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IRREDUCIBLE FACTORS OF A POLYNOMIAL

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Abstract Let (K, v) be a valued field and $\phi \in K[x]$ be any key polynomial for a residue-transcendental extension w of v to $K(x)$. In this article, using the ϕ-Newton polygon of a polynomial $f \in K[x]$ (with respect to w), we give a lower bound for the degree of an irreducible factor of f. This generalizes the result given in Jakhar and Srinivas (On the irreducible factors of a polynomial II, *J. Algebra* **556** (2020), 649–655).

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1. Introduction

The problem of finding the irreducibility of a given polynomial with integer coefficients has fascinated several mathematicians. Some well-known classical irreducibility criteria such as Schönemann–Eisenstein–Dumas irreducibility criteria have seen various extensions and generalizations over the years. Recently, in [8, 9], Jakhar and Srinivas also extended some of these criteria and provided information about the lower bounds on the degrees of irreducible factors of polynomials over valued fields. In this paper, we improve and generalize these results. To state the main results, we first recall some notation and definitions.

Let (K, v) be a valued field with value group Γ_v, valuation ring O_v having maximal ideal M_v and residue field $k_v = O_v/M_v$. An extension w of v to a simple transcendental extension $K(x)$ is called **residue-transcendental** (abbreviated r. t.) if the corresponding residue field extension $k_w|k_v$ is transcendental.

Definition 1.1. *Let $\bar{v}$ be an extension of v to a fixed algebraic closure $\overline{K}$ of K with value group $\Gamma_{\bar{v}}$. A pair (α, δ) in $\overline{K} \times \Gamma_{\bar{v}}$ is called a (K, v)-**minimal pair** if for every β in $\overline{K}$ satisfying $\bar{v}(\alpha - \beta) \geq \delta$, we have $\deg \beta \geq \deg \alpha$, where by $\deg \alpha$ we mean the degree of the extension $K(\alpha)|K$.*

An extension $\overline{w}$ of w to $\overline{K}(x)$ which is also an extension of $\bar{v}$ is called a **common extension** of w and $\bar{v}$.

For a (K, v)-minimal pair $(\alpha, \delta) \in \overline{K} \times \Gamma_{\bar{v}}$, the map $\overline{w}_{\alpha,\delta} : \overline{K}[x] \longrightarrow \Gamma_{\bar{v}}$, given by

$$\overline{w}_{\alpha,\delta}\left(\sum_{i\geq 0} c_i(x-\alpha)^i\right) := \min_{i\geq 0}\{\bar{v}(c_i) + i\delta\},\ c_i \in \overline{K},$$

is a valuation on $\overline{K}[x]$ and extends uniquely to $\overline{K}(x)$ [3, Theorem 2.2.1]. The valuation $\overline{w}_{\alpha,\delta}$ is said to be defined by min, $\bar{v}$, α and δ.

Definition 1.2. *If $\overline{w} = \overline{w}_{\alpha,\delta}$, then we say that (α, δ) is a* ***minimal pair of definition for*** *$\overline{w}$ and denote its restriction to K(x) by $w := w_{\alpha,\delta}$.*

Let $w = w_{\alpha,\delta}$ and ϕ be the minimal polynomial of α over K, then by [2, Theorem 2.1], for any polynomial $f \in K[x]$ with ϕ-expansion $\sum_i a_i\phi^i$, $a_i \in K[x]$, $\deg a_i < \deg \phi$, we have

$$w(f) = \min_i\{\bar{v}(a_i(\alpha)) + iw(\phi)\}. \tag{1.1}$$

Such a valuation w is an r. t. extension and any r. t. extension can be obtained in this way (see [2]). In particular, $w = w_{0,0}$ is called the **Gaussian extension** of v to $K(x)$ and is denoted by v^x.

For an r. t. extension w of v to $K(x)$ and polynomials f, g in $K[x]$, we say that f and g are ***w*-equivalent** if $w(f - g) > w(f) = w(g)$; g is w-**divisible** by f (denoted $f|_w g$) if there exists some polynomial $h \in K[x]$ such that g is w-equivalent to fh.

Let us now recall the definition of key polynomials, introduced by Mac Lane [11] in 1936 for discrete rank-one valuations and generalized to arbitrary valued fields by Vaquié [14] in 2007. Key polynomials have been used to classify all possible extensions of v to $K(x)$ (see [12]). Over the years, key polynomials have been extensively used to find irreducible factors of polynomials over valued fields (for example see [1], [4] and [7]).

