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ON THE DIVISIBILITY AMONG POWER LCM MATRICES ON GCD-CLOSED SETS

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Abstract

Let a, b and n be positive integers and let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. For $x \in S$, define $G_S(x) = \{d \in S : d < x,\ d \mid x \text{ and } (d \mid y \mid x, y \in S) \Rightarrow y \in \{d, x\}\}$. Denote by $[S^a]$ the $n \times n$ matrix having the ath power of the least common multiple of x_i and x_j as its (i, j)-entry. We show that the bth power matrix $[S^b]$ is divisible by the ath power matrix $[S^a]$ if $a \mid b$ and S is gcd closed (that is, $\gcd(x_i, x_j) \in S$ for all integers i and j with $1 \le i, j \le n$) and $\max_{x \in S}\{|G_S(x)|\} = 1$. This confirms a conjecture of Shaofang Hong ['Divisibility properties of power GCD matrices and power LCM matrices', *Linear Algebra Appl.* **428** (2008), 1001–1008].

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1. Introduction

For arbitrary integers x and y, we denote by (x, y) the greatest common divisor of x and y and by $[x, y]$ their least common multiple. Let a, b and n be positive integers. Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Let ξ_a be the arithmetic function defined by $\xi_a = x^a$ for any positive integer x. Let (S^a) and $[S^a]$ stand for the $n \times n$ matrices whose (i, j)-entry is $\xi_a((x_i, x_j))$ and $\xi_a([x_i, x_j])$ respectively. We call (S^a) the *ath power GCD matrix* and $[S^a]$ the *ath power LCM matrix*. The set S is *factor closed* (FC) if $(x \in S, d \mid x) \Rightarrow d \in S$ and *gcd closed* if $(x_i, x_j) \in S$ for all integers i and j with $1 \le i, j \le n$. Obviously, an FC set must be gcd closed but the converse is not true. Nearly 150 years ago, Smith [15] proved that

$$\det([x_i, x_j]) = \prod_{k=1}^{n} \varphi(x_k)\pi(x_k) \tag{1.1}$$

if S is FC, where φ is Euler's totient function and π is the multiplicative function defined for the prime power p^r by $\pi(p^r) = -p$. There are many generalisations of Smith's determinant (1.1) and related results (see, for instance, [1–14, 16–21]). In particular, an elegant result was achieved by Hong *et al.* [8] stating that for any integer $n \geq 2$,

$$\det([i,j])_{2\leq i,j\leq n} = \Big(\prod_{k=1}^{n} \varphi(k)\pi(k)\Big) \sum_{\substack{t=1 \\ t \text{ is square free}}}^{n} \frac{t\mu(t)}{\varphi(t)},$$

where μ is the Möbius function and an integer $x \geq 1$ is called *square free* if x is not divisible by the square of any prime number.

As usual, $\mathbb{Z}$ and $|S|$ denote the ring of integers and the cardinality of the set S. Hong [9] introduced the concept of greatest-type divisor when he solved the Bourque–Ligh conjecture. For any integer $x \in S$, y is called a *greatest-type divisor* of x if

$$(y < x,\ y \mid z \mid x \text{ and } y, z \in S) \Rightarrow z \in \{y, x\}.$$

Let $G_S(x) := \{y \in S : y \text{ is a greatest-type divisor of } x \text{ in } S\}$ and let $M_n(\mathbb{Z})$ stand for the ring of $n \times n$ matrices over the integers. Bourque and Ligh [4] proved that (S) divides $[S]$ in the ring $M_n(\mathbb{Z})$ (that is, $[S] = B(S)$ or $[S] = (S)B$ for some $B \in M_n(\mathbb{Z})$) if S is FC. Hong [10] showed that such a factorisation is not true when S is gcd closed and $\max_{x\in S}\{|G_S(x)|\} = 2$. The results of Bourque–Ligh and Hong were generalised by Korkee and Haukkanen [14] and by Chen *et al.* [6]. Feng *et al.* [7], Zhao [17], Altinisik *et al.* [1] and Zhao *et al.* [18] used the concept of greatest-type divisor to characterise the gcd-closed sets S with $\max_{x\in S}\{|G_S(x)|\} \leq 3$ such that $(S^a) \mid [S^a]$ which partially solved an open problem of Hong [10].

