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Regularity criterion on the energy conservation for the compressible Navier–Stokes equations

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This paper concerns the energy conservation for the weak solutions of the compressible Navier–Stokes equations. Assume that the density is positively bounded, we work on the regularity assumption on the gradient of the velocity, and establish a L^p–L^s type condition for the energy equality to hold in the distributional sense in time. We mention that no regularity assumption on the density derivative is needed any more.

Keywords: Compressible Navier–Stokes; energy conservation; weak solutions

2010 *Mathematics subject classification:* 35Q30, 76B03, 76N10

1. Introduction

The time-evolutionary compressible fluids play an important role in many fields of applications, including astrophysics (star-formation, interstellar/intergalactic medium), engineering (supersonic aircraft, gas turbines, combustion engines), and so on. In this paper, we focus on the isentropic compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \triangle u + (\mu + \lambda)\nabla \operatorname{div} u. \end{cases} \tag{1.1}$$

Here, $t > 0$, $x \in \Omega \subseteq \mathbb{R}^3$, functions $\rho(x,t)$ and $u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ are the density and the velocity fields, respectively; the pressure $P(\rho)$ is determined from the equation of state

$$P(\rho) = \rho^\gamma, \tag{1.2}$$

where $\gamma > 1$ is the adiabatic exponent; the viscosity coefficients μ and λ are constant and satisfy physical restrictions $\mu > 0$ and $2\mu + 3\lambda \geqslant 0$.

We limit ourselves to the periodic domain $\Omega = \mathbb{T}^3$, and impose equations (1.1) with the initial functions

$$(\rho,\, u)(x, t = 0) = (\rho_0,\, u_0)(x). \tag{1.3}$$

Definition 1.1. For a given $T \in (0, \infty)$, we call $(\rho,\, u)$ a weak solution in $(0, T)$ to the problem (1.1)–(1.3) if the following assertions are fulfilled:

- equations (1.1) hold in $\mathcal{D}'(0, T; \Omega)$; and
$$\rho^\gamma,\, \rho|u|^2 \in L^\infty\left(0, T; L^1(\Omega)\right), \quad \nabla u \in L^2\left(0, T; L^2(\Omega)\right) \tag{1.4}$$
- $(\rho,\, u)$ is a renormalized solution of $(1.1)_1$ in the sense of [12] by DiPerna-Lions;
- (1.3) holds in $\mathcal{D}'(\Omega)$;
- the energy inequality holds
$$\frac{d}{dt}\int_\Omega \mathcal{E}(x, t)dx + \int_\Omega \left(\mu|\nabla u|^2 + (\mu + \lambda)(\mathrm{div} u)^2\right) dx \leqslant 0 \quad \text{in} \quad \mathcal{D}'(0, T), \tag{1.5}$$
where
$$\mathcal{E}(x, t) = \frac{1}{2}\rho|u|^2 + \frac{\rho^\gamma}{\gamma - 1}. \tag{1.6}$$

Equations (1.1) is one of the most important mathematical models in continuity mechanism. Lions [26] and Feireisl *et al.* [17] proved that the problem (1.1)–(1.3) admits a weak solution in the sense of definition 1.1, as long as the adiabatic exponent $\gamma > \frac{3}{2}$. The possibility of strict inequality sign in (1.5) is mainly due to the wildness of the weak solutions. The energy dissipation is one of basic properties of equations (1.1) related to its physical origin, and it is also reminiscent of the Leray–Hopf weak solutions to the incompressible Navier–Stokes equations. Having the energy conservation in Navier–Stokes equations would rule out the possibility of interior anomalous energy dissipation (system will possess an energy balance law in vanishing viscosity limit procedure). It is very well motivated from a physical perspective, but still an open problem up to now.

Motivated from Kolmogorov's theory of turbulence (cf. [20]), relationship between regularity of weak solution and conservation of energy is the subject of the celebrated Onsager conjecture [28]: *every weak solution with spatial Hölder continuity exponent $\alpha > \frac{1}{3}$ conserves its energy, and anomalous dissipation of energy occurs when $\alpha < \frac{1}{3}$*. To date, the progress toward both directions on this conjecture are satisfactory. Please refer to the papers [2, 10, 11] and [6, 19], as well as the references therein. There are also related studies on inhomogeneous incompressible flows and compressible flows, see, e.g., [1, 7, 13, 18].

