqslant i, j \leqslant 2$. We call $\mathfrak{R}_{\mathbb{Q}}(F_n)$ the ring of component functions of $\mathrm{SL}(2,\mathbb{C})$-representations of F_n over $\mathbb{Q}$. The ring $\mathfrak{R}_{\mathbb{Q}}(F_n)$ contains the ring of Fricke characters of F_n as a subring. For any $x \in F_n$, the map $\mathrm{tr}\, x := a_{11}(x) + a_{22}(x)$ is the Fricke character of x. Let $\mathfrak{X}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-subalgebra of $\mathcal{F}(R(F_n),\mathbb{C})$ generated by all $\mathrm{tr}\, x$ for $x \in F_n$. The ring $\mathfrak{X}_{\mathbb{Q}}(F_n)$ is called the ring of Fricke characters of F_n. Classically, Fricke characters were first studied by Fricke with respect to the classification problem of Riemann surfaces (see [**5**]). In the 1970s, Horowitz investigated algebraic properties of $\mathfrak{X}_{\mathbb{Q}}(F_n)$ by using combinatorial group theory [**9**, **10**]. In particular, he described a set of finite generators of $\mathfrak{X}_{\mathbb{Q}}(F_n)$ as a ring. Let $\operatorname{Aut} F_n$ be the automorphism group of F_n. In 1980, Magnus [**18**] studied the action of $\operatorname{Aut} F_n$ on $\mathfrak{X}_{\mathbb{Q}}(F_n)$ from a representation-theoretic viewpoint. Using it he constructed faithful representations of braid groups. On the other hand, the ring structure of $\mathfrak{X}_{\mathbb{Q}}(F_n)$ itself is not well understood. One reason why the ring structure of $\mathfrak{X}_{\mathbb{Q}}(F_n)$ is complicated is that the number $n+\binom{n}{2}+\binom{n}{3}$ of Horowitz's generators of $\mathfrak{X}_{\mathbb{Q}}(F_n)$, which is minimal, is too large to handle in most situations. Due to this combinatorial complexity, various computations in the study of $\mathfrak{X}_{\mathbb{Q}}(F_n)$ do not work well enough. To avoid this difficulty, we work with the larger ring $\mathfrak{R}_{\mathbb{Q}}(F_n)$ of component functions.

First, we show that $a_{ij}(x_l)$ for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant l \leqslant n$ generate $\mathfrak{R}_{\mathbb{Q}}(F_n)$ as a ring. Hence, $\mathfrak{R}_{\mathbb{Q}}(F_n)$ is finitely generated, and is therefore Noetherian. Then we consider the polynomial ring

$$\mathfrak{P} := \mathbb{Q}[t_{ij,l} \mid 1 \leqslant i, j \leqslant 2,\ 1 \leqslant l \leqslant n]$$

of $4n$ indeterminates and the surjective homomorphism $\pi \colon \mathfrak{P} \to \mathfrak{R}_{\mathbb{Q}}(F_n)$ defined by

$$\pi(t_{ij,l}) := a_{ij}(x_l).$$

Let I be the kernel of π. Set

$$s_{ii,l} := t_{ii,l} - 1 \quad \text{and} \quad s_{ij,l} := t_{ij,l}$$

for any $1 \leqslant i \neq j \leqslant 2$ and $1 \leqslant l \leqslant n$. Consider the $s_{ij,l}$ as new indeterminates of the polynomial ring $\mathfrak{P}$. Let $\tilde{J}$ be the ideal of $\mathfrak{P}$ generated by all $s_{ij,l}$ for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant l \leqslant n$. We will see later that $I \subset \tilde{J}$. Set $J := \tilde{J}/I$ and consider J as an ideal of $\mathfrak{P}/I \cong \mathfrak{R}_{\mathbb{Q}}(F_n)$. Then we have the descending filtration $J \supset J^2 \supset J^3 \supset \cdots$ of $\mathfrak{R}_{\mathbb{Q}}(F_n)$. For any $k \geqslant 1$, denote by $\mathrm{gr}^k(J)$ the kth graded quotient J^k/J^{k+1} of the filtration. In the early part of this paper, we investigate the ring structure of $\mathfrak{R}_{\mathbb{Q}}(F_n)$ through the filtration $\{J^k\}$ and the graded quotients $\mathrm{gr}^k(J)$. For any $k \geqslant 1$, set

$$T_k := \left\{ \prod_{l=1}^{n} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} \pmod{I} \;\middle|\; e_{ij,l} \geqslant 0,\ \sum_{l=1}^{n} (e_{11,l} + e_{12,l} + e_{21,l}) = k \right\} \subset J^k.$$

We show the following.

Theorem 1.1.

(1) *For each* $k \geqslant 1$, $T_k \pmod{J^{k+1}}$ *forms a basis of* $\mathrm{gr}^k(J)$ *as a* $\mathbb{Q}$*-vector space.*

(2) $\bigcap_{k \geqslant 1} J^k = \{0\}$.

(3) *The ring* $\mathfrak{R}_{\mathbb{Q}}(F_n)$ *is an integral domain. That is, the ideal* I *is prime.*

(4) *The ring* $\mathfrak{R}_{\mathbb{Q}}(F_n)$ *is naturally isomorphic to the universal* $\mathrm{SL}(2, \mathbb{C})$*-representation ring of* F_n.

From part (3), we see that the algebraic set $V(I)$ is an algebraic variety over $\mathbb{Q}$, and $\mathfrak{R}_{\mathbb{Q}}(F_n)$ is its affine coordinate ring. From part (4), $\mathfrak{R}_{\mathbb{Q}}(F_n)$ is one of the explicit realizations of the universal SL_2-representation ring of F_n over $\mathbb{Q}$. In general, the universal SL_2-representation ring of a group G plays an important role in the study of the classification of $\mathrm{SL}(2,R)$-representations of G for any $\mathbb{Q}$-algebra R. It is characterized by the universality, and is constructed by generators and relations in a universal way. (For details about the universal representation ring, see, for example, [**16**].) We remark that, to the best of our knowledge, the problem of whether the ring $\mathfrak{X}_{\mathbb{Q}}(F_n)$ of Fricke characters of F_n is isomorphic to the universal SL_2-character ring of F_n over $\mathbb{Q}$ or not is still open. (See also the end of § 3 in [**23**].)

Now, $\mathrm{Aut}\,F_n$ naturally acts on the ideal J. For any $k \geqslant 1$, let

$$\mathcal{D}_n(k) := \mathrm{Ker}(\mathrm{Aut}\,F_n \to \mathrm{Aut}(J/J^{k+1})).$$

The groups $\mathcal{D}_n(k)$ define a descending filtration of $\mathrm{Aut}\,F_n$. In the latter part of this paper, we study the difference between the filtration $\{\mathcal{D}_n(k)\}$ and the Andreadakis–Johnson filtration $\{\mathcal{A}_n(k)\}$ of $\mathrm{Aut}\,F_n$. Historically, the Andreadakis–Johnson filtration of $\mathrm{Aut}\,F_n$ was introduced by Andreadakis [**1**] in 1965. (For the definition, see § 3.) In the 1980s, Johnson studied a descending filtration of the mapping class group of a surface in order to investigate the group structure of the Torelli group in a series of works [**11**–**14**]. Johnson's filtration is nothing but the intersections of $\mathcal{A}_n(k)$s with the mapping class group. In particular, he determined the abelianization of the Torelli group by introducing a certain homomorphism. Today, his homomorphism is called the first Johnson homomorphism, and it is generalized to higher degrees. Over the last two decades, the Johnson homomorphisms of the mapping class groups have been studied from various viewpoints by many authors including Morita [**20**], Hain [**6**] and others. In [**8**], we introduced a descending filtration $\{\mathcal{E}_n(k)\}$ of $\mathrm{Aut}\,F_n$ and certain homomorphisms. They are the Fricke character analogues of the Andreadakis–Johnson filtration and the Johnson homomorphisms from the graded quotients of the $\mathcal{E}_n(k)$s. In particular, we showed that $\mathcal{E}_n(1) = \mathcal{A}_n(2)\cdot\mathrm{Inn}\,F_n$, and that $\mathcal{A}_n(2k) \subset \mathcal{E}_n(k)$ for any $k \geqslant 1$. However, the group structure of $\mathcal{E}_n(k)$ is quite complicated to handle in general. In this paper, we study an $\mathfrak{R}_{\mathbb{Q}}(F_n)$ version of our previous works. We prove the following.

Theorem 1.2.

(1) *The filtration $\{\mathcal{D}_n(k)\}$ is central.*

(2) *For each $k \geqslant 1$, $\mathcal{A}_n(k) \subset \mathcal{D}_n(k)$. In particular, for $1 \leqslant k \leqslant 4$, $\mathcal{D}_n(k) = \mathcal{A}_n(k)$.*

In order to show part (2), we give a sufficient condition for $\mathcal{D}_n(k) = \mathcal{A}_n(k)$. At the present stage, we do not know whether the condition always holds or not for any $k \geqslant 5$. On the other hand, by introducing Johnson homomorphism analogues η_k (for definitions, see § 5.2), we verify the following.

Proposition 1.3. *For any $n \geqslant 2$,*

(1) *each $\mathrm{gr}^k(\mathcal{D}_n)$ is torsion-free,*

(2) $\dim_{\mathbb{Q}}(\mathrm{gr}^k(\mathcal{D}_n) \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$.

Finally, we consider an extension of the homomorphism η_1 to $\mathrm{Aut}\,F_n$ as a crossed homomorphism. In [**21**] Morita showed that the first rational Johnson homomorphism of the mapping class group, whose initial domain is the Torelli group, can be extended to the mapping class group as a crossed homomorphism. He also showed that this extension is unique up to one coboundary. Similar results for $\mathrm{Aut}\,F_n$ were obtained by Kawazumi [**15**]. Furthermore, Kawazumi constructed higher twisted cocycles of $\mathrm{Aut}\,F_n$ with the crossed homomorphism. By restricting them to the mapping class group, he investigated relations between the higher cocycles and the Morita–Mumford classes. In [**27**], we studied the Fricke character analogue of these works. However, due to the combinatorial complexity of the graded quotients of the filtration of $\mathfrak{X}_{\mathbb{Q}}(F_n)$, we cannot study the crossed homomorphisms of $\mathrm{Aut}\,F_n$ well. In order to enhance the knowledge of twisted cohomology theory in the study of $\mathfrak{X}_{\mathbb{Q}}(F_n)$, it would be better to investigate those of $\mathfrak{R}_{\mathbb{Q}}(F_n)$ first since $\mathfrak{R}_{\mathbb{Q}}(F_n)$ is much easier to handle than $\mathfrak{X}_{\mathbb{Q}}(F_n)$.

In [**26**] we computed $H^1(\mathrm{Aut}\,F_n, (H^* \otimes_{\mathbb{Z}} \Lambda^2 H) \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathbb{Q}^{\oplus 2}$ by using Nielsen's presentation for $\mathrm{Aut}\,F_n$, and described two cocycles that generate it. One of them is f_{K}, that being Kawazumi's extension of the first rational Johnson homomorphism. The other cocycle is f_{M}, which is essentially constructed by Morita with the Magnus representation. (For details, see [**26**].) From these results and the fact that $\mathcal{D}_n(k) = \mathcal{A}_n(k)$ for $1 \leqslant k \leqslant 2$, we see that there exist crossed homomorphisms g_{M} and g_{K} from $\mathrm{Aut}\,F_n$ to the target $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))$ of η_1, corresponding to f_{M} and f_{K}, respectively, under the equality $\mathcal{D}_n(1)/\mathcal{D}_n(2) = \mathcal{A}_n(1)/\mathcal{A}_n(2)$. In § 5.3, we show that the cohomology classes of g_{M} and g_{K} are linearly independent in $H^1(\mathrm{Aut}\,F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J)))$. At the end of the paper, we introduce twisted higher cohomology classes $\overline{\zeta_q^*}(g_{\mathrm{K}}^{\otimes q}) \in H^q(\mathrm{Aut}\,F_n, \mathrm{gr}^1(J)^{\otimes q})$ for each $q \geqslant 1$ according to Kawazumi's construction of higher cohomology classes with the first rational Johnson homomorphism of $\mathrm{Aut}\,F_n$. In [**15**], by restricting them to the mapping class group of a surface, Kawazumi studied a relation between these higher cohomology classes and the Morita–Mumford classes. Here, we prove the following theorem.

Theorem 1.4. *For any $n \geqslant 4$, the restrictions of $\overline{\zeta_1^*}(g_{\mathrm{K}}) \cup \overline{\zeta_1^*}(g_{\mathrm{K}})$ and $\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})$ to IA_n are linearly independent in $H^2(\mathrm{IA}_n, \mathrm{gr}^1(J)^{\otimes 2})$.*

2. Notation and conventions

Throughout the paper, we use the following notation and conventions. Let F_n be the free group of rank n with a basis $x_1, \ldots, x_n$, and let H be its abelianization $H_1(F_n, \mathbb{Z})$. Then H is a free abelian group of rank n, and the coset classes of $x_1, \ldots, x_n$ form a basis of H as a free abelian group. We also use the following notation.

- Let G be a group. The automorphism group $\mathrm{Aut}\,G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\,G$ and $x \in G$, the action of σ on x is denoted by x^{σ}.

- Let N be a normal subgroup of a group G. For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no risk of confusion. Similarly, for a ring R, an element $f \in R$ and an ideal I of R, we also denote by f the coset class of f in R/I if there is no risk of confusion.

- For elements x and y in G, the commutator bracket $[x, y]$ of x and y is defined to be $xyx^{-1}y^{-1}$.