Definition 1.3. *A monic polynomial f is called a* ***key polynomial*** *for w if it is*

(i) *w-**irreducible**, i.e. for any $h, q \in K[x]$, whenever $f \mid_w hq$, then either $f \mid_w h$ or $f \mid_w q$,*
(ii) *and w-**minimal**, i.e. for every non-zero polynomial $h \in K[x]$, whenever $f \mid_w h$, then $\deg h \geq \deg f$.*

We denote by $\mathrm{KP}(w)$ the set of all key polynomials for w. If $w = w_{\alpha,\delta}$ and ϕ the minimal polynomial of α over K, then $\phi \in \mathrm{KP}(w)$ (see [13, Theorem 4.4]).

For any $\phi \in \mathrm{KP}(w)$, consider the subset

$$\Gamma_w^\circ := \{w(a) \mid a \in K[x],\ \deg a < \deg \phi\} \subset \Gamma_w.$$

Then Γ_w° is a subgroup of Γ_w and $\Gamma_w = \langle \Gamma_w^\circ, w(\phi)\rangle$. Since w is r. t., so there is a smallest positive integer e such that $ew(\phi) \in \Gamma_w^\circ$.

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Let w be an r. t. extension of v to $K(x)$ and let Γ be the divisible closure of Γ_w. Any $\phi \in \mathrm{KP}(w)$ determines a Newton-polygon operator

$$N_{\phi,w} : K[x] \longrightarrow \mathcal{P}(\mathbb{Q} \times \Gamma),$$

where $\mathcal{P}(\mathbb{Q} \times \Gamma)$ is the power set of the rational vector space $\mathbb{Q} \times \Gamma$. For any $A \subseteq \mathbb{Q} \times \Gamma$, the set

$$L = \{(a, b) \in A \mid b \leq b' \text{ for every } (a, b') \in A\}$$

is called the **lower part** of A.

Definition 1.4. *For any non-zero polynomial $f \in K[x]$, not divisible by ϕ, with ϕ-expansion $\sum_{i=0}^{n} a_i\phi^i$, $a_n \neq 0$, the ϕ-**Newton polygon** $N_{\phi,w}(f)$ is defined as the lower part of the convex hull in $\mathbb{Q} \times \Gamma$ of the finite set $\{(i, w(a_i)) \mid a_i \neq 0,\ 0 \leq i \leq n\}$.*

Therefore, $N_{\phi,w}(f)$ is either a single point or a chain of sides (or edges), $S_1, \ldots, S_r$, ordered from left to right by increasing slopes.

Definition 1.5. *The **principal Newton polygon** $N^+_{\phi,w}(f)$ of f is the polygon formed by the sides of $N_{\phi,w}(f)$ of slope less than $-w(\phi)$.*

In 2020, Jakhar and Srinivas proved the following.

Theorem 1.6. ([8], Theorem 1.3). *Let $\phi \in O_v[x]$ be a monic polynomial of degree m which is irreducible modulo M_v. Let $f \in O_v[x]$ be a polynomial not divisible by ϕ. Assume that the ϕ-Newton polygon of f with respect to v^x has l many sides with positive slopes λ_j, $1 \leq j \leq l$. If e_j is the smallest positive integer such that $e_j\lambda_j \in \Gamma_v$ for $1 \leq j \leq l$, then f has an irreducible factor of degree at least $\max_{1\leq j\leq l}\{e_j m\}$ over K.*

Remark 1.7. It may be pointed out that any monic polynomial which is irreducible modulo M_v is a key polynomial for v^x and conversely (see [13, Theorem 4.6, Corollary 4.7]). However, if $w(\neq v^x)$ is r. t., then not every key polynomial for w is irreducible modulo M_v (see Example 1.10).

In view of Remark 1.7, Theorem 1.6 follows from our following result.

Theorem 1.8. *Let (K, v) be a valued field and w an r. t. extension of v to $K(x)$. Let $\phi \in \mathrm{KP}(w)$ be of degree m. Let $f \in K[x]$ be a polynomial not divisible by ϕ. Assume that $N^+_{\phi,w}(f)$ has l many sides with slopes $-\lambda_j < -w(\phi)$, $1 \leq j \leq l$. If e_j is the smallest positive integer such that $e_j\lambda_j \in \Gamma^\circ_w$, then f has an irreducible factor of degree at least $\max_{1\leq j\leq l}\{e_j m\}$ over K.*

For $(K, v) = (\mathbb{Q}, v_p)$ and $w = v^x$, the next result immediately follows from the above theorem, which also generalizes [5, Theorem 1.1].