Hong [12] investigated divisibility among power GCD matrices and among power LCM matrices. It was proved in [12] that $(S^a) \mid (S^b), (S^a) \mid [S^b]$ and $[S^a] \mid [S^b]$ if $a \mid b$ and S is a divisor chain (that is, $x_{\sigma(1)} | \cdots | x_{\sigma(n)}$ for a permutation σ of $\{1, \ldots, n\}$), and such factorisations are no longer true if $a \nmid b$ and $|S| \geq 2$. Evidently, a divisor chain is gcd closed but not conversely. Recently, Zhu [19] confirmed two conjectures of Hong raised in [12] stating that if $a \mid b$ and S is a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\} = 1$, then both the bth power GCD matrix (S^b) and the bth power LCM matrix $[S^b]$ are divisible by the ath power GCD matrix (S^a). At the end of [12], Hong also conjectured that if $a \mid b$ and $S = \{x_1, \ldots, x_n\}$ is gcd closed and $\max_{x\in S}\{|G_S(x)|\} = 1$, then $[S^a] \mid [S^b]$ in the ring $M_n(\mathbb{Z})$. Tan and Li [16] partially confirmed this conjecture by proving that $[S^a] \mid [S^b]$ in the ring $M_{|S|}(\mathbb{Z})$ if $a \mid b$ and S consists of finitely many coprime divisor chains with $1 \in S$ and that such a divisibility relation is not true if $a \nmid b$. However, the conjecture still remains open.

Our goal is to present a proof of Hong's conjecture. The main result of the paper is the following theorem.

THEOREM 1.1. *If a and b are positive integers such that $a \mid b$ and S is a gcd-closed set such that* $\max_{x\in S}\{|G_S(x)|\} = 1$, *then the ath power LCM matrix $[S^a]$ divides the bth power LCM matrix $[S^b]$ in the ring $M_{|S|}(\mathbb{Z})$.*

The proof of Theorem 1.1 is similar to that of Feng *et al.* [7] in character, but it is more complicated. This paper is organised as follows. In Section 2, we supply several preliminary lemmas needed in the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.1.

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One can easily check that for any permutation σ on the set $\{1,\ldots,n\}$, $[S^a] \mid [S^b] \Leftrightarrow [S^a_\sigma] \mid [S^b_\sigma]$, where $S_\sigma := \{x_{\sigma(1)},\ldots,x_{\sigma(n)}\}$. Without loss of any generality, we can always assume that the set $S = \{x_1,\ldots,x_n\}$ satisfies $x_1 < \cdots < x_n$.

2. Auxiliary results

In this section, we provide several lemmas that will be needed in the proof of Theorem 1.1. We begin with a result due to Hong which gives the formula for the determinant of the power LCM matrix on a gcd-closed set.

LEMMA 2.1 [11, Lemma 2.1]. *If S is gcd closed, then*

$$\det[S^a] = \prod_{k=1}^{n} x_k^{2a}\alpha_{a,k}, \tag{2.1}$$

where

$$\alpha_{a,k} := \sum_{\substack{d \mid x_k \\ d \nmid x_t,\, x_t < x_k}} \Big(\frac{1}{\xi_a} * \mu\Big)(d) \tag{2.2}$$

and $1/\xi_a$ is the arithmetic function defined for any positive integer x by $(1/\xi_a)(x) := x^{-a}$.