In the context of Navier–Stokes equations, the pioneering work can be traced to Lions [24] and Prodi [29], where they proved that, for homogeneous incompressible flow, the energy balance law holds true if the velocity field satisfy

$$u \in L^4(0, T; L^4(\Omega)).$$

This was reproduced by Ladyžzenskaja *et al.* [21] in the general context of parabolic equations. Serrin [30] derived a dimension-dependent regularity condition on weak

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solutions:

$$u \in L^p(0,T;L^q(\Omega)), \quad \frac{2}{p}+\frac{n}{q}\leqslant 1, \quad n<q,$$

where n is the space dimensionality. With minor changes Shinbrot [31] improved Serrin's result to

$$u \in L^p(0,T;L^q(\Omega)), \quad \frac{1}{p}+\frac{1}{q}\leqslant \frac{1}{2} \quad q>4.$$

Recently, Berselli and Chiodaroli [5] worked on the gradient of the velocity instead of velocity itself, and therefrom, they received a high level on regularity criterion. We also mention the density-dependent incompressible flow. By choosing the momentum as test function and assuming some extra restrictions on pressure, Leslie and Shvydkoy [22] proved the energy conservation for the weak solutions. Besides the results mentioned above, we also refer readers to the papers [1, 3–5, 8, 9, 14, 15, 23, 27, 33, 34] and so on.

For compressible Navier–Stokes equations, the appearance of ρ makes $\partial_t(\rho u)$ is nonlinear, and therefore some density regularity is required in using commutator estimates. In case when the viscosity is either density-dependent or constant, Yu [32] discussed the energy conservation for weak solutions of equation (1.1) in periodic domain. The approach in [32] is using the commutator estimates developed in DiPerna and Lions [12], and thus requires the integrability of the derivative of the density. We developed in [8] a global approximation technique in general bounded domain, and proved that the weak solution of (1.1) with physical boundaries conserves its energy, provided

$$\begin{cases} \sqrt{\rho}\in L^\infty(0,T;L^\infty(\Omega))\cap L^\infty(0,T;H^1(\Omega)), \\ u\in L^p(0,T;L^q(\Omega)), \quad \dfrac{2}{p}+\dfrac{3}{2q}\leqslant\dfrac{3}{4}, \quad 6\leqslant q. \end{cases} \tag{1.7}$$

It is worthy to mention that the assumption on the density assumed in (1.7) is not optimal. Recently, Nguyen *et al.* [27] proved that if the weak solutions satisfy

$$\begin{cases} 0<c_1\leqslant\rho\leqslant c_2<\infty, \quad \sup\limits_{t\in(0,T)}\sup\limits_{|h|<\epsilon}|h|^{-\frac{1}{2}}\|\rho(\cdot+h,t)-\rho(\cdot,t)\|_{L^{\frac{12}{5}}(\Omega)}<\infty, \\ u\in L^p(0,T;L^q(\Omega)), \quad \dfrac{1}{p}+\dfrac{1}{q}\leqslant\dfrac{1}{2}, \ q\geqslant 4, \end{cases} \tag{1.8}$$

then the energy balance law holds true.

Unfortunately, all the regularity criterions claimed above have yet excluded the classical weak solutions. Taking the compressible fluid flows for example, the regularity condition in (1.7) and (1.8) requires

$$\frac{2}{p}+\frac{3}{q}\leqslant\frac{3}{4}+\frac{3}{2q}\leqslant\frac{3}{4}+\frac{3}{12}=1, \tag{1.9}$$

or

$$\frac{2}{p}+\frac{3}{q}\leqslant\frac{2}{p}+\frac{2}{q}\leqslant 1+\frac{1}{q}=\frac{5}{4}. \tag{1.10}$$

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However, by interpolation, we see that the weak solution in definition 1.1 satisfies (if the density is positive bounded from below)

$$u \in L^p\left(0, T; L^q(\Omega)\right), \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2},\ q \in [2, 6]. \tag{1.11}$$

Obviously, there are still interval gaps from exponents assumptions (1.9), (1.10) to (1.11).

The goal of this paper is connected to the relationship between the energy conservation and the degree of regularity for weak solutions to (1.1)–(1.3). Assume that the density is positive bounded from above and below, we establish regularity condition on the level of the gradient of the velocity to ensure that the energy equality holds true in the distributional sense in time. We state the result in detail in the theorem below.