Corollary 1.9. ([8], Theorem 1.5). *Let $f = \sum_{i=0}^{n} a_i x^i$ with $a_0 \neq 0$ be a polynomial over $\mathbb{Z}$. Assume that $N^+_{x,v^x}(f)$ has l many sides with slopes $-\lambda_j = -\frac{r_j}{s_j} < -v^x(x) = 0$,*

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where $gcd(r_j, s_j) = 1$, $1 \leq j \leq l$, *then f has an irreducible factor of degree at least* $\max_{1\leq j\leq l}\{s_j\}$ *over* $\mathbb{Q}$.

The following examples illustrate the importance of our result.

Example 1.10. Let $\mathbb{Q}$ be the field of rational numbers with the p-adic valuation v_p, p an odd prime, having value group $\mathbb{Z}$, valuation ring $\mathbb{Z}_{(p)}$ and maximal ideal $p\mathbb{Z}_{(p)}$. Let $w = w_{0,1/2}$ be the valuation on $\mathbb{Q}(x)$ defined by the minimal pair $(0, 1/2)$ with $\Gamma_w = \frac{1}{2}\mathbb{Z}$. Take the polynomial $\phi = x^2 + px + p$. Then in view of [13, Theorem 4.6], it can be easily shown that $\phi \in \mathrm{KP}(w)$. However, ϕ is not irreducible modulo $p\mathbb{Z}_{(p)}$. Clearly $\Gamma_w^\circ = \frac{1}{2}\mathbb{Z}$. Let

$$f = \frac{1}{p^5}\phi^7 + \left(x + \frac{1}{p^5}\right)\phi^6 + \frac{1}{p^4}\phi^5 + \frac{1}{p}\phi^3 + (px+p)\phi^2 + p^6x + p^6.$$

Then $N_{\phi,w}(f)$ has slopes

$$-\lambda_1 = -\frac{5}{2} < -\lambda_2 = -2 < -\lambda_3 = -\frac{3}{2} < -\lambda_4 = -1 < -\lambda_5 = 0.$$

Since $w(\phi) = 1$, so $N^+_{\phi,w}(f)$ has three sides with slope strictly less than $-w(\phi) = -1$, and $e_1 = 2$, $e_2 = 1$ and $e_3 = 2$. Therefore, by Theorem 1.8, f has an irreducible factor of degree at least $\max\{4, 2, 4\} = 4$ over $\mathbb{Q}$. However, Theorem 1.6 provides no information about the degrees of irreducible factors of f.

Example 1.11. Let $(\mathbb{Q}, v_p)$, w and ϕ be as in the above example. Let a, b and c be polynomials of degree at most one over $\mathbb{Z}$ with w valuation 0, 1 and 6, respectively. Take the polynomial $f = a\phi^4 + b\phi^3 + c \in \mathbb{Z}[x]$. Then $N^+_{\phi,w}(f)$ has a single side with slope $-5/3 < -w(\phi)$, which implies that $e = 3$. Hence by Theorem 1.8, f has an irreducible factor of degree at least 6 over $\mathbb{Q}$.

Our next result gives a lower bound on the degrees of all irreducible factors of f over K.

Theorem 1.12. *Let* $\phi \in K[x]$ *be a monic polynomial of degree m which is irreducible modulo* M_v. *Let* $f = \sum_{i=0}^n a_i\phi^i$, $a_0 \neq 0$, *be the* ϕ*-expansion of f with* $v^x(a_n) = 0$ *and* $v^x(a_i) > 0$ *for* $0 \leq i \leq n-1$. *Assume that* $N_{\phi,v^x}(f)$ *has l many sides with slope* $-\lambda_j$, $1 \leq j \leq l$. *If* e_j *is the smallest positive integer such that* $e_j\lambda_j \in \Gamma_v$ *for* $1 \leq j \leq l$, *then each irreducible factor of f has degree at least* $\min_{1\leq j\leq l}\{e_j m\}$ *over K.*

Remark 1.13. It may be pointed out that the above theorem extends [9, Theorem 1.2], which in turn also generalizes Eisenstein–Dumas irreducibility criterion [9, Theorem 1.5]. Further, on taking $\phi = x$ in the above theorem, if $N_{x,v^x}(f)$ has a single side, then we obtain the main result of [6].

