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ON THE ZEROS OF THE POWER SERIES $\sum_{n=0}^{\infty}(-1)^n(1-c^{-n-1})^\kappa z^n$ WITH AN APPLICATION TO DISCONTINUOUS RIESZ-SUMMABILITY

BY
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1. **On the zeros of $\sum_{n=0}^{\infty}(-1)^n(1-c^{-n-1})^\kappa z^n$.** If not stated otherwise, we assume throughout that $\kappa>0$, $c>1$, and that $k<\kappa\leq k+1$ where $k=0,1,2,\ldots$. We reserve the symbol x to denote real numbers, and define $\mathbf{C}^*=\mathbf{C}-\{x:x<-1\}$, $\mathbf{C}$ being the complex plane. Let

$$\phi(z)=\phi(z,c,\kappa)=\sum_{n=0}^{\infty}(-1)^n(1-c^{-n-1})^\kappa z^n.$$

The series defining $\phi(z)$ is only convergent for $|z|<1$, but Lemma 1 (1) (below) shows that $\phi(z)$ is a meromorphic function in $\mathbf{C}$ with simple poles at $z=-c^n$, $n=0,1,2,\ldots$. The zeros of $\phi(z)$ have been investigated by Peyerimhoff [3], and the following theorem is due to him.

THEOREM P. *$\phi(z)$ has exactly k zeros in the region $\mathbf{C}^*$, and they are all positive and simple.* [3, Theorem 5].

REMARK. We denote the zeros of $\phi(z)=\phi(z,c,\kappa)$ by $r_i(c,\kappa)$, $i=1,\ldots,k$ with $0<r_1(c,\kappa)<\cdots<r_k(c,\kappa)$. Since the zeros are simple, we have $\phi'(r_i(c,\kappa))\neq 0$; and therefore every $r_i(c,\kappa)$ is an analytic function of c and κ for $c>1$, $\kappa>0$, by implicit function theory [1, 10.2].

In this part of the paper we prove the following theorem on the monotonicity of the zeros $r_i(c,\kappa)$.

THEOREM 1. *Every zero $r_i(c,\kappa)$ is a strictly increasing, unbounded function of c with $(\partial/\partial c)r_i(c,\kappa)>0$.*

Wirsing [4] proved:

THEOREM W. *Every zero $r_i(c,\kappa)$ is a strictly decreasing function of κ with $(\partial/\partial\kappa)r_i(c,\kappa)<0$.*

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We shall use the following notation:

$$A_n^{-\kappa-1}=\binom{n-\kappa-1}{n}=(-1)^n\binom{\kappa}{n}$$

for $n=0,1,2,\ldots$, where $\binom{\kappa}{n}$ denotes the binomial coefficient;

$$\psi(z)=\psi(z,c,\kappa)=-z\phi(z,c,\kappa);$$

$$\psi'(z)=\psi_1(z,c,\kappa)=\frac{\partial}{\partial z}\psi(z,c,\kappa);$$

$$\psi_2(z,c,\kappa)=\frac{\partial}{\partial c}\psi(z,c,\kappa);$$

$$\theta(z)=\theta(z,c,\kappa)=\psi'(z)\prod_{\nu=0}^{k+1}(c^\nu+z).$$

We need some auxiliary results:

LEMMA 1. *For* $z\neq -c^n$, $n=0,1,2,\ldots,$

$$\phi(z,c,\kappa)=\sum_{n=0}^{\infty}A_n^{-\kappa-1}\frac{1}{c^n+z};\tag{1}$$

$$\psi(z,c,\kappa+1)=\psi(z,c,\kappa)-\psi\left(\frac{z}{c},c,\kappa\right);\tag{2}$$

$$\psi_2(z,c,\kappa+1)=\frac{(\kappa+1)z}{c^2}\psi_1\left(\frac{z}{c},c,\kappa\right).\tag{3}$$

Proof. Expanding $(1-c^{-n-1})^\kappa$ into a binomial series we get (1). We can derive (2) and (3) directly from the power series representation of $\psi(z,c,\kappa)$.

