Dendritations of surfaces

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Abstract. In this paper we develop a generalization of foliated manifolds in the context of metric spaces. In particular we study dendritations of surfaces that are defined as maximal atlases of compatible upper semicontinuous local decompositions into dendrites. Applications are given in modeling stable and unstable sets of topological dynamical systems. For this purpose new forms of expansivity are defined.

1. *Introduction*

In this paper we develop a *Theory of Foliations* from the viewpoint of *Continuum Theory*. We define and study what we call *continuumwise foliations* or simply *cw-foliations* and as a special case *dendritations*. These are generalizations of foliations of smooth manifolds, laminations, singular pseudo-Anosov foliations and the generalized foliations used by Hiraide to study expansive homeomorphisms. The idea is to consider monotone upper semicontinuous decompositions as *local charts*. We will not assume that the *plaques* are distributed as a product structure as in standard foliation theory. Moreover, two plaques in a common local chart have not even to be homeomorphic.

As we will show in Theorem 5.1.2, cw-foliations are a conceptual framework to understand the distribution of local stable and unstable continua in the dynamical systems that we will consider. A motivation is to classify cw-expansive surface homeomorphisms. We say that f is *cw-expansive* (*continuum-wise expansive*) [23] if there is $\delta > 0$ such that if $\mathrm{diam}(f^n(A)) < \delta$ for all $n \in \mathbb{Z}$ and A is connected then $\mathrm{card}(A) = 1$, i.e. A is a singleton. Important examples of cw-expansive dynamics are Anosov diffeomorphisms of a compact manifold of arbitrary dimension and pseudo-Anosov maps of compact surfaces with singular points and 1-prongs.

For an Anosov diffeomorphism, local stable sets form foliated charts (at least C^0, see [19]). If we consider an expansive homeomorphism of a compact surface, via [18, 30], we know that local stable sets form a singular foliation. Recall that a homeomorphism f is *expansive* if there is $\delta > 0$ such that if $\mathrm{dist}(f^n(x), f^n(y)) < \delta$ for all $n \in \mathbb{Z}$ then $y = x$.

In fact, Hiraide and Lewowicz independently proved that expansive homeomorphisms of compact surfaces are conjugate to pseudo-Anosov diffeomorphisms. If we consider cw-expansivity on a compact surface we have that local stable sets may be far from determining foliations in the standard sense. Moreover, a local stable set may not even be a finite union of arcs. For example, in [**5**], a cw-expansive homeomorphism is constructed on a compact surface with a fixed point whose local stable set is connected but not locally connected, see §2.2.4. This example suggests that the goal of classifying all the cw-expansive surface homeomorphisms requires new technology.

In §2.1, we introduce some definitions located between expansivity and cw-expansivity that will be called *cwN-expansivity*, which is, in a sense, a version of N-expansivity from the viewpoint of continuum theory. Recall that, given $N \geq 1$, we say that f is N-*expansive* [**36**] if there is $\delta > 0$ such that if $A \subset X$ (an arbitrary subset) and diam$(f^n(A)) \leq \delta$ for all $n \in \mathbb{Z}$ then card$(A) \leq N$ (i.e. A has at most N points). We will say, see Definition 2.1.2, that a homeomorphism is *cwN-expansive* if there is $\delta > 0$ such that if $A, B \subset X$ are continua, diam$(f^n(A)) < \delta$ for all $n \geq 0$ and diam$(f^n(B)) < \delta$ for all $n \leq 0$ then card$(A \cap B) \leq N$. Also, a homeomorphism is *cw_F-expansive* (where the F means *finite*) if there is $\delta > 0$ such that if $A, B \subset X$ are continua, diam$(f^n(A)) < \delta$ for all $n \geq 0$ and diam$(f^n(B)) < \delta$ for all $n \leq 0$ then $A \cap B$ is a finite set. These new forms of expansivity allow us to prove the local connection of stable sets and to conclude that they are dendrites. See Theorem 6.7.1. Recall that a *dendrite* is a Peano continuum containing no simple closed curve, a *continuum* is a compact connected metric space and a *Peano continuum* is a locally connected continuum. It is known, see Theorem 2.3.3, that such dendrites (the stable and unstable local sets mentioned above) have a uniform size, i.e. there is $\delta > 0$ such that for all x in the surface the stable and the unstable dendrites of x meet the boundary of the disk centered at x with radius δ. In this way we arrive naturally to the concept of dendritic decomposition, see §3.2.1.

The topic of topological decompositions has a long literature, see for example [**13**, **14**, **38**, **43**, **45**]. For applications in dynamical systems the reader is referred to [**5**, **7**, **12**, **25**, **41**]. A celebrated result proved by Moore in [**34**] states that the quotient space of an upper semicontinuous decomposition of the two-dimensional sphere in non-separating continua is again the sphere (assuming that the decomposition has at least two elements, in other cases the quotient is a point). In some sense, Moore's decompositions are a generalization of zero-dimensional foliations, i.e. the decomposition in singletons. A standard foliated chart of $[0, 1] \times [0, 1]$ by horizontal lines has two properties: each plaque is an arc and the quotient space is an arc. By Proposition 6.2.2, a local chart of a dendritation has the following properties: each plaque is a dendrite and the quotient space is a dendrite. The mentioned properties of a standard foliated chart do not characterize the foliated chart, see Example 6.4.6. In §6.4 we give necessary and sufficient conditions for a decomposition to be a standard foliated chart. In Corollary 6.6.3 we conclude that a continuous and $\mathcal{C}$-smooth dendritation is a standard foliation. The terms *continuous* and $\mathcal{C}$-*smooth* have a special meaning in this paper, see Definitions 6.6.1 and 3.2.14.

In [**17**] Hiraide considered *generalized foliations* for the study of expansivity from a topological viewpoint (see §4.4). However, this definition seems to be useful jointly with the pseudo orbit tracing property, and not in our case. For example, pseudo-Anosov

singular foliations are not generalized foliations in the sense of Hiraide and they are our main examples of dendritations. Our generalization of a foliation is designed to accompany cw-expansivity.

The concept of cw$N$-expansivity appears naturally in the deep study of the articles [**18**, **30**]. In these papers, expansive homeomorphisms of surfaces are classified. A careful reading reveals that several arguments can be made assuming cw1-expansivity instead of expansivity. The concept of cw1-expansivity was previously considered in [**44**] where some topological properties of stable sets were proved. In Theorem 6.8.5 we show that every cw1-expansive homeomorphism of a compact surface is expansive. The key of this proof is Theorem 5.3.4 where we give sufficient conditions for a cw1-expansive homeomorphism of a compact metric space to be expansive. In Example 5.3.2 we show that cw1-expansivity does not imply expansivity on arbitrary compact metric spaces (this space is not locally connected). In §2.2.1 we show that there is a cw2-expansive homeomorphism of the two-dimensional sphere that is not 2-expansive. It is a pseudo-Anosov with 1-prongs.

We obtain some general results on surface dendritations. In Theorem 6.2.7 we show that for every dendritation of a closed surface there is a residual set of points without ramifications. In particular, generic leaves are one-dimensional submanifolds. This is a consequence of another result by Moore [**35**], saying that at most a countable number of disjoint triods can be embedded in a plane. In Theorem 6.7.1 we consider a cw$_F$-expansive homeomorphism of a compact surface. We show that stable and unstable continua form dendritations. We also prove that: no leaf is a Peano-continuum; generic leaves are non-compact one-dimensional manifolds; and, in a dense subset of the surface, stable and unstable leaves are topologically transverse. One can feel that this result gives a nice *generic picture* of what a cw$_F$-expansive homeomorphism of a compact surface is. However, we think that the goal of classifying a surface cw$_F$-expansive homeomorphism is far from the scope of the present paper, not to mention cw-expansivity.

A brief sketch of the paper is as follows. In §2 we introduce new forms of expansivity and we study the topology of stable and unstable sets. In §3 we recall the main results from continuum theory that will be needed and study decompositions, the local charts of our foliations. In §4 we define and study cw-foliations on metric spaces. They are defined via atlases of upper semicontinuous decompositions. We study the induced partition of the space into leaves. The contents of §§2 and 4 are independent. In §5 the results of the previous sections are joined to study the stable and the unstable cw-foliations defined by a cw-expansive homeomorphism of a metric space. In §6 we study some special cases of dendritations of surfaces. We give sufficient conditions to prove that they are (singular) foliations. Also, some properties of the stable and the unstable dendritations of a homeomorphism with some kind of expansivity are derived. Throughout the paper several open problems are given.

2. *Variations of expansivity*

We start by presenting general results of cw-expansive homeomorphisms on Peano continua. In §2.1 we introduce cw$N$-expansivity and we summarize the main variations of expansivity that will be considered in this paper. In §2.2 we present some examples

that will be used to illustrate our results in subsequent sections. In particular, in §2.2.1 we give an example of a cw2-expansive homeomorphism on the two-dimensional sphere that is not 2-expansive. In §2.3 basic topological properties of stable sets are stated. We recall the Invariant Continuum Theorem for such dynamics. In §2.4 we introduce capacitors as a tool to understand what happens with unstable continua between two close stable plates. In §2.5 we give another form of expansivity, called partial expansivity, that generalizes partial hyperbolicity of diffeomorphisms. We do not develop this concept in the paper, however we think that it could be of interest. In §2.6 we consider the relationship between stable sets and stable continua. Also, we give sufficient conditions for a cw$N$-expansive homeomorphism to be N-expansive. For these purposes we introduce expansivity modulo an equivalence relation.

2.1. *CwN-expansivity.* Let $f : X \to X$ be a homeomorphism of a compact metric space (X, dist). We say that $A \subset X$ is a *subcontinuum* if A is compact and connected. Denote by $\mathcal{C}(X)$ the space of subcontinua of X

$$\mathcal{C}(X) = \{A \subset X : A \text{ is a non-empty continuum}\}.$$

In $\mathcal{C}(X)$ we consider the Hausdorff metric. It is usually called a *hyperspace* of X and has the remarkable properties of being compact (if X is compact) and arc-connected (provided that X is connected), see [**37**, **38**]. For $\delta > 0$ define the sets

$$\mathcal{C}_\delta = \{A \in \mathcal{C}(X) : \text{diam}(A) \leq \delta\},$$
$$\mathcal{C}_\delta^s = \{A \in \mathcal{C}(X) : f^n(A) \in \mathcal{C}_\delta \text{ for all } n \geq 0\},$$
$$\mathcal{C}_\delta^u = \{A \in \mathcal{C}(X) : f^n(A) \in \mathcal{C}_\delta \text{ for all } n \leq 0\}.$$

The continua in $\mathcal{C}_\delta^s$ are called δ-*stable* and those in $\mathcal{C}_\delta^u$ are δ-*unstable*. We also consider the sets

$$
\begin{aligned}
\mathcal{C}^s &= \{A^s \in \mathcal{C}(X) : \lim_{n \to +\infty} \text{diam}(f^n(A^s)) = 0\}, \\
\mathcal{C}^u &= \{A^u \in \mathcal{C}(X) : \lim_{n \to -\infty} \text{diam}(f^n(A^u)) = 0\}.
\end{aligned}
\tag{1}
$$

The continua in $\mathcal{C}^s$ are called *stable* and those in $\mathcal{C}^u$ are *unstable*.

Remark 2.1.1. The sets $\mathcal{C}_\delta$, $\mathcal{C}_\delta^s$ and $\mathcal{C}_\delta^u$ are closed in $\mathcal{C}(X)$.
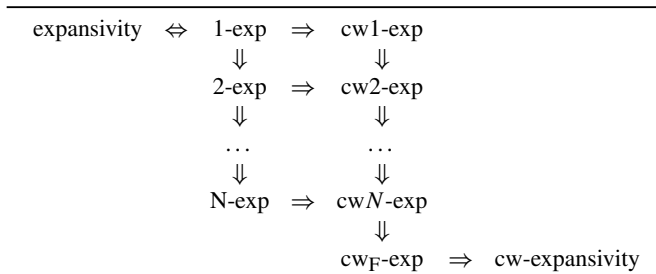
Definition 2.1.2. Given $N \geq 1$ we say that f is *cwN-expansive* if there is $\delta > 0$ such that if $A^s \in \mathcal{C}_\delta^s$ and $A^u \in \mathcal{C}_\delta^u$ then $\text{card}(A^s \cap A^u) \leq N$. In this case δ is a *cwN-expansivity constant*. We say that f is *cw$_\text{F}$-expansive* if there is $\delta > 0$ such that if $A^s \in \mathcal{C}_\delta^s$ and $A^u \in \mathcal{C}_\delta^u$ then $A^s \cap A^u$ is a finite set.

In Table 1 the main variations of expansivity considered in this paper are summarized.

2.2. *Examples.* In this section we explain the examples that we had in mind while developing this paper. We start with the following three classical families for which there are well established theories for modeling stable and unstable sets.
(1) *Anosov diffeomorphisms*. These diffeomorphisms are defined on smooth manifolds and are characterized by the uniform hyperbolicity on the tangent bundle. Considering the distribution of stable and unstable sets, they are the most regular

TABLE 1. Hierarchy of some generalizations of expansivity. The implications indicated by the arrows are easy to prove and hold for homeomorphisms on metric spaces.

expansivity	$\Leftrightarrow$	1-exp	$\Rightarrow$	cw1-exp		
		$\Downarrow$		$\Downarrow$		
		2-exp	$\Rightarrow$	cw2-exp		
		$\Downarrow$		$\Downarrow$		
		$\ldots$		$\ldots$		
		$\Downarrow$		$\Downarrow$		
		N-exp	$\Rightarrow$	cw$N$-exp		
				$\Downarrow$		
				cw$_\mathrm{F}$-exp	$\Rightarrow$	cw-expansivity

kind of expansive homeomorphisms because they form continuous foliations (see [**19**]).

(2) *Smale spaces*. These are expansive homeomorphisms of compact metric spaces with local product structure (equivalently, canonical coordinates or the pseudo-orbit tracing property). The category includes Smale's basic sets of Axiom A diffeomorphisms. Stable and unstable sets can be modeled with *Hiraide's generalized foliations* (see §4.4).

(3) *Pseudo-Anosov diffeomorphisms*. These diffeomorphisms are defined on compact surfaces. They have local product structure except for a finite number of points called *singular*. Stable and unstable sets form *singular foliations* (see for example [**18**]).

Next we describe other examples that will be essential in the development of the paper. Except for the first one, we assume that the reader is familiar with Smale's *derived from Anosov diffeomorphisms* (which in particular are an interesting example of a Smale space).

2.2.1. *A pseudo-Anosov diffeomorphism with 1-prongs.* The dynamics of pseudo-Anosov diffeomor-phisms is not simple, at least from this author's viewpoint. In this section we wish to discuss in detail some properties of a special example, a pseudo-Anosov diffeomorphism with 1-prongs of the sphere. This example has been considered several times in the literature. It has some properties that may not be easy to predict at first sight. In [**47**, Example 1, p. 140] Walters considered it to show that a factor of an expansive homeomorphism may not be expansive. In [**39**, §2.4] it is proved that the local stable set of some points is not locally connected. In [**40**], Pacifico and Vieitez show that it is not entropy expansive; in fact they show that there are arbitrarily small horseshoes. We will show that it is cw2-expansive but not N-expansive, for all $N \geq 1$.

The example is as follows. Let $T^2 = S^1 \times S^1$ be the two-dimensional torus where $S^1 = \mathbb{R}/\mathbb{Z}$. Consider the equivalence relation $p \sim -p$ for $p \in T^2$. The quotient space is a two-dimensional sphere $S^2 = T^2/\sim$. Denote by $\pi : T^2 \to S^2$ the canonical projection. On the torus consider the Anosov diffeomorphism $\tilde{f} : T^2 \to T^2$ defined by $\tilde{f}(x, y) = (2x + y, x + y)$. Define $f : S^2 \to S^2$ by $f(\pi(p)) = \pi(\tilde{f}(p))$ for all $p \in T^2$. For a more detailed construction the reader is referred to the works mentioned above.

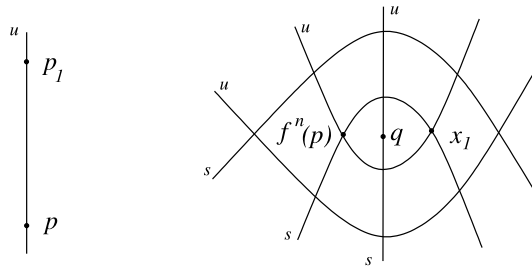PROPOSITION 2.2.1. *The homeomorphism f is cw2-expansive.*

FIGURE 1. The unstable arc of p (left) and a neighborhood of the 1-prong q (right). Stable and unstable arcs are indicated with s and u, respectively.

Proof. Denote by $\mathcal{F}^s$ and $\mathcal{F}^u$ the stable and the unstable singular foliations of f, respectively. These are transverse foliations except at the singularities. Singular points are 1-prongs and the foliations look as in Figure 1. Then, a small arc of the stable foliation intersects in at most two points an unstable arc. Thus, the proof is reduced to showing that every stable continuum is contained in a stable leaf. Arguing by contradiction, let $C \subset S^2$ be a stable continuum that it is not contained in a stable leaf. Then there is a hyperbolic periodic point $p \in S^2$ such that $\mathcal{F}^s(p) \cap C \neq \emptyset$, where $\mathcal{F}^s(p)$ denotes the stable leaf through p. Since p is hyperbolic we see that $\operatorname{diam}(f^n(C))$ cannot go to zero as $n \to +\infty$ because C is not contained in $\mathcal{F}^s(p)$. This contradiction proves that every stable continuum is contained in a stable leaf. $\square$

As usual we define *local stable* and *unstable sets* as

$$W^s_\varepsilon(p) = \{x \in S^2 : \operatorname{dist}(f^n(p), f^n(x)) \leq \varepsilon \text{ for all } n \geq 0\},$$
$$W^u_\varepsilon(p) = \{x \in S^2 : \operatorname{dist}(f^n(p), f^n(x)) \leq \varepsilon \text{ for all } n \leq 0\},$$

respectively. The next result can be derived from [**40**], however, since we think that more details can be given, a proof is included. The author learned this proof from J. Vieitez and J. Lewowicz. We will use the notation $\overline{A}$ for the closure of A.

PROPOSITION 2.2.2. *For all $\varepsilon > 0$ there is a Cantor set C such that* $\operatorname{diam}(f^n(C)) \leq \varepsilon$ *for all $n \in \mathbb{Z}$, in particular f is not N-expansive for all $N \geq 1$.*

Proof. Let $q \in S^2$ be a 1-prong of f. Take $p \in S^2$ such that the orbit $\{f^n(p) : n \geq 0\}$ is dense in S^2. A point with this property will be called *transitive*. Consider $\varepsilon > 0$. We will show that $W^s_\varepsilon(p)$ contains a Cantor set contained in the unstable arc of p. Consider $n \geq 0$ such that there is $x_1 \neq f^n(p)$ with $x_1 \in W^s_{\varepsilon/2}(f^n(p)) \cap W^u_{\varepsilon/2}(f^n(p))$. We have that $p_1 = f^{-n}(x_1) \in W^s_{\varepsilon/2}(p) \cap W^u_{\varepsilon/2}(p)$. Also we can assume that p_1 is in the unstable arc of p, see Figure 1. Since $f^n(p)$ and $f^n(p_1)$ are in a stable arc and p is transitive we have that p_1 is transitive too.

Denote $C_1 = \{p, p_1\}$. For each $x \in C_1$ take a transitive point y in the unstable arc of p with $y \in W^s_{\varepsilon/4}(x)$. Define C_2 as the set containing C_1 and these two new points. Again, for each $x \in C_2$ take a transitive point y in the unstable arc of p with $y \in W^s_{\varepsilon/8}(x)$. Inductively we define a sequence of sets $\{C_n : n \geq 1\}$ with $|C_n| = 2^{n+1}$. By construction each C_n is contained in $W^s_\varepsilon(p)$. Denote $C = \overline{\bigcup_{n \geq 1} C_n}$. Since $W^s_\varepsilon(p)$ is closed we have

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that $C \subset W_\varepsilon^s(p)$. Also, C has no isolated point and it cannot contain an arc because f is cw-expansive (in fact cw2-expansive as we proved). Then, C is a Cantor set contained in the unstable arc of p and in $W_\varepsilon^s(p)$. □

2.2.2. *Quasi-Anosov diffeomorphisms.* By definition, a *quasi-Anosov* diffeomorphism is a C^1 diffeomorphism $f : M \to M$ of a compact manifold such that $\{\|df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded for every non-vanishing tangent vector v, where df^n is the differential of f^n and $\|\cdot\|$ is the norm induced by a Riemannian metric on M. They were characterized by Mañé [**31**] as Axiom A diffeomorphisms with quasi-transverse stable and unstable spaces. Recall that *Axiom A* means that the non-wandering set is hyperbolic and periodic points are dense in the non-wandering set. For an Axiom A diffeomorphism, at every point $x \in M$ the tangent space contains a contracting subspace E_x^s and an expanding subspace E_x^u. The *quasi-transversality* condition means that $E_x^s \cap E_x^u = 0$ for all $x \in M$.

We proceed to sketch the construction of a particular quasi-Anosov diffeomorphism that is not Anosov [**16**]. Consider two derived from Anosov diffeomorphisms $f_i : M_i \to M_i$, $i = 1, 2$, where M_i is an n-torus, f_1 is conjugate to f_2^{-1} and f_1 presents a codimension-one shrinking repeller and a sink fixed point p_1. Consequently, f_2 presents a codimension-one expanding attractor and a source fixed point p_2. Let B_i, $i = 1, 2$, be an open ball around p_i such that $\overline{B_1} \subset f_1^{-1}(B_1)$ and $\overline{B_2} \subset f_2(B_2)$. Consider the manifolds with boundary $N_i = M_i \setminus B_i$ and a diffeomorphism $\varphi : \overline{f_1^{-1}(B_1)} \setminus B_1 \to \overline{f_2(B_2)} \setminus B_2$ such that $M = N_1 \cup N_2 / x \sim \varphi(x)$ is a closed manifold and there is a diffeomorphism $f : M \to M$ extending the dynamics of f_1 and f_2.

In [**16**] it is proved that, for $n = 3$, there is a diffeomorphism φ making f quasi-Anosov. As we said, the non-wandering set is hyperbolic. At wandering points stable and unstable manifolds are one-dimensional and their tangent lines are quasi-transverse. Since M is three-dimensional, they are not transverse and f is not Anosov.