Example 1.14. Consider the valued field $(\mathbb{Q}, v_p)$ and $\phi \in \mathbb{Q}[x]$ be a monic polynomial of degree m which is irreducible modulo $p\mathbb{Z}_{(p)}$. For any odd positive integer n, take the polynomial $f = a_{2n}\phi^{2n} + pa_n\phi^n + p^3a_0$, where a_{2n}, a_n and a_0 are polynomials over $\mathbb{Q}$

having degree at most $m-1$ with Gaussian valuation zero. Then $N_{\phi,v^x}(f)$ has slopes $-\lambda_1=-\frac{2}{n}<-\lambda_2=-\frac{1}{n}$, and therefore, by Theorem 1.12, either f is irreducible or f has two irreducible factors each of degree mn. However, if we take $p=3$ and $\phi=x^2+1$, then [9, Theorem 1.2] provides no information about the degrees of irreducible factors of f.

The following example highlights the importance of Theorems 1.8 and 1.12.

Example 1.15. Let $(\mathbb{Q},v_p)$ and ϕ be as in the above example. Let $q\geq 5$ be prime such that $2(q-p)>p\geq 3$. Then clearly, $q>p$. Now consider the polynomial $f=a_q\phi^q+pa_p\phi^p+p^3a_0$, where a_q, a_p and a_0 be polynomials over $\mathbb{Q}$ of degree at most $m-1$ with Gaussian valuation zero. Then for the vertices $(0,3)$, $(p,1)$ and $(q,0)$, $N_{\phi,v^x}(f)$ has slopes $-\lambda_1=-\frac{2}{p}<-\lambda_2=-\frac{1}{q-p}$, which in view of Theorems 1.8 and 1.12 implies that either f is irreducible or f has two irreducible factors of degree mp and $m(q-p)$.

2. Proof of Theorems 1.8 and 1.12

Let (K,v) and $(\overline{K},\bar{v})$ be as in the previous section. Let w be an r. t. extension of v to $K(x)$.

Definition 2.1. *Let $\phi\in K[x]$ be a key polynomial for a valuation w' on $K(x)$, and let $\gamma>w'(\phi)$ be an element of a totally ordered abelian group Γ containing $\Gamma_{w'}$ as an ordered subgroup. The map $w:K[x]\longrightarrow\Gamma\cup\{\infty\}$ defined by*

$$w(f):=\min_{i\geq 0}\{w'(a_i)+i\gamma\},$$

*where $\sum_{i\geq 0}a_i\phi^i$, $\deg a_i<\deg\phi$, is the ϕ-expansion of $f\in K[x]$, gives a valuation on K(x) (see [14, Section 1.1]) called the **ordinary augmentation** of w' and is denoted by $[w';\phi,\gamma]$.*

Remark 2.2. If $\phi\in O_v[x]$ is a monic polynomial of degree $m\geq 1$ such that ϕ is irreducible modulo M_v and α is a root of ϕ, then (α,δ) is a (K,v)-minimal pair for each positive $\delta\in\Gamma_{\bar{v}}$. Moreover, for any polynomial $g=\sum_{i\geq 0}a_ix^i\in K[x]$ having degree less than m, we have

$$\bar{v}(g(\alpha))=v^x(g). \tag{2.1}$$

Let (α,δ) be the minimal pair of definition for $\overline{w}$, $w=\overline{w}|_{K(x)}$, and ϕ the minimal polynomial of α having degree m over K. Keeping in mind (1.1), for any non-zero polynomial $f\in K[x]$ with ϕ-expansion $\sum_{i\geq 0}a_i\phi^i$, denote

$$I_{\phi,w}(f):=\min\{i\mid w(f)=\bar{v}(a_i(\alpha))+iw(\phi)\} \tag{2.2}$$

$$S_{\phi,w}(f):=\max\{i\mid w(f)=\bar{v}(a_i(\alpha))+iw(\phi)\}. \tag{2.3}$$

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Theorem 2.3. ([10], Lemma 2.1). *For any non-zero polynomials $f, g \in K[x]$, we have*

(i) $I_{\phi,w}(fg) = I_{\phi,w}(f) + I_{\phi,w}(g)$,
(ii) $S_{\phi,w}(fg) = S_{\phi,w}(f) + S_{\phi,w}(g)$.

The main idea in the proof of Theorems 1.8 and 1.12 is as follows: For the given valuation w (or v^x), we first construct suitable augmentations w_j with respect to ϕ and slopes λ_j and show that if λ_j is the slope of $N^+_{\phi,w}(f)$ (or $N_{\phi,v^x}(f)$) connecting the vertices $(k_{j-1}, w(a_{k_{j-1}}))$ and $(k_j, w(a_{k_j}))$ (or $(k_{j-1}, v^x(a_{k_{j-1}}))$ and $(k_j, v^x(a_{k_j}))$) then $I_{\phi,w_j}(f) = k_{j-1}$ and $S_{\phi,w_j}(f) = k_j$. On applying Theorem 2.3 on the factorization of f, we then obtain the desired lower bounds.