The proof of Theorem 1 is based largely on the following lemma:

LEMMA 2. *For all* $x>-1$, $(-1)^k\theta^{(k+1)}(x)>0$.

Proof. Using formula (1) we get

$$\theta(x)=-\sum_{n=0}^{\infty}A_n^{-\kappa-1}c^n\mu_n(x)$$

where

$$\mu_n(x)=\frac{1}{(c^n+x)^2}\prod_{\nu=0}^{k+1}(c^\nu+x)=\frac{1}{w^2}\prod_{\nu=0}^{k+1}(w+c^\nu-c^n)$$

with $w=c^n+x$. We consider two cases.

First, let $n\leq k+1$. Then

$$\mu_n(x)=(-1)^n\frac{M_n}{w}+P_k(w)$$

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where $P_k(w)$ is a polynomial of degree k in w, and

$$M_n=\prod_{\nu=0}^{n-1}(c^n-c^\nu)\prod_{\nu=n+1}^{k+1}(c^\nu-c^n)>0.$$

Hence

$$(-1)^{k+1}A_n^{-\kappa-1}c^n\left(\frac{d}{dx}\right)^{k+1}\mu_n(x)=(-1)^nA_n^{-\kappa-1}c^n(k+1)!\,\frac{M_n}{w^{k+1}}>0$$

for $x>-1$.

Next, let $n>k+1$. Expanding $\prod_{\nu=0}^{k+1}(w+c^\nu-c^n)$ in powers of w, we get

$$\mu_n(x)=(-1)^{k+2}M_n\left(\frac{1}{w^2}-\frac{1}{w}\sum_{\nu=0}^{k+1}\frac{1}{c^n-c^\nu}\right)+P_k(w)$$

where $P_k(w)$ is a polynomial of degree k in w and

$$M_n=\prod_{\nu=0}^{k+1}(c^n-c^\nu)>0.$$

Hence

$$(-1)^{k+1}A_n^{-\kappa-1}c^n\left(\frac{d}{dx}\right)^{k+1}\mu_n(x)$$
$$=(-1)^{k+1}A_n^{-\kappa-1}c^n\frac{(k+1)!}{w^{k+3}}M_n\left(\sum_{\nu=0}^{k+1}\frac{c^n+x}{c^n-c^\nu}-k-2\right)\geq 0$$

for $x>-1$, since $(-1)^{k+1}A_n^{-\kappa-1}\geq 0$ when $n>k+1$.

It follows that

$$(-1)^{k+1}\theta^{(k+1)}(x)=(-1)^{k+1}\left(-\sum_{n=0}^{\infty}A_n^{-\kappa-1}c^n\left(\frac{d}{dx}\right)^{k+1}\mu_n(x)\right)<0 \text{ for } x>-1.$$

Proof of Theorem 1. By Lemma 2, $\theta(x)$ has at most $k+1$ zeros in the range $x>-1$, and consequently the same holds for $\psi'(x)$. By (2), with $z=r_i(c,\kappa+1)=r_i$, we have $\psi(r_i,c,\kappa)=\psi(r_i/c,c,\kappa)$ and hence $\psi'(x_i)=0$ for some $x_i\in(r_i/c,r_i)$, $i=1,\ldots,k+1$.