2.2.3. *Q^r-Anosov diffeomorphisms.* In [**4**] the construction of §2.2.2 was considered for the simpler case $n = 2$. On a surface there is not enough space to construct a quasi-Anosov diffeomorphism (not being Anosov). However, the construction can be performed. In this case the map φ will introduce tangencies between stable and unstable manifolds at wandering points. Assuming that f is of class C^r, if these tangencies are of order at most $r \geq 2$ we say that f is *Q^r-Anosov*. In [**4**] it is shown that the set of Q^r-Anosov diffeomorphisms of a closed surface is an open set (in the C^r topology) of r-expansive diffeomorphisms, where *r-expansive* means N-expansive with $N = r$. These examples are Axiom A and show that $(N + 1)$-expansivity does not imply N-expansivity.

2.2.4. *Anomalous cw-expansivity.* In [**5**], another variation of quasi-Anosov cw-expansivity was considered. On a surface, as in §2.2.3, a cw-expansive homeomorphism is defined with what we called an *anomalous saddle*. This is a fixed point whose stable set is connected but not locally connected. Let us give a brief description. Let $E \subset \mathbb{R}^2$ be the set $(\{1\} \cup \{1 + 1/n\}_{n \geq 2}) \times [0, 1]$. It is a countable union of vertical segments. Consider the auxiliary map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(p) = p/2$. Define $F^s = (\mathbb{R} \times$

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$\{0\}$) $\bigcup_{n \geq 0} T^n(E)$ and $F^u = \{0\} \times \mathbb{R}$. The anomalous saddle in this case is the origin. In [**5**], a homeomorphism is constructed around $(0, 0)$ such that its local stable set is F^s and the unstable set is F^u (intersecting with a neighborhood of the origin). This anomalous saddle is *inserted* in a Q^r-Anosov diffeomorphism of the previous section. In the example, the intersection of local stable and unstable sets is totally disconnected, thus implying cw-expansivity.

2.3. *Invariant continua.*

Our interest is centered on dynamical systems at surfaces, but some fundamental results on cw-expansivity hold for homeomorphisms on Peano continua, i.e. a connected, locally connected and compact metric space. In this setting, Theorem 2.3.3 plays the role of the Invariant Manifold Theorem [**19**] for hyperbolic diffeomorphisms on smooth manifolds.

The following result is a characterization of cw-expansivity in terms of stable and unstable continua. The direct part is known [**23**, §2]. The converse may be new, however it is quite direct from the definitions.

PROPOSITION 2.3.1. *A homeomorphism* $f : X \to X$ *of a compact metric space is cw-expansive if and only if there is* $\varepsilon^* > 0$ *such that:*

(1) $\mathcal{C}_{\varepsilon^*}^{\sigma} \subset \mathcal{C}^{\sigma}$; *and*

(2) *for all* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\mathcal{C}^{\sigma} \cap \mathcal{C}_{\delta} \subset \mathcal{C}_{\varepsilon}^{\sigma}$

for $\sigma = s$ *and* $\sigma = u$.

Proof. Direct. Let ε^* be such that if $A \subset X$ is connected and $\mathrm{diam}(f^n(A)) \leq \varepsilon^*$ for all $n \in \mathbb{Z}$ then A is a singleton. Suppose that $\sigma = s$ and take $A \in \mathcal{C}_{\varepsilon^*}^{s}$, that is, $\mathrm{diam}(f^n(A)) \leq \varepsilon^*$ for all $n \geq 0$. If this diameter does not converge to zero, then there are $r > 0$ and $n_k \to +\infty$ such that $\mathrm{diam}(f^{n_k}(A)) > r$ for all $k \geq 0$. Since $\mathcal{C}(X)$ is compact we can assume that $f^{n_k}(A) \to C$ in the Hausdorff metric. We will show that

$$\mathrm{diam}(f^n(C)) \leq \varepsilon^* \quad \text{for all } n \in \mathbb{Z}. \tag{2}$$

Since $n_k \to +\infty$, given $n \in \mathbb{Z}$ there is k_0 such that $n_k + n \geq 0$ for all $k \geq k_0$. Then, $\mathrm{diam}(f^{n_k+n}(A)) \leq \varepsilon^*$ for all $k \geq k_0$. Since $f^{n_k+n}(A) = f^n(f^{n_k}(A))$ and $f^{n_k}(A) \to C$ we conclude (2). Since $\mathrm{diam}(C) \geq r > 0$ we have that C is not a singleton. This contradicts the cw-expansivity of f and proves that $\mathcal{C}_{\varepsilon^*}^{s} \subset \mathcal{C}^{\sigma}$.

To prove the next part we argue by contradiction. Suppose that there is $\varepsilon > 0$ and a sequence $A_k \in \mathcal{C}^s$ such that $\mathrm{diam}(A_k) \to 0$ and $\mathrm{diam}(f^{m_k}(A_k)) > \varepsilon$ for all $k \geq 0$ where $m_k \geq 0$. Note that $m_k \to +\infty$ and that we can assume that $\varepsilon < \varepsilon^*$. Since $A_k \in \mathcal{C}^s$ there is $n_k \geq m_k$ such that $\mathrm{diam}(f^n(A_k)) \leq \varepsilon$ for all $n \geq n_k$. We can take a subcontinuum $C_k \subset A_k$ and another divergent sequence m_k' such that $\mathrm{diam}(f^{m_k'}(C_k)) = \varepsilon$ and $\mathrm{diam}(f^n(C_k)) \leq \varepsilon$ for all $n \geq 0$. Then, arguing as in the proof of (2), we conclude that a limit continuum of $f^{m_k'}(C_k)$ contradicts the cw-expansivity.

Converse. Let us show that ε^* is a cw-expansivity constant. Suppose that $\mathrm{diam}(f^n(A)) \leq \varepsilon^*$ for all $n \in \mathbb{Z}$. Assume, by contradiction, that there is $\varepsilon \in (0, \mathrm{diam}(A))$. For this value of ε there is $\delta > 0$ such that $\mathcal{C}^s \cap \mathcal{C}_{\delta} \subset \mathcal{C}_{\varepsilon}^{s}$. Since $A \in \mathcal{C}_{\varepsilon^*}^{s}$ and $\mathcal{C}_{\varepsilon^*}^{u} \subset \mathcal{C}^u$ there is $m \leq 0$ such that $\mathrm{diam}(f^m(A)) < \delta$. Then $f^m(A) \in \mathcal{C}^s \cap \mathcal{C}_{\delta}$. As $\mathrm{diam}(A) > \varepsilon$ we have that $f^m(A) \notin \mathcal{C}_{\varepsilon}^{s}$. This is a contradiction that finishes the proof. $\square$

The next theorem states the existence of non-trivial stable and unstable continua through each point of the space. Moreover, these continua have diameter bounded away from zero. Their invariance is in fact given by Proposition 2.3.1. The result was first proved independently by Hiraide and Lewowicz in [**18**, **30**] for the classification of expansive homeomorphisms on compact surfaces. It was generalized by Kato in [**24**, Theorem 1.6] for cw-expansivity. A partial result was previously given by Mañé in [**32**] to prove that minimal expansive homeomorphisms can only exist on totally disconnected spaces.

Remark 2.3.2. For the study of cw-expansive homeomorphisms it is essential to assume some kind of connection of the space. To illustrate this point consider that every homeomorphism of a Cantor set is cw-expansive. It turns out that local connection is a good property to exploit the cw-expansivity. There is a minor loss of generality if, in addition, we assume that the space is a Peano continuum. Indeed, if X is a locally connected compact metric space then it has a finite number of components. Therefore, a homeomorphism f of X will permute these components and, taking a power of f, we will have a finite number of homeomorphisms of Peano continua.

For a set $\mathcal{C}^* \subset \mathcal{C}(X)$, as $\mathcal{C}_\delta$, $\mathcal{C}^\sigma$ and $\mathcal{C}_\varepsilon^\sigma$, define $\mathcal{C}^*(x) = \{A \in \mathcal{C}^* : x \in A\}$.

THEOREM 2.3.3. (Invariant Continuum Theorem) *If f is a cw-expansive homeomorphism of a Peano continuum X then for all $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\mathcal{C}_\varepsilon^\sigma(x) \backslash \mathcal{C}_\delta(x) \neq \emptyset \tag{3}$$

for all $x \in X$ and $\sigma = s, u$.

This theorem has the following direct consequences.

Remark 2.3.4. (Uniform size of stable continua) Note that (3) implies that for all $x \in X$ there are stable and unstable continua through x of diameter greater than δ. Consequently, these continua meet the boundary of the ball $B_{\delta/2}(x)$.

Remark 2.3.5. (No stable points) From the Invariant Continuum Theorem we see that if f is a cw-expansive homeomorphism of a Peano continuum X then neither stable nor unstable continua have interior points. This is because if $A \subset X$ is a stable set with an interior point x then we can take an unstable continuum contained in A that contradicts the cw-expansivity. In particular there are no Lyapunov stable trajectories.

Remark 2.3.6. (Surfaces with a boundary) Let us explain why we do not consider surfaces with a boundary. Suppose that $f : S \to S$ is a cw-expansive homeomorphism of a compact surface with a boundary. By Brouwer's theorem on the invariance of a domain [**21**] we know that ∂S is invariant by f. Then, the restriction $f : \partial S \to \partial S$ is cw-expansive. This gives a contradiction because, on one hand there are non-trivial stable continua in ∂S, and on the other hand every non-trivial continuum of ∂S has interior points (relative to ∂S). This is the argument proving that the circle admits no expansive homeomorphisms that the author learned from Lewowicz.

Problem 2.3.7. Do cw-expansive homeomorphisms of compact manifolds with a non-empty boundary exist? For example, does there exist a cw-expansive homeomorphism of the 3-ball? Does the 3-sphere admit expansive or cw-expansive homeomorphisms?

We say that $C \subset X$ *separates* X if $X \backslash C$ is not connected. We will show in Theorem 2.3.9, under a *natural* assumption on the Peano space X, that no stable set separates. First, we prove a topological lemma.

LEMMA 2.3.8. *If X is a continuum and every point has arbitrarily small neighborhoods with connected boundaries, then for all $\varepsilon > 0$ there is $\delta > 0$ such that if $A \subset X$ is a closed set that separates X and $\mathrm{diam}(A) < \delta$ then there is a component V of $X \backslash A$ with $\mathrm{diam}(V) < \varepsilon$.*

Proof. Given $\varepsilon > 0$ consider a finite open cover $\mathcal{U}$ of X such that $\mathrm{diam}(U) < \varepsilon$ for all $U \in \mathcal{U}$ and ∂U is connected. Take $\delta > 0$ such that if $\mathrm{diam}(A) < \delta$ then there is $U \in \mathcal{U}$ such that $A \subset U$. Suppose that $A \subset X$ is closed, separates X and $\mathrm{diam}(A) < \delta$. Take $U_* \in \mathcal{U}$ such that $A \subset U_*$.

Let us show that $C = X \backslash U_*$ is connected. By contradiction, suppose that $C = A \cup B$ is the union of disjoint, closed, non-empty sets. Since ∂U_* is connected, we can assume that $\partial U_* \subset A$. Then, B and $A \cup \overline{U_*}$ disconnect X, a contradiction that proves that $X \backslash U_*$ is connected.

Let V_* be the component of $X \backslash A$ containing $X \backslash U_*$. Since A separates X, there is at least another component. This component is contained in U_* and has diameter smaller than ε. $\qquad\square$

THEOREM 2.3.9. *If f is a cw-expansive homeomorphism of the Peano continuum X and every point of X has arbitrarily small neighborhoods with connected boundary then no stable closed set separates X.*

Proof. Arguing by contradiction assume that $A \subset X$ is a closed stable set separating X. As f is a homeomorphism, every iterate of A separates X. Since it is stable, taking a positive iterate, we can suppose that it is as small as we want. By Lemma 2.3.8 there is a small component U of its complement. By Theorem 2.3.3 each point of U has a stable continuum meeting ∂U. Since $\partial U \subset A$ we have that ∂U is stable, therefore $\overline{U}$ is a stable set. Since $\overline{U}$ has interior points we have a contradiction with Remark 2.3.5. $\qquad\square$

Remark 2.3.10. If a Peano continuum has no locally separating points then every point has arbitrarily small neighborhoods with connected boundaries. See [**22**] for a proof.

Remark 2.3.11. Theorem 2.3.9 holds if X is a compact manifold with or without a boundary. In the one-dimensional case, intervals do not have connected boundaries, but since there are no cw-expansive homeomorphisms on one-dimensional manifolds the theorem holds true.

The following example shows that in Theorem 2.3.9 we need to assume that every point has arbitrarily small neighborhoods with connected boundaries.

Example 2.3.12. Consider two copies of an Anosov diffeomorphism of the two-dimensional torus identifying two fixed points. The gluing point has no arbitrarily small neighborhoods with connected boundaries. Also, this point forms a stable set and it (locally and globally) separates.
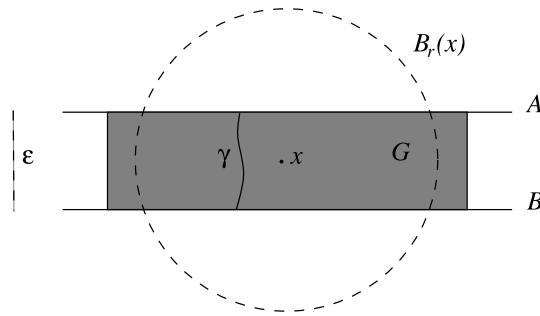
FIGURE 2. A capacitor of separation ε and radius r in the plane.

2.4. *Capacitors.* The following technical definition is based on the arguments of the proof of [**30**, Lemma 2.3].

Definition 2.4.1. Given $x \in X$, $\varepsilon, r > 0$, an (ε, r, x)-*capacitor* is a triple (A, G, B) such that:

(1) $A, B, G \subset X$, A and B are disjoint continua, $x \in \overline{G}$, G is open;
(2) $(\partial G) \cap B_r(x) \subset A \cup B$;
(3)   there is a continuum $\gamma \subset \overline{G} \cap B_{r/2}(x)$ meeting A and B;
(4) $G \subset B_\varepsilon(A)$ or $G \subset B_\varepsilon(B)$.

In this case A and B are the *plates* of the capacitor, ε is the *separation* of the plates and r is the *radius*. See Figure 2. We say that a capacitor has *stable plates* if A, B are stable sets for the homeomorphism $f : X \to X$.

In the next result we study unstable continua between two close stable plates. It generalizes [**30**, Lemma 2.3] and it will be applied in Theorem 6.7.1 for the study of cw$_\mathrm{F}$-expansivity on surfaces.

THEOREM 2.4.2. *Assume that $f : X \to X$ is a cw-expansive homeomorphism of a Peano continuum X. Then, for all $r > 0$ small there is $\varepsilon > 0$ such that if (A, G, B) is an (ε, r, x)-capacitor with stable plates then:*

(1)   *for all $y \in B_{r/2}(x) \cap \overline{G}$ there is an unstable continuum from y to $A \cup B$ contained in $\overline{G} \cap B_r(x)$;*
(2)   *there is an unstable continuum $C \subset \overline{G} \cap B_r(x)$ meeting A and B.*

Proof. Arguing by contradiction assume that there are $r > 0$, a sequence of $(1/n, r, x_n)$-capacitors (A_n, G_n, B_n) with stable plates and $y_n \in B_{r/2}(x_n) \cap \overline{G_n}$ with no unstable continuum from y_n to $A_n \cup B_n$ contained in $\overline{G_n} \cap B_r(x_n)$. Since X is a Peano continuum, by Theorem 2.3.3 for each n there is an unstable continuum $C_n \subset B_r(x_n)$ containing y_n and intersecting the boundary of $B_r(x_n)$. Since $(\partial G_n) \cap B_r(x_n) \subset A_n \cup B_n$ we conclude that $C_n \subset G_n$. Taking subsequences, we can assume that $A_n \to A$ and $C_n \to C$ in the Hausdorff metric. By definition of capacitor we have that $G_n \subset B_{1/n}(A_n)$ which implies that $C \subset A$. Therefore C is a continuum that is stable and unstable. Since $\mathrm{diam}(C) > 0$ we have a contradiction with the cw-expansivity of f.

Given $r > 0$ consider $\varepsilon > 0$ satisfying the first item. Let (A, G, B) be an (ε, r, x)-capacitor. By definition, there is a continuum $\gamma \subset \overline{B_{r/2}(x)}$ and points p^A, p^B with $p^A \in A \cap \gamma$ and $p^B \in B \cap \gamma$. Define

$$\gamma^A = \{y \in \gamma : \exists \text{ an unstable continuum from } y \text{ to } A \text{ contained in } \overline{G} \cap B_r(x)\},$$

$$\gamma^B = \{y \in \gamma : \exists \text{ an unstable continuum from } y \text{ to } B \text{ contained in } \overline{G} \cap B_r(x)\}.$$

The sets γ^A and γ^B are non-empty because they contain p^A and p^B, respectively. They are closed sets and, as we have shown, they cover γ. Since γ is connected, there is a point $z \in \gamma^A \cap \gamma^B$. □

Remark 2.4.3. Let $f : M \to M$ be an expansive homeomorphism of a compact three-manifold. In [**46**] it is shown that local stable sets are locally connected if f is smooth and without wandering points. By Theorem 2.4.2 we know that a stable set of f cannot be homeomorphic to the closure of the set

$$C = \{(x, y, z) : -1 \leq x \leq 1, 0 < z \leq 1, y = \sin(1/z)\}.$$

2.5. *Partial expansivity.* Consider a homeomorphism $f : X \to X$ of a compact metric space. We will use some concepts of topological dimension. We refer the reader to [**21**] for the definitions and basic properties.

Definition 2.5.1. Given an integer $d \geq -1$, we say that f is *partially expansive* with *central dimension d* and *expansivity constant* $\varepsilon > 0$ if for every non-trivial compact set $C \subset X$ with $\dim(C) > d$ there is $k \in \mathbb{Z}$ such that $\operatorname{diam}(f^k(C)) \geq \varepsilon$.

As usual, we say that f is *sensitive to initial conditions* if there is $\rho > 0$ such that for all $x \in M$ and for all $r > 0$ there are $y \in B_r(x)$ and $n \in \mathbb{Z}$ such that $\operatorname{dist}(f^n(y), f^n(x)) > \rho$.

PROPOSITION 2.5.2. *For a homeomorphism $f : X \to X$ the following hold:*
(1)   *f is expansive if and only if it is partially expansive with $d = -1$;*
(2)   *f is cw-expansive if and only if it is partially expansive with $d = 0$;*
(3)   *if in addition X is a compact manifold of dimension n then f is sensitive to initial conditions if and only if f is partially expansive with central dimension $d = n - 1$.*

Proof. Since the arguments are quite direct we only give the details that we consider more relevant. To prove the first part, note that by definition (see [**21**]) the condition $\dim(C) > -1$ means $C \neq \emptyset$. Then, $C \subset X$ with $\dim(C) > -1$ is non-trivial if and only if it has at least two points. To conclude the stated equivalence one has to note that $\operatorname{diam}(\{x, y\}) = \operatorname{dist}(x, y)$.

The statement related to cw-expansivity follows because positive dimension is equivalent to containing a non-trivial continuum.

For the last part we recall [**21**, Corollary 1, p. 46] that if X is a compact n-dimensional manifold and $C \subset X$ then $\dim(C) = n$ if and only if C has non-empty interior. □

Let $\phi : \mathbb{R} \times X \to X$ be a continuous flow. We consider the following weak form of expansivity. We say that a flow ϕ is *separating* [**3**] if there is $\delta > 0$ (a *separating constant*) such that if $\operatorname{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ then $y = \phi_s(x)$ for some $s \in \mathbb{R}$. Examples

of separating flows are expansive flows in the sense of Bowen and Walters [**8**] and k^*-expansive flows as defined by Komuro [**26**].

PROPOSITION 2.5.3. *If ϕ is a separating flow then for all $T \neq 0$ the homeomorphism $\phi_T : X \to X$ is partially expansive with $d = 1$.*

Proof. Let $\sigma > 0$ be a separating constant and take $\delta > 0$ such that if $C \subset X$ is a continuum and $\text{diam}(C) < \delta$ then $\text{diam}(\phi_s(C)) < \sigma$ for all $s \in [-T, T]$. Then, if $\text{diam}(\phi_{nT}(C)) < \delta$ for all $n \in \mathbb{Z}$ we have that $\text{diam}(\phi_s(C)) < \sigma$ for all $s \in \mathbb{R}$. Therefore, C is an orbit segment and consequently $\dim(C) \leq 1$. □

2.6. *Relative expansivity.* Given a homeomorphism $f : X \to X$ it is usual to define the *stable set* of $x \in X$ as

$$W^s(x) = \left\{ y \in X : \lim_{n \to +\infty} \text{dist}(f^n(x), f^n(y)) = 0 \right\}$$

and the *unstable set*

$$W^u(x) = \left\{ y \in X : \lim_{n \to -\infty} \text{dist}(f^n(x), f^n(y)) = 0 \right\}.$$

Consider the equivalence relation $x \sim_s y$, $x, y \in X$, if there is a continuum $C \subset X$ such that $x, y \in C$ and $\text{diam}(f^n(C)) \to 0$ as $n \to +\infty$. Similarly we define $\sim_u$ (taking $n \to -\infty$). The equivalence class of x will be denoted as $\mathcal{F}^s(x)$ (and $\mathcal{F}^u(x)$). We will give conditions that allow us to prove that $W^s = \mathcal{F}^s$ and $W^u = \mathcal{F}^u$.

Definition 2.6.1. Let $\mathcal{F}$ be a partition of X. We say that f is *separating mod $\mathcal{F}$* if there is $\delta > 0$ such that if $\text{dist}(f^n(x), f^n(y)) < \delta$ for all $n \geq 0$ then $y \in \mathcal{F}(x)$

PROPOSITION 2.6.2. *If f is separating mod $\mathcal{F}^s$ then $W^s = \mathcal{F}^s$.*

Proof. Assume that f is separating mod $\mathcal{F}^s$ with δ as in the definition. Suppose that $\text{dist}(f^n(x), f^n(y)) \to 0$ as $n \to +\infty$. Take k such that for all $n \geq k$ it holds that $\text{dist}(f^n(x), f^n(y)) < \delta$. Then there is a stable continuum C containing $f^k(x), f^k(y)$. This implies that x, y are in the stable continuum $f^{-k}(C)$. Then $W^s = \mathcal{F}^s$. □

Problem 2.6.3. Assuming that f is cw-expansive, does the condition $\mathcal{F}^s = W^s$ imply that it is separating mod $\mathcal{F}^s$?