3. The Andreadakis–Johnson filtration of $\mathrm{Aut}\, F_n$

In this section, we review the Andreadakis–Johnson filtration of $\mathrm{Aut}\, F_n$ without proofs. The main purpose of the section is to fix the notation. For basic material concerning the Andreadakis–Johnson filtration and the Johnson homomorphisms, see, for example, [**25**] or [**28**].

For the free group F_n on n generators, we define the lower central series of F_n by the rule

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), n], \quad k \geqslant 2.$$

For any $y_1, \dots, y_k \in F_n$, the left-normed commutator

$$[[\cdots [[y_1, y_2], y_3], \dots], y_k]$$

of weight k is denoted by

$$[y_1, y_2, \dots, y_k]$$

for simplicity. Then we have the following lemma.

Lemma 3.1 (see [19, § 5.3]). *For any $k \geqslant 1$, the group $\Gamma_n(k)$ is generated by all left-normed commutators of weight k.*

Let $\rho\colon \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. Magnus [**17**] showed that for any $n \geqslant 3$, IA_n is finitely generated by automorphisms

$$K_{ij}\colon x_t \mapsto \begin{cases} {x_j}^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i, \end{cases}$$

for distinct $1 \leqslant i, j \leqslant n$, and

$$K_{ijl}\colon x_t \mapsto \begin{cases} x_i [x_j, x_l], & t = i, \\ x_t, & t \neq i, \end{cases}$$

for distinct $1 \leqslant i, j, l \leqslant n$ and $j < l$. For any $k \geqslant 1$, the action of $\mathrm{Aut}\, F_n$ on each nilpotent quotient group $G/\Gamma_G(k+1)$ induces the homomorphism

$$\rho^k\colon \mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

We denote the kernel of ρ^k by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$. We call it the Andreadakis–Johnson filtration of $\mathrm{Aut}\, F_n$. Then we have

Theorem 3.2 (Andreadakis [1]). *For any $k, l \geqslant 1$, $[\mathcal{A}_n(k), \mathcal{A}_n(l)] \subset \mathcal{A}_n(k+l)$.*

4. The rings of component functions of SL(2, $\mathbb{C}$)-representations of free groups

Let F_n be a free group of rank n generated by $x_1, \ldots, x_n$. We denote by $R(F_n)$ the set $\mathrm{Hom}(F_n, \mathrm{SL}(2,\mathbb{C}))$ of all $\mathrm{SL}(2,\mathbb{C})$-representations of F_n. Let $\mathcal{F}(R(F_n), \mathbb{C})$ be the set $\{\chi\colon R(F_n) \to \mathbb{C}\}$ of all complex-valued functions on $R(F_n)$. Then $\mathcal{F}(R(F_n), \mathbb{C})$ has a $\mathbb{C}$-algebra structure by the operations defined by

$$(\chi + \chi')(\rho) := \chi(\rho) + \chi'(\rho),$$
$$(\chi\chi')(\rho) := \chi(\rho)\chi'(\rho),$$
$$(\lambda\chi)(\rho) := \lambda(\chi(\rho))$$

for any $\chi, \chi' \in \mathcal{F}(R(F_n), \mathbb{C})$, $\lambda \in \mathbb{C}$, and $\rho \in R(F_n)$. The automorphism group $\mathrm{Aut}\, F_n$ of F_n naturally acts on $R(F_n)$ and $\mathcal{F}(R(F_n), \mathbb{C})$ from the right by

$$\rho^{\sigma}(x) := \rho(x^{\sigma^{-1}}), \quad \rho \in R(F_n) \quad \text{and} \quad x \in F_n$$

and

$$\chi^{\sigma}(\rho) := \chi(\rho^{\sigma^{-1}}), \quad \chi \in \mathcal{F}(R(F_n), \mathbb{C}) \quad \text{and} \quad \rho \in R(F_n)$$

for any $\sigma \in \mathrm{Aut}\, F_n$.

For any $x \in F_n$ and any $1 \leqslant i, j \leqslant 2$, we define an element $a_{ij}(x)$ of $\mathcal{F}(R(F_n), \mathbb{C})$ to be

$$(a_{ij}(x))(\rho) := \text{an } (i,j)\text{-component of } \rho(x)$$

for any $\rho \in R(F_n)$. The action of an element $\sigma \in \mathrm{Aut}\, F_n$ on $a_{ij}(x)$ is given by $a_{ij}(x^{\sigma})$. We have the relations

$$a_{11}(x^{-1}) = a_{22}(x), \quad a_{12}(x^{-1}) = -a_{12}(x), \quad a_{21}(x^{-1}) = -a_{21}(x), \quad a_{22}(x^{-1}) = a_{11}(x) \tag{4.1}$$

and

$$\left.\begin{aligned} a_{11}(xy) &= a_{11}(x)a_{11}(y) + a_{12}(x)a_{21}(y), \\ a_{12}(xy) &= a_{11}(x)a_{12}(y) + a_{12}(x)a_{22}(y), \\ a_{21}(xy) &= a_{21}(x)a_{11}(y) + a_{22}(x)a_{21}(y), \\ a_{22}(xy) &= a_{21}(x)a_{12}(y) + a_{22}(x)a_{22}(y) \end{aligned}\right\} \tag{4.2}$$

for any $x, y \in F_n$. Let $\mathfrak{R}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-subalgebra of $\mathcal{F}(R(F_n), \mathbb{C})$ generated by all $a_{ij}(x)$ for $x \in F_n$ and $1 \leqslant i, j \leqslant 2$. We call $\mathfrak{R}_{\mathbb{Q}}(F_n)$ the ring of component functions of $\mathrm{SL}(2,\mathbb{C})$-representations of F_n over $\mathbb{Q}$. We remark that for any $x \in F_n$, the map $\mathrm{tr}\, x := a_{11}(x) + a_{22}(x)$ is the Fricke character of x. Let $\mathfrak{X}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-subalgebra of $\mathcal{F}(R(F_n), \mathbb{C})$ generated by all $\mathrm{tr}\, x$ for $x \in F_n$. The algebra $\mathfrak{R}_{\mathbb{Q}}(F_n)$ contains $\mathfrak{X}_{\mathbb{Q}}(F_n)$. In [**8**], we investigated the behaviour of the natural action of $\mathrm{Aut}\, F_n$ on $\mathfrak{X}_{\mathbb{Q}}(F_n)$. The purpose of the paper is to study an $\mathfrak{R}_{\mathbb{Q}}(F_n)$-analogue of our previous results.

Let $\mathfrak{P}$ be a rational polynomial ring

$$\mathbb{Q}[t_{ij,l} \mid 1 \leqslant i, j \leqslant 2,\ 1 \leqslant l \leqslant n]$$

of $4n$ indeterminates. Consider the ring homomorphism $\pi_n\colon \mathfrak{P} \to \mathfrak{R}_{\mathbb{Q}}(F_n)$ defined by

$$\pi_n(t_{ij,l}) := a_{ij}(x_l).$$

We usually omit the subscript n, and write π for π_n for simplicity. Then we have the following proposition.

Proposition 4.1. *The ring homomorphism* $\pi\colon \mathfrak{P} \to \mathfrak{R}_{\mathbb{Q}}(F_n)$ *is surjective.*

Proof. Let $\mathfrak{R}$ be the $\mathbb{Q}$-subalgebra of $\mathfrak{R}_{\mathbb{Q}}(F_n)$ generated by all $a_{ij}(x_l)$ for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant l \leqslant n$. It suffices to show that $\mathfrak{R} = \mathfrak{R}_{\mathbb{Q}}(F_n)$. Let x be a reduced word $x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_r}^{e_r}$ in F_n for some $1 \leqslant i_m \leqslant n$ and $e_m = \pm 1$. We show that $a_{ij}(x) \in \mathfrak{R}_{\mathbb{Q}}(F_n)$ by induction on $r \geqslant 1$. For $r = 1$, it is obvious that $a_{ij}(x) \in \mathfrak{R}$ from (4.1). For $r \geqslant 2$, from (4.2) we have

$$a_{11}(x) = a_{11}(x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_{r-1}}^{e_{r-1}}) a_{11}(x_{i_r}^{e_r}) + a_{12}(x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_{r-1}}^{e_{r-1}}) a_{21}(x_{i_r}^{e_r}).$$

Hence, by the inductive hypothesis, we obtain $a_{11}(x) \in \mathfrak{R}$. By the same argument, we see that $a_{ij}(x) \in \mathfrak{R}$ for any $1 \leqslant i, j \leqslant n$. □

Now set

$$I_n := \mathrm{Ker}(\pi) = \{f \in \mathfrak{P} \mid f((a_{ij}(x_l))(\rho)) = 0 \text{ for any } \rho \in R(F_n)\}.$$

For simplicity, we usually write I for I_n if there is no confusion. Then we have an isomorphism $\mathfrak{P}/I \cong \mathfrak{R}_{\mathbb{Q}}(F_n)$ induced from the homomorphism π. We identify them through this isomorphism. The ideal I is non-trivial since

$$t_{11,l} t_{22,l} - t_{12,l} t_{21,l} - 1 \in I \tag{4.3}$$

for any $1 \leqslant l \leqslant n$. In order to investigate the ideal I and the algebra $\mathfrak{R}_{\mathbb{Q}}(F_n)$, we introduce a descending filtration of $\mathfrak{R}_{\mathbb{Q}}(F_n)$. Set

$$s_{ii,l} := t_{ii,l} - 1 \quad \text{and} \quad s_{ij,l} := t_{ij,l}$$

for any $1 \leqslant i \neq j \leqslant 2$ and $1 \leqslant l \leqslant n$. Consider the $s_{ij,l}$ as new indeterminates of the polynomial ring $\mathfrak{P}$. We can write any polynomial $f \in \mathfrak{P}$ as a polynomial of the $s_{ij,l}$ by substituting $t_{ii} = s_{ii} + 1$ and $t_{ij} = s_{ij}$ for $1 \leqslant i \neq j \leqslant 2$. Then the polynomial (4.3) is rewritten as

$$s_{11,l} s_{22,l} - s_{12,l} s_{21,l} + s_{11,l} + s_{22,l}. \tag{4.4}$$

Let $\tilde{J}_n$ be the ideal of $\mathfrak{P}$ generated by all $s_{ij,l}$ for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant l \leqslant n$. For simplicity, we usually write $\tilde{J}$ for $\tilde{J}_n$ if there is no risk of confusion.

Lemma 4.2. $I \subset \tilde{J}$.

Proof. For any $f \in I$, by rewriting f as a polynomial of the $s_{ij,l}$, we have

$$f = a_0 + \sum_{l=1}^{n} (a_{11,l} s_{11,l} + a_{22,l} s_{22,l} + a_{12,l} s_{12,l} + a_{21,l} s_{21,l}) + (\text{terms with degree} \geqslant 2)$$

for some $a_0, a_{ij,l} \in \mathbb{Q}$. By considering $f(\varepsilon)$ for the trivial representation $\varepsilon\colon F_n \to \mathrm{SL}(2, \mathbb{C})$, we see that $a_0 = 0$, and hence $f \in \tilde{J}$. □

https://doi.org/10.1017/S0013091516000456 Published online by Cambridge University Press

Set $J := \tilde{J}/I$ and consider J as an ideal of $\mathfrak{P}/I$. If we emphasize n, we write J_n instead of J. For any $n \geqslant 2$, let F_{n-1} be the free group of rank $n-1$ with basis $x_1, \dots, x_{n-1}$, and consider F_{n-1} as a subgroup of F_n.

Lemma 4.3. *The natural map $\bar{\iota}\colon J_{n-1} \to J_n$ induced from the inclusion map $\iota\colon \tilde{J}_{n-1} \to \tilde{J}_n$ is injective.*

Proof. For some $f \in \tilde{J}_{n-1}$, assume that $\bar{\iota}(f \pmod{I_{n-1}}) = 0$. Then $\iota(f) \in I_n$, and hence $\pi_n(\iota(f))(\rho) = 0$ for any representation $\rho\colon F_n \to \mathrm{SL}(2,\mathbb{C})$. For any representation $\mu\colon F_n \to \mathrm{SL}(2,\mathbb{C})$, define the representation $\rho\colon F_n \to \mathrm{SL}(2,\mathbb{C})$ by

$$x_i \mapsto \begin{cases} \mu(x_i) & \text{if } 1 \leqslant i \leqslant n-1, \\ E_2 & \text{if } i = n, \end{cases}$$

where E_2 denotes the 2×2 identity matrix. Then we have

$$0 = \pi_n(\iota(f))(\rho) = \pi_{n-1}(f)(\mu),$$

and hence $f \in I_{n-1}$. This shows that $\bar{\iota}$ is injective. □

Consider the descending filtration $J \supset J^2 \supset J^3 \supset \cdots$. For any $k \geqslant 1$, denote by $\mathrm{gr}^k(J)$ the kth graded quotient J^k/J^{k+1} of the filtration. We give a basis of $\mathrm{gr}^k(J)$ as a $\mathbb{Q}$-vector space. For any $k \geqslant 1$, set

$$T_k := \left\{ \prod_{l=1}^{n} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} \pmod{I} \;\middle|\; e_{ij,l} \geqslant 0, \sum_{l=1}^{n} (e_{11,l} + e_{12,l} + e_{21,l}) = k \right\} \subset J^k.$$

Proposition 4.4. *For each $k \geqslant 1$, $T_k \pmod{J^{k+1}}$ forms a basis of $\mathrm{gr}^k(J)$ as a $\mathbb{Q}$-vector space.*

Proof. Since $\mathrm{gr}^k(J)$ is generated by

$$\prod_{l=1}^{n} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} s_{22,l}^{e_{22,l}} \pmod{I} \quad \text{for} \quad \sum_{l=1}^{n} (e_{11,l} + e_{12,l} + e_{21,l} + e_{22,l}) = k,$$

by using (4.4) we see that $T_k \pmod{J^{k+1}}$ generates $\mathrm{gr}^k(J)$. In order to show that the T_k are linearly independent, assume that

$$\sum{}' a(e_{11,1}, e_{11,2}, \dots, e_{21,n}) \left(\prod_{l=1}^{n} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} \pmod{I} \right) \equiv 0 \pmod{J^{k+1}},$$

where the above sum runs over all tuples $(e_{11,1}, e_{11,2}, \dots, e_{21,n})$ such that $e_{11,1} + e_{11,2} + \cdots + e_{21,n} = k$. Denote by f the left-hand side of the above equation, and consider f as an element in $\pi(\tilde{J}^{k+1})$ through the identification $\mathfrak{P}/I \cong \mathfrak{R}_{\mathbb{Q}}(F_n)$ induced from the homomorphism π.