Peyerimhoff has shown that $0<r_1/c<r_1<r_2/c<\cdots<r_k<r_{k+1}/c<r_{k+1}$ [3, p. 210]. Thus the $k+1$ numbers $x_1,x_2,\ldots,x_{k+1}$ are distinct and they yield all the zeros of $\psi'(x)$ in the range $x>-1$. Hence

$$\psi_1(r_i(c,\kappa+1)/c,c,\kappa)\neq 0. \tag{4}$$

Next, by (3) and (4) with $\kappa-1$ in place of κ, we have

$$c^2\psi_2(r_i(c,\kappa),c,\kappa)=\kappa r_i(c,\kappa)\psi_1(r_i(c,\kappa)/c,c,\kappa-1)\neq 0.$$

We also have that $\psi_1(r_i(c,\kappa),c,\kappa)\neq 0$, since the zeros of $\phi(z)$ are simple in $\mathbf{C}^*$

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by Theorem P. Hence

$$\frac{\partial r_i(c,\kappa)}{\partial c}=-\frac{\psi_2(r_i(c,\kappa),c,\kappa)}{\psi_1(r_i(c,\kappa),c,\kappa)}\neq 0 \text{ for } c>1,\ i=1,\ldots,k.$$

In order to prove that $(\partial/\partial c)r_i(c,\kappa)>0$ it suffices to show that

$$\lim_{c\to\infty} r_i(c,\kappa)=\infty.$$

For fixed $\kappa>0$, we have

$$\lim_{c\to\infty}\phi(x,c,\kappa)=\lim_{c\to\infty}\sum_{n=0}^{\infty}A_n^{-\kappa-1}\frac{1}{c^n+x}=\frac{1}{1+x}$$

uniformly for $x\geq 0$. Therefore, given $r>0$, there exists s such that

$$\phi(x,c,\kappa)\geq\frac{1}{2(1+r)}>0$$

whenever $c>s$ and $0\leq x\leq r$. It follows that $r_i(c,\kappa)>r$ whenever $c>s$, and hence that $\lim_{c\to\infty} r_i(c,\kappa)=\infty$.

2. On the equivalence of discontinuous Riesz-summability with convergence. Let $\{\lambda_n\}$ be an unbounded increasing sequence of non-negative numbers. Given a series $\sum_1^\infty a_n$, and a number $\kappa\geq 0$, let

$$A_\lambda^\kappa(x)=\sum_{\lambda_n<x}(x-\lambda_n)^\kappa a_n.$$

If $x^{-\kappa}A_\lambda^\kappa(x)\to s$ as $x\to\infty$, the series $\sum_1^\infty a_n$ is said to be summable (R,λ_n,κ) to s. The series is said to be summable by the discontinuous Riesz method (R^*,λ_n,κ) to s if $\lambda_n^{-\kappa}A_\lambda^\kappa(\lambda_n)\to s$ as $n\to\infty$.

We shall discuss the equivalence of (R^*,λ_n,κ) with convergence in the special case $\lambda_n=c^n$ for some $c>1$. The following results on the equivalence of (R^*,λ_n,κ) with convergence are known.

THEOREM K 1. *If* $\liminf_{n\to\infty}\lambda_{n+1}/\lambda_n>1$, *then* (R^*,λ_n,κ) *is equivalent to convergence for* $0\leq\kappa\leq 1$ *and for* $\kappa=2$; *so that* (R^*,c^n,κ) *is equivalent to convergence for every* $c>1$ *when* $0\leq\kappa\leq 1$ *and when* $\kappa=2$. (*See Kuttner* [2, *Theorem* 2].)

In the same paper Kuttner proved the following results:

THEOREM K 2. *If* $1<\kappa<2$, *then* (R^*,c^n,κ) *is equivalent to convergence for every* $c>1$. [2, *Theorem* 4].

THEOREM K 3. *If* $\kappa>2$, *then there is a* $c_0=c_0(\kappa)$ *such that* (R^*,c^n,κ) *is not equivalent to convergence whenever* $1<c\leq c_0$. [2, *Theorem* 3].

THEOREM K 4. *In order that* (R^*,c^n,κ) *be equivalent to convergence for* $\kappa>1$, $c>1$, *it is necessary and sufficient that* $\phi(z,c,\kappa)\neq 0$ *for* $|z|\leq 1$, $z\neq -1$. [2, *Lemma* 3].

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We shall prove the following theorem.