PROPOSITION 2.6.4. *The pseudo-Anosov diffeomorphism with 1-prongs on the two-dimensional sphere given in §2.2.1 is not separating mod $\mathcal{F}^s$.*

Proof. By Proposition 2.2.2, for all $\varepsilon > 0$ there is a Cantor set C contained in an unstable arc and contained in $W^s_\varepsilon(p)$ for some p in this unstable arc. Since $\mathcal{F}^s(p)$ cuts the unstable arc in a countable set, there is $a \in C$ that is not in $\mathcal{F}^s(p)$. This proves that f is not separating mod $\mathcal{F}^s$. □

Problem 2.6.5. Does the example in §2.2.1 satisfy $W^s = \mathcal{F}^s$? The solution could be simple but we were not able to solve it.

Definition 2.6.6. We say that f is *expansive mod* $\mathcal{F}^s$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\text{dist}(f^n(x), f^n(y)) < \delta$ for all $n \geq 0$ then x and y are in a common ε-stable continuum.

Note that every homeomorphism expansive mod $\mathcal{F}^s$ is separating mod $\mathcal{F}^s$.

PROPOSITION 2.6.7. *If $f : X \to X$ is a cwN-expansive homeomorphism of a Peano continuum and f is expansive mod $\mathcal{F}^s$ and f^{-1} is expansive mod $\mathcal{F}^u$ then f is N-expansive.*

Proof. Let $\alpha > 0$ be a cw$N$-expansivity constant for f and take $\varepsilon = \alpha/2$. Consider δ from Definition 2.6.6 and suppose that $\text{diam}(f^n(C)) < \delta$ for all $n \in \mathbb{Z}$. For $x, y \in C$ we have that $\text{dist}(f^n(x), f^n(y)) < \delta$ for all $n \geq 0$. Then, there is an ε-stable continuum containing x, y. We have that there is a 2ε-stable continuum containing C. Similarly, there is a 2ε-unstable continuum containing C. Since $2\varepsilon = \alpha$ is a cw$N$-expansivity constant for f we have that $\text{card}(C) \leq N$. This proves that δ is an N-expansivity constant for f. □

Problem 2.6.8. It seems that pseudo-Anosov diffeomorphisms of surfaces without 1-prongs are expansive mod $\mathcal{F}^s$. We know that the pseudo-Anosov diffeomorphism with 1-prongs of §2.2.1 is not expansive mod $\mathcal{F}^s$. Are the examples in §§2.2.3 and 2.2.4 expansive mod $\mathcal{F}^s$? It would be interesting to understand which cw-expansive surface homeomorphisms are expansive mod $\mathcal{F}^s$.

3. *Continuum theory and decompositions*

In this section we review some results from continuum theory that will be used throughout the article. Also, we study decompositions that will play the role of *foliated charts* in the next section.

3.1. *Background to continuum theory.* We recall that a *continuum* is a compact connected metric space. General references for continuum theory are [**27, 28, 38**].

3.1.1. *Partitions and monotone restrictions.* Let (X, dist) be a compact metric space and denote by $\mathcal{P}(X)$ the set of subsets of X. A *partition* of X is a function $Q : X \to \mathcal{P}(X)$ such that:
(1) $x \in Q(x)$ for all $x \in X$;
(2) $y \in Q(x)$ if and only if $x \in Q(y)$; and
(3) $x \in Q(y)$ and $y \in Q(z)$ imply $x \in Q(z)$.
A partition Q is *monotone* if each $Q(x)$ is connected. Given $Y \subset X$ and $x \in Y$, denote by $\text{Comp}_x(Y)$ the component of Y containing x. For a partition $Q : X \to \mathcal{P}(X)$ and $Y \subset X$, define the *monotone restriction* $Q|_Y^m : Y \to \mathcal{P}(Y)$ as

$$Q|_Y^m(x) = \text{Comp}_x(Q(x) \cap Y) \tag{4}$$

for all $x \in Y$.

PROPOSITION 3.1.1. *If Q is a partition of X and $Z \subset Y \subset X$ then*

$$(Q|_Y^m)|_Z^m = Q|_Z^m.$$

Proof. From the definition (4) we see that we have to show that

$$\mathrm{Comp}_x(\mathrm{Comp}_x(Q(x) \cap Y) \cap Z) = \mathrm{Comp}_x(Q(x) \cap Z)$$

for all $x \in Z$. To prove the inclusion $\supset$ consider a connected set $C \subset Q(x) \cap Z$ such that $x \in C$. Since $Z \subset Y$ we have that $C \subset Q(x) \cap Y$. Then, $C \subset \mathrm{Comp}_x(Q(x) \cap Y)$ because $x \in C$. Given that $C \subset Z$ we conclude that $C \subset \mathrm{Comp}_x(\mathrm{Comp}_x(Q(x) \cap Y) \cap Z)$. Then $\mathrm{Comp}_x(Q(x) \cap Z) \subset \mathrm{Comp}_x(\mathrm{Comp}_x(Q(x) \cap Y) \cap Z)$. The converse inclusion is easier to prove. It follows from the fact that $\mathrm{Comp}_x(Q(x) \cap Y) \subset Q(x)$. □

We say that a partition Q is *upper semicontinuous* if for all $x \in X$ and every open set U containing $Q(x)$ there is a neighborhood V of x such that if $y \in V$ then $Q(y) \subset U$.

Remark 3.1.2. Q is upper semicontinuous if and only if given $x_n \to x$ such that $Q(x_n) \to C$ in the Hausdorff metric then $C \subset Q(x)$. The upper semicontinuity of Q implies that each $Q(x)$ is a closed subset of X.

We say that a partition Q is *continuous at* $x \in X$ if for every $x_n \to x$ we have that $Q(x_n) \to Q(x)$ in the Hausdorff metric. We say that Q is *continuous* if it is continuous at every $x \in X$. A set $G \subset X$ is *residual* if it is a countable intersection of open and dense subsets of X.

PROPOSITION 3.1.3. *If Q is an upper semicontinuous partition of X then:*
(1) [**28**, pp. 70–71] *there is a residual subset $G \subset X$ such that Q is continuous at every $x \in G$;*
(2) [**38**, Theorem 3.9] *if in addition X is a continuum then X/Q with its quotient topology is a continuum*†.

3.1.2. *Local connection.* A continuum is *hereditarily locally connected* if every subcontinuum is locally connected. A *convergence continuum A* of a compact metric space X is a non-trivial subcontinuum of X for which there is a sequence of continua $A_i \subset X$ such that $A_i \to A$ in the Hausdorff metric, $A_i \cap A = \emptyset$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

THEOREM 3.1.4. [**38**, Theorem 10.4] *A continuum X is hereditarily locally connected if and only if X contains no convergence continuum.*

THEOREM 3.1.5. [**20**, Theorem 3-17] *Every connected, locally connected, complete metric space is arc-connected.*

THEOREM 3.1.6. (Sierpiński's theorem [**27**, p. 218]) *A continuum C is Peano if and only if for all $\varepsilon > 0$ there is a finite cover $C_1, \ldots, C_n$ of C by connected sets of diameter less than ε.*

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† It is clear that X/Q is compact and connected since it is the quotient of the continuum X. In [**38**, Theorem 3.9] it is shown that X/Q is metrizable.

3.1.3. *Unicoherence.* A continuum C is *unicoherent* if given subcontinua A, $B \subset C$ such that $A \cup B = C$ then $A \cap B$ is connected. A continuum is *hereditarily unicoherent* if every subcontinuum is unicoherent. To show the difference between these concepts, and for future reference, let us recall the following result.

THEOREM 3.1.7. (Janisewski's theorem [**28**, p. 506]) *The union of two subcontinua A, B of the 2-sphere disconnects the sphere if and only if $A \cap B$ is disconnected.*

By Janisewski's theorem we see that the 2-sphere is a unicoherent continuum. It is not hereditarily unicoherent because it contains a circle, which is not unicoherent.

3.1.4. *Dendrites.* A *dendrite* is a Peano continuum X containing no simple closed curve. The points of a dendrite are classified as: an *end point* if its complement is connected, a *ramification* or a *branch* point if its complement has at least three components, a *regular point* if its complement has two components. The following results from [**38**, §10] summarize several properties of dendrites.

THEOREM 3.1.8. *Every subcontinuum of a dendrite is a dendrite.*

THEOREM 3.1.9. *The set of all the ramification points of a dendrite is countable.*

THEOREM 3.1.10. *If X is a compact metric space then the following statements are equivalent:*
(1) *X is a dendrite;*
(2) *X is an hereditarily unicoherent Peano continuum;*
(3) *X is connected and any two points of the continuum are separated by a third point.*

THEOREM 3.1.11. *Every dendrite can be embedded in the plane. Moreover, Wazewski's universal dendrite is a dendrite in $\mathbb{R}^2$, which contains a topological copy of any dendrite.*

3.2. *Decompositions.* The standard theory of foliations is based on a special kind of local partition in plaques, i.e. *foliated charts*. On a surface S, a C^0 foliated chart is a homeomorphism $\varphi : U \subset S \to (0, 1) \times (0, 1)$ where $U \subset S$ is an open subset. If $\varphi = (\varphi_1, \varphi_2)$ then the plaques of U are $\varphi_2^{-1}(y)$ for all $y \in (0, 1)$. We can define a map $Q : U \to \mathcal{C}(S)$ as $Q(p) = \varphi_2^{-1}(\varphi(p))$. This map Q is a continuous monotone partition of U.

For a pseudo-Anosov diffeomorphism, the stable and unstable sets form *singular foliations*. At a singular point, an n-prong with $n > 2$, it is not possible to define a local chart as above where the plaques are the local stable sets. The partition in local stable sets in a neighborhood of a singularity is monotone and upper semicontinuous. The (full) continuity is lost at the singularity. This is the idea we have in mind for the next definition of *decomposition*, which is a standard concept in continuum theory.

Given a compact metric space X, we will consider a compact subset $Y \subset X$. We can think that Y is the closure of the open set U in the surface considered above. However, the theory is developed in such a way that Y may not be connected and may have empty interior.

Definition 3.2.1. A *decomposition* of Y is a monotone upper semicontinuous partition $Q : Y \to \mathcal{C}(Y)$. The sets $Q(x)$ are the *plaques* of the decomposition.

Example 3.2.2. (Extremal examples) For a compact metric space Y define $Q_{\min}(x) = \{x\}$ and $Q_{\max}(x) = \mathrm{Comp}_x(Y)$. It holds that $Q_{\min}$ and $Q_{\max}$ are decompositions of Y, a proof can be found in [**34**, Theorem 24]. For every decomposition Q of Y we have that $Q_{\min}(x) \subset Q(x) \subset Q_{\max}(x)$ for all $x \in Y$.

For standard foliations, the restriction of a foliated chart to an arbitrary subset may not be a foliated chart. This is because the product structure may be lost. Next we show that the restriction of a decomposition is a decomposition.

PROPOSITION 3.2.3. *If Q is a decomposition of the compact set Y and $Z \subset Y$ is compact then $Q|_Z^m$ is a decomposition.*

Proof. By definition (4) $Q|_Z^m$ is monotone. Suppose that $x_n \to x$ and $Q|_Z^m(x_n) \to C$. Taking a subsequence we can also assume that $Q(x_n) \to D$ for some subcontinuum $D \subset X$. Since $Q|_Z^m(x_n) \subset Q(x_n)$ and Q is upper semicontinuous we have that $C \subset D \subset Q(x)$ and, as C is a continuum contained in $Z \cap Q(x)$, we conclude that $C \subset Q|_Z^m(x)$. This proves that $Q|_Z^m$ is upper semicontinuous. □

Example 3.2.4. Let P be a continuum and consider a compact metric space Z. Define $Y = P \times Z$ and $Q : Y \to \mathcal{C}(Y)$ as $Q(p, z) = P \times \{z\}$. If we consider the product topology on Y then Q is a continuous decomposition of Y.

Definition 3.2.5. Given two decompositions $Q_i : Y_i \to \mathcal{C}(Y_i)$, $i = 1, 2$, we say that Q_1 and Q_2 are *equivalent* if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(Q_1(p)) = Q_2(h(p))$ for all $p \in Y_1$. We say that a decomposition is a *product structure* if it is equivalent to a decomposition as in Example 3.2.4.

PROPOSITION 3.2.6. *A decomposition Q_1 of a continuum Y is a product structure if and only if there is a decomposition Q_2 of Y such that*

$$\mathrm{card}(Q_1(x) \cap Q_2(y)) = 1 \tag{5}$$

for all $x, y \in Y$.

Proof. If Q_1 is a product structure, we can assume that $Y = P \times Z$. Since Y is a continuum we have that P and Z are continua. If $Q_1(p, z) = P \times \{z\}$ we can define $Q_2(p, z) = \{p\} \times Z$. Then

$$Q_1(p_1, z_1) \cap Q_2(p_2, z_2) = \{(p_2, z_1)\}$$

for all $(p_1, z_1), (p_2, z_2) \in P \times Z$.

To show the converse note that by Proposition 3.1.3 we know that Y/Q_1 and Y/Q_2 are continua. Consider $h : Y \to Y/Q_1 \times Y/Q_2$ defined by

$$h(x) = (Q_1(x), Q_2(x)).$$

The condition (5) implies that h is bijective. Since we consider quotient topology, the projections onto Y/Q_1 and Y/Q_2 are continuous and h is continuous. Since Y is compact we have that h^{-1} is continuous. Given that $h(Q_1(x)) = \{Q_1(x)\} \times Y/Q_2$ the proof ends. □

Definition 3.2.7. We say that a compact subset $K \subset Y$ is a *representative set* of the decomposition Q if $Q(x) \cap K \neq \emptyset$ for all $x \in Y$. If K is a representative set we say that Q is a decomposition of (Y, K).

We will consider decompositions of $(\overline{U}, \partial U)$, with U an open set, because for a cw-expansive homeomorphism we know that the diameter of the stable and the unstable sets are bounded away from zero. Therefore, if U is small, each stable plaque in U meets the boundary of U.

PROPOSITION 3.2.8. *If Q is a decomposition of (Y, K), $U \subset Y$ is open and $U \cap K = \emptyset$ then $Q|_{\overline{U}}^m$ is a decomposition of $(\overline{U}, \partial U)$.*

Proof. From Proposition 3.2.3 we know that $Q' = Q|_{\overline{U}}^m$ is a decomposition. Given $x \in U$ the fact that $Q'(x) \cap \partial U \neq \emptyset$ follows by [**20**, Theorem 2-16] or [**15**, 6.1.25]. $\square$

In the following subsections we study some particular types of decompositions. Applications will be given in §6 for the study of decompositions of two-dimensional disks.

3.2.1. *Dendritic decompositions.* Let $Q : Y \to \mathcal{C}(Y)$ be a decomposition of the continuum Y.

Definition 3.2.9. We say that Q is *dendritic* if $Q(x)$ is a dendrite for all $x \in Y$. We say that Q is *codendritic* if Y/Q is a dendrite with the quotient topology.

Dendritic decompositions are a generalization of one-dimensional foliated charts. Codendritic decompositions generalize codimension-one foliated charts. In §6.1 we will study codendritic and dendritic decompositions of a two-dimensional disk. Some general properties can be derived in the generality of a continuum Y.

PROPOSITION 3.2.10. *If $Q : Y \to \mathcal{C}(Y)$ is a codendritic decomposition whose plaques have empty interior then there is a residual subset $G \subset Y$ such that $Y \backslash Q(x)$ has 1 or 2 components for all $x \in G$.*

Proof. From Theorem 3.1.9 we have that the set of ramification points of the quotient dendrite Y/Q is at most countable. Let $Q_1, Q_2, \ldots$ be the ramification points of Y/Q. We have that $Y \backslash Q_i$ is open in Y. It is dense in Y because no Q_i has interior points. Define $G = Y \backslash \cup Q_i$. Since for every point x of the residual set G we know that $Q(x)$ is not a ramification point, we conclude that $Y \backslash Q(x)$ has one or two components. $\square$

In the hypothesis of Proposition 3.2.10 we cannot conclude that $Y \backslash Q(x)$ has two components for all x in a residual subset of Y (as could seem natural). See Example 6.2.5.

With respect to the next proposition, if we think that Q^1 represents stable continua and Q^2 unstable continua, then the condition $Q^1(z) \cap Q^2(z) = \{z\}$ is related with cw1-expansivity (recall that for cw1-expansivity we require that local stable and unstable continua intersect in at most one point).

PROPOSITION 3.2.11. *Let Q^1, Q^2 be decompositions of the continuum Y such that $Q^1(z) \cap Q^2(z) = \{z\}$ for all $z \in Y$. If Q^1 is codendritic then Q^2 is dendritic.*

Proof. Given $x \in Y$ consider $f : Q^2(x) \to Y/Q^1$ defined as $f(y) = Q^1(y)$. With the quotient topology on Y/Q^1 we have that f is continuous because it is the restriction of the decomposition map Q^1. The condition $Q^1(z) \cap Q^2(z) = \{z\}$ for all $z \in Y$ gives us that f is injective. Since $Q^2(x)$ is compact, f is a homeomorphism onto its image. The image of f is a subcontinuum of the quotient dendrite Y/Q^1. Then, the result follows by Theorem 3.1.8 (every subcontinuum of a dendrite is a dendrite).   □

Remark 3.2.12. The converse of Proposition 3.2.11 is not true. Take two one-dimensional foliations on a three-dimensional ball. Both foliations are dendritic but neither is codendritic.

3.2.2. *Smooth and unicoherent decompositions.* Our motivation for including the following kind of decompositions is to include the example of §2.2.4 (the anomalous saddle) in the theory. The example presents local stable sets that are not locally connected but hereditarily unicoherent.

Definition 3.2.13. A decomposition is *hereditarily unicoherent* if each plaque is hereditarily unicoherent.

   If P is an hereditarily unicoherent continuum then given $x, y \in P$ there is a unique minimal continuum containing x and y. If Q is an hereditarily unicoherent decomposition and $y \in Q(x)$, this minimal continuum will be denoted by $Q(x, y)$.
   We remark that in continuum theory the term *smooth* has a particular meaning that is not related with any class of differentiability (or at least the author cannot see any connection). To avoid confusions we will call it $\mathcal{C}$-*smooth*.

Definition 3.2.14. A decomposition Q of a continuum Y is $\mathcal{C}$-*smooth* if it is hereditarily unicoherent and if $x_n \to x$, $y_n \in Q(x_n)$ and $y_n \to y$ then $Q(x_n, y_n) \to Q(x, y)$ in the Hausdorff metric.

PROPOSITION 3.2.15. *Every $\mathcal{C}$-smooth decomposition is dendritic.*

Proof. By Theorem 3.1.10 it is sufficient to prove that each plaque is locally connected. If P is a non-locally connected plaque then there is $x \in P$ without arbitrarily small and connected neighborhoods. Let $\varepsilon > 0$ be such that Comp_x $(B_\varepsilon(x) \cap P)$ is not a neighborhood of x in P. Then we can take $x_n \in P$ such that $x_n \to x$ and $\mathrm{Comp}_{x_n}(B_\varepsilon(x) \cap P)$ is disjoint from $\mathrm{Comp}_x(B_\varepsilon(x) \cap P)$. Then, $\mathrm{diam}(Q(x, x_n))$ is bounded away from zero. But $x_n \to x$ and $Q(x, x) = x$. This contradicts that Q is $\mathcal{C}$-smooth and finishes the proof.   □

PROPOSITION 3.2.16. *If Q is a $\mathcal{C}$-smooth decomposition of a continuum Y and $Z \subset Y$ is compact then $Q|_Z^m$ is $\mathcal{C}$-smooth.*

Proof. Since each plaque of Q is hereditarily unicoherent we have that each plaque of $Q|_Z^m$ is hereditarily unicoherent. Note that if $Q(x, y) \subset Z$ then $Q(x, y) = Q|_Z^m(x, y)$. Suppose that $x_n \to x$, $y_n \in Q|_Z^m(x_n)$ and $y_n \to y$. We know that $Q(x_n, y_n) \to Q(x, y)$. Since $Q(x_n, y_n) = Q|_Z^m(x_n, y_n)$ and $Q(x, y) = Q|_Z^m(x, y)$ we have that $Q|_Z^m$ is $\mathcal{C}$-smooth.   □

TABLE 2. Hierarchy of $\mathcal{C}$-foliations. Dendritations, see §6, are Peano cw-foliations of surfaces.

dendritation	$\Rightarrow$	Peano foliation
$\Downarrow$		$\Downarrow$
cw-foliation	$\Rightarrow$	$\mathcal{C}$-foliation
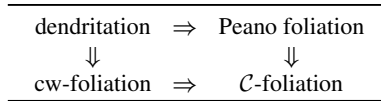
4. *Foliations from continuum theory*

In this section we present a detailed exposition of our *foliations* from the viewpoint of continuum theory. Since we were not able to find this concept in the literature we develop the theory from the most basic concepts.

In the *standard theory of foliations*† there are at least two main viewpoints for the definition of foliation.

(1)   A partition of the manifold into leaves (satisfying certain conditions). This approach seems to be preferred in dynamical systems as it appears in [**9**, §5.13] and [**2**, §6.7].

(2)   A maximal atlas of foliated charts. This definition is usual in texts on foliations such as [**10**, **11**].

Leaves–atlas equivalence. [**11**, Theorem 1.2.18] Both definitions of foliation coincide.

Roughly speaking, the proof is as follows. Given an atlas, we construct chains of plaques, and then the leaves. For the converse implication, one needs to recover the plaques from the leaves. A plaque is defined as the component of the intersection of a leaf with a local chart.

In this section we introduce some levels of generalizations of foliations. Table 2 summarizes the definitions and their implications.

As we will see in §4.2.1, $\mathcal{C}$-foliations do not satisfy leaves–atlas equivalence. This depends on the local connection of plaques. Then, we introduce Peano foliations as $\mathcal{C}$-foliations with locally connected plaques. In Theorem 4.2.2 we show that Peano foliations satisfy the leaves–atlas equivalence, see Equation (6). For the study of cw-expansive homeomorphisms we introduce cw-foliations.