Proof of Theorem 1.8. Let $f = \sum_{i=0}^n a_i\phi^i$ be the ϕ-expansion of f and let

$$\{(k_0 = 0, w(a_{k_0})), (k_1, w(a_{k_1})), \ldots, (k_l, w(a_{k_l}))\}$$

denote the successive vertices corresponding to the sides of $N^+_{\phi,w}(f)$ with slopes $-\lambda_1, -\lambda_2, \ldots, -\lambda_l$. Then $k_0 < k_1 < \cdots < k_l$ and $-\lambda_1 < -\lambda_2 < \cdots < -\lambda_l < -w(\phi)$. For each $\lambda_j > w(\phi)$, $1 \le j \le l$, consider the ordinary augmentation of w, $w_j = [w; \phi, \lambda_j]$ so that $w_j(\phi) = \lambda_j$. Then

$$w_j(f) = w_j\left(\sum_{i=0}^n a_i\phi^i\right) = \min_{0\le i\le n}\{w(a_i) + i\lambda_j\}. \tag{2.4}$$

Let $I_{\phi,w_j}(f)$ and $S_{\phi,w_j}(f)$ be as in (2.2) and (2.3). Since $-\lambda_j$ is the slope of $N^+_{\phi,w}(f)$ connecting the vertices $(k_{j-1}, w(a_{k_{j-1}}))$ and $(k_j, w(a_{k_j}))$, so by definition

$$\min_{0\le i\le k_{j-1}}\left\{\frac{w(a_{k_j}) - w(a_i)}{k_j - i}\right\} \le -\lambda_j, \tag{2.5}$$

$$\min_{k_{j-1}< i\le n}\left\{\frac{w(a_i) - w(a_{k_{j-1}})}{i - k_{j-1}}\right\} \ge -\lambda_j. \tag{2.6}$$

The smallest and largest index i for which equality holds in (2.5) and (2.6) is k_{j-1} and k_j, respectively. Consequently, by (2.4), we have

$$w_j(f) = w(a_{k_j}) + k_j\lambda_j = w(a_{k_{j-1}}) + k_{j-1}\lambda_j.$$

i.e. $I_{\phi,w_j}(f) = k_{j-1}$ and $S_{\phi,w_j}(f) = k_j$. Let $f = f_1f_2\cdots f_t$ be the factorization of f into irreducible factors over K. Denote $I_{\phi,w_j}(f_r)$ by $k^{(r)}_{j-1}$ and $S_{\phi,w_j}(f_r)$ by $k^{(r)}_j$ for $1 \le r \le t$.

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Then by Theorem 2.3,

$$k_j = k_j^{(1)} + \cdots + k_j^{(t)} \text{ and } k_{j-1} = k_{j-1}^{(1)} + \cdots + k_{j-1}^{(t)}.$$

Since $k_j > k_{j-1}$, so $k_j - k_{j-1} = (k_j^{(1)} - k_{j-1}^{(1)}) + \cdots + (k_j^{(t)} - k_{j-1}^{(t)}) > 0$, implies that $k_j^{(r)} - k_{j-1}^{(r)} > 0$ for some $1 \le r \le t$. Without loss of generality, we can assume that $k_j^{(1)} - k_{j-1}^{(1)} > 0$. Let $f_1 = \sum_{u=0}^{d_1} b_u \phi^u$ be the ϕ-expansion of f_1. Then

$$w_j(f_1) = w\left(b_{k_j^{(1)}}\right) + k_j^{(1)}\lambda_j = w\left(b_{k_{j-1}^{(1)}}\right) + k_{j-1}^{(1)}\lambda_j,$$

which implies that $(k_j^{(1)} - k_{j-1}^{(1)})\lambda_j = w(b_{k_j^{(1)}}) - w(b_{k_{j-1}^{(1)}}) \in \Gamma_w^\circ$. Now e_j being the smallest positive integer such that $e_j\lambda_j \in \Gamma_w^\circ$, so $k_j^{(1)} - k_{j-1}^{(1)} \ge e_j$. Since, $S_{\phi,w_j}(f_1) = k_j^{(1)}$, so

$$\deg f_1 \ge \deg b_{k_j^{(1)}} + k_j^{(1)} m \ge k_j^{(1)} m \ge (k_j^{(1)} - k_{j-1}^{(1)})m \ge e_j m.$$

As j is arbitrary, so f has an irreducible factor of degree at least $\max_{1\le j\le l}\{e_j m\}$ over K. □

Remark 2.4. It may be pointed out that the proof of the above theorem is motivated by [8, Theorem 1.3]. However, the main difference is the presence of key polynomials and corresponding augmented valuations with their respective value groups.