LEMMA 2.2 [5, Theorem 3]. *If S is a gcd-closed set and $(f((x_i,x_j)))$ is invertible, then $(f((x_i,x_j)))^{-1} = (a_{ij})$, where*

$$a_{ij} := \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{c_{ik}c_{jk}}{\delta_k}$$

with

$$\delta_k := \sum_{\substack{d \mid x_k \\ d \nmid x_t,\, x_t < x_k}} (f * \mu)(d) \quad \text{and} \quad c_{ij} := \sum_{\substack{dx_i \mid x_j \\ dx_i \nmid x_t,\, x_t < x_j}} \mu(d). \tag{2.3}$$

LEMMA 2.3 [11, Lemma 2.3]. *Let m be a positive integer. Then*

$$\sum_{d \mid m} \Big(\frac{1}{\xi_a} * \mu\Big)(d) = m^{-a}.$$

LEMMA 2.4 [7, Lemma 2.2]. *Let S be gcd closed and $\max_{x\in S}\{|G_S(x)|\} = 1$. Let $\alpha_{a,k}$ be defined as in (2.2). If $G_S(x_k) = \{x_{k_1}\}$ for $2 \le k \le |S|$, then $\alpha_{a,k} = x_k^{-a} - x_{k_1}^{-a}$.*

LEMMA 2.5. *Let S be gcd closed and $\max_{x\in S}\{|G_S(x)|\} = 1$. Let $\alpha_{a,k}$ and c_{ij} be defined as in (2.2) and (2.3), respectively. Then $[S^a]$ is nonsingular and $[S^a]^{-1} = (s_{ij})_{1\le i,j\le n}$ with*

$$s_{ij} := \frac{1}{x_i^a x_j^a} \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{c_{ik}c_{jk}}{\alpha_{a,k}}.$$

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PROOF. Since $[x_i, x_j]^a = x_i^a x_j^a/(x_i, x_j)^a$,

$$[S^a] = D\Big(\frac{1}{\xi_a}(x_i, x_j)\Big)D, \tag{2.4}$$

where $D := \mathrm{diag}(x_1^a, \ldots, x_n^a)$. By (2.1) and (2.4),

$$\det\Big(\frac{1}{\xi_a}((x_i, x_j))\Big) = \prod_{k=1}^{n} \alpha_{a,k}.$$

By Lemma 2.3, $\alpha_{a,1} = x_1^{-a}$. For $2 \le k \le n$, since $\max_{x\in S}\{|G_S(x)|\} = 1$, one may let $G_S(x_k) = \{x_{k_1}\}$. By Lemma 2.4, $\alpha_{a,k} = x_k^{-a} - x_{k_1}^{-a} \neq 0$. So the matrix $((1/\xi_a)((x_i, x_j)))$ is nonsingular. Now applying Lemma 2.2 gives

$$\Big(\frac{1}{\xi_a}((x_i, x_j))\Big)^{-1} = (h_{ij}), \tag{2.5}$$

where

$$h_{ij} := \sum_{\substack{x_i|x_k\\ x_j|x_k}} \frac{c_{ik}c_{jk}}{\alpha_{a,k}}.$$

The desired result follows immediately from (2.4) and (2.5). □

We next recall some basic results on gcd-closed sets.

LEMMA 2.6 [7, Lemma 2.3]. *Let S be a gcd-closed set with $|S| \ge 2$. Let c_{ij} be defined as in (2.3). Then*

$$c_{w1} = \begin{cases} 1 & \text{if } w = 1,\\ 0 & \text{otherwise.}\end{cases}$$

Further, if $G_S(x_m) = \{x_{m_1}\}$ for $2 \le m \le |S|$, then

$$c_{wm} = \begin{cases} -1 & \text{if } w = m_1,\\ 1 & \text{if } w = m,\\ 0 & \text{otherwise.}\end{cases}$$

LEMMA 2.7 [7, Lemma 3.1]. *Let S be gcd closed and $x, z \in S$ such that $x \nmid z$. If $G_S(x) = \{y\}$, then $(x, z) = (y, z)$.*

LEMMA 2.8. *Let S be gcd closed and $x, y \in S$ with $G_S(x) = \{y\}$. If $a \mid b$, then for any $z, r \in S$ with $r \mid x$, $y^a[z, x]^b - x^a[z, y]^b$ is divisible by each of $x^a(y^a - x^a)$ and $r^a(y^a - x^a)$.*

PROOF. We divide the proof into two cases.