4.1. *$\mathcal{C}$-foliations.*

Let (X, dist) be a compact metric space and denote by τ the topology of X. Given two closed sets $Y_1, Y_2 \subset X$ and we say that two decompositions Q_i of Y_i, $i = 1, 2$, are *compatible* if

$$Q_1|^m_{Y_1 \cap Y_2} = Q_2|^m_{Y_1 \cap Y_2}.$$

Recall that $\overline{U}$ denotes the closure of U.

Definition 4.1.1. An *atlas* is a collection of compatible decompositions

$$\mathcal{A} = \{Q_{\overline{U}} : \overline{U} \to \mathcal{C}(\overline{U})\}_{U \in \mathcal{U}},$$

where $\mathcal{U}$ is an open cover of X. In this case we say that $\mathcal{A}$ is an atlas *over $\mathcal{U}$*.

Atlases can be ordered by inclusion, and by Zorn's lemma we have that every atlas is contained in a maximal atlas.

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† For precision, fix the meaning of this expression to 'the theory developed in [**11**]'.

Definition 4.1.2. A $\mathcal{C}$*-foliation* is a maximal atlas.

A basis $\mathcal{U}$ of the topology of X is *complete* if $V \in \mathcal{U}$ and $U \subset V$ is open, then $U \in \mathcal{U}$.

PROPOSITION 4.1.3. *Every $\mathcal{C}$-foliation is defined over a complete basis.*

Proof. Let $\mathcal{F}$ be a $\mathcal{C}$-foliation over $\mathcal{U}$. Let U be an open set contained in $V \in \mathcal{U}$. Define $Q_{\overline{U}} = Q_{\overline{V}}|_{\overline{U}}^m$. To show that $U \in \mathcal{U}$ we will show that $Q_{\overline{U}}$ is compatible with the decompositions of $\mathcal{F}$. If $W \in \mathcal{U}$ then

$$Q_{\overline{U}}|_{\overline{U} \cap \overline{W}}^m = (Q_{\overline{V}}|_{\overline{U}}^m)|_{\overline{U} \cap \overline{W}}^m = Q_{\overline{V}}|_{\overline{U} \cap \overline{W}}^m = Q_{\overline{W}}|_{\overline{U} \cap \overline{W}}^m,$$

where the second equality follows by Proposition 3.1.1 and the last one is the compatibility of the decompositions of V and W. □

Notice that the decompositions $Q_{\overline{U}}$ are defined on the closures of the open sets of $\mathcal{U}$. We will consider $Q_U = Q_{\overline{U}}|_U^m$ for $U \in \mathcal{U}$.

Definition 4.1.4. An *open plaque* is a set of the form $Q_U(x)$ for $x \in U \in \mathcal{U}$.

PROPOSITION 4.1.5. *If $\mathcal{A}$ is an atlas over the complete basis $\mathcal{U}$ then the set of open plaques $\{Q_U(x) : x \in U \in \mathcal{U}\}$ is a basis of a topology $\tau_{\mathcal{A}}$ of X. If $\mathcal{A}_1 \subset \mathcal{A}_2$ are two of such atlases then $\tau_{\mathcal{A}_1} = \tau_{\mathcal{A}_2} \subset \tau$.*

Proof. Since $\mathcal{U}$ is a cover of X, the open plaques cover X. Given $x \in Q_U(y) \cap Q_V(z)$, $U, V \in \mathcal{U}$, notice that $Q_U(y) \cap Q_V(z) = Q_U(x) \cap Q_V(x)$ and $Q_{U \cap V}(x) \subset Q_U(x) \cap Q_V(x)$. Since $\mathcal{U}$ is complete $U \cap V \in \mathcal{U}$, and $Q_{U \cap V}(x)$ is an open plaque. This proves that the open plaques form a basis of a topology. Since $U = \bigcup_{x \in U} Q_U(x)$ for all $U \in \mathcal{U}$, it holds that $\tau_{\mathcal{A}} \subset \tau$.

Suppose that $\mathcal{A}_1 \subset \mathcal{A}_2$ are atlases over $\mathcal{U}_1 \subset \mathcal{U}_2$, respectively. It is clear that $\tau_{\mathcal{A}_1} \subset \tau_{\mathcal{A}_2}$. Given $x \in U \in \mathcal{U}_2$ take $V \in \mathcal{U}_1$ such that $x \in V \subset U$. Then, $Q_V(x) \subset Q_U(x)$ and $\tau_{\mathcal{A}_2} \subset \tau_{\mathcal{A}_1}$. □

Definition 4.1.6. The topology $\tau_{\mathcal{A}}$ will be called *the plaque topology*. Since it does not depend on the atlas $\mathcal{A}$ (over a complete basis) representing the $\mathcal{C}$-foliation $\mathcal{F}$ it will also be denoted as $\tau_{\mathcal{F}}$.

4.1.1. *Chains of plaques.* Let $\mathcal{A}$ be an atlas over a complete basis $\mathcal{U}$ of the compact metric space (X, dist). A *chain of $\mathcal{A}$-plaques* is a sequence $P_1, \ldots, P_n$ of open plaques of $\mathcal{A}$ such that $P_i \cap P_{i+1} \neq \emptyset$ for all $i = 1, \ldots, n-1$. Given $x \in P_1$ and $y \in P_n$ we say that the chain goes from x to y.

Definition 4.1.7. For a point $x \in X$ the *$\mathcal{A}$-leaf* of x is the set $\mathcal{A}(x)$ defined by $y \in \mathcal{A}(x)$ if there is a chain of open plaques from x to y.

PROPOSITION 4.1.8. *Every component of $(X, \tau_{\mathcal{A}})$ is contained in a $\mathcal{A}$-leaf.*

Proof. This follows because the $\mathcal{A}$-leaves form a partition of X in $\tau_{\mathcal{A}}$-open sets. □

We will define a metric on X defining the topology $\tau_{\mathcal{A}}$. Define $\mathrm{dist}_{\mathcal{A}} : X \times X \to \mathbb{R}$ as

$$\mathrm{dist}_{\mathcal{A}}(x, y) = \inf \sum_{i=1}^{n} \mathrm{diam}(P_i),$$

where the infimum is taken over all the chains of open plaques $P_1, \ldots, P_n$, $n \geq 1$, with $x \in P_1$ and $y \in P_n$. If x and y are in different leaves we set $\mathrm{dist}_{\mathcal{A}}(x, y) = +\infty$. It is standard to prove that $\mathrm{dist}_{\mathcal{A}}$ is a metric on X.

LEMMA 4.1.9. *If $U \in \mathcal{U}$, $P_1, \ldots, P_n$ is a chain of open plaques contained in U and $x \in P_1$ then each $P_i \subset Q_U(x)$.*

Proof. Suppose that $P_i = Q_{U_i}(x_i)$ with $U_i \in \mathcal{U}$. Define $V_i = U \cap U_i$. Since $P_i \subset U$, by Proposition 3.1.1 we have that $P_i = Q_{V_i}(x_i)$ and $P_1 \subset Q_U(x)$. As $P_1 \cap P_2 \neq \emptyset$ we conclude that $P_2 \subset Q_U(x)$. By induction, each $P_i \subset Q_U(x)$. $\qquad\square$

PROPOSITION 4.1.10. *The topology defined by $\mathrm{dist}_{\mathcal{A}}$ is $\tau_{\mathcal{A}}$.*

Proof. Given $x \in U \in \mathcal{U}$, consider $\delta > 0$ such that $B_\delta(x) \subset U$. Take $y \in X$ such that $\mathrm{dist}_{\mathcal{A}}(x, y) < \delta$. We will show that $y \in Q_U(x)$. Since $\mathrm{dist}_{\mathcal{A}}(x, y) < \delta$ there is a chain of plaques P_i, $i = 1, \ldots, n$, such that $x \in P_1$, $y \in P_n$ and $\sum_{i=1}^{n} \mathrm{diam}(P_i) < \delta$. Then $\mathrm{diam}(\bigcup_{i=1}^{n} P_i) < \delta$ and $\bigcup_{i=1}^{n} P_i \subset U$. By Lemma 4.1.9 we conclude that $\bigcup_{i=1}^{n} P_i \subset Q_U(x)$. Consequently $y \in Q_U(x)$. To prove the converse consider a ball $B = \{y \in X : \mathrm{dist}_{\mathcal{A}}(x, y) < r\}$, $r > 0$ and $x \in X$. Taking $U \in \mathcal{U}$ with $x \in U$ and $\mathrm{diam}(U) < r$ we see that $Q_U(x) \subset B$ and $B \in \tau_{\mathcal{A}}$. $\qquad\square$

PROPOSITION 4.1.11. *Every leaf and X itself are complete (as metric spaces) with respect to $\mathrm{dist}_{\mathcal{A}}$.*

Proof. Let x_n be a Cauchy sequence with respect to dist_A. The definition of the metric allows us to assume that the sequence is contained in a leaf. Also, its limit (which exists, as we shall prove) must be in this leaf. Thus, the proof is reduced to showing that X is complete.

Since $\mathrm{dist} \leq \mathrm{dist}_{\mathcal{A}}$, x_n is a Cauchy sequence with respect to dist. Then, there is $x \in X$ such that $\mathrm{dist}(x_n, x) \to 0$. Take $\varepsilon > 0$ such that $U = B_\varepsilon(x) \in \mathcal{U}$. Take $l \geq 1$ such that $\mathrm{dist}_{\mathcal{A}}(x_n, x_l) < \varepsilon/2$ and $\mathrm{dist}(x_n, x) < \varepsilon/2$ for all $n \geq l$. Consider a sequence of open plaques P_n such that $x_l, x_n \in P_n$ and $\mathrm{diam}(P_n) < \varepsilon/2$. Take $U_n \in \mathcal{U}$ such that $U_n \subset U$ and $Q_{U_n}(x_l) = P_n$. We know that $Q_{U_n}(x_l) \subset Q_U(x_l)$. Since $x_n \in Q_{U_n}(x_l)$ and $x_n \to x$ (in dist) we conclude that $x \in Q_U(x_l)$. Therefore, $x_n \in Q_U(x)$ for all $n \geq l$. This proves that $\mathrm{dist}_{\mathcal{A}}(x_n, x) \to 0$. $\qquad\square$

4.2. Peano foliations.

Let X be a compact metric space with topology τ.

Definition 4.2.1. An atlas $\mathcal{A}$ is a *Peano atlas* if its open plaques are locally connected in τ. A *Peano foliation* is a maximal Peano atlas.

By *maximal Peano atlas* we mean maximal among all Peano atlases, see Example 5.1.7 for a maximal Peano atlas that is not maximal among all the atlases. Every Peano atlas is

contained in a Peano foliation and every Peano foliation is defined over a complete basis. Given a topology τ_* on X and $Y \subset X$, denote by $\tau_*|_Y$ the relative topology on Y induced by τ_*.

THEOREM 4.2.2. *If $\mathcal{F}$ is a Peano foliation over $\mathcal{U}$ then:*
(1) *for every open plaque P it holds that $\tau|_P = \tau_\mathcal{F}|_P$;*
(2) *every open plaque is connected in $\tau_\mathcal{F}$;*
(3) *$(X, \tau_\mathcal{F})$ is locally connected;*
(4) *every leaf is arc-connected in both topologies $\tau_\mathcal{F}$ and τ;*
(5) *every leaf is a component of $(X, \tau_\mathcal{F})$;*
(6) *for every leaf L, $U \in \mathcal{U}$ and $x \in L \cap U$ it holds that*

$$\mathrm{Comp}_x(L \cap U) = Q_U(x). \tag{6}$$

Proof. Let P be an open plaque. By Proposition 4.1.5 we know that $\tau \subset \tau_\mathcal{F}$ and then $\tau|_P \subset \tau_\mathcal{F}|_P$. To show that $\tau_\mathcal{F}|_P \subset \tau|_P$, it is sufficient to prove that if P' is another open plaque then $P \cap P' \in \tau|_P$. Take $x \in P \cap P'$ and suppose that $P = Q_U(x)$ and $P' = Q_V(x)$ with $U, V \in \mathcal{U}$. Since $\mathcal{U}$ is a complete basis of τ there is $W \in \mathcal{U}$ such that $x \in W \subset U \cap V$ and $P \cap W$ is connected (P is locally connected at x). Then $P \cap W = \mathrm{Comp}_x(P \cap W) = \mathrm{Comp}_x(Q_U(x) \cap W) = Q_W(x)$. Since $W \subset U \cap V$ we can apply Proposition 3.1.1 to conclude that $Q_W(x) \subset P \cap P'$. Given that $Q_W(x) = P \cap W \in \tau|_P$ we conclude that $\tau_\mathcal{F}|_P \subset \tau|_P$. Then $\tau|_P = \tau_\mathcal{F}|_P$.

By definition, every open plaque is connected in τ and then we have that P is connected in $\tau_\mathcal{F}$. Given that open plaques form a basis for $\tau_\mathcal{F}$, we conclude that $(X, \tau_\mathcal{F})$ is locally connected.

To prove that the leaves are arc-connected we first show that they are $\tau_\mathcal{F}$-connected. Take x, y in a leaf L. We know that there is a chain of plaques $P_1, \ldots, P_n$ from x to y. Since each plaque is $\tau_\mathcal{F}$-connected, $\bigcup_{i=1,\ldots,n} P_i$ is $\tau_\mathcal{F}$-connected. Fixing x and varying $y \in L$ we conclude that L is $\tau_\mathcal{F}$-connected. Then, we know that each leaf L is connected and locally connected in $\tau_\mathcal{F}$. In addition, by Proposition 4.1.11, we know that $(L, \mathrm{dist}_\mathcal{F})$ is a complete metric space. Then, we can apply Theorem 3.1.5 to conclude the arc-connection of L. This implies that every leaf is contained in a component of $(X, \tau_\mathcal{F})$. The converse inclusion follows by Proposition 4.1.8.

To prove (6), take a leaf L, $U \in \mathcal{U}$, $x \in L \cap U$ and define $C = \mathrm{Comp}_x(L \cap U)$. By the definition of a leaf, it is clear that $Q_U(x) \subset C$. We will show the converse inclusion. For this purpose, since C is a union of open plaques, we can write

$$C = \bigcup_{j \in J} P_j$$

with $P_j = Q_{U_j}(x_j)$ and some index set J. By Proposition 3.1.1 we can suppose that $U_j \subset U$ for all $j \in J$. Define

$$J_1 = \{j \in J : P_j \cap Q_U(x) \neq \emptyset\}$$

and $J_2 = J \backslash J_1$. Since $Q_{U_j} = Q_U|_{U_j}^m$, if $j \in J_1$ then $P_j \subset Q_U(x) \subset C$. Define $V_1 = \bigcup_{j \in J_1} P_j$ and $V_2 = \bigcup_{j \in J_2} P_j$. Define $V_1' = V_1 \cap Q_U(x)$ and $V_2' = V_2 \cap Q_U(x)$. Then $Q_U(x) \subset V_1' \cup V_2'$. Since $Q_U(x)$ is locally connected with respect to τ, we conclude that
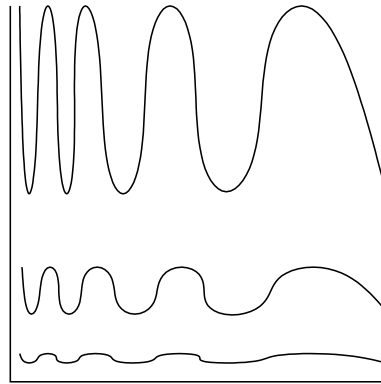
FIGURE 3. In this continuum the topology $\tau_{\mathcal{A}}$ is not locally connected around the origin.

V_1', $V_2' \in \tau|_{Q_U(x)}$. Since $V_1' \cap V_2' = \emptyset$, $V_1' \neq \emptyset$ and $Q_U(x)$ is connected in τ we conclude that $V_2' = \emptyset$. Then $Q_U(x) \subset \bigcup_{j \in J_1} Q_{U_j}(x_j)$. Since $Q_{U_j}(x_j) \subset Q_U(x)$ for all $j \in J_1$ we conclude that $Q_U(x) = \bigcup_{j \in J_1} Q_{U_j}(x_j)$. That is, $Q_U(x) = C = \mathrm{Comp}_x(L \cap U)$. □

We will give two examples of $\mathcal{C}$-foliations with particular properties to show the necessity of the hypothesis of Theorem 4.2.2. First we show that a leaf may not be connected in the plaque topology (for a non-Peano foliation).

Example 4.2.3. Consider the plane continuum $X = X_1 \cup X_2$ where $X_1 = \{0\} \times [-1, 1]$ and $X_2 = \{(x, \sin(1/x)) : x \in (0, 1]\}$. On X consider the complete basis $\mathcal{U} = \tau$ (the relative open subsets of the plane with its usual topology). Let $\mathcal{F}$ be the $\mathcal{C}$-foliation defined as $Q_{\overline{U}}(x) = \mathrm{Comp}_x(\overline{U})$ for every $U \in \tau$. We have only one leaf $L = X$. We see that this leaf is not connected in $\tau_{\mathcal{F}}$ since it has two components X_1 and X_2. Note that X itself is an open plaque that is not locally connected. Then $\mathcal{F}$ is not a Peano foliation. It is an exercise for the interested reader to check which conclusions of Theorem 4.2.2 are satisfied in this example. In Example 6.3.3 we give a decomposition of a square with a leaf like the set X.

In the following variation of the previous example we show that (X, τ_A) may not be locally connected.

Example 4.2.4. For n a non-negative integer define

$$X_n = \{(x, (3 + \sin(1/x))/4^n) \in \mathbb{R}^2 : x \in (0, 1]\}$$

and $X = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}) \cup \bigcup_{n \geq 0} X_n$. See Figure 3. As in the previous example consider the $\mathcal{C}$-foliation defined by $Q_{\overline{U}}(x) = \mathrm{Comp}_x(\overline{U})$ for every $U \in \tau$. We have that $(0, 0)$ has no connected neighborhoods in the plaque topology.

We think that the following result is interesting because it allows us to characterize Peano foliations in terms of the plaque topology.

COROLLARY 4.2.5. *For an atlas $\mathcal{A}$ over a complete basis the following statements are equivalent:*
(1) $\mathcal{A}$ *is a Peano atlas;*
(2) $\tau|_P = \tau_{\mathcal{A}}|_P$ *for every open plaque P.*

Proof. The direct part follows by Theorem 4.2.2. To prove the converse, let P be an open plaque. We have to prove that P is locally connected in τ. Given $x \in P$ we can take a small neighborhood $U \in \mathcal{U}$ of x. By definition, the plaque $Q_U(x)$ is connected in τ. Also, $Q_U(x)$ is a neighborhood of x in the plaque topology. Since, by hypothesis, $\tau|_P = \tau_{\mathcal{A}}|_P$, we conclude that $Q_U(x)$ is a (connected) neighborhood of x in $\tau|P$. Then, P is locally connected with respect to τ.   □

We say that a leaf is *plaque-compact* if it is compact in the plaque topology.

THEOREM 4.2.6. *Let $\mathcal{F}$ be a Peano foliation. An $\mathcal{F}$-leaf is plaque-compact if and only if it is a Peano continuum in the relative topology of τ.*

Proof. Assume that L is a plaque-compact leaf. Since the plaque topology is finer than the relative topology, Proposition 4.1.5, we have that L is compact in the relative topology. Given $\varepsilon > 0$ we define the complete basis

$$\mathcal{U}_\varepsilon = \{U \in \tau : \operatorname{diam}(U) < \varepsilon\}. \qquad (7)$$

We assume that ε is so small that $\mathcal{U}_\varepsilon \subset \mathcal{U}$, where $\mathcal{U}$ is the complete basis of $\mathcal{F}$. Consider the following cover of L:

$$\mathcal{U}_L = \{Q_U(x) : x \in L \cap U, \, U \in \mathcal{U}_\varepsilon\}.$$

Since L is plaque-compact, there is a finite subcover. The plaques of this subcover have diameter smaller than ε. Applying Sierpiński's Theorem 3.1.6 we conclude the local connection of L in the relative topology.

Converse. Assume that the leaf L is a Peano continuum in τ. Take a cover $\mathcal{U}_L = \{P_i\}_{i \in I}$ of L by open plaques. Given $x \in L$ take $U_x \in \mathcal{U}$ such that $x \in P_i = Q_{U_x}(x)$. Since L is locally connected and $\mathcal{U}$ is a complete basis there is $V_x \in \mathcal{U}$ such that $x \in V_x \subset U_x$ and $V_x \cap L$ is connected. Then, by Theorem 4.2.2, $Q_{V_x}(x) = V_x \cap L$. Since $\{V_x \cap L : x \in L\}$ is a cover of L by relative open sets of L and L is compact we can take $x_1, \ldots, x_n$ such that $V_{x_1}, \ldots, V_{x_n}$ cover L. Since $V_x \subset U_x$, we have that $\{Q_{U_{x_1}}(x_1), \ldots, Q_{U_{x_n}}(x_n)\}$ is a finite subcover of $\mathcal{U}_L$. This proves that L is compact in the plaque topology.   □

Remark 4.2.7. In Example 4.2.3 we see that a compact leaf (in the relative topology) may not be plaque-compact (it fails to be locally connected).

Remark 4.2.8. A locally connected leaf may not be compact. Take a plane flow (with X a compact invariant annulus) with limit cycles. An orbit converging to a cycle is locally connected but it is not compact.

Note that the examples given above (Examples 4.2.3 and 4.2.4) are not Peano foliations but they satisfy (6) in Theorem 4.2.2. The next subsection is devoted to describing an example of a $\mathcal{C}$-foliation not satisfying (6).