Now consider the interior of the unit disk in $\mathbb{C}$:

$$D := \{ z \in \mathbb{C} \mid z\bar{z} < 1 \}.$$

For any $z_{ij,l} \in D \setminus \{0\}$ for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant l \leqslant n$ except for $(i,j) = (2,2)$, we define the representation $\rho\colon F_n \to \mathrm{SL}(2,\mathbb{C})$ by

$$\rho(x_l) := \begin{pmatrix} z_{11,l}+1 & z_{12,l} \\ z_{21,l} & (z_{11,l}+1)^{-1}(1+z_{12,l}z_{21,l}) \end{pmatrix}.$$

Then, from $f \in \pi(\tilde{J}^{k+1})$, we see that

$$f(\rho) = {\sum}' a(e_{11,1}, e_{11,2}, \ldots, e_{21,n}) \prod_{l=1}^{n} z_{11,l}^{e_{11,l}} z_{12,l}^{e_{12,l}} z_{21,l}^{e_{21,l}}$$

can be written as a polynomial of $z_{11,l}$, $z_{12,l}$ and $z_{21,l}$ with degree greater than k. Since we can take $z_{ij,l}$ arbitrary, $a(e_{11,1}, e_{11,2}, \ldots, e_{21,n}) = 0$ for any tuple $(e_{11,1}, e_{11,2}, \ldots, e_{21,n})$. □

In order to show that the filtration J^k has trivial intersection, we prepare some lemmas.

Lemma 4.5. *For any* $f \in \mathfrak{P}$*,* f *can be written as*

$$\begin{aligned} f \equiv{}& a^{(q)}s_{22,n}^{q} + a^{(q-1)}s_{22,n}^{q-1} + \cdots + a^{(1)}s_{22,n} \\ &+ b^{(q)}s_{11,n}^{q} + b^{(q-1)}s_{11,n}^{q-1} + \cdots + b^{(1)}s_{11,n} + c \pmod{I}, \end{aligned}$$

where $a^{(m)}$*,* $b^{(m)}$ *and* c *are polynomials among* $s_{ij,l}$ *such that* $(i,j,l) \neq (1,1,n), (2,2,n)$*.*

Proof. First, for any $f \in \mathfrak{P}$, write f as

$$f = C^{(q)}s_{22,n}^{q} + C^{(q-1)}s_{22,n}^{q-1} + \cdots + C^{(1)}s_{22,n} + C^{(0)}$$

with $C^{(m)}$ polynomials among $s_{ij,l}$ for $(i,j,l) \neq (2,2,n)$. The coefficient $C^{(q)}$ can be written as

$$C^{(q)} = D^{(q)}(s_{11,n}+1) + a^{(q)}$$

with $D^{(q)}$ polynomials among $s_{ij,l}$ for $(i,j,l) \neq (2,2,n)$, and with $a^{(q)}$ a polynomial among $s_{ij,l}$ for $(i,j,l) \neq (1,1,n), (2,2,n)$. Since

$$(s_{11,n}+1)s_{22,n} = -s_{11,n} + s_{12,n}s_{21,n},$$

we have

$$f \equiv a^{(q)}s_{22,n}^{q} + C'^{(q-1)}s_{22,n}^{q-1} + \cdots + C'^{(1)}s_{22,n} + C'^{(0)} \pmod{I}$$

with $C'^{(m)}$ polynomials among $s_{ij,l}$ for $(i,j,l) \neq (2,2,n)$. By an inductive argument as above, we obtain

$$f \equiv a^{(q)}s_{22,n}^{q} + a^{(q-1)}s_{22,n}^{q-1} + \cdots + a^{(1)}s_{22,n} + a^{(0)} \pmod{I}$$

with $a^{(0)}$ a polynomial among $s_{ij,l}$ for $(i,j,l) \neq (2,2,n)$. Then $a^{(0)}$ can be written as

$$a^{(0)} = b^{(r)}s_{11,n}^{r} + b^{(r-1)}s_{11,n}^{r-1} + \cdots + b^{(1)}s_{11,n} + c$$

with $b^{(m)}$, c polynomials among $s_{ij,l}$ for $(i,j,l) \neq (1,1,n), (2,2,n)$. If $q > r$ (respectively, $r > q$), by setting $b^{(m)} = 0$ (respectively, $a^{(m)} = 0$) for $r+1 \leqslant m \leqslant q$ (respectively, $q+1 \leqslant m \leqslant r$), we obtain the required result. □

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Lemma 4.6. *For any $1 \leqslant l \leqslant n$ and $m \geqslant 1$, we have*

$$\begin{aligned} s_{22,l} &\equiv -s_{11,l} + s_{11,l}^2 - \cdots + (-1)^m s_{11,l}^m \\ &\quad + s_{12,l}s_{21,l} - s_{11,l}s_{12,l}s_{21,l} + s_{11,l}^2 s_{12,l}s_{21,l} - \cdots + (-1)^{m-1} s_{11,l}^{m-1} s_{12,l}s_{21,l} \\ &\quad + (-1)^m s_{11,l}^m s_{22,l} \pmod{I}. \end{aligned}$$

Proof. We can show this by induction on $m \geqslant 1$ using $s_{22,l} = -s_{11,l} + s_{12,l}s_{21,l} - s_{11,l}s_{22,l}$. □

Lemma 4.7. *Let n be an integer greater than 1. For some $f \in \tilde{J}_{n-1}$ and $m \geqslant 2$, if $\iota(f) \pmod{I_n} \in (J_n)^m$, then $f \pmod{I_{n-1}} \in (J_{n-1})^m$.*

Proof. By using Lemmas 4.5 and 4.6, we can write f as

$$f \equiv \sum_{k \geqslant 1} \sum_{e_{11,1}+\cdots+e_{21,n-1}=k} \alpha(e_{11,1}, e_{12,1}, e_{21,1}, \ldots, e_{21,n-1}) \times \prod_{l=1}^{n-1} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} \pmod{(J_{n-1})^m}$$

for some $\alpha(e_{11,1}, e_{12,1}, e_{21,1}, \ldots, e_{21,n-1}) \in \mathbb{Q}$. Then, by considering $\iota(f)$ and observing the above equation modulo $(J_n)^m$ through ι, we obtain that all the $\alpha(e_{11,1}, e_{12,1}, e_{21,1}, \ldots, e_{21,n-1})$ are equal to zero by applying Proposition 4.4 recursively. Therefore, $f \in (J_{n-1})^m$. □

Lemma 4.8. *For any $1 \leqslant l \leqslant n$, $q \geqslant 1$ and $m \geqslant 2(q+1)$, if we set*

$$\begin{aligned} s_{22,l}^q &\equiv A_{q,0}^{(q)} s_{11,l}^q + A_{q+1,0}^{(q)} s_{11,l}^{q+1} + \cdots + A_{m+1,0}^{(q)} s_{11,l}^{m+1} \\ &\quad + A_{q-1,1}^{(q)} s_{11,l}^{q-1} s_{12,l}s_{21,l} + A_{q,1}^{(q)} s_{11,l}^{q} s_{12,l}s_{21,l} + \cdots + A_{m-1,1}^{(q)} s_{11,l}^{m-1} s_{12,l}s_{21,l} \\ &\quad + A_{q-2,2}^{(q)} s_{11,l}^{q-2} s_{12,l}^2 s_{21,l}^2 + A_{q-1,2}^{(q)} s_{11,l}^{q-1} s_{12,l}^2 s_{21,l}^2 + \cdots + A_{m-3,1}^{(q)} s_{11,l}^{m-3} s_{12,l}^2 s_{21,l}^2 \\ &\quad \vdots \\ &\quad + A_{1,q-1}^{(q)} s_{11,l} s_{12,l}^{q-1} s_{21,l}^{q-1} + A_{2,q-1}^{(q)} s_{11,l}^2 s_{12,l}^{q-1} s_{21,l}^{q-1} \\ &\qquad + \cdots + A_{m-2q+3,q-1}^{(q)} s_{11,l}^{m-2q+3} s_{12,l}^{q-1} s_{21,l}^{q-1} \pmod{J^{m+2}}, \end{aligned}$$

then the coefficients $A_{*,*}^{(q+1)}$ of $s_{22,l}^{q+1} \pmod{J^{m+2}}$ are written as

$$\begin{aligned} A_{q+k,0}^{(q+1)} &= -A_{q+k-1,0}^{(q)} + A_{q+k-2,0}^{(q)} - \cdots + (-1)^k A_{q,0}^{(q)}, \\ A_{q-1+k,1}^{(q+1)} &= -A_{q+k-2,1}^{(q)} + A_{q+k-3,1}^{(q)} - \cdots + (-1)^k A_{q-1,1}^{(q)} \\ &\quad + (A_{q+k-1,0}^{(q)} - A_{q+k-2,0}^{(q)} + \cdots + (-1)^{k-1} A_{q,0}^{(q)}), \\ &\ \vdots \\ A_{k,q}^{(q+1)} &= -A_{k-1,q}^{(q)} + A_{k-2,q}^{(q)} - \cdots + (-1)^k A_{0,q}^{(q)} \\ &\quad + (\text{polynomial among } A_{i,j}^{(q)} \text{ with } j \leqslant q-1) \end{aligned}$$

for any $1 \leqslant k \leqslant m - 2q + 1$. In particular, $A_{q,0}^{(q)} = (-1)^q$.

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Proof. By observing the coefficients of the product $s_{22,l}^q \times s_{22,l}$, we obtain this lemma from a direct computation. □

As a corollary, we obtain the following.

Corollary 4.9. *For any* $1 \leqslant l \leqslant n$, $q \geqslant 1$, $1 \leqslant u \leqslant q$ *and* $q+1 \leqslant r \leqslant m+1$, *we have*

$$A_{r+1,0}^{(q+1)} = -A_{r,0}^{(q)} - A_{r,0}^{(q+1)},$$
$$A_{r+1,u}^{(q+1)} = -A_{r,u}^{(q)} - A_{r,u}^{(q+1)} + (\text{a polynomial among } A_{i,j}^{(q)} \text{ with } j \leqslant u-1).$$

Lemma 4.10. *For any* $1 \leqslant l \leqslant n$, $k \geqslant 1$, *we have*

$$\begin{vmatrix} A_{k+1,0}^{(1)} & A_{k+2,0}^{(1)} & A_{k+3,0}^{(1)} & \cdots & A_{2k,0}^{(1)} \\ A_{k+1,0}^{(2)} & A_{k+2,0}^{(2)} & A_{k+3,0}^{(2)} & \cdots & A_{2k,0}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k+1,0}^{(k)} & A_{k+2,0}^{(k)} & A_{k+3,0}^{(k)} & \cdots & A_{2k,0}^{(k)} \end{vmatrix} = 1.$$

Proof. Recall that $A_{q,0}^{(1)} = (-1)^q$ and $A_{r+1,0}^{(q+1)} = -A_{r,0}^{(q)} - A_{r,0}^{(q+1)}$. By applying the elementary transformations

- add $(k-1)$st column to kth column,
- add $(k-2)$nd column to $(k-1)$st column,

  $\vdots$

- add 1st column to 2nd column

in order, we see that

$$\begin{vmatrix} A_{k+1,0}^{(1)} & A_{k+2,0}^{(1)} & A_{k+3,0}^{(1)} & \cdots & A_{2k,0}^{(1)} \\ A_{k+1,0}^{(2)} & A_{k+2,0}^{(2)} & A_{k+3,0}^{(2)} & \cdots & A_{2k,0}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k+1,0}^{(k)} & A_{k+2,0}^{(k)} & A_{k+3,0}^{(k)} & \cdots & A_{2k,0}^{(k)} \end{vmatrix} = \begin{vmatrix} A_{k+1,0}^{(1)} & 0 & 0 & \cdots & 0 \\ A_{k+1,0}^{(2)} & A_{k+1,0}^{(1)} & A_{k+2,0}^{(1)} & \cdots & A_{2k-1,0}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k+1,0}^{(k)} & A_{k+1,0}^{(k-1)} & A_{k+2,0}^{(k-1)} & \cdots & A_{2k-1,0}^{(k-1)} \end{vmatrix}$$
$$= (-1)^{k+1} \begin{vmatrix} A_{k+1,0}^{(1)} & A_{k+2,0}^{(1)} & \cdots & A_{2k-1,0}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k+1,0}^{(k-1)} & A_{k+2,0}^{(k-1)} & \cdots & A_{2k-1,0}^{(k-1)} \end{vmatrix}.$$

By the inductive argument, we see that the determinant of the above matrix is equal to $(-1)^{(k+1)k} = 1$. □

Now we are ready to prove the main theorem of this section.

Theorem 4.11. *For any* $n \geqslant 1$, *we have* $\bigcap_{k \geqslant 1} J^k = \{0\}$.