Proof of Theorem 1.12. Let $(k_0 = 0, k_1, \ldots, k_{l-1}, k_l = n)$ be integers such that the successive vertices of the sides of $N_{\phi,v^x}(f)$ are given by

$$\{(k_0, v^x(a_{k_0})), (k_1, v^x(a_{k_1})), \ldots, (k_l, v^x(a_{k_l}))\}.$$

Let $-\lambda_1 < -\lambda_2 < \cdots < -\lambda_l$ denote the corresponding slopes in $N_{\phi,v^x}(f)$. In particular, $-\lambda_j = \frac{v^x(a_{k_j}) - v^x(a_{k_{j-1}})}{k_j - k_{j-1}}$, $1 \le j \le l$, i.e.

$$v^x(a_{k_j}) + k_j\lambda_j = v^x(a_{k_{j-1}}) + k_{j-1}\lambda_j. \tag{2.7}$$

For a root α of ϕ, let $\phi = c_m(x-\alpha)^m + \cdots + c_1(x-\alpha)$, $c_m = 1$ and set

$$\delta_j = \max_{1\le i\le m}\left\{\frac{\lambda_j - \bar{v}(c_i)}{i}\right\} \in \Gamma_{\bar{v}}.$$

Since $\lambda_j > 0$, $\forall j$, so (α, δ_j) is a (K, v)-minimal pair. Let $\overline{w}_{\alpha,\delta_j}$ be the valuation of $\overline{K}(x)$ defined by the pair (α, δ_j) and let $w_j := w_{\alpha,\delta_j}$ be its restriction to $K(x)$. Then by the choice of δ_j, we have

$$w_j(\phi) = \min_i \{\bar{v}(c_i) + i\delta_j\} = \lambda_j. \tag{2.8}$$

As $\deg a_i < \deg \phi$, by Remark 2.2, we have $v^x(a_i) = \bar{v}(a_i(\alpha))$ and hence $w_j(f) = \min_{1\le i\le n}\{v^x(a_i) + i\lambda_j\}$. Again as $-\lambda_j$ is the slope of $N_{\phi,v^x}(f)$ connecting the vertices $(k_{j-1}, v^x(a_{k_{j-1}}))$ and $(k_j, v^x(a_{k_j}))$, by definition

$$\min_{0\le i\le k_{j-1}} \left\{\frac{v^x(a_{k_j}) - v^x(a_i)}{k_j - i}\right\} \le -\lambda_j, \tag{2.9}$$

$$\min_{k_{j-1}<i\le n} \left\{\frac{v^x(a_i) - v^x(a_{k_{j-1}})}{i - k_{j-1}}\right\} \ge -\lambda_j. \tag{2.10}$$

The smallest index i for which equality holds in (2.9) is k_{j-1} and the largest index i for which equality holds in (2.10) is k_j. This implies that

$$w_j(f) = v^x(a_{k_{j-1}}) + k_{j-1}\lambda_j = v^x(a_{k_j}) + k_j\lambda_j. \tag{2.11}$$

Therefore, $I_{\phi,w_j}(f) = k_{j-1}$ and $S_{\phi,w_j}(f) = k_j$. Let $f = f_1 f_2 \cdots f_t$ be the factorization of f into irreducible factors over K and let $f_s = \sum_{u=0}^{d_s} b_{su}\phi^u$, $b_{sd_s} \ne 0$, be the ϕ-expansion of f_s. Fix any s, $1 \le s \le t$, and assume that for $2 \le j \le l$, $S_{\phi,w_{j-1}}(f_s) = k_{j-1}^{(s)}$, with $0 \le k_{j-1}^{(s)} \le d_s$. We claim that $I_{\phi,w_j}(f_s) = k_{j-1}^{(s)}$, and

$$w_j(f_s) = v^x\left(b_{sk_{j-1}^{(s)}}\right) + k_{j-1}^{(s)}\lambda_j = v^x\left(b_{sk_j^{(s)}}\right) + k_j^{(s)}\lambda_j,\ 1 \le j \le l, \tag{2.12}$$

with $0 \le k_{j-1}^{(s)} \le k_j^{(s)} \le d_s$ and $k_l^{(s)} = d_s$, $k_0^{(s)} = 0$.