4.2.1. *Devil's backgammon.* In Theorem 4.2.2 we proved that, for Peano foliations, if L is a leaf then for every $U \in \mathcal{U}$ and $x \in L \cap U$ it holds that $\operatorname{Comp}_x(L \cap U)$ is the plaque of x in U. That is, knowing the leaves and the basis $\mathcal{U}$ we can recover the plaques. This is the *leaves–atlas equivalence* mentioned at the beginning of this section. The next example
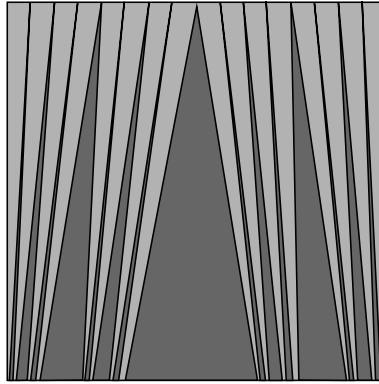
FIGURE 4. The Devil's backgammon continuum is obtained from the square by deleting the dark open triangles. Each triangle is determined by I_x and I_y if $g(x) = g(y)$ and $x \neq y$.

shows that a maximal atlas (with non-locally connected plaques) can contain strictly more information than the leaves and the basis, i.e. does not satisfy (6).

We start constructing a continuum $X \subset [0, 1] \times [0, 1]$. Let $K \subset [0, 1]$ be the ternary Cantor set. A point $x \in K$ can be expressed as $x = \sum_i 2/3^{n_i}$ for some increasing sequence n_i of integers. Consider the function $g : K \to [0, 1]$ such that if $x = \sum_i 2/3^{n_i} \in K$ then $g(x) = \sum_i 1/2^{n_i}$. For each $y \in [0, 1]$ there are one or two preimages by g. For $x \in K$ let I_x be the closed segment line from $(x, 0)$ to $(g(x), 1)$. Define

$$X = ([0, 1] \times \{0\}) \cup \bigcup_{x \in K} I_x.$$

We have that X is an arc-connected continuum. Note that it is not locally connected. It can also be obtained by deleting open triangles from the square, as shown in Figure 4.

Consider the open sets

$$U = \{(x, y) \in X : y < 2/3\},$$
$$V = \{(x, y) \in X : y > 1/3\}.$$

Define $Q_U(x, y) = U$ for all $(x, y) \in U$. Given $p \in V$ suppose that $p \in I_a$. Define

$$J_p = \begin{cases} I_a & \text{if } g^{-1}(g(a)) = \{a\}, \\ I_a \cup I_b & \text{if } g^{-1}(g(a)) = \{a, b\} \end{cases}$$

and $Q_V(p) = \text{Comp}_p(J_p \cap V)$. The decompositions Q_U, Q_V define a $\mathcal{C}$-foliation $\mathcal{F}$. It is clear that it is not a Peano foliation since it has non-locally connected open plaques. Note that there is only one leaf, namely $L = X$.

In this example we see that for $p \in V$ we have that $\text{Comp}_p(L \cap V) = L \cap V$ and it is not $Q_V(p)$. As we said, this means that the plaques cannot be recovered by the leaves and the basis.

4.3. *Cw-foliations.*

It must be recalled that we have in mind the study of the distribution of stable and unstable continua of cw-expansive homeomorphisms on surfaces and Peano

continua. This guides us in selecting the properties that we require in the following definition.

Recall that a subset of X is *meagre* if it can be expressed as the union of countably many nowhere dense subsets of X. A set is *nowhere dense* if its closure has empty interior.

Definition 4.3.1. An atlas $\mathcal{A}$ is a *cw-atlas* if its $\mathcal{A}$-leaves:
(1)   are a countable union of open plaques;
(2)   are meagre sets; and
(3)   have diameter bounded away from zero.
A *cw-foliation* is a $\mathcal{C}$-foliation with a cw-atlas.

More observations about this definition must be given.

Remark 4.3.2. Note that requiring that a leaf is a countable union of plaques does not imply that the leaf has a countable basis. The plaques in this countable union may not be small.

Remark 4.3.3. A foliation of a compact manifold M in the standard sense is a cw-foliation if and only if the dimension of the leaves is positive and less than $\dim(M)$. On a surface, this means *one-dimensional*.

In the standard theory of foliations the leaves are immersed submanifolds and have a countable basis for its leaf topology. Also, the intersection of a leaf with a local chart is a countable number of plaques. For Peano cw-foliations these properties can be recovered. A *Peano cw-foliation* is a Peano foliation with a cw-atlas.

THEOREM 4.3.4. *If $\mathcal{F}$ is a Peano cw-foliation then:*
(1)   *$\tau_{\mathcal{F}}|_L$ has a countable basis for every leaf L;*
(2)   *for every leaf L and every $U \in \tau$ the set $L \cap U$ has a countable number of components with respect to both topologies τ and $\tau_{\mathcal{F}}$.*

Proof. Since X is a compact metric space it has a countable basis. By Theorem 4.2.2 we know that $\tau|_P = \tau_{\mathcal{F}}|_P$, for every open plaque P. We conclude that each plaque has a countable basis. Consequently, as every leaf is a countable union of plaques, we have that each leaf has a countable basis.

Consider a leaf L and $U \in \tau$. We know that $L \in \tau_{\mathcal{F}}$. From Proposition 4.1.5 we have that $U \in \tau_{\mathcal{F}}$. Then $U \cap L \in \tau_{\mathcal{F}}$. We know that $\tau_{\mathcal{F}}|_L$ has a countable basis. Therefore, $\tau_{\mathcal{F}}|_{U \cap L}$ has a countable basis. By Theorem 4.2.2, knowing that $U \cap L \in \tau_{\mathcal{F}}$, we conclude that $U \cap L$ is locally connected with respect to $\tau_{\mathcal{F}}$. Then† $U \cap L$ has a countable number of components with respect to $\tau_{\mathcal{F}}$. Since $\tau \subset \tau_{\mathcal{F}}$, the same holds for τ.   □

For future reference (Proposition 6.1.7) we give one more result. Recall from (7) that $\mathcal{U}_\delta$ is the complete basis containing all the open sets of diameter smaller than δ.

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† Every locally connected topological space with a countable basis has at most a countable number of components. This can be proved as follows. Since the space is locally connected, its components are open and given that the space has a countable basis each component contains an open set of the countable basis. As the components are disjoint, there is at most a countable number of components.

PROPOSITION 4.3.5. *If $\mathcal{A}$ is a cw-atlas over the complete basis $\mathcal{U}$ of X then there is $\delta > 0$ such that for all $x \in U \in \mathcal{U}_\delta$ it holds that:*

(1) $Q_{\overline{U}}(x) \cap \partial U \neq \emptyset$; *and*

(2)   *x is not a τ-interior point of $Q_U(x)$.*

Proof. From the definition of cw-atlas we know that there is $\delta > 0$ such that every leaf has diameter greater than δ. Take $x \in U \in \mathcal{U}_\delta$. Since the leaf of x has diameter greater than δ there is $y \notin \overline{U}$ in the leaf of x. Consider a chain of plaques $P_1, \ldots, P_n$ from x to y. Let P_m be the first of these plaques that is not contained in U. By Lemma 4.1.9 the plaques $P_1, \ldots, P_{m-1}$ are contained in $Q_U(x)$. Take $z \in P_{m-1} \cap P_m$. Since P_m is connected we have that $\mathrm{Comp}_z(P_m \cap \overline{U})$ cuts ∂U. Given that $\mathrm{Comp}_z(P_m \cap \overline{U}) \subset Q_{\overline{U}}(x)$ we have that $Q_{\overline{U}}(x) \cap \partial U \neq \emptyset$.

Since each leaf is a meagre set, no plaque has interior points. □

The next example shows that the converse of Proposition 4.3.5 is not true.

Example 4.3.6. Let $K \subset [0, 1]$ be a Cantor set. Define $X = K \times [0, 1] \times [0, 1] \cup [0, 1] \times [0, 1] \times \{0\}$. Define the projection $\pi : X \subset \mathbb{R}^3 \to \mathbb{R}$ given by $\pi(x, y, z) = y$ and the decomposition $\mathcal{F}$ as $\mathcal{F}(x, y, z) = \pi^{-1}(y)$. For $\delta \in (0, 1/2)$ and the Euclidean metric on X, define $Q_U(p) = \mathcal{F}|_U^m(p)$. It defines an atlas over $\mathcal{U}_\delta$. No leaf has interior points and every leaf has diameter greater than δ. But this atlas does not define a cw-foliation because the leaves are not a countable union of plaques.

4.4. *Continuous atlas.*   We say that an atlas $\mathcal{A}$ over a complete basis $\mathcal{U}$ is *continuous* if the set

$$\{U \in \mathcal{U} : Q_{\overline{U}} \text{ is continuous}\}$$

is a basis of the topology of X. The continuity of $Q_{\overline{U}}$ means the continuity of the map $Q_{\overline{U}} : \overline{U} \to \mathcal{C}(Q_{\overline{U}})$ with respect to the Hausdorff metric.

Remark 4.4.1. The stable foliation of a pseudo-Anosov diffeomorphism with 1-prongs (as in §2.2.1) defines a continuous atlas. In Theorem 6.6.2 we will show that continuous Peano cw-foliations of compact surfaces (i.e. continuous dendritations), are in fact foliations (in the standard sense) with a finite number of 1-prongs.

Example 4.4.2. (Hiraide's generalized foliations) Let (X, dist) be a Peano continuum. A partition $\mathcal{F}$ of X is a *generalized foliation* [2, 17] if for all $x \in X$ there are non-trivial arc-connected subsets $D, K \subset X$ with $D \cap K = \{x\}$, a connected open neighborhood N of x in X and a homeomorphism $\phi_x : D \times K \to N$ such that:

(1) $\phi_x(y, x) = y$ for all $y \in D$, $\phi_x(x, z) = z$ for all $z \in K$;

(2)   for every leaf L there is an at most countable set $B \subset K$ such that $N \cap L = \phi_x$ $(D \times B)$.

Generalized foliations are a main concept for the study of expansive homeomorphisms with canonical coordinates. Notice that the stable foliation of the pseudo-Anosov diffeomorphism of §2.2.1 is not a generalized foliation. We can show that every generalized foliation defines a continuous atlas as follows. Given $x \in X$ consider a local coordinate $\phi : D \times K \to N$ around x. Take D' and K' compact neighborhoods of x in D

and K, respectively. Let U be an open set containing x such that $\overline{U} = \phi(D' \times K')$. Since on $D \times K$ we consider the product topology, we have that $\mathcal{F}|_{\overline{U}}^{m}$ is continuous.

5. Expansivity and cw-foliations

The results that we have obtained will now be applied in the study of the stable and unstable cw-foliations determined by a cw-expansive homeomorphism. Also, we will give sufficient conditions for cw1-expansivity to imply expansivity.

5.1. Stable and unstable cw-foliations.

Let $f : X \to X$ be a homeomorphism of a compact metric space. Recall the definitions of stable and unstable continua given in (1).

Definition 5.1.1. A decomposition of a closed subset of X is *stable* if every plaque is stable. An atlas $\mathcal{A}$ is *stable* if each decomposition is stable. A $\mathcal{C}^s$-*foliation* is a maximal stable atlas. The unstable versions of these definitions are analogous.

As before, *maximal stable atlas* means maximal among all the stable atlases, see Example 5.1.7. For a closed subset $Y \subset X$ define $Q_Y^s : Y \to \mathcal{P}(Y)$ by $y \in Q^s(x)$ if there is a stable continuum $C \subset Y$ such that $x, y \in C$. For every homeomorphism f and closed subset $Y \subset X$ we have that Q_Y^s is a monotone partition of Y. Similarly we define Q_Y^u considering unstable continua. Define:

$$\begin{aligned} \mathcal{U}^s &= \big\{ U \in \tau : Q_{\overline{U}}^s \text{ is a stable decomposition} \big\}, \\ \mathcal{A}^s &= \big\{ Q_{\overline{U}}^s : U \in \mathcal{U}^s \big\}. \end{aligned} \tag{8}$$

Note that $\mathcal{U}^s$ and $\mathcal{A}^s$ depend on f. The sets $\mathcal{U}^u$ and $\mathcal{A}^u$ are defined similarly considering the inverse of f.

We will show that if f is cw-expansive then $\mathcal{U}^\sigma$ is a cover of X and that in this case $\mathcal{A}^\sigma$ is a stable atlas of a cw-foliation. By definition it is clear that $\mathcal{A}^\sigma$ is a maximal stable (unstable) atlas. We define the *stable* and *unstable cw-foliations* of f as $\mathcal{F}^s$, $\mathcal{F}^u$, respectively.

An atlas $\mathcal{A}$ over $\mathcal{U}$ is *invariant* if:
- $f^n(U) \in \mathcal{U}$; and
- $Q_{f^n(\overline{U})}(f^n(x)) = f^n(Q_{\overline{U}}(x))$ for all $x \in \overline{U}$

for all $n \in \mathbb{Z}$ and all $U \in \mathcal{U}$. The purpose of this section is to prove the following result.

THEOREM 5.1.2. *If f is a cw-expansive homeomorphism of a Peano continuum then $\mathcal{F}^s$, $\mathcal{F}^u$ are invariant cw-foliations without plaque-compact leaves.*

The proof is developed in some lemmas that do not assume that f is cw-expansive.

LEMMA 5.1.3. *If Q_Y^σ is a stable (unstable) decomposition, $\sigma = s, u$, and $Z \subset Y$ is closed then*

$$Q_Z^\sigma = Q_Y^\sigma|_Z^m.$$

Proof. Take $x \in Z$ and $y \in Q_Z^s(x)$. Then there is a stable continuum C such that $x, y \in C \subset Z$. Since $Z \subset Y$, we have that $C \subset Q_Y^s(x)$. Since $x \in C$ and C is connected we have that $C \subset Q_Y^s|_Z^m(x)$. Consequently $y \in Q_Y^s|_Z^m(x)$ and $Q_Z^s(x) \subset Q_Y^s|_Z^m(x)$.

Let us show the converse inclusion. Take $x \in Z$ and $y \in Q_Y^s|_Z^m(x)$. Recall that $Q_Y^s|_Z^m(x) = \mathrm{Comp}_x(Q_Y^s(x) \cap Z)$. Since Q_Y^s is a stable decomposition we have that $\mathrm{Comp}_x(Q_Y^s(x) \cap Z)$ is a stable continuum. Given that $\mathrm{Comp}_x(Q_Y^s(x) \cap Z) \subset Z$ we have that $\mathrm{Comp}_x(Q_Y^s(x) \cap Z) \subset Q_Z^s(x)$ and $y \in Q_Z^s(x)$. Then $Q_Y^s|_Z^m(x) \subset Q_Z^s(x)$. $\qquad \square$

LEMMA 5.1.4. *If $\mathcal{U}^\sigma$ covers X, $\sigma = s, u$, then:*
(1) $\mathcal{A}^\sigma$ *is a stable atlas over the complete basis $\mathcal{U}^\sigma$;*
(2) *every stable (unstable) continuum is contained in an open $\mathcal{A}^\sigma$-plaque;*
(3) $Q_X^\sigma(x)$ *is the $\mathcal{A}^\sigma$-leaf of x for all $x \in X$;*
(4) *every $\mathcal{A}^\sigma$-leaf is a countable union of open $\mathcal{A}^\sigma$-plaques.*

Proof. Assume that $\mathcal{U}^s$ covers X. By Lemma 5.1.3 we know that $\mathcal{U}^s$ is a complete basis. Consequently, $\mathcal{A}^s$ is a stable atlas. Let $C \subset X$ be a stable continuum. Since $\mathcal{U}^s$ covers X we can take $n \geq 0$ and $U \in \mathcal{U}$ such that $f^n(C) \subset U \in \mathcal{U}^s$. Then $C \subset Q_{f^{-n}(U)}^s(x)$ for all $x \in C$. Since $\mathcal{U}^s$ is f-invariant, $Q_{f^{-n}(U)}^s(x)$ is a stable open plaque containing C. It is clear that the leaf of x is contained in $Q_X^s(x)$. The converse inclusion follows by the previous item.

Let us show that each leaf is a countable union of open plaques. Take $\delta > 0$ such that every open subset of diameter smaller than δ belongs to $\mathcal{U}^s$. Then $B_{\delta/2}(x) \in \mathcal{U}^s$ for all $p \in X$ and we can consider the open plaque $Q_{B_{\delta/2}(x)}^s(x)$. For $x \in X$ and $n \geq 0$ define

$$P_n = f^{-n}(Q_{B_{\delta/2}(f^n(x))}^s(f^n(x))).$$

We have that P_n is a stable open plaque. We have proved that the leaf of x is $Q_X^s(x)$. Then $P_n \subset Q_X^s(x)$ for all $n \geq 0$. Given $y \in Q_X^s(x)$ let us show that there is $k \geq 0$ such that $y \in P_k$. If $y \in Q_X^s(x)$ then there is a stable continuum C containing x and y. Since C is stable, there is $k \geq 0$ such that $\mathrm{diam}(f^k(C)) < \delta/2$. Given that $f^k(x) \in f^k(C)$ and that $f^k(C)$ is a stable continuum we have that $f^k(C) \subset Q_{B_{\delta/2}(f^k(x))}^s(f^k(x))$. Then $y \in P_k$. This proves that $Q_X^s(x) = \bigcup_{n \geq 0} P_n$, a countable union of plaques. $\qquad \square$

LEMMA 5.1.5. *If f is a homeomorphism of a compact metric space X and there are $\varepsilon, \delta > 0$ such that $\mathcal{C}^\sigma \cap \mathcal{C}_\delta \subset \mathcal{C}_\varepsilon^\sigma$ and $\mathcal{C}_{2\varepsilon}^\sigma \subset \mathcal{C}^\sigma$ for $\sigma = s, u$, then $\mathcal{U}^\sigma$ covers X.*

Proof. We give the proof for $\sigma = s$. Let $Y \subset X$ be a compact subset with $\mathrm{diam}(Y) \leq \delta$. We will show that Q_Y^s is a stable decomposition of Y. Take $x \in Y$. By definition we have that $Q_Y^s(x)$ is connected for all $x \in Y$. We have that $Q_Y^s(x)$ is the union of all the stable continua C contained in Y and containing x. Since $\mathcal{C}^s \cap \mathcal{C}_\delta \subset \mathcal{C}_\varepsilon^s$ and $\mathrm{diam}(Y) \leq \delta$ we have that each $C \in \mathcal{C}_\varepsilon^s$. This implies that

$$\mathrm{diam}(f^k(Q_Y^s(x))) \leq 2\varepsilon \quad \text{for all } k \geq 0. \tag{9}$$

Since $\mathcal{C}_{2\varepsilon}^s \subset \mathcal{C}^s$ we have that $Q_Y^s(x)$ is a stable continuum. Take $x_n \to x$ in Y with $Q_Y^s(x_n) \to C$. From (9) we know that $\mathrm{diam}(f^k(C)) \leq 2\varepsilon$ for all $k \geq 0$. Therefore, we conclude that C is stable. Then, $C \subset Q_Y^s(x)$ and the proof ends. $\qquad \square$

Proof of Theorem 5.1.2. By Proposition 2.3.1 we know that f satisfies the hypothesis of Lemma 5.1.5. Then $\mathcal{U}^\sigma$ covers X. Applying Lemma 5.1.4 we conclude that $\mathcal{U}^\sigma$ is a

complete basis, that $\mathcal{A}^\sigma$ is an atlas and that every $\mathcal{A}^\sigma$-leaf is a countable union of open $\mathcal{A}^\sigma$-plaques. By Theorem 2.3.3 we know that $\mathcal{C}^\sigma_\varepsilon(x)\backslash\mathcal{C}_\delta(x) \neq \emptyset$. Then, every leaf has diameter greater than δ. The cw-expansivity and the condition $\mathcal{C}^\sigma_\varepsilon(x)\backslash\mathcal{C}_\delta(x) \neq \emptyset$ imply that each stable plaque has empty interior. Therefore, stable and unstable leaves are meagre sets. This proves that $\mathcal{F}^\sigma$ are cw-foliations.

Suppose that L is a plaque-compact leaf of $\mathcal{A}^s$. Let $\{P_i\}_{i \in I}$ be a cover of L by stable plaques. Since L is plaque-compact we can assume that I is finite. Then $\mathrm{diam}(f^n(L)) \to 0$ as $n \to +\infty$. This gives us stable leaves of arbitrarily small diameter, a contradiction to Theorem 2.3.3. $\square$

Remark 5.1.6. The quasi-Anosov diffeomorphism §2.2.2 is expansive and Theorem 5.1.2 can be applied. In this example we see that the plaques of $\mathcal{F}^s$ and $\mathcal{F}^u$ may not have constant dimension. There are stable plaques of dimension one and two.

Example 5.1.7. (One-dimensional expanding attractor) Let $g : T^2 \to T^2$ be a derived from Anosov diffeomorphism of the two-dimensional torus and denote by $X \subset T^2$ the non-trivial basic set (an expanding attractor). Let $f : X \to X$ be the restriction of g. We have that X is locally a product of a Cantor set and an arc and every leaf is dense in X. The dynamics of f on X expands the length of each arc contained in X and every proper subcontinuum is an unstable arc. That is, given an open set $U \subset X$, such that $\overline{U} \neq X$, the unstable plaque of $x \in U$ is, simply, its component in U (an arc). Then, the unstable cw-foliation is a Peano foliation. It is remarkable that in this case, the unstable atlas is compatible with the trivial decomposition $Q : X \to \mathcal{C}(X)$ given by $Q(x) = X$ for all $x \in X$. However, X, as a plaque of this decomposition Q, is neither unstable nor locally connected. Then, a maximal atlas extending a Peano unstable atlas may not be neither unstable nor Peano.

Problem 5.1.8. For a cw-expansive homeomorphism of a Peano continuum, is it true that every continuum contained on a stable leaf is stable?

Problem 5.1.9. Can a stable leaf of a cw-expansive homeomorphism of a Peano continuum be compact in the relative topology? By Theorem 5.1.2 we only know that it cannot be plaque-compact.

5.2. *Generating pairs.* If $\mathcal{U}^1$ and $\mathcal{U}^2$ are complete bases then $\mathcal{U}^1 \cap \mathcal{U}^2$ is a complete basis. If $\mathcal{A}^i = \{Q^i_U : U \in \mathcal{U}^i\}$ is an atlas over the complete basis $\mathcal{U}^i$, $i = 1, 2$, then both atlases are defined in a common complete basis, namely $\mathcal{U} = \mathcal{U}^1 \cap \mathcal{U}^2$.

The following concept is the key to proving in §6.8.5 that cw1-expansivity implies expansivity on a compact surface.