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Proof. We prove this theorem by induction on $n \geqslant 1$. Assume that $n = 1$ and $f \in \bigcap_{k\geqslant 1} J^k$. By Lemma 4.5, we may assume that

$$\begin{aligned} f \equiv\, & a^{(q)} s_{22,1}^q + a^{(q-1)} s_{22,1}^{q-1} + \cdots + a^{(1)} s_{22,1} \\ & + b^{(q)} s_{11,1}^q + b^{(q-1)} s_{11,1}^{q-1} + \cdots + b^{(1)} s_{11,1} + c \pmod I \end{aligned}$$

for some $q \geqslant 1$ and $a^{(r)}$, $b^{(r)}$, $c \in \mathbb{Q}[s_{12,1}, s_{21,1}]$. We show that $a^{(r)} = b^{(r)} = c = 0$ for any $1 \leqslant r \leqslant q$.

For any $1 \leqslant r \leqslant q$, set

$$a^{(r)} := \sum_{i,j\geqslant 0} a_{i,j}^{(r)} s_{12,1}^i s_{21,1}^j$$

for $a_{i,j}^{(r)} \in \mathbb{Q}$. We remark that the sum in the above definition runs over finitely many tuples (i, j). Similarly, define $b_{ij}^{(r)}$ and $c_{ij} \in \mathbb{Q}$. Then we have

$$\begin{aligned} f \equiv & \sum_{r=1}^{q} \left(\sum_{i,j\geqslant 0} a_{i,j}^{(r)} s_{12,1}^i s_{21,1}^j \right) s_{22,1}^r + \sum_{r=1}^{q} \left(\sum_{i,j\geqslant 0} b_{i,j}^{(r)} s_{12,1}^i s_{21,1}^j \right) s_{11,1}^r \\ & + \sum_{i,j\geqslant 0} c_{i,j} s_{12,1}^i s_{21,1}^j \pmod I. \end{aligned}$$

For sufficiently large $m \gg 2q$, consider $f \equiv 0 \pmod{J^{m+2}}$. Rewrite $s_{22,1}^r$ as a polynomial among $s_{11,1}$, $s_{12,1}$, $s_{21,1}$ by Lemma 4.8. By Proposition 4.4, we see that the coefficients of $s_{11,1}^r s_{12,1}^i s_{21,1}^j$ are equal to zero for any r, i, j such that $r + i + j \leqslant m + 2$.

Consider the linear order $\leqslant'$ of $\mathbb{N} \times \mathbb{N}$ defined by

$$(i, j) \leqslant' (k, l) \iff i + j < k + l, \text{ or } i + j = k + l \text{ and } j \leqslant l.$$

For example, we have

$$(0, 0) \leqslant' (1, 0) \leqslant' (0, 1) \leqslant' (2, 0) \leqslant' (1, 1)' \leqslant' (0, 2) \leqslant' \cdots .$$

We show that all $a_{i,j}^{(r)}$ are equal to zero by the induction on (i, j) with respect to the above order. First, by observing the coefficients of $s_{11,1}^{q+1}, s_{11,1}^{q+2}, \ldots, s_{11,1}^{2q}$, we have

$$\begin{aligned} a_{0,0}^{(1)} A_{q+1,0}^{(1)} + a_{0,0}^{(2)} A_{q+1,0}^{(2)} + \cdots + a_{0,0}^{(q)} A_{q+1,0}^{(q)} &= 0, \\ a_{0,0}^{(1)} A_{q+2,0}^{(1)} + a_{0,0}^{(2)} A_{q+2,0}^{(2)} + \cdots + a_{0,0}^{(q)} A_{q+2,0}^{(q)} &= 0, \\ &\vdots \\ a_{0,0}^{(1)} A_{2q,0}^{(1)} + a_{0,0}^{(2)} A_{2q,0}^{(2)} + \cdots + a_{0,0}^{(q)} A_{2q,0}^{(q)} &= 0. \end{aligned}$$

From Lemma 4.10, we obtain

$$a_{0,0}^{(1)} = a_{0,0}^{(2)} = \cdots = a_{0,0}^{(q)} = 0.$$

Assume that $(i,j) >' (0,0)$. For any $q+1 \leqslant r \leqslant 2q$, the coefficients of $s_{11,1}^r s_{12,1}^i s_{21,1}^j$ satisfy

$$\begin{aligned} a_{i,j}^{(1)} A_{r,0}^{(1)} &+ a_{i,j}^{(2)} A_{r,0}^{(2)} + \cdots + a_{i,j}^{(q)} A_{r,0}^{(q)} \\ &+ a_{i-1,j-1}^{(1)} A_{r,1}^{(1)} + a_{i-1,j-1}^{(2)} A_{r,1}^{(2)} + \cdots + a_{i-1,j-1}^{(q)} A_{r,1}^{(q)} \\ &+ a_{i-2,j-2}^{(1)} A_{r,2}^{(1)} + a_{i-2,j-2}^{(2)} A_{r,2}^{(2)} + \cdots + a_{i-2,j-2}^{(q)} A_{r,2}^{(q)} + \cdots = 0, \end{aligned}$$

and hence

$$a_{i,j}^{(1)} A_{r,0}^{(1)} + a_{i,j}^{(2)} A_{r,0}^{(2)} + \cdots + a_{i,j}^{(q)} A_{r,0}^{(q)} = 0$$

by the inductive hypothesis. Thus we have

$$a_{i,j}^{(1)} = a_{i,j}^{(2)} = \cdots = a_{i,j}^{(q)} = 0$$

by Lemma 4.10. Therefore, we see that $a^{(r)} = 0$ for any $1 \leqslant r \leqslant q$. Thus we have

$$f \equiv b^{(q)} s_{11,1}^q + b^{(q-1)} s_{11,1}^{q-1} + \cdots + b^{(1)} s_{11,1} + c \pmod I \in \bigcap_{k \geqslant 1} J^k.$$

By using Proposition 4.4 recursively, we can see that $b^{(r)} = c = 0$ for any $1 \leqslant r \leqslant q$, and hence $f = 0$.

Next, assume that $n \geqslant 2$ and $f \in \bigcap_{k \geqslant 1} J^k$. Then we have $f \equiv 0 \pmod{J^k}$ for any $k \geqslant 1$. By Lemma 4.5, we may assume that

$$\begin{aligned} f = a^{(q)} s_{22,n}^q &+ a^{(q-1)} s_{22,n}^{q-1} + \cdots + a^{(1)} s_{22,n} \\ &+ b^{(q)} s_{11,n}^q + b^{(q-1)} s_{11,n}^{q-1} + \cdots + b^{(1)} s_{11,n} + c \pmod I \end{aligned} \tag{4.5}$$

for some $q \geqslant 1$ and $a^{(r)}$, $b^{(r)}$, $c \in \mathbb{Q}[s_{ij,l} \mid (i,j,l) \neq (1,1,n),(2,2,n)]$ as above. For any $1 \leqslant r \leqslant q$, set

$$a^{(r)} := \sum_{i,j \geqslant 0} a_{i,j}^{(r)} s_{12,n}^i s_{21,n}^j$$

for some $a_{i,j}^{(r)} \in \mathbb{Q}[s_{ij,l} \mid l \neq n]$. Similarly, define $b_{ij}^{(r)}$ and $c_{ij} \in \mathbb{Q}$.

Take any $m \geqslant 1$, and fix it. The indeterminates $s_{22,l}$ for $1 \leqslant l \leqslant n-1$ might appear in the coefficients $a_{i,j}^{(r)}$. By using Lemma 4.6 and Proposition 4.4, we have

$$\begin{aligned} a_{i,j}^{(r)} &\equiv \sum_{e_{11,1}+\cdots+e_{21,n-1}<m} \alpha_{i,j}^{(r)}(e_{11,1},\ldots,e_{21,n-1}) \prod_{l=1}^{n-1} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} \pmod{(J_{n-1})^m}, \\ b_{i,j}^{(r)} &\equiv \sum_{e_{11,1}+\cdots+e_{21,n-1}<m} \beta_{i,j}^{(r)}(e_{11,1},\ldots,e_{21,n-1}) \prod_{l=1}^{n-1} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} \pmod{(J_{n-1})^m}, \\ c_{i,j} &\equiv \sum_{e_{11,1}+\cdots+e_{21,n-1}<m} \gamma_{i,j}(e_{11,1},\ldots,e_{21,n-1}) \prod_{l=1}^{n-1} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}} \pmod{(J_{n-1})^m} \end{aligned}$$

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for some $\alpha_{i,j}^{(r)}(e_{11,1},\dots,e_{21,n-1}), \beta_{i,j}^{(r)}(e_{11,1},\dots,e_{21,n-1}), \gamma_{i,j}(e_{11,1},\dots,e_{21,n-1}) \in \mathbb{Q}$. Set

$$\tilde{a}_{i,j}^{(r)} := \sum_{e_{11,1}+\cdots+e_{21,n-1}<m} \alpha_{i,j}^{(r)}(e_{11,1},\dots,e_{21,n-1}) \prod_{l=1}^{n-1} s_{11,l}^{e_{11,l}} s_{12,l}^{e_{12,l}} s_{21,l}^{e_{21,l}}.$$

Similarly, define $\tilde{b}_{i,j}^{(r)}$ and $\tilde{c}_{i,j}$.

We show that all $\tilde{a}_{i,j}^{(r)}$ are equal to 0. Consider $f \equiv 0 \pmod{J_n^{m+q+1}}$. By observing the coefficients of $s_{11,n}^{q+1}$, we obtain

$$(\tilde{a}_{0,0}^{(1)}A_{q+1,0}^{(1)} + \tilde{a}_{0,0}^{(2)}A_{q+1,0}^{(2)} + \cdots + \tilde{a}_{0,0}^{(q)}A_{q+1,0}^{(q)})s_{11,n}^{q+1} \equiv 0 \pmod{(J_n)^{m+q+1}}.$$

On the other hand, since $\tilde{a}_{0,0}^{(1)}A_{q+1,0}^{(1)} + \tilde{a}_{0,0}^{(2)}A_{q+1,0}^{(2)} + \cdots + \tilde{a}_{0,0}^{(q)}A_{q+1,0}^{(q)}$ is a polynomial of degree at most $m-1$ in $\mathbb{Q}[s_{ij,l} \mid l \neq n]$, we see that

$$\tilde{a}_{0,0}^{(1)}A_{q+1,0}^{(1)} + \tilde{a}_{0,0}^{(2)}A_{q+1,0}^{(2)} + \cdots + \tilde{a}_{0,0}^{(q)}A_{q+1,0}^{(q)} \equiv 0 \pmod{(J_{n-1})^m}$$

by using Proposition 4.4. By the same argument, we obtain

$$\begin{aligned} \tilde{a}_{0,0}^{(1)}A_{q+1,0}^{(1)} + \tilde{a}_{0,0}^{(2)}A_{q+1,0}^{(2)} + \cdots + \tilde{a}_{0,0}^{(q)}A_{q+1,0}^{(q)} &= 0, \\ \tilde{a}_{0,0}^{(1)}A_{q+2,0}^{(1)} + \tilde{a}_{0,0}^{(2)}A_{q+2,0}^{(2)} + \cdots + \tilde{a}_{0,0}^{(q)}A_{q+2,0}^{(q)} &= 0, \\ &\vdots \\ \tilde{a}_{0,0}^{(1)}A_{2q,0}^{(1)} + \tilde{a}_{0,0}^{(2)}A_{2q,0}^{(2)} + \cdots + \tilde{a}_{0,0}^{(q)}A_{2q,0}^{(q)} &= 0. \end{aligned}$$

This shows that

$$\tilde{a}_{0,0}^{(1)} = \tilde{a}_{0,0}^{(2)} = \cdots = \tilde{a}_{0,0}^{(q)} = 0,$$

and hence

$$a_{0,0}^{(1)} \equiv a_{0,0}^{(2)} \equiv \cdots \equiv a_{0,0}^{(q)} \equiv 0 \pmod{(J_{n-1})^m}.$$

By using the same argument as that for the $n=1$ case, we obtain $a_{i,j}^{(r)} \equiv 0 \pmod{(J_{n-1})^m}$ for any $i,j \geqslant 0$. Since we can take $m \geqslant 1$ arbitrarily, we see that

$$a_{i,j}^{(r)} \in \bigcap_{k \geqslant 1} (J_{n-1})^m.$$

By the inductive hypothesis, we obtain $a_{i,j}^{(r)} = 0$, and hence $a^{(r)} = 0$. Thus,

$$f = b^{(q)}s_{11,n}^q + b^{(q-1)}s_{11,n}^{q-1} + \cdots + b^{(1)}s_{11,n} + c \pmod{I}.$$

By using Proposition 4.4 recursively, we can show that $b^{(r)} = c = 0$. This means that $f = 0$. Therefore, the induction proceeds. □

Theorem 4.12. *The ideal I is generated by (4.3) as an ideal of $\mathfrak{P}$.*

Proof. Let I' be the ideal generated by (4.3). It is clear that $I' \subset I$. For any $f \in I$, by using (4.3) we can write $f \pmod{I'}$ as a form in (4.5). Since $f \pmod{I} \in \bigcap_{k \geqslant 1} J^k$, by using the same argument as in the proof of Theorem 4.11, we can see that all coefficients $a^{(r)}$, $b^{(r)}$, c are equal to zero by using elements in I'. Thus $f \in I'$. □

For any $f \in \mathfrak{R}_{\mathbb{Q}}(F_n) \setminus \{0\}$, there exists some $k \geqslant 1$ such that $f \in J^k$ and $f \notin J^{k+1}$. We call this k the weight of f, and denote it by $\mathrm{wt}(f)$. By using the weight, we show the following.