THEOREM 1.2. *Assume that* (ρ, u) *is a weak solution to system* (1.1)–(1.3) *in the sense of definition* 1.1. *Assume in addition that*

$$0 < \underline{\rho} \leqslant \rho(x, t) \leqslant \overline{\rho} < \infty, \quad \nabla u \in L^p(0, T; L^s), \tag{1.12}$$

where the constants $\underline{\rho}$, $\overline{\rho}$ *are given, the exponents* p *and* s *satisfy*

$$\begin{cases} \dfrac{1}{p} + \dfrac{3}{s} < 2, & \text{if} \quad \dfrac{3}{2} < s \leqslant \dfrac{9}{5}, \\ \dfrac{5}{p} + \dfrac{6}{s} < 5, & \text{if} \quad \dfrac{9}{5} < s \leqslant 3, \\ \dfrac{1}{p} + \dfrac{2}{s+2} < 1, & \text{if} \quad 3 < s < \infty. \end{cases} \tag{1.13}$$

Then, the energy is conserved in the sense of distribution in $(0, T)$, *i.e.*,

$$\frac{d}{dt}\int_\Omega \mathcal{E}(x, t)dx + \int_\Omega \left(\mu|\nabla u|^2 + (\mu + \lambda)(\mathrm{div} u)^2\right) dx = 0 \quad \text{in} \quad \mathcal{D}'(0, T), \tag{1.14}$$

where $\mathcal{E}(x, t)$ *is defined in* (1.6).

Some remarks are in order:

REMARK 1.3. In accordance with [**18**], we say that energy equality (1.14) is local in time.

REMARK 1.4. Compared with previous results in [**8**, **27**, **32**] for compressible flows, we make no any regularity assumption on the derivative of the density.

REMARK 1.5. We relaxed the regularity criterion for the energy conservation of the weak solutions. In particular, from (1.13) and the standard Sobolev embedding (in

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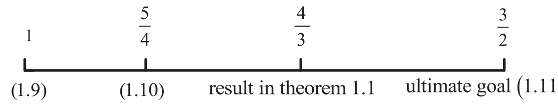

Figure 1. The regularity level

dimension three), one has (e.g., $\frac{3}{2} < s \leqslant 2$)

$$\frac{2}{p}+\frac{3}{q}=\frac{2}{p}+\frac{3}{s}-1=\begin{cases} 3-\dfrac{3}{s}\leqslant\dfrac{4}{3} & \text{if} \quad \dfrac{3}{2}<s\leqslant\dfrac{9}{5},\\ 1+\dfrac{3}{5s}<\dfrac{4}{3} & \text{if} \quad \dfrac{9}{5}<s\leqslant 2.\end{cases}$$

In this connection, the range of $2/p + 3/q$ is wider than the previous ones obtained in [**8**, **27**, **32**].

The following diagram (Fig. 1) gives an explicit understand and comparison.

Remark 1.6. Our results work for bounded domains with C^1 smooth boundaries, by slightly modifying the argument of [**8**].

Remark 1.7. It is interesting to discuss the case when the vacuum state is allowed.

The basic strategy used in theorem 1.2 is mollification approximation and commutator estimates. Observe that the nonlinearity of $\partial_t(\rho u)$ in (ρ, u) usually requires commutator in time, and therefore, the additional regularity assumption on the derivative of the density appears essential. So, new difficulty arises if the integrability of the derivative of the density is unavailable.

In the light of [**5**], we consider the energy conservation of the weak solution of (1.1)–(1.3) in terms of the gradient of the velocity fields Let us briefly analyse the proof of theorem 1.2 in paragraphs below.

Time evolution term. We first mollify the momentum equations and mass equation in space variables only, and then test them against $((\rho u)^\epsilon/\rho^\epsilon)\psi(t)$ and $(((\rho u)^\epsilon/\rho^\epsilon)^2)\psi(t)$, respectively. Doing this allows us to avoid the commutator involved in the time variable.

Convective term. In dealing with the error terms induced from the convective term, we need to guarantee that the value of $\nabla((\rho u)^\epsilon/\rho^\epsilon)$ in L^p–L^s Sobolev space depends only on ∇u, but not on $\nabla\rho$. This can be overcome by lemma 2.3, whose proof is based on the commutator technique originally due to DiPerna and Lions [**12**, **25**].

Pressure term. Since the pressure is nonlinear in density and no derivative information is known, we proceed by decomposing it as

$$\int_0^T\int(\rho^\gamma)^\epsilon \mathrm{div}\left(\frac{(\rho u)^\epsilon}{\rho^\epsilon}\right)\psi=\int_0^T\int(\rho^\epsilon)^\gamma \mathrm{div} u\psi+\text{error terms}.$$

For the right-hand side terms, we reformulate the mass equation (see (3.18) below), and therefrom, estimate the resulting expressions by properties of mollification and classical commutator estimates.