Since $S_{\phi,w_{j-1}}(f_s) = k_{j-1}^{(s)}$, so

$$w_{j-1}(f_s) = \min_{0\le u\le d_s}\{v^x(b_{su}) + u\lambda_{j-1}\} = v^x\left(b_{sk_{j-1}^{(s)}}\right) + k_{j-1}^{(s)}\lambda_{j-1}. \tag{2.13}$$

We first show that

$$k_{j-1} = k_{j-1}^{(1)} + \cdots + k_{j-1}^{(t)}. \tag{2.14}$$

By the choice of $k_{j-1}^{(s)}$, we have

$$w_{j-1}(f_s) \le v^x(b_{su}) + u\lambda_{j-1}, \tag{2.15}$$

for $0 \le u \le d_s$ with strict inequality if $u > k_{j-1}^{(s)}$. Keeping in mind (2.15) together with the fact that k_{j-1} is the largest index for which $w_{j-1}(f) = v^x(a_{k_{j-1}}) + k_{j-1}\lambda_{j-1}$, we

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have that $k_{j-1}^{(1)} + \cdots + k_{j-1}^{(t)}$ is the largest at which $w_{j-1}(f) = w_{j-1}(f_1) + \cdots + w_{j-1}(f_t)$ is attained and hence (2.14) follows. Now on using Equations (2.11), (2.13) and (2.14), we obtain

$$v^x\left(a_{k_{j-1}}\right) = v^x\left(b_{1k_{j-1}^{(1)}}\right) + \cdots + v^x\left(b_{tk_{j-1}^{(t)}}\right). \tag{2.16}$$

Recall that k_{j-1} is the smallest at which $w_j(f) = v^x(a_{k_{j-1}}) + k_{j-1}\lambda_j$ is attained and $w_j(f_s) \leq v^x(b_{su}) + u\lambda_j$ for $0 \leq u \leq d_s$. By (2.14) and (2.16), it can be easily varified that $k_{j-1}^{(s)}$ is the smallest index at which $w_j(f_s)$ is attained which proves that $I_{\phi,w_j}(f_s) = k_{j-1}^{(s)}$. Therefore, we have for $1 \leq j \leq l$, $w_1(f_s) = v^x(b_{sk_1^{(s)}}) + k_1^{(s)}\lambda_1$,

$$w_j(f_s) = v^x\left(b_{sk_{j-1}^{(s)}}\right) + k_{j-1}^{(s)}\lambda_j = v^x\left(b_{sk_j^{(s)}}\right) + k_j^{(s)}\lambda_j,\ 2 \leq j \leq l-1, \tag{2.17}$$

$$w_l(f_s) = v^x\left(b_{sk_{l-1}^{(s)}}\right) + k_{l-1}^{(s)}\lambda_l, \tag{2.18}$$

where $0 \leq k_{j-1}^{(s)} \leq k_j^{(s)} \leq d_s$ for $2 \leq j \leq l-1$, $k_1^{(s)}$ is the largest index at which $w_1(f_s)$ is attained and $k_{l-1}^{(s)}$ is the smallest at which $w_l(f_s)$ is attained. Let $I_{\phi,w_1}(f_s) = k_0^{(s)}$ and $S_{\phi,w_l}(f_s) = k_l^{(s)}$. Clearly, $0 \leq k_0^{(s)} \leq k_1^{(s)} \leq d_s$ and $0 \leq k_{l-1}^{(s)} \leq k_l^{(s)}$. We now show that

$$k_l^{(s)} = d_s \text{ and } k_0^{(s)} = 0. \tag{2.19}$$

Since, $k_l = n$ and $v^x(a_n) = 0$, so we may assume that $v^x(b_{sd_s}) = 0$ for $1 \leq s \leq t$. By definition, we have $w_l(f_s) = \min_{0\leq u\leq d_s}\{v^x(b_{su}) + u\lambda_l\} \leq d_s\lambda_l$, with

$$w_l(f) = w_l(f_1) + \cdots + w_l(f_t). \tag{2.20}$$

Keeping in mind that $v^x(a_n) = 0$, we have $w_l(f) = n\lambda_l$. Since $n = d_1 + d_2 + \cdots + d_t$, it follows from (2.20) that $w_l(f_s) = \min_{0\leq u\leq d_s}\{v^x(b_{su}) + u\lambda_l\} = d_s\lambda_l$, which implies that $k_l^{(s)} = d_s$. Now on using (2.11), we have