Corollary 4.13. *The ring* $\mathfrak{R}_{\mathbb{Q}}(F_n)$ *is an integral domain. That is, the ideal* I *is prime.*

Proof. For any $f, g \in \mathfrak{R}_{\mathbb{Q}}(F_n) \setminus \{0\}$, let k_1 and k_2 be the weights of f and g, respectively. Then it is easily seen that the weight of fg is $k_1 + k_2$. This means that $fg \neq 0$. □

Finally, we discuss the relation between $\mathfrak{R}_{\mathbb{Q}}(F_n)$ and the universal SL_2-representation ring of F_n. Let $A_{\mathbb{Q}}(F_n, \mathrm{SL}(2,\mathbb{C}))$ be the quotient ring of $\mathbb{Q}[\alpha_{ij}(x) \mid 1 \leqslant i,j \leqslant 2,\ x \in F_n]$ with the ideal I_0 generated by

$$\alpha_{ij}(1) = \delta_{ij}, \quad \alpha_{ij}(xy) = \sum_{l=1}^{2} \alpha_{il}(x)\alpha_{lj}(y), \det(\alpha_{ij}(x)) = 1$$

for any $1 \leqslant i,j \leqslant 2$ and $x, y \in F_n$. The ring $A_{\mathbb{Q}}(F_n, \mathrm{SL}_2)$ is called the universal SL_2-representation ring of F_n over $\mathbb{Q}$.

Theorem 4.14. *The ring* $\mathfrak{R}_{\mathbb{Q}}(F_n)$ *is naturally isomorphic to* $A(F_n, \mathrm{SL}_2)$*.*

Proof. From Theorem 4.12, the natural isomorphism $\bar{\pi} \colon \mathfrak{P}/I \to \mathfrak{R}_{\mathbb{Q}}(F_n)$ factors through $A_{\mathbb{Q}}(F_n, \mathrm{SL}_2)$ with the natural homomorphism $\psi \colon \mathfrak{P}/I \to A_{\mathbb{Q}}(F_n, \mathrm{SL}_2)$ defined by $s_{ij,l} \mapsto \alpha_{ij}(x_l)$. Since ψ is surjective, it must be the isomorphism. □

5. On the action of Aut F_n on $\mathfrak{R}_{\mathbb{Q}}(F_n)$

5.1. A descending filtration $\{\mathcal{D}_n(k)\}$ of Aut F_n

In this section, we always identify the ideal J in $\mathfrak{P}/I$ with its image in $\mathfrak{R}_{\mathbb{Q}}(F_n)$ through the isomorphism $\mathfrak{P}/I \cong \mathfrak{R}_{\mathbb{Q}}(F_n)$ induced from π unless otherwise noted. Set $a'_{ii}(x) := a_{ii}(x) - 1 \in \mathfrak{R}_{\mathbb{Q}}(F_n)$ for any $1 \leqslant i \leqslant 2$ and $x \in F_n$. The indeterminates $s_{ii,l}$, $s_{12,l}$ and $s_{21,l}$ correspond to $a'_{ii}(x_l)$, $a_{12}(x_l)$ and $a_{21}(x_l)$, respectively. For any $k \geqslant 1$, let

$$\mathcal{D}_n(k) := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{Aut}(J/J^{k+1}))$$

be the kernel of the homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}(J/J^{k+1})$ induced from the action of $\mathrm{Aut}\, F_n$ on J/J^{k+1}. Then the groups $\mathcal{D}_n(k)$ define a descending filtration

$$\mathcal{D}_n(1) \supset \mathcal{D}_n(2) \supset \cdots \supset \mathcal{D}_n(k) \supset \cdots$$

of $\mathrm{Aut}\, F_n$. In this section, we study this filtration. First, we show that this is a central filtration of $\mathrm{Aut}\, F_n$. For any $f \in J$ and $\sigma \in \mathrm{Aut}\, F_n$, set

$$s_\sigma(f) := f^\sigma - f \in J.$$

Then we have the following lemma.

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Lemma 5.1. *For any $f \in J$ and $\sigma, \tau \in \operatorname{Aut} F_n$,*

(1) $s_{\sigma\tau}(f) = (s_\sigma(f))^\tau + s_\tau(f)$,

(2) $s_{1_{F_n}}(f) = 0$,

(3) $s_{\sigma^{-1}}(f) = -(s_\sigma(f))^{\sigma^{-1}}$,

(4) $s_{[\sigma,\tau]}(f) = \{s_\tau(s_\sigma(f)) - s_\sigma(s_\tau(f))\}^{\sigma^{-1}\tau^{-1}}$.

Proof. Parts (1), (2) and (3) are straightforward. Here we prove part (4). Using parts (1) and (3), we obtain

$$\begin{aligned} s_{[\sigma,\tau]}(f) &\overset{(1)}{=} (s_{\sigma\tau}(f))^{\sigma^{-1}\tau^{-1}} + s_{\sigma^{-1}\tau^{-1}}(f) \\ &\overset{(3)}{=} (s_{\sigma\tau}(f))^{\sigma^{-1}\tau^{-1}} - (s_{\tau\sigma}(f))^{\sigma^{-1}\tau^{-1}} \\ &\overset{(1)}{=} \{(s_\sigma(f))^\tau + s_\tau(f) - (s_\tau(f))^\sigma - s_\sigma(f)\}^{\sigma^{-1}\tau^{-1}} \\ &= \{s_\tau(s_\sigma(f)) - s_\sigma(s_\tau(f))\}^{\sigma^{-1}\tau^{-1}}. \end{aligned}$$

This completes the proof of Lemma 5.1. □

Lemma 5.2. *For any $k, l \geqslant 1$, $f \in J^m$ and $\sigma \in \mathcal{D}_n(k)$, we have $s_\sigma(f) \in J^{k+m}$.*

Proof. It suffices to show the lemma for the case in which f is (the coset class of) a monomial $s_{i_1 j_1, l_1} s_{i_2 j_2, l_2} \cdots s_{i_m j_m, l_m}$. Then we have

$$\begin{aligned} s_\sigma(f) &= f^\sigma - f \\ &= (s_{i_1 j_1, l_1})^\sigma \cdots (s_{i_m j_m, l_m})^\sigma - s_{i_1 j_1, l_1} \cdots s_{i_m j_m, l_m} \\ &= (s_{i_1 j_1, l_1} + s_\sigma(s_{i_1 j_1, l_1})) \cdots (s_{i_m j_m, l_m} + s_\sigma(s_{i_m j_m, l_m})) - s_{i_1 j_1, l_1} \cdots s_{i_m j_m, l_m}. \end{aligned}$$

By the definition of $\mathcal{D}_n(k)$, the elements $s_\sigma(s_{i_1 j_1, l_1}), \ldots, s_\sigma(s_{i_m j_m, l_m})$ belong to J^{k+1}. Therefore, we obtain $s_\sigma(f) \in J^{k+m}$. This completes the proof of Lemma 5.2. □

Proposition 5.3. *For any $k, m \geqslant 1$, $[\mathcal{D}_n(k), \mathcal{D}_n(m)] \subset \mathcal{D}_n(k+m)$.*

Proof. For any $\sigma \in \mathcal{D}_n(k)$, $\tau \in \mathcal{D}_n(m)$ and $f \in J$, by Lemmas 5.1 and 5.2, we see that

$$\begin{aligned} s_{[\sigma,\tau]}(f) &= \{s_\tau(s_\sigma(f)) - s_\sigma(s_\tau(f))\}^{\sigma^{-1}\tau^{-1}} \\ &\equiv 0 \pmod{J^{k+m+1}}. \end{aligned}$$

Hence, $[\sigma, \tau] \in \mathcal{D}_n(k+m)$. This completes the proof of Proposition 5.3. □

This proposition shows that the filtration $\mathcal{D}_n(k)$ is a central filtration of $\operatorname{Aut} F_n$. Next, we consider how different the filtration $\mathcal{D}_n(k)$ is from the Andreadakis–Johnson filtration $\mathcal{A}_n(k)$ of $\operatorname{Aut} F_n$.

Proposition 5.4. *For any* $n \geqslant 2$, $\mathcal{D}_n(1) = \mathrm{IA}_n$.

Proof. We remark that

$$\{a'_{11}(x_l), a_{12}(x_l), a_{21}(x_l) \pmod{J^2} \mid 1 \leqslant l \leqslant n\}$$

is a basis of J/J^2 as a $\mathbb{Q}$-vector space by Proposition 4.4. On the other hand, by using (4.1) and (4.2), we can see that

$$\begin{aligned} a'_{11}(w) &\equiv e_1 a'_{11}(x_{i_1}) + \cdots + e_r a'_{11}(x_{i_r}) \pmod{J^2}, \\ a_{12}(w) &\equiv e_1 a_{12}(x_{i_1}) + \cdots + e_r a_{12}(x_{i_r}) \pmod{J^2}, \\ a_{21}(w) &\equiv e_1 a_{21}(x_{i_1}) + \cdots + e_r a_{21}(x_{i_r}) \pmod{J^2} \end{aligned}$$

for any $w = x_{i_1}^{e_1} \cdots x_{i_r}^{e_r} \in F_n$. Hence, for any $\sigma \in \mathcal{D}_n(1)$ and any $1 \leqslant l \leqslant n$, we have

$$x_l^\sigma = x_l c_l$$

for some $c_l \in \Gamma_n(2)$. Thus, $\sigma \in \mathrm{IA}_n$. It is obvious that $\mathrm{IA}_n \subset \mathcal{D}_n(1)$. □

From Proposition 5.4, we see that the filtration $\{\mathcal{D}_n(k)\}$ contains the lower central series $\{\mathcal{A}'_n(k)\}$ of IA_n. Below, we show that the filtration $\{\mathcal{D}_n(k)\}$ contains the Andreadakis–Johnson filtration $\{\mathcal{A}_n(k)\}$. Namely, we will see that $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k) \subset \mathcal{D}_n(k)$.

Lemma 5.5. *For any* $k \geqslant 1$ *and* $y \in \Gamma_n(k)$, *we have* $a'_{11}(y), a_{12}(y), a_{21}(y), a'_{22}(y) \in J^k$.

Proof. Since $\Gamma_n(k)$ is generated by all left-normed commutators, it suffices to show the lemma in the case where $y = [y_1, y_2, \ldots, y_k]$ for some $y_1, \ldots, y_k \in F_n$. We use induction on $k \geqslant 1$. If $k = 1$, the lemma is clear. Assume that $k \geqslant 2$ and set $z := [y_1, \ldots, y_{k-1}]$. Then $a'_{ii}(z), a_{12}(z), a_{21}(z) \in J^{k-1}$ by the inductive hypothesis. Furthermore, from (4.1), we see that $a'_{ii}(z^{-1}), a_{12}(z^{-1}), a_{21}(z^{-1}) \in J^{k-1}$. By using (4.2), we have

$$\begin{aligned} a'_{11}([z, y_k]) &= a'_{11}(z y_k z^{-1} y_k^{-1}) \\ &= a'_{11}(z y_k) a'_{11}(z^{-1} y_k^{-1}) + a_{12}(z y_k) a_{21}(z^{-1} y_k^{-1}) + a'_{11}(z y_k) + a'_{11}(z^{-1} y_k^{-1}) \\ &\equiv a'_{11}(y_k) a'_{11}(y_k^{-1}) + a_{12}(y_k) a_{21}(y_k^{-1}) \\ &\qquad + a'_{11}(z) + a'_{11}(y_k) + a'_{11}(z^{-1}) + a'_{11}(y_k^{-1}) \pmod{J^k} \\ &\equiv a'_{11}(z) + a'_{11}(z^{-1}) \pmod{J^k} \\ &\equiv 0 \pmod{J^k}. \end{aligned}$$

The last equation follows from

$$0 = a'_{11}(1) = a'_{11}(z z^{-1}) \equiv a'_{11}(z) + a'_{11}(z^{-1}) \pmod{J^k}.$$

Similarly, we can obtain that $a_{12}(y), a_{21}(y), a'_{22}(y) \in J^k$. □

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Lemma 5.6. *For any $k \geqslant 1$, $z \in F_n$ and $y \in \Gamma_n(k)$, we have*

$$\begin{aligned} a'_{11}(zy) &\equiv a'_{11}(y) \pmod{J^k}, \\ a_{12}(zy) &\equiv a_{12}(y) \pmod{J^k}, \\ a_{21}(zy) &\equiv a_{21}(y) \pmod{J^k}, \\ a'_{22}(zy) &\equiv a'_{22}(y) \pmod{J^k}. \end{aligned}$$

Proof. This lemma follows from (4.2) and Lemma 5.5 immediately. □

From Lemma 5.6, we obtain the following theorem.

Theorem 5.7. *For any $k \geqslant 1$, $\mathcal{A}_n(k) \subset \mathcal{D}_n(k)$.*

Next, we give a sufficient condition for $\mathcal{A}_n(k) = \mathcal{D}_n(k)$. Consider the homomorphisms $\alpha_{ii}^{(k)}$ and $\alpha_{ij}^{(k)} \colon \Gamma_n(k) \to \mathrm{gr}^k(J)$ defined by

$$x \mapsto a'_{ii}(x) \quad \text{and} \quad a_{ij}(x),$$

respectively. By Lemma 5.5, $\alpha_{ii}^{(k)}$ and $\alpha_{ij}^{(k)}$ naturally induce the $\mathrm{GL}(n,\mathbb{Z})$-equivariant homomorphisms $\mathcal{L}_n(k) \to \mathrm{gr}^k(J)$, which are also denoted by $\alpha_{ii}^{(k)}$ and $\alpha_{ij}^{(k)}$, respectively, with some abuse of notation.

Proposition 5.8. *Let k be a positive integer. For any $1 \leqslant m \leqslant k$, assume that*

$$\mathrm{Ker}(\alpha_{11}^{(m)}) \cap \mathrm{Ker}(\alpha_{12}^{(m)}) \cap \mathrm{Ker}(\alpha_{21}^{(m)}) \cap \mathrm{Ker}(\alpha_{22}^{(m)}) = \{0\}.$$

Then $\mathcal{D}_n(k) \subset \mathcal{A}_n(k)$.