THEOREM 2. *There exists a function* $c(\kappa)$, *defined on* $[0,\infty)$, *such that*

(a) (R^*, c^n, κ) *is equivalent to convergence if and only if* $c > c(\kappa)$;

(b) $c(\kappa)$ *is continuous and monotonic non-decreasing on* $[0,\infty)$ *with* $c(\kappa)=1$ *for* $0 \le \kappa \le 2$, *and* $c'(\kappa)>0$ *for* $\kappa>2$;

(c) $c(\kappa)$ *is analytic for* $\kappa>2$, *and for sufficiently large* κ,

$$c(\kappa)=\sum_{n=-1}^{\infty} c_n \kappa^{-n},$$

where $c_{-1}=1/\log 2$, $c_0=-\frac{3}{2}$ *and* $c_1=-6+(\frac{73}{12}-\log 2)\log 2$; *so that*

$$c(\kappa)=\frac{\kappa}{\log 2}-\frac{3}{2}+o(1) \text{ as } \kappa\to\infty.$$

We write $f_\kappa(c)=\phi(1, c, \kappa)$. Since

$$\phi(z, c, \kappa)=\sum_{n=0}^{\infty}(-1)^n((1-c^{-n-1})^\kappa-1)z^n+\frac{1}{1+z} \quad \text{for} \quad |z|<1,$$

we have

$$f_\kappa(c)=\tfrac{1}{2}+\sum_{n=0}^{\infty}(-1)^n((1-c^{-n-1})^\kappa-1). \tag{5}$$

For $\kappa \ge 2$, let

$$S_\kappa=\{c\ge 1: \phi(r, c, \kappa)=0 \text{ for some } r\in[0, 1]\};$$
$$c(\kappa)=\sup S_\kappa;$$
$$\tilde{c}(\kappa)=\sup\{c\ge 1: f_\kappa(c)=0\}.$$

Note that $S_\kappa \neq \phi$ (since $1\in S_\kappa$ for every $\kappa\ge 2$) and that $\tilde{c}(\kappa)\le c(\kappa)$.

For the proof of Theorem 2 we need two lemmas.

LEMMA 3.

$$S_\kappa=[1, c(\kappa)]; \tag{6}$$

$$c(\kappa)>1 \text{ for all } \kappa>2 \quad \text{and} \quad \lim_{\kappa\to 2+} c(\kappa)=1; \tag{7}$$

$$r_1(c(\kappa), \kappa)=1; \tag{8}$$

$$c(\kappa)=\tilde{c}(\kappa). \tag{9}$$

Proof. Since

$$\phi(r, c, 2)=\frac{(c-1)^2(c-r)}{(1+r)(c+r)(c^2+r)}, \tag{10}$$

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it follows that $c(2)=1$. By Theorem K3, we have $c(\kappa)>1$ for all $\kappa>2$. Assume $\kappa>2$. Since $\phi(r, c, \kappa)$ is continuous in r and c, it follows that $c(\kappa)\in S_\kappa$, i.e., $\phi(r, c(\kappa), \kappa)=0$ for some $r\in[0,1]$. Hence $r_1(c(\kappa),\kappa)\le 1$. Since $r_1(c,\kappa)>0$ for $c>1$, $\kappa>1$ by Theorem P, and $r_1(c,\kappa)$ is an increasing function of c by Theorem 1, we have

$$0<r_1(c,\kappa)\le r_1(c(\kappa),\kappa)\le 1 \text{ for } 1<c\le c(\kappa).$$

Therefore $S_\kappa=[1, c(\kappa)]$.

Let $c'=\limsup_{\kappa\to 2+} c(\kappa)$. Then $\phi(r, c', 2)=0$ for some $r\in[0,1]$, and hence $c'\le 1$ by (10). Since $c(\kappa)>1$ for $\kappa>2$, it follows that $\lim_{\kappa\to 2+} c(\kappa)=1$.