Case 1: $x \nmid z$. By Lemma 2.7, $(x, z) = (y, z)$, which implies

$$y^a[z, x]^b - x^a[z, y]^b = y^a\frac{z^b x^b}{(z, x)^b} - x^a\frac{z^b y^b}{(z, y)^b} = \frac{z^b}{(z, x)^b}x^a y^a(x^{b-a} - y^{b-a}). \tag{2.6}$$

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Since $a \mid b$,

$$x^{b-a} - y^{b-a} = (x^a - y^a) \sum_{i=0}^{(b/a)-2} (x^a)^{(b/a)-2-i} y^{ai} \quad \text{and} \quad \sum_{i=0}^{(b/a)-2} (x^a)^{(b/a)-2-i} y^{ai} \in \mathbb{Z}.$$

Hence, $(x^a - y^a) \mid (x^{b-a} - y^{b-a})$. Then by (2.6), we deduce that $y^a[z,x]^b - x^a[z,y]^b$ is divisible by each of $x^a(y^a - x^a)$ and $r^a(y^a - x^a)$.

Case 2: $x \mid z$. Then $[x,z] = [y,z] = z$. It follows that

$$y^a[z,x]^b - x^a[z,y]^b = y^a z^b - x^a z^b = z^b(y^a - x^a).$$

Since $a \mid b$, the desired results follow immediately. □

LEMMA 2.9. *Let S be gcd closed and $\max_{x\in S}\{|G_S(x)|\} = 1$. If $a \mid b$, then all the elements of the nth column and the nth row of $[S^b][S^a]^{-1}$ are integers.*

PROOF. The proof of Lemma 2.9 is divided into two cases.

Case 1: $1 \le i \le n$ and $j = n$. By Lemmas 2.5 and 2.6,

$$\begin{aligned}([S^b][S^a]^{-1})_{in} &= \sum_{m=1}^{n} [x_i, x_m]^b \frac{1}{x_m^a x_n^a} \sum_{\substack{x_m \mid x_k \\ x_n \mid x_k}} \frac{c_{mk} c_{nk}}{\alpha_{a,k}} \\ &= \frac{1}{x_n^a} \sum_{m=1}^{n} \frac{[x_i, x_m]^b c_{mn}}{x_m^a \alpha_{a,n}} = \frac{1}{x_n^a \alpha_{a,n}} \sum_{m=1}^{n} \frac{[x_i, x_m]^b c_{mn}}{x_m^a}.\end{aligned}$$

Since $\max_{x\in S}\{|G_S(x)|\} = 1$, we may let $G_S(x_n) = \{x_{n_1}\}$. Then by Lemmas 2.4, 2.6 and 2.8,

$$([S^b])[S^a]^{-1})_{in} = \frac{x_{n_1}^a [x_i, x_n]^b - x_n^a [x_i, x_{n_1}]^b}{x_n^a (x_{n_1}^a - x_n^a)} \in \mathbb{Z}$$

as required.

Case 2: $i = n,\ 1 \le j \le n-1$. Then

$$([S^b][S^a]^{-1})_{nj} = \sum_{m=1}^{n} [x_n, x_m]^b \frac{1}{x_m^a x_j^a} \sum_{\substack{x_m \mid x_k \\ x_j \mid x_k}} \frac{c_{mk} c_{jk}}{\alpha_{a,k}} = \sum_{x_j \mid x_k} \frac{c_{jk}}{x_j^a \alpha_{a,k}} \sum_{x_m \mid x_k} \frac{1}{x_m^a} c_{mk} [x_m, x_n]^b.$$

We claim that

$$\gamma_k := \frac{1}{x_j^a \alpha_{a,k}} \sum_{x_m \mid x_k} \frac{1}{x_m^a} c_{mk} [x_m, x_n]^b \in \mathbb{Z}$$

for any positive integer k with $x_j \mid x_k$.