Proof. Assume that $\sigma \in \mathcal{D}_n(k)$ and $\sigma \notin \mathcal{A}_n(k)$. Since $\mathcal{D}_n(k) \subset \mathcal{D}_n(1) = \mathcal{A}_n(1)$, there exists some $1 \leqslant m \leqslant k-1$ such that $\sigma \in \mathcal{A}_n(m) \setminus \mathcal{A}_n(m+1)$. Thus, there exists some $1 \leqslant l \leqslant n$ such that $x_l^{-1}x_l^{\sigma} \in \Gamma_n(m+1)$ and $x_l^{-1}x_l^{\sigma} \notin \Gamma_n(m+2)$. By the assumption, at least one of $a'_{ii}(x_l^{-1}x_l^{\sigma})$ and $a_{ij}(x_l^{-1}x_l^{\sigma})$ for $i \neq j$ does not belong to J^{m+2}. Without loss of generality, we may assume that $a'_{11}(x_l^{-1}x_l^{\sigma}) \notin J^{m+2}$. If we set $\gamma := x_l^{-1}x_l^{\sigma} \in \Gamma_n(m+1)$, then

$$\begin{aligned} a'_{11}(x_l^{\sigma}) &= a'_{11}(x_l\gamma) \\ &= a'_{11}(x_l)a'_{11}(\gamma) + a_{12}(x_l)a_{21}(\gamma) + a'_{11}(x_l) + a'_{11}(\gamma) \\ &\equiv a'_{11}(x_l) + a'_{11}(\gamma) \pmod{J^{m+2}}. \end{aligned}$$

On the other hand, since $\sigma \in \mathcal{D}_n(k)$, we have $a'_{11}(x_l^{\sigma}) - a'_{11}(x_l) \in J^{k+1} \subset J^{m+2}$. This is a contradiction. Therefore, $\sigma \in \mathcal{A}_n(k)$. □

Using Proposition 5.8, we can show that $\mathcal{A}_n(k) = \mathcal{D}_n(k)$ for $1 \leqslant k \leqslant 4$ by a straightforward calculation. Below, we give this by showing that $\alpha_{11}^{(k)}$ is injective for $1 \leqslant k \leqslant 4$. At this stage, we do not know whether $\alpha_{ii}^{(k)}$ and $\alpha_{ij}^{(k)} \colon \mathcal{L}_n(k) \to \mathrm{gr}^k(J)$ are injective or not in general.

Table 1. *Basic commutators of weight less than* 5.

m	basic commutators	
1	x_i	$1 \leqslant i \leqslant n$
2	$[x_i, x_j]$	$1 \leqslant j < i \leqslant n$
3	$[x_i, x_j, x_k]$	$i > j \leqslant k$
4	$[x_i, x_j, x_k, x_l]$	$i > j \leqslant k \leqslant l$
	$[[x_i, x_j], [x_k, x_l]]$	$i > k > l,\ i > j$
	$[[x_i, x_j], [x_i, x_l]]$	$i > j > l$

Lemma 5.9. *For any* $x, y, z, w \in F_n$, *we have*

$$\begin{aligned}
\alpha_{11}^{(2)}([x,y]) &= a_{12}(x)a_{21}(y) - a_{12}(y)a_{21}(x),\\
\alpha_{11}^{(3)}([x,y,z]) &= 2(a'_{11}(x)a_{12}(y)a_{21}(z) + a'_{11}(x)a_{12}(z)a_{21}(y)\\
&\qquad - a'_{11}(y)a_{12}(x)a_{21}(z) - a'_{11}(y)a_{12}(z)a_{21}(x)),\\
\alpha_{11}^{(4)}([x,y,z,w]) &= 2a'_{11}(x)a_{12}(y)a_{12}(z)a_{21}(w) - 2a'_{11}(y)a_{12}(x)a_{12}(z)a_{21}(w)\\
&\quad - 4a'_{11}(x)a'_{11}(z)a_{12}(y)a_{21}(w) + 4a'_{11}(y)a'_{11}(z)a_{12}(x)a_{21}(w)\\
&\quad - 4a'_{11}(y)a'_{11}(z)a_{12}(w)a_{21}(x) + 4a'_{11}(x)a'_{11}(z)a_{12}(w)a_{21}(y)\\
&\quad + 2a_{12}(x)a_{12}(w)a_{21}(y)a_{21}(z) - 2a_{12}(y)a_{12}(w)a_{21}(x)a_{21}(z),\\
\alpha_{11}^{(4)}([[x,y],[z,w]]) &= 4a'_{11}(x)a'_{11}(w)a_{12}(y)a_{21}(z) - 4a'_{11}(x)a'_{11}(z)a_{12}(y)a_{21}(w)\\
&\quad - 4a'_{11}(y)a'_{11}(w)a_{12}(x)a_{21}(z) + 4a'_{11}(y)a'_{11}(z)a_{12}(x)a_{21}(w)\\
&\quad - 4a'_{11}(y)a'_{11}(z)a_{12}(w)a_{21}(x) + 4a'_{11}(x)a'_{11}(z)a_{12}(w)a_{21}(y)\\
&\quad + 4a'_{11}(y)a'_{11}(w)a_{12}(z)a_{21}(x) - 4a'_{11}(x)a'_{11}(w)a_{12}(z)a_{21}(y).
\end{aligned}$$

Proof. We can obtain these by direct computation. □

Proposition 5.10. *For any* $1 \leqslant m \leqslant 4$, *the homomorphism* $\alpha_{11}^{(m)} \colon \mathcal{L}_n(m) \to \mathrm{gr}^m(J)$ *is injective.*

Proof. First, we consider the Hall basis of $\mathcal{L}_n(m)$. By the theory of commutator calculus due to Hall, the basic commutators of weight m form a basis of $\mathcal{L}_n(m)$. In Table 1 we give a list of basic commutators of weight $m \leqslant 4$. (See also [**7**] for details on the basic commutators of the free groups.) It suffices to show that the images of basic commutators of weight m under the map $\alpha_{11}^{(m)}$ are linearly independent. It is clear for the case in which $m = 1$.

Case 1 (*m* = 2). Assume that

$$\sum_{i>j} c_{ij}\alpha_{11}^{(2)}([x_i, x_j]) = \sum_{i>j} c_{ij}(a_{12}(x_i)a_{21}(x_j) - a_{12}(x_j)a_{21}(x_i)) = 0$$

https://doi.org/10.1017/S0013091516000456 Published online by Cambridge University Press

for some $c_{ij} \in \mathbb{Q}$. We remark that $a_{12}(x_i)a_{21}(x_j)$ and $a_{12}(x_j)a_{21}(x_i)$ for $i > j$ are members of the basis of $\mathrm{gr}^2(J)$ given in Proposition 4.4 such that there is no overlap between them. Then we see that $c_{ij} = 0$ for any $i > j$.

Case 2 ($m = 3$). Assume that

$$\begin{aligned}\sum_{i>j\leqslant k} c_{ijk}\alpha_{11}^{(3)}([x_i, x_j, x_k]) &= \sum_{i>j\leqslant k} 2c_{ijk}(a'_{11}(x_i)a_{12}(x_j)a_{21}(x_k) + a'_{11}(x_i)a_{12}(x_k)a_{21}(x_j) \\ &\qquad - a'_{11}(x_j)a_{12}(x_i)a_{21}(x_k) - a'_{11}(x_j)a_{12}(x_k)a_{21}(x_i)) \\ &= 0\end{aligned}$$

for some $c_{ijk} \in \mathbb{Q}$. We remark that $a'_{11}(x_i)a_{12}(x_j)a_{21}(x_k)$, $a'_{11}(x_i)a_{12}(x_k)a_{21}(x_j)$, $a'_{11}(x_j)$ $a_{12}(x_i)a_{21}(x_k)$ and $a'_{11}(x_j)a_{12}(x_k)a_{21}(x_i)$ for any $i > j \leqslant k$ are members of the basis of $\mathrm{gr}^3(J)$ given in Proposition 4.4. On the left-hand side of the above equation, for any distinct i, j, k such that $i > j < k$, the coefficient of $a'_{11}(x_i)a_{12}(x_j)a_{21}(x_k)$ is $2c_{ijk}$. Similarly, for any $i > j$, the coefficients of $a'_{11}(x_i)a_{12}(x_j)a_{21}(x_i)$ and $a'_{11}(x_i)a_{12}(x_j)a_{21}(x_j)$ are $2c_{iji}$ and $4c_{ijj}$. Hence, $c_{ijk} = 0$ for any $i > j \leqslant k$.

Case 3 ($m = 4$). Assume that

$$\sum_{i>j\leqslant k\leqslant l} c_{ijkl}\alpha_{11}^{(4)}([x_i, x_j, x_k, x_l]) + \sum_{i>k>l,\, i>j} d_{ijkl}\alpha_{11}^{(4)}([[x_i, x_j], [x_k, x_l]]) + \sum_{i>j>l} d_{ijil}\alpha_{11}^{(4)}([[x_i, x_j], [x_i, x_l]]) = 0$$

for some $c_{ijkl}, d_{ijkl} \in \mathbb{Q}$. By an argument similar to above, we see the following. For any $i > j \leqslant k \leqslant l$, the coefficient of $a'_{11}(x_i)a_{12}(x_j)a_{12}(x_k)a_{21}(x_l)$ on the left-hand side of the above equation is $2c_{ijkl}$. Hence, all c_{ijkl} are equal to zero. Next, for any distinct i, j, k, l such that $i > k > l$ and $i > j$, the coefficient of $a'_{11}(x_j)a'_{11}(x_l)a_{12}(x_i)a_{21}(x_k)$ is $-4d_{ijkl}$. For any $i > k > j$, the coefficient of $a'_{11}(x_j)a'_{11}(x_j)a_{12}(x_i)a_{21}(x_k)$ is $-4d_{ijkj}$. For any $i > j > l$, the coefficient of $a'_{11}(x_i)a'_{11}(x_i)a_{12}(x_j)a_{21}(x_l)$ is $-4d_{ijil}$. Hence, all d_{ijkl} are equal to zero. This completes the proof of Proposition 5.10. □

Then we have the following.

Corollary 5.11. *For any* $1 \leqslant k \leqslant 4$, $\mathcal{A}_n(k) = \mathcal{D}_n(k)$.

5.2. Graded quotients $\mathrm{gr}^k(\mathcal{D}_n)$

In this section we study some properties of the graded quotients $\mathrm{gr}^k(\mathcal{D}_n) := \mathcal{D}_n(k)/\mathcal{D}_n(k+1)$. Since each $\mathcal{D}_n(k)$ is a normal subgroup of $\mathrm{Aut}\, F_n$, the group $\mathrm{Aut}\, F_n$ naturally acts on $\mathrm{gr}^k(\mathcal{D}_n)$ by conjugation from the right. Furthermore, since $\{\mathcal{D}_n(k)\}$ is a central filtration, the action of $\mathcal{D}_n(1) = \mathrm{IA}_n$ on $\mathrm{gr}^k(\mathcal{D}_n)$ is trivial. Hence, we can consider each $\mathrm{gr}^k(\mathcal{D}_n)$ as an $\mathrm{Aut}\, F_n/\mathcal{D}_n(1) = \mathrm{GL}(n, \mathbb{Z})$-module.

https://doi.org/10.1017/S0013091516000456 Published online by Cambridge University Press

To begin with, we introduce analogues of Johnson homomorphisms to study the $\mathrm{GL}(n,\mathbb{Z})$-module structure of $\mathrm{gr}^k(\mathcal{D}_n)$. To begin with, for any $k \geqslant 1$ and $\sigma \in \mathcal{D}_n(k)$, define the map $\eta_k(\sigma)\colon \mathrm{gr}^1(J) \to \mathrm{gr}^{k+1}(J)$ to be

$$\eta_k(\sigma)(f) := s_\sigma(f) = f^\sigma - f \in \mathrm{gr}^{k+1}(J)$$

for any $f \in J$. That the map $\eta_k(\sigma)$ is well defined follows from Lemma 5.2. It is easily seen that $\eta_k(\sigma)$ is a homomorphism. Then we have the map

$$\eta_k\colon \mathrm{gr}^k(\mathcal{D}_n) \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^{k+1}(J))$$

defined by $\sigma \mapsto \eta_k(\sigma)$. For any $\sigma, \tau \in \mathcal{D}_n(k)$, from Lemma 5.1 (1), and from Lemma 5.2, we see that

$$s_{\sigma\tau}(f) = (s_\sigma(f))^\tau + s_\tau(f) \equiv s_\sigma(f) + s_\tau(f) \pmod{J^{k+2}}.$$

This shows that η_k is a homomorphism. By definition, each η_k is injective. Furthermore, we have the following lemma.

Lemma 5.12. *For each* $k \geqslant 1$*, the homomorphism* η_k *is a* $\mathrm{GL}(n,\mathbb{Z})$*-equivariant homomorphism.*

Proof. It suffices to show that η_k is an $\mathrm{Aut}\,F_n$-equivariant homomorphism. For any $\sigma \in \mathrm{Aut}\,F_n$ and $\tau \in \mathcal{D}_n(k)$, we see that

$$\begin{aligned}\eta_k(\tau\cdot\sigma)(f) &= \eta_k(\sigma^{-1}\tau\sigma)(f) = s_{\sigma^{-1}\tau\sigma}(f),\\ (\eta_k(\tau)\cdot\sigma)(f) &= (\eta_k(\tau)(f^{\sigma^{-1}}))^\sigma = s_\tau(f^{\sigma^{-1}})^\sigma\\ &= (f^{\sigma^{-1}\tau} - f^{\sigma^{-1}})^\sigma = f^{\sigma^{-1}\tau\sigma} - f = s_{\sigma^{-1}\tau\sigma}(f)\end{aligned}$$

for any $f \in J$. Hence, we have $\eta_k(\tau\cdot\sigma) = \eta_k(\tau)\cdot\sigma$. This means that η_k is an $\mathrm{Aut}\,F_n$-equivariant homomorphism. □

By using the homomorphisms η_k, we see that $\mathrm{gr}^k(\mathcal{D}_n)$ is an $\mathrm{GL}(n,\mathbb{Z})$-submodule of the $\mathbb{Q}$-vector space $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^{k+1}(J))$, and hence we obtain the following proposition.