Definition 5.2.1. Given two atlases $\mathcal{A}^1$ and $\mathcal{A}^2$ of the continuum X, we say that they *generate (the topology of X)* if for all $B_\varepsilon(x) \in \mathcal{U}$ there is $\delta > 0$ such that if $y \in B_\delta(x)$ then $Q^i_{B_\varepsilon(x)}(x) \cap Q^j_{B_\varepsilon(x)}(y) \neq \emptyset$ for $\{i, j\} = \{1, 2\}$.

Example 5.2.2. A pair of transverse foliations of a smooth manifold (with the standard meaning of these concepts) form a generating pair of cw-foliations. Another example is
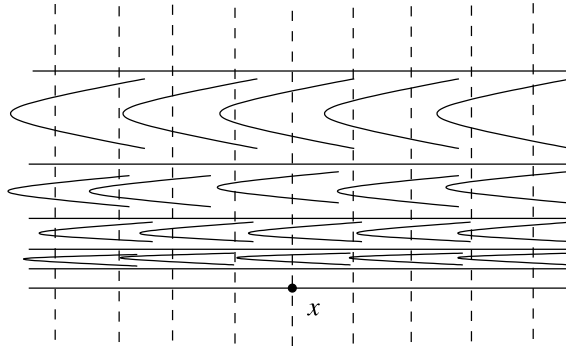
FIGURE 5. A non-generating pair of plane cw-foliations.

formed by the stable and the unstable singular foliations of a pseudo-Anosov map of a compact surface, even with singularities and 1-prongs†.

Remark 5.2.3. For $r > 0$ and $x \in X$ define

$$su_r(x) = \{y \in X : \exists A^s \in \mathcal{C}^s_r(y), \ A^u \in \mathcal{C}^u_r(x) \text{ with } A^u \cap A^s \neq \emptyset\}.$$

Also define $y \in us_r(x)$ if $x \in su_r(y)$. If $f : X \to X$ is cw-expansive then the cw-foliations $\mathcal{A}^s$ and $\mathcal{A}^u$ generate if and only if $su_\varepsilon(x) \cap us_\varepsilon(x)$ is a neighborhood of x, for all $x \in X$ and for all $\varepsilon > 0$. The proof follows by the definitions.

Remark 5.2.4. Notice that, for pseudo-Anosov singular foliations, the size of the ball covered by $su_r(x)$ is not uniform. For x close to a singularity, the maximal ball contained in $su_r(x)$ is small.

Example 5.2.5. (Quadratic tangencies) Let $\mathcal{F}^1$ be the foliation of $\mathbb{R}^2$ whose leaves are horizontal lines $y = c$ for $c \in \mathbb{R}$. Define $\mathcal{F}^2$ by $y = x^2 + c$ for $c \in \mathbb{R}$. It is easy to prove that these foliations do not generate at tangency points of the foliations. These kinds of tangencies are present in the wandering set of the Q^r-Anosov diffeomorphisms of §2.2.3.

Example 5.2.6. Consider the vertical cw-foliation $\mathcal{F}^1$ of the plane and a cw-foliation $\mathcal{F}^2$ as in Figure 5. In this case, for a point x as in the figure, it holds that $\mathcal{F}^2|^m_{B_\varepsilon(x)}(x) \cap \mathcal{F}^1|^m_{B_\varepsilon(x)}(y) \neq \emptyset$ for all y close to x. But, exchanging the foliations, we can find points y arbitrarily close to x such that $\mathcal{F}^1|^m_{B_\varepsilon(x)}(x) \cap \mathcal{F}^2|^m_{B_\varepsilon(x)}(y) = \emptyset$. This kind of non-generating point appears: (1) at the non-wandering set of the examples in §2.2.3; and (2) at wandering points in the example of §2.2.4.

PROPOSITION 5.2.7. *If $su_r(x)$ is a neighborhood of x, for all $x \in X$ and for all $r > 0$ then X is locally connected.*

Proof. Given $x \in X$ and a neighborhood U of x, take $\rho > 0$ such that $B_\rho(x) \subset U$. Take $r \in (0, \rho/2)$. We know that $su_r(x)$ is a neighborhood of x. For each $y \in su_r(x)$ there are

† We were tempted to say *transversal* instead of *generating pair* in Definition 5.2.1, but at 1-prongs the foliations look closer to a tangency than to a transversal point, at least from this author's viewpoint.
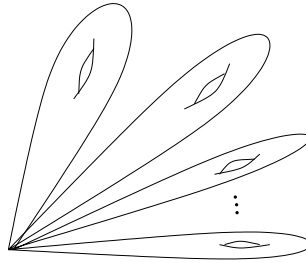
FIGURE 6. A non-expansive but cw1-expansive homeomorphism on a continuum that is not locally connected. An Anosov diffeomorphism acts on each torus.

$C^s \in \mathcal{C}_r^s$ and $C^u \in \mathcal{C}_r^u$ such that $x \in C^u$, $y \in C^s$ and $C^u \cap C^s \neq \emptyset$. Then, $\text{diam}(C^s \cup C^u) \leq 2r$ and $\text{dist}(x, y) < \rho$. Then $su_r(x) \subset B_\rho(x)$. Since $su_r(x)$ is a union of connected sets $C^s \cup C^u$, with a common point x, we conclude that $su_r(x)$ is connected. This proves the local connection of X at x. Since x is arbitrary, X is locally connected. □

5.3. *Cw1-expansivity versus expansivity.* We now investigate the following question.

Problem 5.3.1. Does cw1-expansivity imply the expansivity of a homeomorphism f on a Peano continuum X?

The answer is affirmative in the examples known to the author and in Proposition 2.6.7 we gave some information on this problem. In Theorem 6.8.5 we will prove it for X a compact surface. On a Cantor set X the identity is (trivially) cw1-expansive but not expansive. Even assuming the connection of X, the local connection of X is needed in the question because of the following example.

Example 5.3.2. Let $f_1 : T^2 \to T^2$ be an Anosov diffeomorphism with a fixed point p. Denote by X_2 the closure of $\{1/n : n \in \mathbb{Z}^+\}$ in the usual topology of $\mathbb{R}$. Define $Y = T^2 \times X_2$ and $g : Y \to Y$ as $g(x, y) = (f_1(x), y)$. Now, collapsing the fixed points $\{(p, 0), (p, 1), (p, 1/2), \dots\}$ to a single point we obtain a continuum X, see Figure 6. Also, we have a homeomorphism $f : X \to X$ induced by g. We have that f is not expansive because the distance $\text{dist}(f^n(x, 1/k), f^n(x, 1/l))$ is small for all $n \in \mathbb{Z}$ whenever k and l are sufficiently large. By construction, f is cw1-expansive. Notice that X is a non-locally connected continuum.

Under a certain hypothesis, see Theorem 5.3.4, we can prove that cw1-expansivity implies expansivity. The following lemma is a part of its proof, it is separated in order to simplify the notation. We assume that f is not expansive because we will argue by contradiction.

LEMMA 5.3.3. *If f is not expansive and $us_r(x)$ is a neighborhood of x for all $x \in X$ and for all $r > 0$, then for all $\varepsilon > 0$ there are $x, y \in X$ such that $x \neq y$, $\text{dist}(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$ and $x, y \in A$ with $A \in \mathcal{C}_\varepsilon^s \cup \mathcal{C}_\varepsilon^u$.*

Proof. For $\varepsilon > 0$ given, consider $\delta_1 > 0$ and a finite set $Z \subset X$ such that $B_{\delta_1}(z) \subset us_{\varepsilon/3}(z)$ for each $z \in Z$ and $X = \bigcup_{z \in Z} B_{\delta_1}(z)$. Take $\delta_2 \in (0, \varepsilon/3)$ such that if $v, w \in X$ and

dist$(v, w) < \delta_2$ then $v, w \in B_{\delta_1}(z)$ for some $z \in Z$. Since f is not expansive there are $v, w \in X$ such that dist$(f^n(v), f^n(w)) < \delta_2$ for all $n \in \mathbb{Z}$ and $v \neq w$. Take $z \in Z$ such that $v, w \in B_{\delta_1}(z)$. Then $v, w \in su_{\varepsilon/3}(z)$. By the definitions, we can take $A_v^s, A_w^s \in \mathcal{C}_{\varepsilon/3}^s$ and $A_v^u, A_w^u \in \mathcal{C}_{\varepsilon/3}^u$ such that $z \in A_v^s \cap A_w^s$, $A_v^s \cap A_v^u \neq \emptyset$, $A_w^s \cap A_w^u \neq \emptyset$, $v \in A_v^u$ and $w \in A_w^u$.

Take $v_1 \in A_v^s \cap A_v^u$ and $w_1 \in A_w^s \cap A_w^u$. Since $A_v^s, A_w^s \in \mathcal{C}_{\varepsilon/3}^s$ with a common point, namely z, their union is a $2\varepsilon/3$-stable continuum. Since $v_1, w_1 \in A_v^s \cup A_w^s$ we have that

$$\text{dist}(f^n(v_1), f^n(w_1)) < 2\varepsilon/3 \quad \text{for all } n \geq 0. \tag{10}$$

Also dist$(f^n(v_1), f^n(w_1))$ is less than or equal to

$$\text{dist}(f^n(v_1), f^n(v)) + \text{dist}(f^n(v), f^n(w)) + \text{dist}(f^n(w), f^n(w_1)).$$

For $n \leq 0$ we have that

$$\text{dist}(f^n(v_1), f^n(v)) \leq \varepsilon/3,$$
$$\text{dist}(f^n(v), f^n(w)) < \delta_2 < \varepsilon/3 \quad \text{and}$$
$$\text{dist}(f^n(w), f^n(w_1)) \leq \varepsilon/3.$$

Therefore dist$(f^n(v_1), f^n(w_1)) < \varepsilon$ for all $n \leq 0$. Recalling (10) we conclude that

$$\text{dist}(f^n(v_1), f^n(w_1)) < \varepsilon$$

for all $n \in \mathbb{Z}$.

Recall that $A_v^s \cup A_w^s$ is a $2\varepsilon/3$-stable continuum containing v_1 and w_1. If $v_1 \neq w_1$ then we take $x = v_1$, $y = w_1$ and the ε-stable continuum $A = A_v^s \cup A_w^s$. If $v_1 = w_1$ then we take $x = v$, $y = w$ and the $2\varepsilon/3$-unstable continuum $A = A_v^u \cup A_w^u$. □

Notice that by Remark 5.2.3 and Proposition 5.2.7 the hypothesis of the next result implies that X is locally connected.

THEOREM 5.3.4. *If $f : X \to X$ is a cw1-expansive homeomorphism of a continuum X and $\mathcal{F}^s$, $\mathcal{F}^u$ is a generating pair, then f is expansive.*

Proof. Take $\gamma > 0$ such that

$$\text{if } A^s \in \mathcal{C}_\gamma^s \text{ and } A^u \in \mathcal{C}_\gamma^u \text{ then } \text{card}(A^s \cap A^u) \leq 1. \tag{11}$$

Consider $\delta \in (0, \gamma/3)$ such that

$$\text{if } \text{diam}(A) \leq 2\delta \text{ then } \text{diam}(f^{\pm 1}(A)) < \gamma/3 \tag{12}$$

for every $A \subset X$. Consider $\varepsilon \in (0, \delta)$ such that

$$\text{if } \text{dist}(a, b) < \varepsilon \text{ then there is } z \in X \text{ such that } a, b \in su_\delta(z) \cap us_\delta(z) \tag{13}$$

for any $a, b \in X$. Assume by contradiction that f is not expansive. From Lemma 5.3.3 there are $x, y \in X$ and $A_0^s \in \mathcal{C}_\varepsilon^s$ such that $x \neq y$, dist$(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$ and $x, y \in A_0^s$. If the lemma gives an ε-unstable continuum then change f to f^{-1} in what follows.

In this paragraph we will show that there is $n \geq 0$ such that $f^{-n}(x)$ and $f^{-n}(y)$ are not in a common 2δ-stable continuum. Arguing by contradiction, assume that for each

$n \geq 0$ there is $A_n^s \in \mathcal{C}_{2\delta}^s$ containing $f^{-n}(x)$ and $f^{-n}(y)$. Since f is cw-expansive and δ is small, we have that $\mathrm{diam}(f^n(A_n^s)) \to 0$ as $n \to \infty$. Since $x, y \in f^n(A_n^s)$ we conclude that $\mathrm{dist}(x, y) = 0$, contradicting that $x \neq y$.

Let n_0 be the first positive number satisfying the condition proved in the previous paragraph. Then $f^{-n_0+1}(x)$ and $f^{-n_0+1}(y)$ are in a common 2δ-stable continuum $A_{n_0-1}^s$. From (12) we have that $\mathrm{diam}(f^{-1}(A_{n_0-1}^s)) < \gamma/3$. Define $a = f^{-n_0}(x), b = f^{-n_0}(y)$ and $A_*^s = f^{-1}(A_{n_0-1}^s)$. We have that A_*^s is a $\gamma/3$-stable continuum containing a and b. Also, a and b are not in a common 2δ-stable continuum.

Since $\mathrm{dist}(a, b) < \varepsilon$ we can apply condition (13) to conclude that there is $z \in X$ such that $a, b \in su_\delta(z)$. Then, there are $A_a^s, A_b^s \in \mathcal{C}_\delta^s$ and $A_a^u, A_b^u \in \mathcal{C}_\delta^u$ such that $z \in A_a^u \cap A_b^u$, $a \in A_a^s, b \in A_b^s, A_a^s \cap A_a^u \neq \emptyset$ and $A_b^s \cap A_b^u \neq \emptyset$. We have that $A_a^s \cap A_b^s = \emptyset$ because a and b are not in a common 2δ-stable continuum. Therefore, we can take $p \in A_a^s \cap A_a^u$ and $q \in A_b^s \cap A_b^u$ such that $p \neq q$. Since $\delta < \gamma/3$ we have that p and q are in a γ-stable continuum, namely $A_a^s \cup A_*^s \cup A_b^s$. Also, p and q are in the 2δ-unstable continuum $A_a^u \cup A_b^u$. This contradicts (11) and proves that f is expansive. $\qquad\square$

Remark 5.3.5. The quasi-Anosov diffeomorphism of §2.2.2 is expansive but it does not satisfy the hypothesis of Theorem 5.3.4. In this case, the set $su_\delta(x) \cap us_\delta(x)$ is not a neighborhood of a wandering point x. This means, if it is true that cw1-expansivity implies expansivity on a Peano continuum, then new arguments will be needed for its proof.

6. *Dendritations of surfaces*
In this section we will study cw-foliations of compact surfaces. Applications to cw-expansive surface homeomorphisms will be given.

6.1. *Cw-decompositions of disks.*   Let D be a metric space homeomorphic to the Euclidean disk
$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$
As usual, ∂D is the boundary of the disk.

Definition 6.1.1. A decomposition of D is a *cw-decomposition* if it is codendritic and hereditarily unicoherent. That is, the quotient space is a dendrite and each plaque is hereditarily unicoherent.

We say that Q is *n-dimensional* if $\dim Q(x) = n$ for all $x \in D \setminus \partial D$. As in §2.5, dim denotes the topological dimension.

PROPOSITION 6.1.2. *If Q is a decomposition of D then the following statements are equivalent:*
(1)   *Q is a cw-decomposition;*
(2)   *$Q(x) \cap \partial D \neq \emptyset$ and Q is one-dimensional for all $x \in D$.*

Proof. $(1 \to 2)$. Arguing by contradiction, suppose that $Q(x) \cap \partial D = \emptyset$ for some $x \in D$. Since Q is codendritic there is $y \in D$ close to x such that the plaque $Q(y)$ separates D and $Q(y) \cap \partial D = \emptyset$. By Janisewski's Theorem 3.1.7 $Q(y)$ is not unicoherent, contradicting that Q is a cw-decomposition. To prove that Q is one-dimensional, note that each plaque
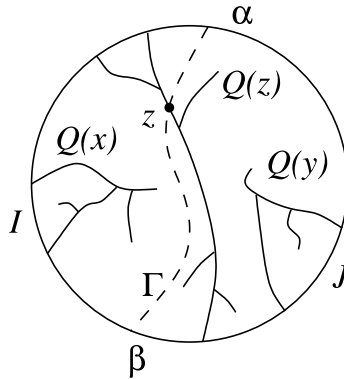
FIGURE 7. The plaque $Q(z)$ separates $Q(x)$ from $Q(y)$.

$Q(x)$ has dimension zero, one or two. If some plaque has dimension two then it has interior points and then it is not hereditarily unicoherent. For an interior point x of the disk it holds that $\dim(Q(x)) \neq 0$, because it is a continuum meeting ∂D.

$(2 \to 1)$. Let us first show that Q is hereditarily unicoherent. Arguing by contradiction suppose that $Q(x)$ is not hereditarily unicoherent. Then there are two continua A, B such that $Q(x) = A \cup B$ and $A \cap B$ is disconnected. By Janisewski's theorem, $Q(x)$ separates D. Let U be a component of $D \backslash Q(x)$ disjoint from ∂D. This implies that $U \subset Q(x)$, contradicting that $Q(x)$ is one-dimensional.

Now we prove that Q is codendritic. First we show that if $x, y \in D$ and $Q(x) \cap Q(y) = \emptyset$ then there is $z \in D$ such that $Q(z)$ separates $Q(x)$ from $Q(y)$. Let I, J be two arcs contained in ∂D such that $Q(x) \cap \partial D \subset I$ and $Q(y) \cap \partial D \subset J$. Suppose that I, J are minimal with this property. Since $Q(x)$ and $Q(y)$ are disjoint we have that $I \cap J = \emptyset$. Denote by α and β the open arcs in $\partial D \backslash (I \cup J)$. Take an arc $\Gamma \subset D$ from $a \in \alpha$ to $b \in \beta$ separating $Q(x)$ and $Q(y)$ as in Figure 7.

Define the sets

$$\Gamma_\alpha = \{u \in \Gamma : Q(u) \cap \alpha \neq \emptyset\},$$
$$\Gamma_\beta = \{u \in \Gamma : Q(u) \cap \beta \neq \emptyset\}.$$

We have that $a \in \Gamma_\alpha$ and $b \in \Gamma_\beta$, which implies that they are non-empty. Since Γ is disjoint from $Q(x)$ and $Q(y)$, the upper semicontinuity of Q implies that Γ_α and Γ_β are closed, recall Remark 3.1.2. As every plaque intersects ∂D we have that $\Gamma = \Gamma_\alpha \cup \Gamma_\beta$. Given that Γ is connected, we find $z \in \Gamma_\alpha \cap \Gamma_\beta$. Therefore, $Q(z)$ separates $Q(x)$ from $Q(y)$.

By Proposition 3.1.3 we know that D/Q is a continuum and from Theorem 3.1.10 we conclude that Q is codendritic. □

PROPOSITION 6.1.3. *If Q is a cw-decomposition and x_1, x_2, $x_3 \in D$ are such that $Q(x_i)$ does not separate $Q(x_j)$ from $Q(x_k)$ for $\{i, j, k\} = \{1, 2, 3\}$ then there is $x \in D$ such that $Q(x_1)$, $Q(x_2)$, $Q(x_3)$ are in different components of $D \backslash Q(x)$.*

Proof. Knowing that D/Q is a dendrite, the result follows by general properties of such spaces. □
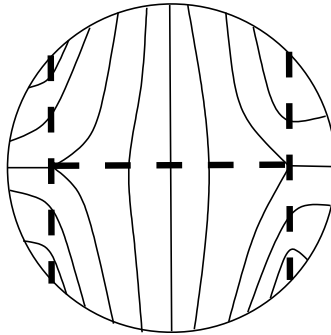
FIGURE 8. The quotient dendrite has the shape of the letter H.

Remark 6.1.4. On a dendrite there can be four points that cannot be separated simultaneously with a single point. Consider for example a dendrite with the shape of the letter H and try to separate with one single point the four ends. It is easy to construct a decomposition Q such that D/Q has this topology. See Figure 8.

Remark 6.1.5. It seems that given any tree T (i.e. a dendrite with a finite number of ramification points) there is a decomposition Q of D such that D/Q is homeomorphic to T. It also seems that this is still true for an arbitrary dendrite T, but a proof is not so clear to this author.

Remark 6.1.6. In the study of *complex dynamics*, several definitions of laminations can be found. Following [**33**], a *lamination* is a closed equivalence relation on ∂D such that the convex hull of different equivalence classes is disjoint. The convex hull is considered with respect to (a differentiable structure and) a Riemannian metric on D that makes it isometric with a Euclidean disk. There is a strong link between cw-decompositions and laminations: if Q is a cw-decomposition of D then the restriction of Q to the boundary of D is a lamination.

PROPOSITION 6.1.7. *If $\mathcal{A}$ is a cw-foliation over the complete basis $\mathcal{U}$ of a closed surface S, then there is $\delta > 0$ such that $Q_{\overline{U}}$ is a cw-decomposition for all $U \in \mathcal{U}$ with $\mathrm{diam}(U) < \delta$.*

Proof. It follows by Propositions 4.3.5 and 6.1.2. □

6.2. *Dendritations and generic leaves.* The following is the main definition of the paper. It combines the properties of Peano foliations, cw-foliations and the plane topology.

Definition 6.2.1. A *dendritation* is a Peano cw-foliation of a compact surface.

By Remark 2.3.6 we know that compact surfaces with non-empty boundaries admit no cw-expansive homeomorphisms. Then, we will assume that the surface has empty boundary. Recall from (7) the definition of the complete basis $\mathcal{U}_\delta$.

PROPOSITION 6.2.2. *If $\mathcal{A} = \{Q_U : U \in \mathcal{U}\}$ is an atlas of a dendritation, $\delta > 0$ is sufficiently small, $U \in \mathcal{U}_\delta$ and $D = \overline{U}$ is a disk then each plaque $Q_D(x)$ is a dendrite and the quotient D/Q_D is a dendrite.*

Proof. By definition, we know that $\mathcal{A}$ is a cw-atlas. Take δ from Proposition 6.1.7 such that Q_D is a cw-decomposition. By the definition of cw-decomposition we know that D/Q_D is a dendrite and that each plaque is an hereditarily unicoherent continuum. By hypothesis, the plaques are locally connected. Applying Theorem 3.1.10 we conclude the plaques are dendrites. □

Remark 6.2.3. Knowing that each plaque is a dendrite it is natural to look for more topological information given that they are contained in the plane. Unfortunately, we must deal with the worst kind of dendrites because every dendrite can be embedded in the plane. Recall Wazewski's universal dendrite, Theorem 3.1.11. However, generic dendrites in a dendritation are nice, as we will now explain.