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If $k = 1$, then $m = j = 1$. In this case,

$$\gamma_1 = \frac{1}{\alpha_{a,1}} \cdot \frac{1}{x_1^{2a}} \cdot c_{11} \cdot [x_1, x_n]^b = \frac{[x_1, x_n]^b}{x_1^a} = \frac{x_1^{b-a} x_n^b}{(x_1, x_n)^b} \in \mathbb{Z}.$$

Now let $k > 1$. We can set $G_S(x_k) = \{x_{k_1}\}$ since $|G_S(x_k)| = 1$. By Lemmas 2.4, 2.6 and 2.8,

$$\gamma_k = \frac{1}{x_j^a \alpha_{a,k}} \sum_{x_m | x_k} \frac{1}{x_m^a} c_{mk} [x_m, x_n]^b = \frac{x_{k_1}^a [x_k, x_n]^b - x_k^a [x_{k_1}, x_n]^b}{x_j^a (x_{k_1}^a - x_k^a)} \in \mathbb{Z}$$

as desired. This concludes the proof of the claim and of Lemma 2.9. □

Finally, we can use Lemma 2.9 to establish the main result of this section.

Lemma 2.10. *Let S be gcd closed and $\max_{x \in S}\{|G_S(x)|\} = 1$. Let $S_1 := S \setminus \{x_n\} = \{x_1, \ldots, x_{n-1}\}$. If $a \mid b$, then $[S^b][S^a]^{-1} \in M_n(\mathbb{Z})$ if and only if $[S_1^b][S_1^a]^{-1} \in M_{n-1}(\mathbb{Z})$.*

Proof. First, it follows from the hypothesis and Lemma 2.9 that all the elements of the nth column and the nth row of $[S^b][S^a]^{-1}$ are integers. So it suffices to show that

$$\mathcal{A}_{ij} := ([S^b][S^a]^{-1})_{ij} - ([S_1^b][S_1^a]^{-1})_{ij} \in \mathbb{Z} \tag{2.7}$$

for all integers i and j with $1 \le i, j \le n - 1$.

To see this, define

$$e_{uv} := \begin{cases} 1 & \text{if } x_v \mid x_u, \\ 0 & \text{if } x_v \nmid x_u, \end{cases}$$

for all integers u and v between 1 and n. Then $e_{nj} = 1$ if $x_j \mid x_n$ and $e_{nj} = 0$ otherwise. Furthermore, for any integer m with $1 \le m \le n - 1$, one has $e_{nm} = 1$ if $x_m \mid x_n$ and $e_{nm} = 0$ otherwise. We then deduce that

$$\begin{aligned} \mathcal{A}_{ij} &= \sum_{m=1}^{n} [x_i, x_m]^b \sum_{\substack{x_m | x_k \\ x_j | x_k}} \frac{c_{mk} c_{jk}}{x_m^a x_j^a \alpha_{a,k}} - \sum_{m=1}^{n-1} [x_i, x_m]^b \sum_{\substack{x_m | x_k \\ x_j | x_k,\, x_k \neq x_n}} \frac{c_{mk} c_{jk}}{x_m^a x_j^a \alpha_{a,k}} \\ &= \frac{c_{nn} c_{jn}}{x_n^a x_j^a \alpha_{a,n}} [x_i, x_n]^b e_{nj} + \sum_{m=1}^{n-1} \frac{c_{mn} c_{jn}}{x_m^a x_j^a \alpha_{a,n}} [x_i, x_m]^b e_{nj} e_{nm} \\ &= e_{nj} \frac{c_{jn}}{x_j^a \alpha_{a,n}} \bigg(\frac{[x_i, x_n]^b}{x_n^a} + \sum_{m=1}^{n-1} \frac{[x_i, x_m]^b c_{mn} e_{nm}}{x_m^a} \bigg) := e_{nj} A_{ij}. \end{aligned} \tag{2.8}$$