Proposition 5.13. *For any* $n \geqslant 2$*,*

(1) *each* $\mathrm{gr}^k(\mathcal{D}_n)$ *is torsion-free,*

(2) $\dim_{\mathbb{Q}}(\mathrm{gr}^k(\mathcal{D}_n) \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$.

If $\mathcal{D}_n(k) = \mathcal{A}_n(k)$, then the above facts follow immediately from Andreadakis's result for the Andreadakis–Johnson filtration in [**1**].

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Table 2. *Nielsen generators of* $\mathrm{Aut}\,F_n$.

	x_1	x_2	x_3	$\cdots$	x_{n-1}	x_n
P	x_2	x_1	x_3	$\cdots$	x_{n-1}	x_n
Q	x_2	x_3	x_4	$\cdots$	x_n	x_1
S	x_1^{-1}	x_2	x_3	$\cdots$	x_{n-1}	x_n
U	x_1x_2	x_2	x_3	$\cdots$	x_{n-1}	x_n

5.3. An extension of η_1 to $\mathrm{Aut}\,F_n$ as a crossed homomorphism

Here we consider the first homomorphism η_1 and its extension to $\mathrm{Aut}\,F_n$ as a crossed homomorphism. We can easily calculate the images of Magnus generators K_{ij} and K_{ijl} of IA_n by η_1 as

$$\left.\begin{aligned} \eta_1(K_{ij}) &= s^*_{11,i} \otimes (s_{12,i}s_{21,j} - s_{12,j}s_{21,i}) + 2s^*_{12,i} \otimes (s_{11,i}s_{12,j} - s_{11,j}s_{12,i}) \\ &\quad + 2s^*_{21,i} \otimes (s_{21,i}s_{11,j} - s_{11,i}s_{21,j}), \\ \eta_1(K_{ijl}) &= s^*_{11,i} \otimes (s_{12,j}s_{21,l} - s_{12,l}s_{21,j}) + 2s^*_{12,i} \otimes (s_{11,j}s_{12,l} - s_{11,l}s_{12,j}) \\ &\quad + 2s^*_{21,i} \otimes (s_{21,j}s_{11,l} - s_{11,j}s_{21,l}). \end{aligned}\right\} \tag{5.1}$$

Recall that H is the abelianization of F_n. It is easily seen that the image of the first Johnson homomorphism τ_1 is $H^* \otimes_{\mathbb{Z}} \Lambda^2 H$. From independent works of Cohen and Pakianathan [**2**, **3**], Farb [**4**] and Kawazumi [**15**], it is known that τ_1 is the abelianization of IA_n. In other words, $\mathcal{A}_n(2)$ coincides with the commutator subgroup of IA_n. Hence, the abelianization of IA_n is the free abelian group with basis Magnus generators. Let V be the rationalization $(H^* \otimes_{\mathbb{Z}} \Lambda^2 H) \otimes_{\mathbb{Z}} \mathbb{Q}$ of $H^* \otimes_{\mathbb{Z}} \Lambda^2 H$. In [**26**], we computed

$$H^1(\mathrm{Aut}\,F_n, V) = \mathbb{Q}^{\oplus 2}$$

for any $n \geqslant 5$, and showed that $H^1(\mathrm{Aut}\,F_n, V)$ is generated by Morita's cocycle f_{M} and Kawazumi's cocycle f_{K}. They are crossed homomorphisms of $\mathrm{Aut}\,F_n$ defined with the Magnus representation and the Magnus expansion, respectively. In particular, f_{K} is an extension of the rational first Johnson homomorphism

$$\mathrm{IA}_n \to \mathrm{gr}^1(\mathcal{A}_n) \xrightarrow{\tau_1} H^* \otimes_{\mathbb{Z}} \Lambda^2 H \xrightarrow{\otimes_{\mathbb{Z}}\mathbb{Q}} V.$$

Since $\mathcal{D}_n(k) = \mathcal{A}_n(k)$ for $1 \leqslant k \leqslant 2$ from Corollary 5.11, and since $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))$ is a $\mathbb{Q}$-vector space, it turns out that there exist crossed homomorphisms g_{M} and g_{K}, corresponding to f_{M} and f_{K}, respectively, such that g_{K} is an extension of the homomorphism

$$\mathrm{IA}_n \to \mathrm{gr}^1(\mathcal{D}_n) \xrightarrow{\eta_1} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J)).$$

We give the images of Nielsen's generators of $\mathrm{Aut}\,F_n$ by g_{K}. Let P, Q, S and U be automorphisms of F_n given by specifying its images of the basis $x_1, \ldots, x_n$ as shown in Table 2.

In 1924, Nielsen [**22**] showed that $\mathrm{Aut}\,F_n$ is generated by P, Q, S and U, and gave finitely many relations among them. In [**26**], we gave the images of P, Q, S and U by f_{M} and f_{K} explicitly. From this, we see that

$$g_{\mathrm{M}}(\sigma) := \begin{cases} \displaystyle\sum_{i=2}^{n}(s^*_{11,i} \otimes (s_{12,i}s_{21,1} - s_{12,1}s_{21,i}) \\ \qquad + 2s^*_{12,i} \otimes (s_{11,i}s_{12,1} - s_{11,1}s_{12,i}) \\ \qquad\qquad + 2s^*_{21,i} \otimes (s_{11,1}s_{21,i} - s_{11,i}s_{21,1})), & \sigma = S, \\ 0, & \sigma = P, Q, U \end{cases}$$

and

$$g_{\mathrm{K}}(\sigma) := \begin{cases} -\, s^*_{11,1} \otimes (s_{12,1}s_{21,2} - s_{12,2}s_{21,1}) \\ \qquad - 2s^*_{12,1} \otimes (s_{11,1}s_{12,2} - s_{11,2}s_{12,1}) \\ \qquad\qquad - 2s^*_{21,1} \otimes (s_{11,2}s_{21,1} - s_{11,1}s_{21,2}), & \sigma = U, \\ 0, & \sigma = P, Q, S. \end{cases}$$

In the following, we show that g_{M} and g_{K} are linearly independent in the first cohomology group $H^1(\mathrm{Aut}\,F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J)))$. In order to do this, to begin with, we determine the $\mathrm{GL}(n,\mathbb{Z})$-module structure of $\mathrm{gr}^k(J)$ for any $k \geqslant 1$. Recall that $\mathrm{GL}(n,\mathbb{Z})$ is identified with $\mathrm{Aut}\,F_n/\mathrm{IA}_n$ induced from the abelianization of F_n. Let H be the abelianization of F_n, and consider H as an additive group here. We have $\mathcal{L}_n(1) \cong H$, and write α_{ij} for $\alpha_{ij}^{(1)}$ for simplicity. Then, for any $y_1, \ldots, y_l \in H$, we can consider $\alpha_{ij}(y_1)\cdots\alpha_{ij}(y_l)$ as an element in $\mathrm{gr}^l(J)$. Set $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition 5.14. *For any* $k \geqslant 1$ *we have*

$$\mathrm{gr}^k(J) \cong \bigoplus_{e_{11}+e_{12}+e_{21}=k} S^{e_{11}}H_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^{e_{12}}H_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^{e_{21}}H_{\mathbb{Q}}.$$

Proof. Let $\mathfrak{M}$ be the right-hand side of the above equation. First, for any $1 \leqslant i,j \leqslant 2$ and $e \geqslant 1$, the homomorphism $f^e_{ij}\colon S^eH_{\mathbb{Q}} \to \mathrm{gr}^e(J)$ defined by

$$x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n} \mapsto \alpha_{ij}(x_1)^{l_1}\alpha_{ij}(x_2)^{l_2}\cdots\alpha_{ij}(x_n)^{l_n} \pmod{J^{e+1}}$$

for $l_1 + l_2 + \cdots + l_n = e$ is $\mathrm{Aut}\,F_n$-equivariant. In fact, for any Nielsen generators $\sigma = P, Q, S$ and U of $\mathrm{Aut}\,F_n$, we can check that $f^e_{ij}(x^\sigma) = (f^e_{ij}(x))^\sigma$ for any $x = x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n}$. For example, we see that

$$\begin{aligned} f^e_{ij}((x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n})^U) &= f^e_{ij}((x_1+x_2)^{l_1}x_2^{l_2}\cdots x_n^{l_n}) \\ &= f^e_{ij}\left(\sum_{t=0}^{l_1}\binom{l_1}{t}x_1^t x_2^{l_1-t}x_2^{l_2}\cdots x_n^{l_n}\right) \\ &\equiv \sum_{t=0}^{l_1}\binom{l_1}{t}\alpha_{ij}(x_1)^t\alpha_{ij}(x_2)^{l_1-t+l_2}\cdots\alpha_{ij}(x_n)^{l_n} \pmod{J^{e+1}} \end{aligned}$$

$$\equiv \alpha_{ij}(x_1x_2)^{l_1}\alpha_{ij}(x_2)^{l_2}\cdots\alpha_{ij}(x_n)^{l_n} \pmod{J^{e+1}}$$
$$= (f^e_{ij}(x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n}))^U.$$

Hence, we obtain a surjective $\mathrm{GL}(n,\mathbb{Z})$-equivariant homomorphism $F\colon \mathfrak{M} \to \mathrm{gr}^k(J)$ defined by

$$\sum_{e_{11}+e_{12}+e_{21}=k} a_{e_{11},e_{12},e_{21}} X_{e_{11}} \otimes X_{e_{12}} \otimes X_{e_{21}} \mapsto \sum_{e_{11}+e_{12}+e_{21}=k} a_{e_{11},e_{12},e_{21}} f_{11}^{e_{11}}(X_{e_{11}}) f_{12}^{e_{12}}(X_{e_{12}}) f_{21}^{e_{21}}(X_{e_{21}})$$

for any $X_{e_{ij}} \in S^{e_{ij}}H_{\mathbb{Q}}$ and $a_{e_{11},e_{12},e_{21}} \in \mathbb{Q}$. The surjectivity of F follows from Proposition 4.4. In fact, for any element

$$Y := \prod_{l=1}^{n} \alpha_{11}(x_l)^{e_{11,l}} \alpha_{12}(x_l)^{e_{12,l}} \alpha_{21}(x_l)^{e_{21,l}}$$

in the basis T_k of $\mathrm{gr}^k(J)$, if we set

$$X := \sum_{e_{11,1}+\cdots+e_{21,n}=k} (x_1^{e_{11,1}}\cdots x_n^{e_{11,n}}) \otimes \cdots \otimes (x_1^{e_{21,1}}\cdots x_n^{e_{21,n}}) \in \mathfrak{M},$$

then we have $Y = F(X)$.

Next, we prove that F is an isomorphism by showing that the dimensions of $\mathfrak{M}$ and $\mathrm{gr}^k(J)$ as $\mathbb{Q}$-vector spaces are equal. The basis T_k can be rewritten as

$$T_k = \left\{ \prod_{\substack{1\leqslant i,j\leqslant 2\\(i,j)\neq(2,2)}} s_{ij,1}^{e_{ij,1}}\cdots s_{ij,n}^{e_{ij,n}} \;\middle|\; \sum_{l=1}^{n}(e_{11,l}+\cdots+e_{21,l}) = k \right\}$$
$$= \left\{ \prod_{\substack{1\leqslant i,j\leqslant 2\\(i,j)\neq(2,2)}} s_{ij,1}^{e_{ij,1}}\cdots s_{ij,n}^{e_{ij,n}} \;\middle|\; e_{ij,1}+\cdots+e_{ij,n} = e_{ij},\ e_{11}+e_{12}+e_{21} = k \right\}.$$

From the last term of the above equation, we see that the number of elements in T_k is equal to $\dim_{\mathbb{Q}} \mathfrak{M}$. This completes the proof of Proposition 5.14. □

Now we show the following.