If $r_1(c(\kappa),\kappa)<1$, then $r_1(c(\kappa)+\varepsilon,\kappa)<1$ for some $\varepsilon>0$, since $r_1(c,\kappa)$ is a continuous function of c. It follows that $c(\kappa)+\varepsilon\in S_\kappa$, which contradicts the definition of $c(\kappa)$. Hence $r_1(c(\kappa),\kappa)=1$. This implies that $\tilde{c}(\kappa)=c(\kappa)$.

LEMMA 4. *Let* $c^*=(\kappa/\log 2)-\frac{3}{2}+\varepsilon$ *for complex* ε *and* κ. *Then*

$$f_\kappa(c^*)=\frac{\varepsilon\log^2 2}{2\kappa}+0(1/\kappa^2), \tag{11}$$

$$f'_\kappa(c^*)=\frac{\log^2 2}{2\kappa}+0(1/\kappa^2), \tag{12}$$

as $\kappa\to\infty$ *uniformly for* $|\varepsilon|\le 1$; *and*

(13) $f'_\kappa(c)>0$ *when* c *and* κ *are real,* $c\ge\kappa$ *and* κ *is sufficiently large.*

Proof. We have

$$(1-c^{*-1})^\kappa=\exp\left(-\frac{\kappa}{c^*}-\frac{\kappa}{2c^{*2}}+0(1/\kappa^2)\right)$$

$$=\frac{1}{2}\left(1-\frac{\log^2 2}{\kappa}(2-\varepsilon)\right)+0(1/\kappa^2),$$

$$(1-c^{*-2})^\kappa=1-\frac{\log^2 2}{\kappa}+0(1/\kappa^2),$$

and

$$(1-c^{*-n-1})^\kappa=1+0(1/\kappa^n)$$

uniformly for $|\varepsilon|\le 1$ and $n=2,3,\dots$ as $\kappa\to\infty$.

From this and (5) we obtain

$$f_\kappa(c^*)=\tfrac{1}{2}+\sum_{n=0}^{\infty}(-1)^n((1-c^{*-n-1})^\kappa-1)=\frac{\varepsilon\log^2 2}{2\kappa}+0(1/\kappa^2)$$

and

$$f'_\kappa(c^*)=\frac{\kappa}{c^{*2}}\sum_{n=0}^{\infty}(-1)^n(n+1)c^{*-n}(1-c^{*-n-1})^{\kappa-1}=\frac{\log^2 2}{2\kappa}+0(1/\kappa^2)$$

as $\kappa\to\infty$, uniformly for $|\varepsilon|\le 1$.

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For real $c, \kappa, c \geq \kappa$, κ sufficiently large we have

$$f'_\kappa(c) = \frac{\kappa}{c^2}((1-c^{-1})^{\kappa-1} + 0(1/\kappa)) > 0,$$

since $(1-c^{-1})^{\kappa-1} \geq (1-1/\kappa)^{\kappa-1} \geq 1/e$.

Proof of Theorem 2. Conclusion (a) of the theorem follows from Theorem K4 and (6). Since $r_1(c(\kappa), \kappa) = 1$ for $\kappa > 2$ by (8), we obtain, by implicit function theory [1, 10.2], that $c(\kappa)$ is analytic and $c'(\kappa) = -\{(\partial/\partial\kappa)r_1(c(\kappa), \kappa)\}/\{(\partial/\partial c)r_1(c(\kappa), \kappa)\}$ for $\kappa > 2$. Since $(\partial/\partial c)r_1(c, \kappa) > 0$ by Theorem 1, and $(\partial/\partial\kappa)r_1(c, \kappa) < 0$ by Theorem W, we have $c'(\kappa) > 0$. This, together with (7), Theorems K 2 and K 3 establishes (b).