Let us now show that $A_{ij} \in \mathbb{Z}$. Since $\max_{x \in S}\{|G_S(x)|\} = 1$, one may let $G_S(x_n) = \{x_{n_1}\}$. From Lemma 2.4, $\alpha_{a,n} = x_n^{-a} - x_{n_1}^{-a}$. However, by Lemma 2.6, for any integer m with $1 \le m \le n - 1$, $c_{mn} = -1$ if $m = n_1$ and $c_{mn} = 0$ otherwise. It follows from (2.8) and

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Lemma 2.8 that

$$A_{ij} = \frac{x_{n_1}^a [x_i, x_n]^b - x_n^a [x_i, x_{n_1}]^b}{x_j^a (x_{n_1}^a - x_n^a)} \cdot c_{jn} \in \mathbb{Z}. \tag{2.9}$$

Since $e_{nj} \in \{0, 1\}$, (2.8) and (2.9) yield (2.7).

The proof of Lemma 2.10 is complete. □

3. Proof of Theorem 1.1

We prove Theorem 1.1 by using induction on $n = |S|$.

For $n = 1$, the statement is clearly true.

Let $n = 2$. Since $S = \{x_1, x_2\}$ is gcd closed, $(x_1, x_2) = x_1$ and $x_1 \mid x_2$. It follows that

$$[S^b][S^a]^{-1} = \begin{pmatrix} x_1^b & x_2^b \\ x_2^b & x_2^b \end{pmatrix} \cdot \frac{1}{x_2^a (x_1^a - x_2^a)} \begin{pmatrix} x_2^a & -x_2^a \\ -x_2^a & x_1^a \end{pmatrix} = \begin{pmatrix} \mathcal{B} & -x_1^a \mathcal{C} \\ 0 & x_2^{b-a} \end{pmatrix},$$

where

$$\mathcal{B} := \frac{x_2^b - x_1^b}{x_2^a - x_1^a} \quad \text{and} \quad \mathcal{C} := \frac{x_2^{b-a} - x_1^{b-a}}{x_2^a - x_1^a}.$$

Since $a \mid b$, implying that $a \mid (b - a)$, it follows that $\mathcal{B} \in \mathbb{Z}$ and $\mathcal{C} \in \mathbb{Z}$, that is, $[S^b][S^a]^{-1} \in M_2(\mathbb{Z})$. The statement is true for this case.

Let $n = 3$. Since $S = \{x_1, x_2, x_3\}$ is gcd closed, we have $x_1 \mid x_i$ $(i = 2, 3)$ and $(x_2, x_3) = x_1$ or x_2. Consider the following two cases.

Case 1: $(x_2, x_3) = x_1$. Then one computes

$$[S^b][S^a]^{-1} = \begin{pmatrix} x_1^b & x_2^b & x_3^b \\ x_2^b & x_2^b & \dfrac{x_2^b x_3^b}{x_1^b} \\ x_3^b & \dfrac{x_2^b x_3^b}{x_1^b} & x_3^b \end{pmatrix} \cdot \frac{x_1^a}{x_2^a x_3^a (x_2^a - x_1^a)(x_3^a - x_1^a)}$$