Proposition 5.15. *For any $n \geqslant 3$, the natural homomorphism*

$$\iota\colon H^1(\mathrm{Aut}\,F_n, \mathrm{Im}(\eta_1)\otimes_{\mathbb{Z}}\mathbb{Q}) \to H^1(\mathrm{Aut}\,F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J)))$$

induced from the inclusion $\mathrm{Im}(\eta_1)\otimes_{\mathbb{Z}}\mathbb{Q} \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))$ *is injective.*

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Proof. From Proposition 5.14, we have

$$\mathrm{gr}^1(J) \cong H_{\mathbb{Q}}^{\oplus 3}, \qquad \mathrm{gr}^2(J) \cong (S^2 H_{\mathbb{Q}})^{\oplus 3} \oplus (H^{\otimes 2})^{\oplus 3}.$$

Thus, we have a $\mathrm{GL}(n,\mathbb{Q})$-equivariant isomorphism

$$\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J)) \xrightarrow{\cong} (\mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}}, S^2 H_{\mathbb{Q}}))^{\oplus 9} \oplus (\mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes 2}))^{\oplus 9}.$$

Let

$$\mathrm{pr}\colon \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J)) \to \mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes 2})$$

be the projection map defined by

$$\sum{}' \sum_{l,l_1,l_2=1}^{n} \alpha_{ij}(x_l)^* \otimes \alpha_{i_1 j_1}(x_{l_1}) \alpha_{i_2 j_2}(x_{l_2}) \mapsto \sum_{l,l_1,l_2=1}^{n} \alpha_{11}(x_l)^* \otimes \alpha_{12}(x_{l_1}) \alpha_{21}(x_{l_2}),$$

where the $\alpha_{ij}(x_l)^*$s are the dual basis of the $\alpha_{ij}(x_l)$s, and the sum $\sum'$ runs over all $1 \leqslant i, i_1, i_2, j, j_1, j_2 \leqslant 2$ such that

$$(i,j), (i_1,j_1), (i_2,j_2) \neq (2,2) \quad \text{and} \quad (i_1,j_1) \leqslant_{\mathrm{lex}} (i_2,j_2).$$

Here $\leqslant_{\mathrm{lex}}$ denotes the usual lexicographic ordering. Let

$$\overline{\mathrm{pr}}\colon H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))) \to H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes 2}))$$

be the homomorphism induced from pr. Then the cohomology classes of g_{M} and g_{K} in $H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J)))$ are mapped to those of g'_{M} and g'_{K} in $H^1(\mathrm{Aut}\, F_n,$ $\mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes 2}))$ by $\overline{\mathrm{pr}} \circ \iota$ such that

$$g'_{\mathrm{M}}(\sigma) := \begin{cases} \displaystyle\sum_{i=2}^{n} s_{11,i}^* \otimes (s_{12,i} s_{21,1} - s_{12,1} s_{21,i}), & \sigma = S, \\ 0, & \sigma = P, Q, U, \end{cases}$$

and

$$g'_{\mathrm{K}}(\sigma) := \begin{cases} -s_{11,1}^* \otimes (s_{12,1} s_{21,2} - s_{12,2} s_{21,1}), & \sigma = U, \\ 0, & \sigma = P, Q, S. \end{cases}$$

If we identify the submodule of $H_{\mathbb{Q}}^{\otimes 2}$ generated by

$$\{x \otimes y - y \otimes x \mid x, y \in H_{\mathbb{Q}}\}$$

with the exterior product $\Lambda^2 H_{\mathbb{Q}}$ of $H_{\mathbb{Q}}$ of degree 2, then we see that the image of $\overline{\mathrm{pr}} \circ \iota$ is contained in $H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}}, \Lambda^2 H_{\mathbb{Q}}))$ by observing the images of g'_{M} and g'_{K}. Furthermore, it turns out that the cohomology classes of g'_{M} and g'_{K} generate $H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}}, \Lambda^2 H_{\mathbb{Q}})) \cong \mathbb{Q}^{\oplus 2}$ from our previous result in [**26**]. Therefore, we obtain the required result. This completes the proof of Proposition 5.15. □

We remark that in our forthcoming paper [**29**], we compute

$$H^1(\mathrm{Aut}\,F_n, H_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} S^2 H_{\mathbb{Q}}) \cong \mathbb{Q}.$$

Thus, by using the result

$$H^1(\mathrm{Aut}\,F_n, H_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} \Lambda^2 H_{\mathbb{Q}}) \cong \mathbb{Q}^{\oplus 2}$$

in [**26**], we see that

$$\begin{aligned} &H^1(\mathrm{Aut}\,F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))) \\ &\quad \cong H^1(\mathrm{Aut}\,F_n, H_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} S^2 H_{\mathbb{Q}})^{\oplus 9} \oplus H^1(\mathrm{Aut}\,F_n, H_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} H_{\mathbb{Q}}^{\otimes 2})^{\oplus 9} \\ &\quad = \mathbb{Q}^{\oplus 36}. \end{aligned}$$

5.4. Twisted second cohomology classes

In this section we introduce twisted higher cohomology classes by using the crossed homomorphism g_{K}, according to Kawazumi's construction of higher cohomology classes with the first Johnson homomorphism τ_1 in [**15**]. Then, in particular, we study its second cohomology class.

First, we study the image of η_1. For any $1 \leqslant i_1, j_1, i_2, j_2 \leqslant 2$, set

$$t_{(i_1j_1,i_2j_2)}(p,q) := s_{i_1j_1}(x_p)s_{i_2j_2}(x_q) - s_{i_1j_1}(x_q)s_{i_2j_2}(x_p). \tag{5.2}$$

Let T be the submodule of $\mathrm{gr}^1(J)$ generated by

$$\{t_{(i_1j_1,i_2j_2)}(p,q) \mid (i_1,j_1,i_2,j_2) = (1,1,1,2),(1,2,2,1),(2,1,1,1),\ 1 \leqslant q < p \leqslant n\}.$$

Clearly, T is an $\mathrm{Aut}\,F_n$-invariant module. From (5.1), the image of η_1 is contained in $\mathrm{gr}^1(J)^* \otimes_{\mathbb{Q}} T$. From Proposition 4.4, we see that (5.2) is a basis of T as a $\mathbb{Q}$-vector space. Define the $\mathbb{Q}$-linear map $T \to \mathrm{gr}^1(J) \otimes_{\mathbb{Q}} \mathrm{gr}^1(J)$ by

$$t_{(i_1j_1,i_2j_2)}(p,q) \mapsto s_{i_1j_1}(x_p) \otimes s_{i_2j_2}(x_q) - s_{i_1j_1}(x_q) \otimes s_{i_2j_2}(x_p).$$

It is easily seen that this map is $\mathrm{Aut}\,F_n$-equivariant injective. Consider T as an $\mathrm{Aut}\,F_n$-invariant submodule of $\mathrm{gr}^1(J) \otimes_{\mathbb{Q}} \mathrm{gr}^1(J)$ through this map. Then we can consider that $\mathrm{Im}(\eta_1)$ is contained in

$$\mathcal{V} := \mathrm{gr}^1(J)^* \otimes_{\mathbb{Q}} (\mathrm{gr}^1(J))^{\otimes 2},$$

and hence g_{K} is a crossed homomorphism from $\mathrm{Aut}\,F_n$ to $\mathcal{V}$.

For any $q \geqslant 1$, define a map $\zeta_q \colon \mathcal{V}^{\otimes q} \to \mathrm{gr}^1(J)^* \otimes_{\mathbb{Q}} \mathrm{gr}^1(J)^{\otimes q+1}$ by

$$u_1 \otimes \cdots \otimes u_q \mapsto (u_1 \otimes 1^{\otimes q-1}) \circ (u_2 \otimes 1^{\otimes q-2}) \circ \cdots \circ (u_{q-1} \otimes 1) \circ u_q.$$

By considering the cup product, we have the induced homomorphism

$$\zeta_q^* \colon H^1(\mathrm{Aut}\,F_n, \mathcal{V})^{\otimes q} \to H^q(\mathrm{Aut}\,F_n, \mathrm{gr}^1(J)^* \otimes_{\mathbb{Q}} \mathrm{gr}^1(J)^{\otimes q+1})$$

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from ζ_q. Let $C_{12}\colon \mathrm{gr}^1(J)^* \otimes_{\mathbb{Q}} \mathrm{gr}^1(J)^{\otimes q+1} \to \mathrm{gr}^1(J)^{\otimes q}$ be the contraction homomorphism defined by

$$f^* \otimes (f_1 \otimes \cdots \otimes f_{q+1}) \mapsto f^*(f_1) f_2 \otimes \cdots \otimes f_{q+1}.$$

Then C_{12} induces the homomorphism

$$\overline{\zeta_q^*}\colon H^1(\mathrm{Aut}\, F_n, \mathcal{V})^{\otimes q} \to H^q(\mathrm{Aut}\, F_n, \mathrm{gr}^1(J)^{\otimes q}).$$

By using the above homomorphisms, we can obtain higher twisted cohomology classes $\zeta_q^*(g_{\mathrm{K}}^{\otimes q})$ and $\overline{\zeta_q^*}(g_{\mathrm{K}}^{\otimes q})$ for any $q \geqslant 1$, where $\zeta_1^*(g_{\mathrm{K}}) = g_{\mathrm{K}}$. This is an analogue of Kawazumi's construction of higher cohomology classes of $\mathrm{Aut}\, F_n$ with the first Johnson homomorphism.

Next we consider the case in which $q = 2$. We show that $H^2(\mathrm{IA}_n, \mathrm{gr}^1(J)^{\otimes 2})$ has a two-dimensional $\mathbb{Q}$-vector subspace whose generating set contains the restriction of $\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})$ to IA_n. First, consider $H^1(\mathrm{Aut}\, F_n, \mathrm{gr}^1(J))$. In [**24**] we computed $H^1(\mathrm{Aut}\, F_n, H) = \mathbb{Z}$ for any $n \geqslant 2$. This induces $H^1(\mathrm{Aut}\, F_n, H_{\mathbb{Q}}) = \mathbb{Q}$. On the other hand, by Proposition 4.4 we see that the $\mathrm{Aut}\, F_n$-equivariant homomorphism

$$\alpha_{11} \oplus \alpha_{12} \oplus \alpha_{21}\colon H_{\mathbb{Q}}^{\oplus 3} \to \mathrm{gr}^1(J)$$

is an isomorphism. Hence, we have $H^1(\mathrm{Aut}\, F_n, \mathrm{gr}^1(J)) \cong \mathbb{Q}^{\oplus 3}$. By observing the images of Nielsen's generators, we can see that

$$(n-1)\overline{\zeta_1^*}(g_{\mathrm{K}}) + 2\overline{\zeta_1^*}(g_{\mathrm{M}}) = 0 \in H^1(\mathrm{Aut}\, F_n, \mathrm{gr}^1(J)).$$

Now, we have two two-cohomology classes $\overline{\zeta_1^*}(g_{\mathrm{K}}) \cup \overline{\zeta_1^*}(g_{\mathrm{K}})$ and $\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})$ in $H^2(\mathrm{Aut}\, F_n, \mathrm{gr}^1(J)^{\otimes 2})$, where $\cup$ denotes the cup product. We can see that they are linearly independent from the following theorem.

Theorem 5.16. *For any* $n \geqslant 4$*, the restrictions of* $\overline{\zeta_1^*}(g_{\mathrm{K}}) \cup \overline{\zeta_1^*}(g_{\mathrm{K}})$ *and* $\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})$ *to* IA_n *are linearly independent in* $H^2(\mathrm{IA}_n, \mathrm{gr}^1(J)^{\otimes 2})$.

Proof. Assume that

$$\lambda(\overline{\zeta_1^*}(g_{\mathrm{K}}) \cup \overline{\zeta_1^*}(g_{\mathrm{K}})) + \mu\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2}) = \delta^1(\varphi)$$

for some $\lambda, \mu \in \mathbb{Q}$ and some map $\varphi\colon \mathrm{IA}_n \to \mathrm{gr}^1(J)^{\otimes 2}$. Since $n \geqslant 4$, we can take distinct indices i, j, k and l. Then we have

$$\begin{aligned}(\overline{\zeta_1^*}(g_{\mathrm{K}}) \cup \overline{\zeta_1^*}(g_{\mathrm{K}}))(K_{ij}, K_{kl}) &= 4(a'_{11}(x_j) - a'_{11}(x_i)) \otimes (a'_{11}(x_l) - a'_{11}(x_i)),\\ \overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})(K_{ij}, K_{kl}) &= 0.\end{aligned}$$

On the other hand, since $[K_{ij}, K_{kl}] = 1$, we have

$$\begin{aligned}4(a'_{11}(x_j) &- a'_{11}(x_i)) \otimes (a'_{11}(x_l) - a'_{11}(x_i))\\ &= \lambda(\overline{\zeta_1^*}(g_{\mathrm{K}}) \cup \overline{\zeta_1^*}(g_{\mathrm{K}}))(K_{ij}, K_{kl}) + \mu\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})(K_{ij}, K_{kl})\\ &= \varphi(K_{kl}) - \varphi(K_{ij}K_{kl}) + \varphi(K_{ij})\\ &= \varphi(K_{ij}) - \varphi(K_{kl}K_{ij}) + \varphi(K_{kl})\\ &= \lambda(\overline{\zeta_1^*}(g_{\mathrm{K}}) \cup \overline{\zeta_1^*}(g_{\mathrm{K}}))(K_{kl}, K_{ij}) + \mu\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})(K_{kl}, K_{ij})\\ &= 4(a'_{11}(x_l) - a'_{11}(x_i)) \otimes (a'_{11}(x_j) - a'_{11}(x_i)).\end{aligned}$$

This shows that $\lambda = 0$.

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Next, consider the relation $[K_{ik}, K_{ij}K_{kj}]$ for distinct i, j and k. Then we have

$$\begin{aligned}2\mu(a_{12}(x_k)\otimes a_{21}(x_j)+a_{21}(x_k)\otimes a_{12}(x_j)+2a'_{11}(x_k)\otimes a'_{11}(x_j))\\ &= \mu\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})(K_{ik}, K_{ij}K_{kj})\\ &= \varphi(K_{ij}K_{kj}) - \varphi(K_{ik}K_{ij}K_{kj}) + \varphi(K_{ik})\\ &= \varphi(K_{ik}) - \varphi(K_{ij}K_{kj}K_{ik}) + \varphi(K_{ij}K_{kj})\\ &= \mu\overline{\zeta_2^*}(g_{\mathrm{K}}^{\otimes 2})(K_{ij}K_{kj}, K_{ik})\\ &= 2\mu(a_{12}(x_j)\otimes a_{21}(x_k)+a_{21}(x_j)\otimes a_{12}(x_k)+2a'_{11}(x_j)\otimes a'_{11}(x_k)).\end{aligned}$$

This shows that $\mu = 0$. □

As further research, it would be interesting to describe a relation among cup products of $\overline{\zeta_1^*}(g_{\mathrm{K}})$ and $\overline{\zeta_q^*}(g_{\mathrm{K}}^{\otimes q})$ as well as to determine the twisted cohomology groups $H^q(\mathrm{Aut}\, F_n, \mathrm{gr}^1(J)^{\otimes q})$.

Acknowledgements. The author would like to thank the referee for their careful reading and helpful comments on the original paper. The author is supported by a Grant-in-Aid for Young Scientists (B) from JSPS.

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