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Diffusion term. Since the test function $((\rho u)^\epsilon/\rho^\epsilon)\psi(t)$, in stead of u^ϵ, is used in momentum equations, we have difficulty in proving the convergence of $\int\int \nabla u^\epsilon : \nabla((\rho u)^\epsilon/\rho^\epsilon)$ to $\int\int |\nabla u|^2$. Fortunately, it can be overcome by a two-fold approximation procedure. See lemma 3.4 for details.

Notation: During this paper, the capital letter C denotes a positive constant which may vary from line to line and rely on $\gamma, \mu, \lambda, \underline{\rho}, \overline{\rho}, p, s$, as well as the initial functions, particularly, $C(\alpha)$ is used to emphasize that C depends on α. For $p \in [1,\infty]$, we denote by p' the conjugate of p satisfying $1 = 1/p + 1/p'$. The standard Sobolev spaces

$$W^{1,p}(\Omega) := \left\{ f : \begin{array}{l} \displaystyle\sum_{|\alpha|\leqslant 1} \left(\int_\Omega |\partial^\alpha f|^p \right)^{1/p} < \infty \text{ if } p \in [1,\infty), \\ \displaystyle\sum_{|\alpha|\leqslant 1} \operatorname*{ess\,sup}_\Omega |\partial^\alpha f| < \infty \text{ if } p = \infty \end{array} \right\}.$$

For simplicity reason, we denote by

$$W^{1,p}(\Omega) = W^{1,p},\ W^{1,2} = H^1,\ W^{0,p} = L^p, \quad \int f = \int_\Omega f \,\mathrm{d}x.$$

2. Preliminaries

Mollify function $f(x,t)$ in spatial variables with

$$f^\epsilon(x,t) = \int f(y,t)\eta_\epsilon(x-y)\,\mathrm{d}y \tag{2.1}$$

where η_ϵ is the Friedrichs mollifier of width ϵ. See, e.g., [**16**].

The first lemma below follows directly from basic properties of the mollifier.

LEMMA 2.1. *Assume that* $\rho \in L^{p_1}$, $u \in L^{p_2}$ *with* $1 \leqslant p_1, p_2 \leqslant \infty$. *Then,*

$$(\rho u)^\epsilon - \rho^\epsilon u \to 0 \quad \text{in} \quad L^{\tilde p} \quad \text{as} \quad \epsilon \downarrow 0, \tag{2.2}$$

where

$$1 \geqslant \frac{1}{\tilde p} = \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{if} \quad p_1 < \infty, \quad 1 \geqslant \frac{1}{\tilde p} > \frac{1}{p} \quad \text{if} \quad p_1 = \infty. \tag{2.3}$$

LEMMA 2.2 [**25**, lemma 2.3]. *Assume that* $\rho \in L^{p_1}$, $u \in W^{1,p_2}$ *with* $1 \leqslant p_1, p_2 \leqslant \infty$. *Then,*

$$\|\partial(\rho u)^\epsilon - \partial(\rho^\epsilon u)\|_{L^p} \leqslant C\|\rho\|_{L^{p_1}}\|\nabla u\|_{L^{p_2}}, \tag{2.4}$$

with $\partial = \partial_{x_i}$. *Moreover,*

$$\partial(\rho u)^\epsilon - \partial(\rho^\epsilon u) \to 0 \quad \text{in} \quad L^{\tilde p} \quad \text{as} \quad \epsilon \downarrow 0,$$

where the exponents p *and* $\tilde p$ *satisfy* (2.3).

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Lemma 2.3. *In addition to the hypotheses in lemma 2.2, we assume that*

$$0 < \underline{\rho} \leqslant \rho(x,t) \leqslant \overline{\rho} < \infty. \tag{2.5}$$

Then,

$$\left\| \partial \left(\frac{(\rho u)^{\epsilon}}{\rho^{\epsilon}} - u \right) \right\|_{L^p} \leqslant C \|\nabla u\|_{L^p}. \tag{2.6}$$

Moreover,

$$\partial \left(\frac{(\rho u)^{\epsilon}}{\rho^{\epsilon}} - u \right) \to 0 \quad \text{in} \quad L^{\tilde{p}} \quad \text{as} \quad \epsilon \downarrow 0, \tag{2.7}$$

where $\tilde{p}$ satisfies (2.3).