$$\times \begin{pmatrix} \dfrac{x_1^{2a} x_2^a x_3^a - x_2^{2a} x_3^{2a}}{x_1^{2a}} & \dfrac{x_2^a x_3^{2a} - x_1^a x_2^a x_3^a}{x_1^a} & \dfrac{x_2^{2a} x_3^a - x_1^a x_2^a x_3^a}{x_1^a} \\ \dfrac{x_2^a x_3^{2a} - x_1^a x_2^a x_3^a}{x_1^a} & x_1^a x_3^a - x_3^{2a} & 0 \\ \dfrac{x_2^{2a} x_3^a - x_1^a x_2^a x_3^a}{x_1^a} & 0 & x_1^a x_2^a - x_2^{2a} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{B} + x_3^a \mathcal{F} & -x_1^a \mathcal{C} & -x_1^a \mathcal{F} \\ x_3^a \mathcal{D}\mathcal{F} & x_2^{b-a} & -x_1^a \mathcal{D}\mathcal{F} \\ x_2^a \mathcal{E}\mathcal{C} & -x_1^a \mathcal{E}\mathcal{C} & x_3^{b-a} \end{pmatrix},$$

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where $\mathcal{B}$ and $\mathcal{C}$ are as given earlier in this section, $\mathcal{D} := x_2^b/x_1^b$, $\mathcal{E} := x_3^b/x_1^b$ and $\mathcal{F} := (x_3^{b-a} - x_1^{b-a})/(x_3^a - x_1^a)$. Since $x_1 \mid x_2$, $x_1 \mid x_3$ and $a \mid (b-a)$, all of $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ are integers. Hence, $[S^b][S^a]^{-1} \in M_3(\mathbb{Z})$. The statement holds in this case.

Case 2: $(x_2, x_3) = x_2$. Then $x_2 \mid x_3$. We compute

$$
\begin{aligned}
[S^b][S^a]^{-1} &= \begin{pmatrix} x_1^b & x_2^b & x_3^b \\ x_2^b & x_2^b & x_3^b \\ x_3^b & x_3^b & x_3^b \end{pmatrix} \cdot \frac{1}{x_3^a(x_2^a - x_1^a)(x_3^a - x_2^a)} \\
&\quad \times \begin{pmatrix} x_3^a(x_2^a - x_3^a) & x_3^a(x_3^a - x_2^a) & 0 \\ x_3^a(x_3^a - x_2^a) & x_3^a(x_1^a - x_3^a) & x_3^a(x_2^a - x_1^a) \\ 0 & x_3^a(x_2^a - x_1^a) & x_2^a(x_1^a - x_2^a) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{B} & -\mathcal{B} + \mathcal{G} & -x_2^a \mathcal{H} \\ 0 & \mathcal{G} & -x_2^a \mathcal{H} \\ 0 & 0 & x_3^{b-a} \end{pmatrix},
\end{aligned}
$$

where $\mathcal{B}$ is as before, $\mathcal{G} := (x_3^b - x_2^b)/(x_3^a - x_2^a)$ and $\mathcal{H} := (x_3^{b-a} - x_2^{b-a})/(x_3^a - x_2^a)$. Since $a \mid b$ and $a \mid (b-a)$ imply that $\mathcal{G} \in \mathbb{Z}$ and $\mathcal{H} \in \mathbb{Z}$, it follows immediately that $[S^b][S^a]^{-1} \in M_3(\mathbb{Z})$. The statement is true for this case.

Now let $n \geq 4$. Assume that the statement is true for the $n-1$ case. In what follows, we show that the statement is true for the n case. Since S is gcd closed and $\max_{x \in S}\{|G_S(x)|\} = 1$, it follows that $S_1 := \{x_1, \ldots, x_{n-1}\}$ is also gcd closed and $\max_{x \in S_1}\{|G_{S_1}(x)|\} = 1$. Hence by the inductive hypothesis, $[S_1^b][S_1^a]^{-1} \in M_{n-1}(\mathbb{Z})$. Finally, from Lemma 2.10, $[S^b][S^a]^{-1} \in M_n(\mathbb{Z})$ as desired.

This finishes the proof of Theorem 1.1. □

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