We will study the properties of the plaque of a point in a residual subset of the disk. A property is *generic* if it holds in a residual subset. For this purpose we recall that in [**35**] Moore proved that every family of mutually disjoint triods of the plane is countable. An alternative proof can be found in [**42**]. Recall that a *triod* is a union of three arcs, homeomorphic to $([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$ in the plane (three segments with a common end point).

PROPOSITION 6.2.4. *If Q is a cw-decomposition of the disk D then there is a residual set $G \subset D$ such that* $\operatorname{card}(Q(x) \cap \partial D) \leq 2$ *for all $x \in G$.*

Proof. First we show that there is a residual set $G_1 \subset D$ such that $Q(x) \cap \partial D$ is totally disconnected for all $x \in G_1$. Let $A \subset \partial D$ be a countable dense subset. If $Q(x) \cap \partial D$ contains a non-trivial arc then $A \cap Q(x) \neq \emptyset$. Therefore, at most a countable family of plaques intersects ∂D in a non-totally-disconnected set. The complement of the union of such plaques gives our residual set G_1.

By Proposition 6.1.2 we know that Q is codendritic. Then, by Proposition 3.2.10 there is a residual set $G_2 \subset D$ such that $D \setminus Q(x)$ has one or two components for all $x \in G_2$. By Janisewski's theorem this implies that $Q(x) \cap \partial D$ has one or two components. For $x \in G = G_1 \cap G_2$ we have that $Q(x) \cap \partial D$ has one or two points. □

In the examples of decompositions defined from stable sets of cw-expansive homeomorphisms known to the author, it is true that $\operatorname{card}(Q(x) \cap \partial D) = 2$ on a residual set. In general, for arbitrary decompositions, this is not true as the next example shows.

Example 6.2.5. We will construct a cw-decomposition Q of a disk D such that for every residual set $G \subset D$ there is $x \in G$ whose dendrite $Q(x)$ does not separate D. Consider a pseudo-Anosov diffeomorphism on a hyperbolic surface. Denote by $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ the universal cover of the surface and denote by $\tilde{F}^s$ the lifting to U of the stable singular foliation of the pseudo-Anosov diffeomorphism. On the closed disk $D' = \overline{U}$ (the closure in the usual topology of $\mathbb{R}^2$) consider the dendritic decomposition where $Q'(p)$ is the closure of $\tilde{F}^s(p)$. Consider the annulus $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$ and the

disk $D = U \cup A$. We will extend Q' to D. For $p \in \partial U$ define $A_p = \{\lambda p : 1 \leq \lambda \leq 2\}$ a line segment from p to ∂D contained in the line through p and the origin. For $p \in D'$ define

$$Q(p) = Q'(p) \cup \bigcup_{q \in Q'(p) \cap \partial D'} A_q.$$

It is easy to prove that Q defines a dendritic decomposition on D.

Let us show that for a generic point $p \in A$ we have that $Q(p)$ is a non-separating arc. The proof follows the ideas in [1, Lemma 3.1]. Let D_n be the closed disk of radius $1 - 1/n$ centered at the origin. Define $T_n = Q^{-1}(Q(D_n))$. Notice that

$$\bigcup_{}^{\infty} T_n = \{p \in D : Q(p) \text{ separates } D\}.$$

Also $A \setminus T_n$ is open and dense in A. Define $G = A \setminus (\bigcup^{\infty} T_n)$. Therefore, G is residual in A and for all $p \in A$ the plaque $Q(p)$ is an arc of the form A_q (containing p), and this arc does not separate D.

A decomposition is *arc-connected* if each plaque is arc-connected.

PROPOSITION 6.2.6. *If Q is an arc-connected cw-decomposition on D then there is a residual set $G \subset D$ such that $Q(x)$ is an arc for all $x \in G$.*

Proof. If a plaque is not an arc then it has a ramification point and contains a triod. By Moore's theorem [35], at most a countable number of disjoint triods can be embedded in the plane. Then, the set of plaques with ramification points is countable. Denote by $Q_1, Q_2, \ldots$ such plaques. Since each Q_n is closed and has empty interior, its complement in the disk D, $U_n = D \setminus Q_n$, is an open and dense subset of D. Then, $G = \cap_{n \geq 1} U_n$ is a residual set of points x such that $Q(x)$ is an arc. ☐

If $\mathcal{F}$ is a dendritation of a surface we say that $y \in \mathcal{F}(x)$ is a *ramification point* if $\mathcal{F}(x) \setminus \{y\}$ has at least three components in the plaque topology.

THEOREM 6.2.7. *If $\mathcal{F}$ is a dendritation of a compact surface S, then there is a residual set $G \subset S$ such that $\mathcal{F}(x)$ has no ramification for all $x \in G$. For all $x \in G$ the leaf $\mathcal{F}(x)$ with the plaque topology is a one-dimensional manifold: $\mathbb{R}$, $[0, +\infty)$, $[0, 1]$ or the circle.*

Proof. By Proposition 4.1.10 and Theorems 4.3.4, 4.2.2 we know that: $\tau_{\mathcal{F}}$ is a metric topology, the leaves are arc-connected in $\tau_{\mathcal{F}}$ and each leaf has a countable basis with respect to $\tau_{\mathcal{F}}$. Arguing as in the proof of Proposition 6.2.6 we have that the number of leaves with ramification points is at most countable. Since each leaf is a meagre set (by the definition of cw-foliation), the set of points in leaves without ramifications is a residual set G. Then, the result follows by the classification of one-dimensional manifolds, see for example [29]. ☐

Remark 6.2.8. On a residual set, as in Theorem 6.2.7, one-dimensional manifolds can occur. For example, a torus can be foliated by circles. Also, a minimal flow on the torus has all the leaves homeomorphic to $\mathbb{R}$. Using Example 6.2.5 we can define a cw-foliation of the sphere (identifying two disks by the boundary) with a residual set of leaves homeomorphic to one-dimensional manifolds with boundaries.

Remark 6.2.9. It would be interesting to extend Theorem 6.2.7 without assuming local connection of the plaques.

In the next result we will use the stable and unstable cw-foliations $\mathcal{F}^s$, $\mathcal{F}^u$ that were constructed in §5.1 for an arbitrary cw-expansive homeomorphism of a Peano continuum (as our compact surface S).

THEOREM 6.2.10. *If $f : S \to S$ is a cw-expansive homeomorphism and $\mathcal{F}^s$, $\mathcal{F}^u$ are dendritations, then there is a residual set $G \subset S$ such that for all $x \in G$ it holds that $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ are homeomorphic to $\mathbb{R}$ or $[0, +\infty)$.*

Proof. Take from Theorem 6.2.7 two residual sets G^s and G^u corresponding to $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively, and define $G = G^s \cap G^u$. For $x \in G$ we know that the stable and the unstable leaf of x is a one-dimensional manifold. In Theorem 5.1.2 we proved that no leaf $\mathcal{F}^\sigma(x)$ is plaque-compact. By Theorem 4.2.6, this implies that no leaf $\mathcal{F}^\sigma(x)$ is a Peano continuum in the relative topology of τ (the topology of the surface). Since $[0, 1]$ and the circle are Peano continua we know from Theorem 5.1.2 that $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ are homeomorphic to $\mathbb{R}$ or $[0, +\infty)$ for each $x \in G$. $\square$

Remark 6.2.11. In the examples known to the author, for every cw-expansive homeomorphism of a compact surface there is a residual set of points whose stable and unstable leaves are homeomorphic to $\mathbb{R}$ and just a countable number of leaves are homeomorphic to $[0, +\infty)$.

6.3. *Graph-like continua and decompositions.* In this section we develop a technique to construct examples of decompositions.

Let $I, J \subset \mathbb{R}$ be compact intervals. Consider a continuum with empty interior $C \subset I \times J$ disjoint from $I \times \partial J$ and such that for each $x \in I$ the set $C \cap [\{x\} \times J]$ is connected and non-empty. In this case we say that C is a *graph-like continuum*. For such a continuum C, define

$$U^+ = \{(x, y) \in I \times J : \text{if } (x, y') \in C \text{ then } y > y'\},$$
$$U^- = \{(x, y) \in I \times J : \text{if } (x, y') \in C \text{ then } y < y'\}.$$

We have that $I \times J$ is the disjoint union of C, U^+ and U^-.

We will define a decomposition of $I \times J$ associated with a graph-like continuum C. Define $v : I \times \mathbb{R} \to \mathbb{R}^2$ by $v(p) = (0, \text{dist}(p, C))$, where $\text{dist}(p, C) = \min\{\text{dist}(p, q) : q \in C\}$. Since v is Lipschitz, it defines a continuous flow ϕ on $I \times \mathbb{R}$ with equilibrium points at every point of C. For $p \in I \times J$ define $Q_C(p) = C$ if $p \in C$ and

$$Q_C(p) = \{q \in I \times J : \exists t \in \mathbb{R} \text{ such that } \phi_t(p), \phi_t(q) \in I \times \partial J\}$$

if $p \notin C$. Recall from §3.1.1 that the continuity of a decomposition is with respect to the Hausdorff metric.

PROPOSITION 6.3.1. *If $C \subset I \times J$ is a graph-like continuum then Q_C is a cw-decomposition and:*
(1)   *$Q_C(p)$ is the graph of a continuous map $I \to J$ for all $p \notin C$;*
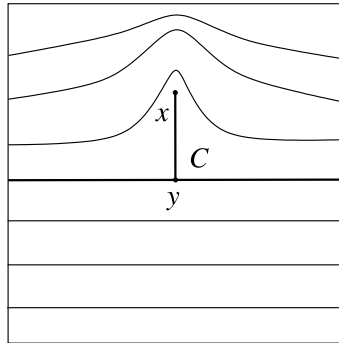(2)   *Q_C is continuous at every $p \notin C$;*

FIGURE 9. The decomposition is continuous at x and discontinuous at y (approximate y from below).

(3) Q_C is continuous (at every point in C) if and only if $C \cap \partial U^+ = C \cap \partial U^-$;
(4) $[I \times J]/Q_C$ is an arc.

Proof. Fix $p \in U^+$. Define $L^{\pm}$ as the components of $I \times \partial J$ contained in $\partial U^{\pm}$, respectively. Take $T(p) \in \mathbb{R}$ such that $\phi_{T(p)}(p) \in L^+$. Since $L^{\pm}$ are transverse to ϕ we have that $T : U^+ \to \mathbb{R}$ is continuous. From the definition of Q_C we have that $Q_C(p) = \phi_{-T(p)}(L^+)$, given that $p \in U^+$. Therefore, $Q_C(p)$ is the graph of a continuous map for all $p \in U^+$. In a similar way this conclusion holds for all $p \in U^-$. This implies that each $Q_C(p)$ is a continuum meeting $\partial(I \times J)$. Also, no $Q_C(p)$ has interior points.

In order to conclude that Q_C is a decomposition it only remains to prove the upper semicontinuity. The continuity of the flow ϕ implies that the functions whose graphs are $Q_C(p)$, for $p \notin C$, vary with continuity. This proves the continuity of Q_C at the points $p \notin C$. Now consider $p \in C$ and $p_n \to p$ and assume without loss of generality that $p_n \in U^+$. We have that $T(p_n) \to -\infty$. Take $q_n \in Q_C(p_n)$ with $q_n \to q$. By the continuity of T we have that $q \in C$. This proves the upper semicontinuity of Q_C at $p \in C$. Therefore Q_C is a decomposition.

With the previous notation, we have that $Q_C(p_n) \to C \cap \partial U^+$. Therefore, Q_C is continuous at points in C if and only if $C \cap \partial U^+ = C \cap \partial U^-$. In order to conclude that the quotient is an arc, note that, except $L^{\pm}$, every plaque separates the rectangle $I \times J$. This also implies that Q_C is codendritic and a cw-decomposition. $\square$

Example 6.3.2. In the square $[-1, 1] \times [-1, 1]$ consider the graph-like continuum

$$C = ([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1/2]).$$

The decomposition Q_C induced by C is shown in Figure 9. We see that it can be continuous at some x while not being continuous at some $y \in Q_C(x)$.

Example 6.3.3. In the square $[-2, 2] \times [-2, 2]$ consider the graph-like continuum

$$C = (\{0\} \times [-1, 1]) \cup \{(x, \sin(1/x)) : 0 < |x| \le 2\}.$$

We have that Q_C is a continuous decomposition. It is illustrated in Figure 10.
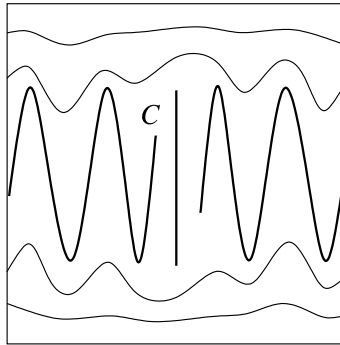
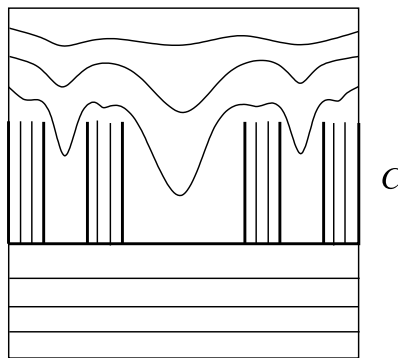FIGURE 10. A continuous decomposition with a plaque that is not arc-connected.

FIGURE 11. The (non-locally connected) leaf C is not a countable union of small plaques.

Example 6.3.4. Consider $K \subset [0, 1]$ a Cantor set and define the graph-like continuum

$$C = (K \times [1/3, 2/3]) \cup ([0, 1] \times \{1/3\}).$$

Consider the decomposition Q_C in the square $[0, 1] \times [0, 1]$. See Figure 11. If we think of Q_C as a cw-foliation of this square and we consider a complete basis of small open sets then we see that C is a leaf that is not a countable union of plaques.

6.4. *The foliated box.* The purpose of the present section is to give a characterization of the standard foliated two-dimensional box (a product structure of a rectangle).

PROPOSITION 6.4.1. *Let $Q : D \to \mathcal{C}(D)$ be a cw-decomposition. If P is an arc-connected plaque that separates D and Q is continuous at every $x \in P$ then P is an arc and $P \cap \partial D$ has exactly two points.*

Proof. By Janisewski's Theorem 3.1.7†, we have that $P \cap \partial D$ is disconnected because P is a continuum separating D. Take $a, b \in \partial D$ in different components of $P \cap \partial D$. Since P is arc-connected there is an arc $\gamma \subset P$ from a to b. We can assume that $\gamma \cap \partial D = \{a, b\}$.

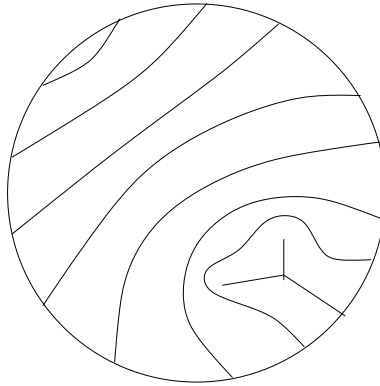† In this case, Jordan's closed curve theorem could also be applied.

FIGURE 12. A non-separating plaque not being an arc in a continuous arc-connected decomposition of a disk.

We will show that $\gamma = P$. Arguing by contradiction assume that there is $z \in P \backslash \gamma$. Denote by D_1, D_2 the components of $D \backslash \gamma$ and suppose that $z \in D_1$. Take $y \in \gamma$ and $x_n \in D_2$ with $x_n \to y$. Since Q is continuous at every point of P, it is continuous at y. But z is not in the limit of $Q(y_n)$. This contradiction proves that $P = \gamma$. $\quad\square$

Remark 6.4.2. We wish to remark the necessity of the hypothesis in Proposition 6.4.1. The arc connection is needed by Example 6.3.3, where C is separating and continuous but not arc-connected. In Example 6.3.2 we see that Q must be continuous at every point of C in order to conclude that it is an arc, even being arc-connected and separating. We need to assume that C is separating in order to conclude that it is an arc. An example is illustrated in Figure 12.

PROPOSITION 6.4.3. *If $Q : D \to \mathcal{C}(D)$ is a continuous arc-connected cw-decomposition, then D/Q is an arc.*

Proof. By definition of cw-decomposition we know that D/Q is a dendrite. In order to prove that it is an arc it is sufficient to prove that if $Q(x)$ separates D/Q then it separates into two components, and this is a direct consequence of Proposition 6.4.1. $\quad\square$

Example 6.4.4. Proposition 6.4.3 is not true if we do not assume the arc-connection of each $Q(x)$. An example can be constructed based on Wada's lakes. See Figure 13. A similar decomposition was previously considered in [**13**, Remark 4.11]. It is a continuous decomposition Q of a disk D such that Q/D is a triod.

For the following proof we recall that a *size function* is a continuous function $\mu : \mathcal{C}(D) \to \mathbb{R}$ such that: $\mu(C) \geq 0$ for all $C \in \mathcal{C}(D)$ with equality if and only if C is a singleton; if $C \subset C'$ and $C \neq C'$ then $\mu(C) < \mu(C')$. Size functions exist [**37**] and in fact were introduced by Whitney in [**48**] for the study of regular families of curves (very similar to our case). The following result characterizes standard foliated boxes (or product structure as defined in §3.2). Its proof uses techniques from [**48**, Theorem 17A].

THEOREM 6.4.5. *If $Q : D \to \mathcal{C}(D)$ is a $\mathcal{C}$-smooth, continuous cw-decomposition with two non-trivial plaques contained in ∂D then it is a product structure.*
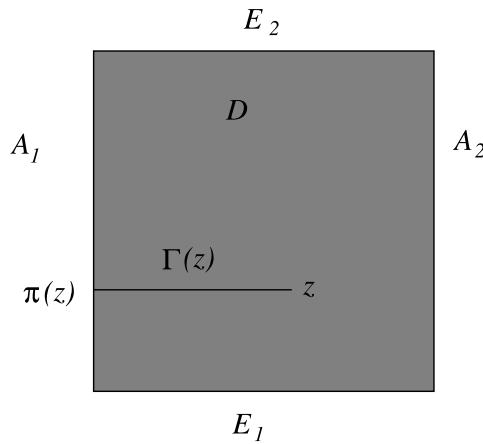
FIGURE 13. Wada's lakes.

FIGURE 14. The construction of suitable coordinates.

Proof. In Proposition 3.2.15 we proved that every $\mathcal{C}$-smooth decomposition is dendritic (in particular, arc-connected). Then, we can apply Proposition 6.4.3 to conclude that the quotient D/Q is an arc. Therefore, it has two non-separating plaques. Let E_1, $E_2 \subset \partial D$ be the non-separating plaques of Q. Take two arcs A_1, $A_2 \subset \partial D$ intersecting E_1 and E_2 in one point to each one, as shown in Figure 14.

Since D/Q is an arc, $Q(z)$ separates E_1 from E_2 for all $z \in D$. Then $Q(z) \cap A_i \neq \emptyset$ for all $z \in D$ and $i = 1, 2$. By Proposition 6.4.1 we have that $Q(z) \cap A_i$ is a singleton. Define $\pi : D \to A_1$ by $\{\pi(z)\} = A_1 \cap Q(z)$. Proposition 6.4.1 also gives us that each $Q(z)$ is an arc. Define $\Gamma : D \to \mathcal{C}(D)$ such that $\Gamma(z) \subset Q(z)$ is the arc from z to $\pi(z)$.

Let $\mu : \mathcal{C}(D) \to \mathbb{R}$ be a size function and define $h : D \to A_1 \times [0, 1]$ by

$$h(z) = (\pi(z), \mu(\Gamma(z))/\mu(Q(z))). \tag{14}$$

Since Q has no trivial plaques, we have that $\mu(Q(z)) \neq 0$ for all $z \in D$. Let us show that h is continuous. As $Q(z) \cap A_1$ is a singleton for all $z \in D$ we have that π is continuous. Given that Q is $\mathcal{C}$-smooth, we conclude that Γ is continuous. Since μ is continuous we conclude that h is continuous.
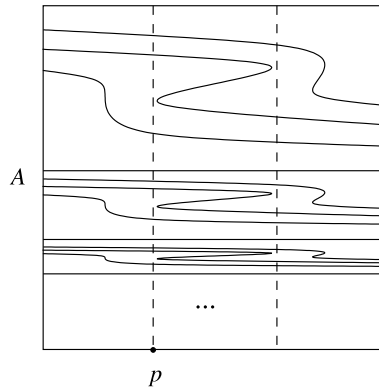
FIGURE 15. Every plaque is an arc from the left side to the right side but Q is not $\mathcal{C}$-smooth. Also, the quotient space is an arc.

To prove that h is injective, suppose that $h(z) = h(z')$. Then $\pi(z) = \pi(z')$ and $\mu(\Gamma(z))/\mu(Q(z)) = \mu(\Gamma(z'))/\mu(Q(z'))$. Therefore z, z' are in the same plaque and $\mu(Q(z)) = \mu(Q(z'))$, which implies that $\mu(\Gamma(z)) = \mu(\Gamma(z'))$. Since every plaque is an arc we conclude that $z = z'$ and the injectivity of h. Similar arguments prove the surjectivity of h and we conclude that h is a homeomorphism. Denote by $\tilde{Q}$ the decomposition of the rectangle $A_1 \times [0, 1]$ in horizontal lines $\{a\} \times [0, 1]$ for $a \in A$. Since it holds that $h(Q(z)) = \tilde{Q}(h(z))$ the proof ends. $\qquad\square$

It is natural to ask if a continuous arc-connected cw-decomposition $Q : D \to \mathcal{C}(D)$ must be a product structure. In the next example we show that this is not always the case.

Example 6.4.6. The cw-decomposition illustrated in Figure 15 is not a product structure because it is not $\mathcal{C}$-smooth.

6.5. *Smooth dendritations.* Let $\mathcal{F}$ be a cw-foliation on a closed surface S with atlas $\mathcal{A} = \{Q_{\overline{U}} : U \in \mathcal{U}\}$.

Definition 6.5.1. We say that cw-foliation $\mathcal{F}$ is $\mathcal{C}$-*smooth* if each $Q_{\overline{U}}$ is a $\mathcal{C}$-smooth decomposition.

By Proposition 3.2.16 we know that monotone restrictions of a $\mathcal{C}$-smooth cw-decomposition are $\mathcal{C}$-smooth. Then, a $\mathcal{C}$-smooth cw-foliation is defined ever a complete basis.