$$w_0(f) = v^x(a_{k_0}) = v^x(a_0) \tag{2.21}$$

and by definition, $w_0(f_s) = \min_{0\leq u\leq d_s}\{v^x(b_{su}) + u\lambda_0\} \leq v^x(b_{s0})$ and

$$w_0(f) = w_0(f_1) + \cdots + w_0(f_t). \tag{2.22}$$

Keeping in mind that $v^x(a_0) = v^x\left(\prod_{s=1}^t b_{s0}\right) = v^x(b_{10}) + \cdots + v^x(b_{t0})$ together with (2.21) and (2.22), we obtain that $w_0(f_s) = v^x(b_{s0})$, i.e. $k_0^{(s)} = 0$, which completes the claim.

If $k_0^{(s)} = k_1^{(s)} = \cdots = k_l^{(s)}$, then $d_s = k_l^{(s)} = k_0^{(s)} = 0$ contradicting the fact that $d_s \geq 1$. Therefore, we must have that $k_j^{(s)} > k_{j-1}^{(s)}$, for some j, $1 \leq j \leq l$, which in view of (2.12) implies that $(k_j^{(s)} - k_{j-1}^{(s)})\lambda_j \in \Gamma_v$. Since e_j is the smallest positive integer such that $e_j\lambda_j \in \Gamma_v$, so $(k_j^{(s)} - k_{j-1}^{(s)}) \geq e_j$. Thus, we have

$$\deg f_s = d_s m \geq d_s m - k_{j-1}^{(s)} \geq k_j^{(s)} m - k_{j-1}^{(s)} m = (k_j^{(s)} - k_{j-1}^{(s)})m \geq e_j m.$$

Since s is arbitrary, so each irreducible factor of f has degree at least $\min_{1\leq j\leq l}\{e_j m\}$ over K. □

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References

(1) M. Alberich-Carraminãna, J. Guàrdia, E. Nart, A. Poteaux, J. Roé and M. Weimann, Polynomial factorization over Henselian fields, *Found. Comput. Math.* (2024). doi:10.1007/s10208-024-09646-x

(2) V. Alexandru, N. Popescu and A. Zaharescu, A theorem of characterization of residual transcendental extensions of a valuation, *J. Math. Kyoto Univ.* **28**(4) (1988), 579–592.

(3) A. J. Engler and A. Prestel, *Valued Fields* (Springer-Verlag, Berlin, 2005).

(4) J. Guàrdia, J. Montes and E. Nart, Newton polygons of higher order in algebraic number theory, *Trans. Amer. Math. Soc.* **364**(1) (2012), 361–416.

(5) A. Jakhar. On the irreducible factors of a polynomial, *Proc. Am. Math. Soc.* **148**(4) (2020), 1429–1437. https://www.ams.org/journals/proc/2020-148-04/home.html?active=allissues

(6) A. Jakhar, On the factors of a polynomial, *Bull. Lond. Math. Soc.* **52**(1) (2020), 158–160.

(7) A. Jakhar, S. K. Khanduja and N. Sangwan, On factorization of polynomials in Henselian valued fields, *Comm. Algebra* **46**(7) (2018), 3205–3221.

(8) A. Jakhar and K. Srinivas, On the irreducible factors of a polynomial II, *J. Algebra* **556** (2020), 649–655.

(9) A. Jakhar and K. Srinivas. On the degrees of irreducible factors of a polynomial, *Proc. Am. Math. Soc.* **150**(5) (2022), 1949–1953. https://www.ams.org/journals/proc/2022-150-05/home.html?active=allissues

(10) S. K. Khanduja and M. Kumar, Prolongations of valuations to finite extensions, *Manuscr. Math.* **131**(3-4) (2010), 323–334.

(11) S. MacLane, A construction for absolute values in polynomial rings, *Trans. Amer. Math. Soc.* **40**(3) (1936), 363–395.

(12) E. Nart, MacLane-Vaquié chains of valuations on a polynomial ring, *Pac. J. Math.* **311**(1) (2021), 165–195.

(13) L. Popescu and N. Popescu, On the residual transcendental extensions of a valuation. Key polynomials and augmented valuation, *Tsukuba J. Math.* **15**(1) (1991), 57–78.

(14) M. Vaquié, Extension d'une valuation, *Trans. Amer. Math. Soc.* **359**(7) (2007), 3439–3481.