Proof. The proof is a modification of lemma 2.2. Direct computation shows

$$\partial \left(\frac{(\rho u)^{\epsilon}}{\rho^{\epsilon}} - u \right) = \frac{\partial \left((\rho u)^{\epsilon} - \rho^{\epsilon} u \right)}{\rho^{\epsilon}} - \frac{\left((\rho u)^{\epsilon} - \rho^{\epsilon} u \right) \partial \rho^{\epsilon}}{(\rho^{\epsilon})^2} := I_1 + I_2. \tag{2.8}$$

Thanks to lemma 2.2 and (2.5), one has

$$\|I_1\|_{L^p} \leqslant C \|\nabla u\|_{L^p}. \tag{2.9}$$

Next to consider I_2. From (2.5), (2.1) and Hölder inequality, we deduce

$$\begin{aligned} I_2 &= \left| \int \eta_{\epsilon}(x-y)\rho(y)(u(y)-u(x))\mathrm{d}y \frac{\int \nabla_x \eta_{\epsilon}(x-y)\rho(y)\,\mathrm{d}y}{\left(\int \eta_{\epsilon}(x-y)\rho(y)\,\mathrm{d}y\right)^2} \right| \\ &\leqslant C\epsilon^{-1} \left| \int \eta_{\epsilon}(x-y)\rho(y)(u(y)-u(x))\,\mathrm{d}y \right| \\ &\leqslant C\epsilon^{-1-n} \int_{B_{\epsilon}(x)} |\rho(y)||u(y)-u(x)|\,\mathrm{d}y \\ &\leqslant C \left(\epsilon^{-n} \int_{B_{\epsilon}(x)} \left| \frac{u(y)-u(x)}{\epsilon} \right|^p \right)^{1/p}, \end{aligned} \tag{2.10}$$

where $B_{\epsilon}(x)$ denotes the ball which centres in x with radius ϵ. However,

$$\begin{aligned} \epsilon^{-n} \int_{B_{\epsilon}(x)} \left| \frac{u(y)-u(x)}{\epsilon} \right|^p \mathrm{d}y &\leqslant \epsilon^{-n} \int_{B_{\epsilon}(x)} \int_0^1 \left| \frac{\nabla u(x+(x-y)s)\cdot(x-y)}{\epsilon} \right|^p \mathrm{d}s\,\mathrm{d}y \\ &= \int_0^1 \int_{B_1(0)} |\nabla u(x+w\epsilon s)|^p |w|^p \,\mathrm{d}w\,\mathrm{d}s \\ &\leqslant \int_0^1 \int_{B_1(0)} |\nabla u(x+w\epsilon s)|^p \,\mathrm{d}w\,\mathrm{d}s \\ &= \int_{\mathbb{R}^n} |\nabla u(x-z)|^p \int_0^1 \frac{\mathbf{1}_{B_{s\epsilon}(0)}(z)}{(s\epsilon)^n} \,\mathrm{d}s\,\mathrm{d}z \\ &= |\nabla u|^p * J, \end{aligned} \tag{2.11}$$

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where $J(z) = \int_0^1 (\mathbf{1}_{B_{s\epsilon}(0)}(z)/(s\epsilon)^n)\,\mathrm{d}s$ belongs to L^1. Therefore, by Young's inequality,

$$\|I_2\|_{L^p} \leqslant C\left\|(|\nabla u|^p * J)^{1/p}\right\|_{L^p} \leqslant C\|\nabla u\|_{L^p}\|J\|_{L^1}^{1/p} \leqslant C\|\nabla u\|_{L^p}. \tag{2.12}$$

In conclusion, (2.6) follows from (2.8)–(2.12).

We prove (2.7) by density argument. Putting $1/p_1 = 1/\tilde{p} - 1/p > 0$, we construct the smooth sequence $\{\rho_n\}$ such that $\rho_n \to \rho$ in L^{p_1} as n tends to infinity. Decompose I_1 as

$$I_1 = \frac{\partial\left(((\rho-\rho_n)u)^\epsilon - (\rho-\rho_n)^\epsilon u\right)}{\rho^\epsilon} + \frac{\partial\left((\rho_n u)^\epsilon - \rho_n^\epsilon u\right)}{\rho^\epsilon}.$$

This and (2.5) imply

$$\|I_1\|_{L^{\tilde{p}}} \leqslant C\|(\rho-\rho_n)\|_{L^{p_1}}\|\nabla u\|_{L^p} + C\|\partial\left((\rho_n u)^\epsilon - \rho_n^\epsilon u\right)\|_{L^{\tilde{p}}}.$$

Since ρ_n is smooth and $\nabla u \in L^p$, for fixed n, it has $\partial(\rho_n u)^\epsilon - \partial(\rho_n)^\epsilon u \to 0$ in $L^{\tilde{p}}$ as ϵ goes to zero, after that, sending n to infinity yields $\|(\rho-\rho_n)\|_{L^{p_1}} \to 0$. Hence, we obtain $\|I_1\|_{L^{\tilde{p}}} \to 0$ by sending ϵ to zero firstly, and then n to infinity.