THEOREM 6.5.2. *Every $\mathcal{C}$-smooth cw-foliation of a closed surface S is a standard foliation with a finite set of prong-singularities and no 1-prongs.*

Proof. Consider a cw-atlas $\mathcal{A} = \{Q_{\overline{U}} : U \in \mathcal{U}\}$ of the $\mathcal{C}$-smooth cw-foliation $\mathcal{F}$. By Proposition 3.2.15 we have that $\mathcal{F}$ is a dendritation. Take $U \in \mathcal{U}$ such that $D = \overline{U}$ is a disk and define $Q = Q_D$.

In this paragraph we will show that if z is an end point of the dendrite $Q(z)$ then $z \in \partial D$. Arguing by contradiction suppose that there is $z \in D \setminus \partial D$ that is an end point of its plaque.
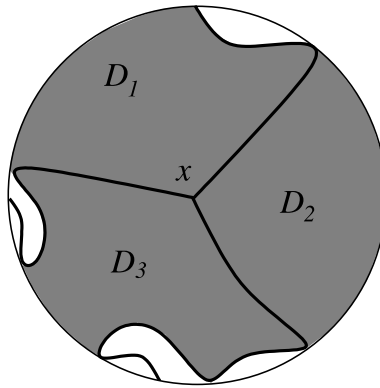
FIGURE 16. A finite number of components of $D \setminus Q(x)$ are disks with x in their boundary.

Take $z_n \in D \setminus Q(z)$ such that $z_n \to z$. Since Q is codendritic, for each $n \geq 1$ there is a plaque P_n separating $Q(z)$ from z_n. Consider an arc $A_n \subset P_n$ separating $Q(z)$ from z_n such that $A_n \cap \partial D = \{x_n, y_n\}$. Consider a Riemannian metric on D and for each $n \geq 1$ a geodesic arc γ_n from z to z_n. Since A_n separates z from z_n, we can take $z'_n \in \gamma_n \cap A_n$. Since γ_n is a geodesic arc we have that $z'_n \to z$. Taking subsequences we can assume that $x_n \to x$ and $y_n \to y$ with $x, y \in Q(z) \cap \partial D$. Given that Q is $\mathcal{C}$-smooth we have that $A_n \to A$ where A is an arc from x to y. Since $z'_n \in A_n$ and $z_n \to z$ we have that $z \in A$. The upper semicontinuity of Q gives us that $A \subset Q(z)$. Since $z \notin \partial D$ we have that z is not an end point of $Q(z)$. This contradiction proves that every end point of each plaque is in the boundary of D.

Now we will show that for all compact sets $K \subset \mathrm{int}(D)$ the set of points $x \in K$ such that x is a ramification point of $Q(x)$ is finite. The result of the previous paragraph implies that for all $x \in D$ every component of $Q(x) \setminus \{x\}$ meets ∂D. Now suppose that there is a sequence $y_n \to y$ such that $y_n \in K$ and each y_n is a ramification point of $Q(y_n)$. Denote $A_n = Q(y_n) \cap \partial D$. We have that each A_n has at least three points. We can take $a_n, b_n \in A_n$ such that $\mathrm{dist}(a_n, b_n) \to 0$ and $y_n \in Q(a_n, b_n)$†. This contradicts that Q is $\mathcal{C}$-smooth and proves that the set of ramification points can only accumulate on ∂D. Then, there is a finite number of ramification points in the whole surface.

Take $x \in S$ and a small disk D around x. Assume that in $D \setminus \{x\}$ there are no ramification points. We know that $Q_D(x)$ separates D in at least two components. Some of these components may be far from x and only a finite number of them contain x in their boundary, see Figure 16. Denote by $D_1, \ldots, D_n$ the closures of such components.

Since the only possible ramification point in D is x, we know that in each D_i every plaque is an arc. As the end points of the plaques are in the boundary, each plaque is an arc starting and ending at the boundary. Given y in the interior of D_i, the arcs $Q_{D_i}(x)$ and $Q_{D_i}(y)$ determine two arcs A, B contained in the boundary of D_i as shown in Figure 17.

Now we will prove that there is a neighborhood V of x in D_i such that every $z \in V$ separates $Q_{D_i}(z)$ in two arcs $\gamma_A(z)$ and $\gamma_B(z)$ such that $\gamma_A(z)$ cuts A and is disjoint from B and $\gamma_B(z)$ cuts B and is disjoint from A. Arguing by contradiction, assume first that

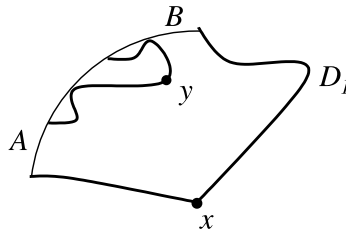† For two points a, b in a common plaque, $Q(a, b)$ is the arc in the plaque from a to b.

FIGURE 17. The arcs A and B cut $Q_{D_1}(x)$ and $Q_{D_1}(y)$ in one point.

there is $z_n \to x$ such that $Q_{D_i}(z_n) \cap B = \emptyset$. We can take two points $a_n, b_n \in A \cap Q_{D_i}(z_n)$ such that $\mathrm{dist}(a_n, b_n) \to 0$. This contradicts that Q_{D_i} is $\mathcal{C}$-smooth. Therefore, if z is close to x then $Q_{D_i}(z)$ cuts A and B. A similar argument proves that if z is sufficiently close to x then one component of $Q_{D_i}(z) \backslash \{z\}$ cuts A and not B while the other component cuts B and not A.

Define K as the union of all the arcs starting and ending in A and contained in a plaque of Q_{D_i}. Considering that the arcs defining K may be trivial, we have that $A \subset K$. In this way K is a continuum. For $u, v \in D_i$ in a common plaque define $\mathrm{arc}(u, v)$ as the arc from u to v inside the plaque. For $u \in V$ define $\pi(u)$ as the first point of $\gamma_A(u)$ in A. Define $\Gamma : V \to \mathcal{C}(D_i)$ as

$$\Gamma(u) = \mathrm{arc}(u, \pi(u)) \cup K.$$

In this paragraph we show that Γ is continuous. Take $u \in V$ and $u_n \to u$. We can assume that $\Gamma(u_n)$ converges and $\pi(u_n) \to v \in \gamma_A(u) \cap A$. Since Q is $\mathcal{C}$-smooth we have that $\mathrm{arc}(u_n, \pi(u_n)) \to \mathrm{arc}(u, v)$. It is clear that $\mathrm{arc}(u, \pi(u)) \subset \mathrm{arc}(u, v)$. Consequently, $\Gamma(u) \subset \lim \Gamma(u_n)$. Since $\mathrm{arc}(\pi(u), v) \subset K$ and

$$\mathrm{arc}(u, v) = \mathrm{arc}(u, \pi(u)) \cup \mathrm{arc}(\pi(u), v)$$

we have that $\lim \Gamma(u_n) \subset \Gamma(u)$. This proves the continuity of Γ.

Consider a size function $\mu : \mathcal{C}(D_i) \to \mathbb{R}$ and define $\varphi : V \to \mathbb{R}$ as $\varphi(u) = \mu(\Gamma(u))$. The function φ is continuous and increases along a plaque from A to B. The level sets of φ and the plaques give coordinates and define a product structure in a neighborhood of x in D_i (contained in V).

Arguing in the same way in the sectors $D_1, \ldots, D_n$ an n-prong structure is defined around x. As we proved, $n \neq 1$ and the set of n-prongs with $n \geq 3$ is finite. This finishes the proof. $\square$

6.6. *Continuous dendritations.*   Let $\mathcal{A} = \{Q_{\overline{U}} : U \in \mathcal{U}\}$ be an atlas of a dendritation $\mathcal{F}$ over the complete basis $\mathcal{U}$ of a surface S.

Definition 6.6.1. We say that a dendritation $\mathcal{F}$ is *continuous* if for all $x \in U \in \mathcal{U}$ there is a closed disk $D \subset U$ such that x is in the interior of D and Q_D is continuous.

This definition of continuity is stronger than the one given in §4.4 because we require the continuity on disks instead of arbitrary neighborhoods.

Denote by Q_1 the decomposition of $D = [-1, 1] \times [-1, 1]$ in horizontal plaques (a foliated box). Consider the equivalence relation on D generated by $(x, 1) \sim (-x, 1)$ for all

$x \in [-1, 1]$. Denote by Q_2 the induced decomposition on the quotient space $\tilde{D} = D/\sim$. The point $\{(0, 1)\}$ in $\tilde{D}$ is called a *1-prong* of Q_2.

THEOREM 6.6.2. *Every continuous dendritation is a foliation possibly with a finite number of* 1*-prongs.*

Proof. By Proposition 6.4.1 there are no ramification points, that is, every plaque is an arc. Take $x \in S$ and a disk D around x with Q_D continuous. From Proposition 6.4.3 we know that D/Q is an arc. Then there are exactly two non-separating plaques that will be called E_1, E_2. From Proposition 6.4.1, every separating plaque cuts the boundary of D at exactly two points. Reducing D with two separating plaques (if needed) we can assume that E_1 and E_2 are non-trivial (are not singletons).

In this paragraph we will show that if $E_1, E_2 \subset \partial D$ then Q is a product structure. For this purpose we will show that Q is $\mathcal{C}$-smooth and apply Theorem 6.4.5. Arguing by contradiction assume that $Q(x_n, y_n)$ does not converge to $Q(x, y)$ assuming that $x_n \to x$, $y_n \to y$ and $y_n \in Q(x_n)$. Taking a subsequence we can assume that $Q(x_n, y_n) \to C$ for a continuum $C \subset D$. Since $Q(x_n, y_n) \subset Q(x_n)$ and Q is continuous we have that $C \subset Q(x)$. Therefore, C is an arc because $Q(x)$ is an arc. Moreover, $x, y \in C$. Since $C \neq Q(x, y)$, we have that y (or x) is an interior point of the arc C. Let $z \in C$ be the end point of C that is not in $Q(x, y)$. Consider $z_n \in Q(x_n, y_n)$, an interior point, such that $z_n \to z$. Since the dendritation is continuous there is a disk D', a neighborhood of z, such that $Q' = Q|_{D'}^m$ is continuous. We can assume that D' is so small that $y, y_n \notin D'$ for all $n \geq 1$. Therefore, $Q'(z_n)$ is a sequence of arcs separating D'. We can find n_1, n_2, n_3 such that z and $Q'(z_{n_i})$ are in the same component of $D' \setminus Q'(z_{n_j})$ if $i \neq j$, $i, j \in \{1, 2, 3\}$. Then Q' has at least three non-separating plaques. This contradicts Proposition 6.4.3 in the disk D' and proves that Q is $\mathcal{C}$-smooth. Since E_1, E_2 are non-trivial we can apply Theorem 6.4.5 to conclude that Q is a product structure.

If $x \in E_1$ (i.e. E_1 is not contained in the boundary of D) we can cut D along E_1 and reduce the proof to the previous case ($E_1, E_2 \subset \partial D$). In this case we obtain that Q is a 1-prong decomposition around x. This also proves that the number of 1-prongs is finite. $\square$

In the following result the word foliation means a one-dimensional C^0 foliation in the standard sense (without singular points).

COROLLARY 6.6.3. *A dendritation is a foliation if and only if it is continuous and $\mathcal{C}$-smooth.*

Proof. It follows by Theorems 6.5.2 and 6.6.2. $\square$

6.7. *cw_F-expansivity and dendritations.* If two arcs α, β in a surface meet at x we say that they are *topologically transverse* at x if the components of $\alpha \setminus \beta$ are in different components of $D \setminus \beta$, where D is a disk separated by β and containing α. See Figure 18.

The following result gives some idea of how the stable and the unstable cw-foliations of a cw$_\mathrm{F}$-expansive homeomorphism are distributed in the surface.

THEOREM 6.7.1. *If f is a cw$_F$-expansive homeomorphism of a compact surface S then:*
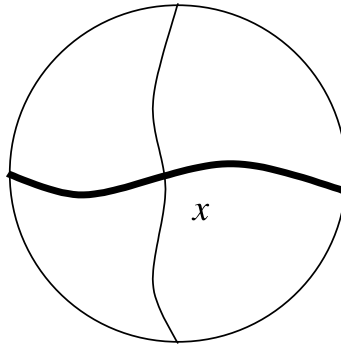
FIGURE 18. Two arcs topologically transverse at the point x.

(1) $\mathcal{F}^s$ and $\mathcal{F}^u$ are f-invariant dendritations and no leaf is a Peano continuum;
(2)   there is a residual set $G \subset S$ such that $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ are non-compact one-dimensional manifolds for all $x \in G$;
(3)   there is a subset $H \subset G$ that is dense in S such that $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ are topologically transverse at x for all $x \in H$.

Proof. By Theorem 5.1.2 we know that $\mathcal{F}^s$ and $\mathcal{F}^u$ are f-invariant cw-foliations without plaque-compact leaves. We start by proving the local connection of stable and unstable plaques to conclude that they are dendritations. Arguing by contradiction, let $P \subset S$ be a non-locally connected stable continuum. By Theorem 3.1.4 we can consider P_n a sequence of subcontinua of P such that $P_i \cap P_j = \emptyset$ if $i \neq j$ and $\mathrm{dist}(P_n, P_*) \to 0$ for some non-trivial continuum $P_* \subset P$. Take $x \in P_*$ and a small disk D around x such that every P_n separates D for every $n \geq n_0$, for some n_0. We can assume that $D = B_r(x)$ for some $r > 0$. Denote by Q^s and Q^u the stable and the unstable cw-decompositions of D, respectively. Consider $x_n \in P_n \cap D$ such that $x_n \to x$ and each $Q^s(x_n)$ separates D. Taking a subsequence if needed we can assume that $Q^s(x_n)$ converges to a stable continuum $C \subset Q^s(x)$. Moreover, we can assume that the sequence of plaques $Q^s(x_n)$ is monotonous (i.e. if $i < j < k$ then $Q^s(x_j)$ separates $Q^s(x_i)$ from $Q^s(x_k)$). For the value of $r > 0$ fixed above ($D = B_r(x)$) take $\varepsilon > 0$ given by Theorem 2.4.2. Denote by G_n the component of D between C and $Q^s(x_n)$. Since $Q^s(x_n) \to C$ in the Hausdorff metric, there is $n_1 > 0$ such that $Q^s(x_n) \subset B_\varepsilon(C)$ for all $n \geq n_1$. Consequently, $G_n \subset B_\varepsilon(C)$ for all $n \geq n_1$. If ε is small, we can take $y_n \in \partial B_{r/2}(x) \cap Q^s(x_n)$ and an arc $\gamma_n \subset B_{r/2}(x)$ from x_n to y_n. Taking a subarc of γ_n we can assume that $\gamma_n \subset \overline{G_n} \cap B_{r/2}(x)$ meeting C and $Q^s(x_n)$. This proves that $(C, G_n, Q^s(x_n))$ is an (r, ε, x)-capacitor for all $n \geq n_1$. Applying Theorem 2.4.2 we can take an unstable plaque $Q^u(z)$ cutting C and $Q^s(x_n)$. This unstable plaque cuts $Q^s(x_m)$ for all $m \geq n$. Since the plaques $Q^s(x_n)$ are contained in the stable plaque P (considered at the beginning of the proof) we have a contradiction with the cw$_F$-expansivity of f. This proves that $\mathcal{F}^s$ and $\mathcal{F}^u$ are dendritations.

Since cw$_F$-expansivity implies cw-expansivity we can now apply Theorem 6.2.10 to obtain the residual set G.

By Theorem 5.1.2 no stable leaf is plaque-compact and, by Theorem 4.2.6, a leaf is plaque-compact if and only if it is a Peano continuum. Therefore, no stable leaf is a Peano continuum.

Let $D \subset S$ be a small disk and take an interior point $x \in D$. We can take a sequence $x_n \in G$ such that $x_n \to x$. Consider a stable arc γ_n^s from x_n to $y_n \in \partial D$. We can assume that $y_n \to y$ and $\gamma_n \to C$ where C is a dendrite containing x and y. Taking a subsequence we can suppose that y_n is monotone in ∂D. Consider four consecutive arcs $\gamma_{n+1}^s, \ldots, \gamma_{n+4}^s$. As before, for n large, we can apply Theorem 2.4.2 to the capacitor determined by γ_{n+1}^s and γ_{n+4}^s to find a point $p \in G$ between γ_{n+2}^s and γ_{n+3}^s whose unstable arc cuts γ_{n+1}^s or γ_{n+4}^s. Suppose that it cuts γ_{n+1}^s. Then, it cuts γ_{n+2}^s. By the cw$_{\mathrm{F}}$-expansivity of f, the unstable arc of p intersects γ_{n+2}^s in finitely many points. By construction, at least one of these cuts is topologically transverse. $\qquad\square$

6.8. *cw1-expansivity.* In this section we show that every cw1-expansive homeomorphism of a compact surface is expansive.

LEMMA 6.8.1. *Let Q^1, Q^2 be two cw-decompositions of D such that $Q^1(x) \cap Q^2(x) = \{x\}$ for all $x \in D$, ∂D is the union of two arcs α and β with extreme points p, q and $\alpha \subset Q^1(a)$ for some $a \in \partial D$. Then for all $x \in \alpha$, $x \notin \{p, q\}$, it holds that $Q^2(x) \cap \beta \neq \emptyset$.*

Proof. Suppose that for some $x \in \alpha$, $x \neq p, q$, we have that $Q^2(x) \cap \beta = \emptyset$. We know that D/Q^2 is a dendrite. Since $Q^2(x) \cap Q^1(x) = \{x\}$ we have that $Q^2(x)$ is an end of D/Q^2. In the quotient dendrite we can find a point say $Q^2(y)$ that separates D/Q^2 and is arbitrarily close to $Q^2(x)$. Then $Q^2(y)$ cuts $Q^1(a)$ in at least two points, contradicting our hypothesis. $\qquad\square$

Definition 6.8.2. Given two cw-decompositions Q^1, Q^2 of a disk D we say that ∂D is a (Q^1, Q^2)-*rectangle* if there are $x_1, y_1, x_2, y_2 \in \partial D$ such that ∂D is contained in the ordered union $Q^1(x_1) \cup Q^2(y_1) \cup Q^1(x_2) \cup Q^2(y_2)$.

PROPOSITION 6.8.3. *Let Q^1, Q^2 be two cw-decompositions of a disk D such that $Q^1(x) \cap Q^2(x) = \{x\}$ for all $x \in D$ and ∂D is a (Q^1, Q^2)-rectangle. Then there is a homeomorphism $h : D \to [0, 1] \times [0, 1]$ sending the plaques of Q^1 and Q^2 onto horizontal and vertical segments, respectively.*

Proof. We know that D/Q^1 and D/Q^2 are dendrites. Let us show that they are arcs. For this purpose we will show that they have no ramification point. Suppose by contradiction that $Q^1(x)$ (similarly for Q^2) is a ramification of the dendrite D/Q^1 for some $x \in D$. Then $Q^1(x)$ separates D into at least three components. Consequently, $Q^1(x) \cap \partial D$ has at least three components. Since ∂D is a (Q^1, Q^2)-rectangle, we have that $Q^1(x)$ cuts in at least two points to a plaque of Q^2 in the boundary of D. This contradiction proves that D/Q^1 is an arc.

Let $h : D \to [D/Q^1] \times [D/Q^2]$ as $h(x) = (Q^1(x), Q^2(x))$. We have that h is continuous and injective. In order to conclude that it is a homeomorphism it is sufficient to prove that it is surjective. The surjectivity in this case means that $Q^1(x)$ cuts $Q^2(y)$ for all $x, y \in D$, and this follows by Lemma 6.8.1. $\qquad\square$

PROPOSITION 6.8.4. *If Q^1, Q^2 is a pair of dendritic cw-decompositions of D and $Q^1(x) \cap Q^2(x) = \{x\}$ for all $x \in D$ then (Q^1, Q^2) is a generating pair.*

Proof. Fix a point $x \in D$. Taking a small subdisk around x we can assume that the intersection of each component of $Q^i(x) \backslash \{x\}$ with ∂D is a single point, $i = 1, 2$. In this way $D \backslash Q^i(x)$ has a finite number of components. By Lemma 6.8.1 we know that each component of $D \backslash Q^1(x)$ is separated by a component of $Q^2(x) \backslash \{x\}$. Therefore, the components of $Q^1(x) \backslash \{x\}$ and $Q^2(x) \backslash \{x\}$ are alternated in the disk. Denote by D' a disk contained in D bounded by an arc $\alpha^1 \subset Q^1(x)$, an arc $\alpha^2 \subset Q^2(x) \backslash \{x\}$ and an arc $\gamma \subset \partial D$. Assume that α^1 and α^2 are consecutive.

By Lemma 6.8.1 we have that for each interior point $y \in \alpha^1$ there is an arc β_y^2 (of Q^2) from y to γ. Also, for each interior point $z \in \alpha^2$ there is a stable arc β_z^u from z to γ. Since α^s and α^u are consecutive, if z and y are close to x we have that β_y^s cuts β_z^u. In this way we obtain a (Q^1, Q^2)-rectangle and we can apply Proposition 6.8.3 to prove that both decompositions generate a neighborhood of x in D'.

Repeating this argument in each sector we conclude that Q^1 and Q^2 generate a (full) neighborhood of x.   □

THEOREM 6.8.5. *Every cw1-expansive homeomorphism of a compact surface is expansive.*

Proof. If f is cw1-expansive then by Proposition 6.8.4 we have that $\mathcal{F}^s$ and $\mathcal{F}^u$ generate. Then, the result follows by Theorem 5.3.4.   □

As usual, $\Omega(f)$ denotes the non-wandering set of f.

COROLLARY 6.8.6. *For a homeomorphism $f : S \to S$ of a compact surface the following statements are equivalent:*

(1)   *f is cw1-expansive;*

(2)   *f is expansive;*

(3)   *f is 2-expansive and $\Omega(f) = S$;*

(4)   *f is conjugate to a pseudo-Anosov diffeomorphism without 1-prongs.*

Proof. We have that $2 \to 1$ on arbitrary metric spaces. The equivalence of 2 and 4 was shown in [**18**, **30**]. The equivalence of 2 and 3 follows by [**6**]. We have that $1 \to 2$ by Theorem 6.8.5.   □

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