Let $\varepsilon > 0$ and $\gamma(\kappa) = (\kappa/\log 2) - \frac{3}{2}$. By (11) and (13), we have $f_\kappa(\gamma(\kappa) - \varepsilon) < 0$, $f_\kappa(\gamma(\kappa) + \varepsilon) > 0$ and $f'_\kappa(c) > 0$ for $c \geq \kappa$ and $\kappa \geq \kappa_0(\varepsilon)$. Hence

$$c(\kappa) = \frac{\kappa}{\log 2} - \frac{3}{2} + \phi(1) \text{ as } \kappa \to \infty. \tag{14}$$

Now consider $f_\kappa(c)$ for complex c, κ with $|c| > 1$. For κ sufficiently large, we have, by (11) and (12), that

$$f_\kappa(\gamma(\kappa) + \varepsilon) \neq 0 \text{ whenever } |\varepsilon| = 1 \tag{15}$$

and

$$f'_\kappa(\gamma(\kappa) + \varepsilon) \neq 0 \text{ whenever } |\varepsilon| \leq 1. \tag{16}$$

Suppose in what follows that ρ is a sufficiently large positive number. Let $K_\rho = \{\kappa \in \mathbf{C} : |\kappa| = \rho\}$, and let $C_\rho = \{\kappa : \kappa \in K_\rho,\ f_\kappa(c) = 0$ for some c such that $|c - \gamma(\kappa)| \leq 1\}$. Since $|c(\rho) - \gamma(\rho)| < 1$, by (14), we have $\rho \in C_\rho$, and hence $C_\rho \neq \phi$. By the continuity of $f_\kappa(c)$ in c and κ, C_ρ is closed. For $\kappa_1 \in C_\rho$, we have $f_{\kappa_1}(c) = 0$ for some c such that $|c - \gamma(\kappa)| \leq 1$; and, for the same c, $f'_{\kappa_1}(c) \neq 0$ by (16). By implicit function theory [1, 10.2], we can conclude that there exists a neighbourhood of κ_1 and an analytic function $c(\kappa)$ such that $f_\kappa(c(\kappa)) = 0$ throughout this neighbourhood; moreover, $|c(\kappa) - \gamma(\kappa)| < 1$ by (15). This shows that C_ρ is non-empty, and is open and closed relative to K_ρ. Therefore $C_\rho = K_\rho$.

We show next that, for every $\kappa \in C_\rho$, there exists a unique $c = c(\kappa)$ such that $f_\kappa(c(\kappa)) = 0$ and $|c(\kappa) - \gamma(\kappa)| < 1$. Assume $f_\kappa(c_1) = f_\kappa(c_2) = 0$, $|c_i - \gamma(\kappa)| < 1$, $i = 1, 2$. Then $0 = f_\kappa(c_2) - f_\kappa(c_1) = \int_{c_1}^{c_2} f'_\kappa(c)\, dc = (c_2 - c_1)\int_0^1 f'_\kappa(u(t))\, dt$, where $u(t) = c_1 + t(c_2 - c_1)$. Since $|u(t) - \gamma(\kappa)| < 1$, we have

$$f'_\kappa(u(t)) = ((\log^2 2)/(2\kappa))(1 + 0(1/\kappa))$$

as $\kappa \to \infty$, uniformly for $t \in [0, 1]$, by (12). Therefore $\int_0^1 f'_\kappa(u(t))\, dt \neq 0$ for large κ, and this implies that $c_2 = c_1$. We thus have a unique function $c(\kappa)$ for $\kappa \in K_\rho$,

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which is analytic on K_ρ by implicit function theory [1, 10.2]. Therefore

$$c(\kappa) = \sum_{n=-1}^{\infty} c_n \kappa^{-n}$$

for large κ. By (14), $c_{-1} = 1/\log 2$ and $c_0 = -\frac{3}{2}$.

Calculations similar to those used in the proof of Lemma 4 show that $c_1 = -6 + (\frac{73}{12} - \log 2)\log 2 < 0$, and therefore $c(\kappa)$ is convex (i.e., $c''(\kappa) < 0$) for large κ.

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