We estimate I_2 as follows:

$$\begin{aligned} I_2 &= \left(((\rho-\rho_n)u)^\epsilon - (\rho-\rho_n)^\epsilon u\right)\frac{\partial\rho^\epsilon}{(\rho^\epsilon)^2} \\ &\quad + \left((\rho_n u)^\epsilon - \rho_n^\epsilon u\right)\frac{\partial(\rho-\rho_n)^\epsilon}{(\rho^\epsilon)^2} + \left((\rho_n u)^\epsilon - \rho_n^\epsilon u\right)\frac{\partial\rho_n^\epsilon}{(\rho^\epsilon)^2} \\ &:= I_2^1 + I_2^2 + I_2^3. \end{aligned}$$

For fixed n, lemma 2.1 and (2.5) guarantee

$$I_2^3 \to 0 \quad \text{in} \quad L^{\tilde{p}} \quad \text{as} \quad \epsilon \downarrow 0.$$

The first two terms can be treated as (2.10)–(2.12). In particular,

$$\begin{aligned} I_2^1 &= \left|\int \eta_\epsilon(x-y)(\rho-\rho_n)(y)(u(y)-u(x))\,\mathrm{d}y \frac{\int \nabla_x\eta_\epsilon(x-y)\rho(y)\,\mathrm{d}y}{\left(\int \eta_\epsilon(x-y)\rho(y)\,\mathrm{d}y\right)^2}\right| \\ &\leqslant \epsilon^{-1-n}\int_{B_\epsilon(x)} |(\rho-\rho_n)(y)||u(y)-u(x)|\mathrm{d}y \\ &\leqslant C\left(\epsilon^{-n}\int_{B_\epsilon(x)} |(\rho-\rho_n)(y)|^{s_1}\right)^{1/s_1}\left(\epsilon^{-n}\int_{B_\epsilon(x)}\left|\frac{u(y)-u(x)}{\epsilon}\right|^s\right)^{1/s} \\ &= C\left(|\rho-\rho_n|^{s_1} * \tilde{J}\right)^{1/s_1}(|\nabla u|^s * J)^{1/s}, \end{aligned}$$

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where $1 \leqslant s \leqslant p$, $1 \leqslant s_1 \leqslant p_1$, $1/s + 1/s_1 = 1$ and $\tilde{J} = \epsilon^{-n}\mathbf{1}_{B_\epsilon(0)} \in L^1$. Hence, by Hölder and Young's inequalities,

$$\begin{aligned}
\|I_2^1\|_{L^{\tilde{p}}} &\leqslant C \left\| \left(|\rho-\rho_n|^{s_1} * \tilde{J}\right)^{1/s_1} \right\|_{L^{p_1}} \left\| (|\nabla u|^s * J)^{1/s} \right\|_{L^p} \\
&\leqslant C \left\|(\rho-\rho_n)\right\|_{L^{p_1}} \|\nabla u\|_{L^p} \\
&\leqslant C \left\|(\rho-\rho_n)\right\|_{L^{p_1}} \to 0 \quad \text{as} \quad n \uparrow \infty, \quad \text{uniformly in } \epsilon.
\end{aligned}$$

By similar argument,

$$\begin{aligned}
I_2^2 &= \left| \int \eta_\epsilon(x-y)\rho_n(y)(u(y)-u(x))\,\mathrm{d}y \frac{\int \nabla_x \eta_\epsilon(x-y)(\rho-\rho_n)(y)\,\mathrm{d}y}{\left(\int \eta_\epsilon(x-y)\rho(y)\,\mathrm{d}y\right)^2} \right| \\
&\leqslant C \left(|\rho-\rho_n|^{s_1} * \tilde{J}\right)^{1/s_1} (|\nabla u|^s * J)^{1/s},
\end{aligned}$$

which leads to

$$\|I_2^2\|_{L^{\tilde{p}}} \leqslant C \|\rho_n - \rho\|_{L^{p_1}} \to 0 \quad \text{as} \quad n \uparrow \infty, \quad \text{uniformly in } \epsilon.$$

Therefore, $\|I_2\|_{L^{\tilde{p}}} \to 0$ as ϵ goes to zero. This proves (2.7). □

3. Proof of theorem 1.2

Define

$$U(x,t) = \frac{(\rho u)^\epsilon}{\rho^\epsilon} \quad \text{and} \quad \psi(t) \in C_0^\infty((0,T)). \tag{3.1}$$

Mollifying the mass equation and the momentum equations respectively, multiplying them by $\frac{1}{2}U^2\psi(t)$ and $U\psi(t)$, respectively, and integrating the resulting expressions, we obtain

$$\frac{1}{2}\int_0^T \int \partial_t \rho^\epsilon U^2 \psi = \frac{1}{2}\int_0^T \int (\rho u)^\epsilon \cdot \nabla U^2 \psi \tag{3.2}$$

and

$$\begin{aligned}
\int_0^T \int \partial_t (\rho u)^\epsilon U\psi &= \int_0^T \int (\rho u \otimes u)^\epsilon : \nabla U \psi + \int_0^T \int (\rho^\gamma)^\epsilon \mathrm{div}\, U\psi \\
&\quad - \int_0^T \int \left(\mu \nabla u^\epsilon : \nabla U\psi + (\mu+\lambda)\mathrm{div}\, u^\epsilon \mathrm{div}\, U\psi\right).
\end{aligned} \tag{3.3}$$

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Subtracting (3.2) from (3.3) yields

$$\begin{aligned}&\int_0^T\int\partial_t(\rho u)^\epsilon U\psi-\frac{1}{2}\int_0^T\int\partial_t\rho^\epsilon U^2\psi\\&-\int_0^T\int(\rho u\otimes u)^\epsilon:\nabla U\psi+\frac{1}{2}\int_0^T\int(\rho u)^\epsilon\cdot\nabla U^2\psi\\&-\int_0^T\int(\rho^\gamma)^\epsilon\text{div}\,U\psi\\&+\int_0^T\int(\mu\nabla u^\epsilon:\nabla U\psi+(\mu+\lambda)\text{div}\,u^\epsilon\text{div}\,U\psi)=0.\end{aligned}\tag{3.4}$$

In the following lemmas 3.1–3.4, we are devoted to proving ϵ-limit for the integral quantities in (3.4).

Lemma 3.1. *Under the assumptions listed in theorem 1.2, we have*

$$\lim_{\epsilon\downarrow 0}\left(\int_0^T\int\partial_t(\rho u)^\epsilon U\psi-\frac{1}{2}\int_0^T\int\partial_t\rho^\epsilon U^2\psi\right)=-\int_0^T\psi'\int\frac{1}{2}\rho|u|^2.\tag{3.5}$$

Proof. Owing to (3.1), we compute

$$\frac{1}{2}\int_0^T\int\partial_t\rho^\epsilon U^2\psi=-\int_0^T\int(\rho u)^\epsilon\partial_t U\psi-\frac{1}{2}\int_0^T\int\frac{|(\rho u)^\epsilon|^2}{\rho^\epsilon}\psi'$$

and

$$\int_0^T\int\partial_t(\rho u)^\epsilon U\psi=-\int_0^T\int(\rho u)^\epsilon\partial_t U\psi-\int_0^T\int\frac{|(\rho u)^\epsilon|^2}{\rho^\epsilon}\psi'.$$

The two equalities above provide us

$$\int_0^T\int\partial_t(\rho u)^\epsilon U\psi-\frac{1}{2}\int_0^T\int\partial_t\rho^\epsilon U^2\psi=-\frac{1}{2}\int_0^T\int\frac{|(\rho u)^\epsilon|^2}{\rho^\epsilon}\psi'.\tag{3.6}$$

Thanks to (1.4), (1.12) and (1.13), one has

$$u\in L^\infty(0,T;L^2)\cap L^p(0,T;L^{s^*})\tag{3.7}$$

with

$$s^*=\begin{cases}\dfrac{3s}{3-s}, & \dfrac{3}{2}<s<3,\\ \infty, & 3\leqslant s.\end{cases}\tag{3.8}$$

Utilizing (1.12), (3.7) and lemma 2.1, we easily check

$$\lim_{\epsilon\downarrow 0}\int_0^T\int\frac{|(\rho u)^\epsilon|^2}{\rho^\epsilon}\psi'=\int_0^T\psi'\int\rho|u|^2.$$

This and (3.6) yield the desire (3.5). □

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Lemma 3.2. *Under the assumptions listed in theorem 1.2, we have*

$$\lim_{\epsilon\downarrow 0}\left(\int_0^T\int(\rho u\otimes u)^\epsilon:\nabla U\psi-\frac{1}{2}\int_0^T\int(\rho u)^\epsilon\cdot\nabla U^2\psi\right)=0. \tag{3.9}$$

Proof. Direct calculation shows

$$\begin{aligned}&\int_0^T\int(\rho u\otimes u)^\epsilon:\nabla U\psi-\frac{1}{2}\int_0^T\int(\rho u)^\epsilon\cdot\nabla U^2\psi\\&=\int_0^T\int(\rho u\otimes u)^\epsilon:\nabla U\psi-\int_0^T\int(\rho u)^\epsilon\otimes U:\nabla U\psi\\&=\int_0^T\int((\rho u\otimes u)^\epsilon-(\rho u)^\epsilon\otimes u):\nabla U\psi+\int_0^T\int(\rho u)^\epsilon\otimes(u-U):\nabla U\psi\\&=I_1+I_2.\end{aligned} \tag{3.10}$$

Observe from (1.12), (1.13) and lemma 2.3 that

$$\begin{aligned}\|\nabla U\|_{L^p(0,T:L^s)}&\leqslant\|\nabla(U-u)\|_{L^p(0,T:L^s)}+\|\nabla u\|_{L^p(0,T:L^s)}\\&\leqslant C\|\nabla u\|_{L^p(0,T:L^s)}\leqslant C.\end{aligned} \tag{3.11}$$

Let us divide (3.10) into three cases, according to (1.13).

- Case I: $\frac{3}{2}<s\leqslant\frac{9}{5}$. It has $s^*\leqslant 2s'<6$, where s^* is defined in (3.8).

By (1.4), (1.12), (1.13), (3.7) and (3.11), it satisfies

$$\begin{aligned}|I_1|&\leqslant C\|(\rho u\otimes u)^\epsilon-(\rho u)^\epsilon\otimes u\|_{L^{p'}(0,T;L^{s'})}\|\nabla U\|_{L^p(0,T;L^s)}\\&\leqslant C\|(\rho u\otimes u)^\epsilon-(\rho u)^\epsilon\otimes u\|_{L^{p'}(0,T;L^{s'})}\\&\leqslant C\|u\|^2_{L^{2p'}(0,T;L^{2s'})}\\&\leqslant C\|u\|^{2\theta}_{L^{\frac{s}{2s-3}}(0,T;L^{s^*})}\|u\|^{2(1-\theta)}_{L^2(0,T;L^6)}\\&\leqslant C\|\nabla u\|^{2\theta}_{L^{\frac{s}{2s-3}}(0,T;L^s)}\|\nabla u\|^{2(1-\theta)}_{L^2(0,T;L^2)}\\&\leqslant C\|\nabla u\|^{2\theta}_{L^p(0,T;L^s)}\\&\leqslant C,\end{aligned} \tag{3.12}$$

where $\theta=(2s-3)/(3(2-s))$. Utilizing (3.12) and lemma 2.1, we obtain

$$\lim_{\epsilon\downarrow 0}I_1=0. \tag{3.13}$$

- Case II: $\frac{9}{5}<s\leqslant 3$. It has $2<2s'<s^*$, where s^* is defined in (3.8).

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From (1.12), (1.13), (3.7) and (3.11), one deduces

$$\begin{aligned} |I_1| &\leqslant C\|(\rho u\otimes u)^\epsilon-(\rho u)^\epsilon\otimes u\|_{L^{p'}(0,T;L^{s'})}\|\nabla U\|_{L^p(0,T;L^s)}\\ &\leqslant C\|(\rho u\otimes u)^\epsilon-(\rho u)^\epsilon\otimes u\|_{L^{p'}(0,T;L^{s'})}\\ &\leqslant C\|u\|^2_{L^{2p'}(0,T;L^{2s'})}\\ &\leqslant C\|u\|^{2\theta}_{L^{\frac{5s}{5s-6}}(0,T;L^{s^*})}\|u\|^{2(1-\theta)}_{L^\infty(0,T;L^2)}\\ &\leqslant C\|\nabla u\|^{2\theta}_{L^p(0,T;L^s)}\\ &\leqslant C, \end{aligned} \tag{3.14}$$

where $\theta=3/(5s-6)$. Hence, (3.14) and lemma 2.1 show that (3.13) holds true if $\frac{9}{5}<s\leqslant 3$.

- Case III: $3<s<\infty$.

Notice that

$$\begin{aligned} |I_1| &\leqslant C\|(\rho u\otimes u)^\epsilon-(\rho u)^\epsilon\otimes u\|_{L^{p'}(0,T;L^{s'})}\|\nabla U\|_{L^p(0,T;L^s)}\\ &\leqslant C\|\rho u\|_{L^{2p'}(0,T;L^{2s'})}\|u\|^2_{L^{2p'}(0,T;L^{2s'})}\\ &\leqslant C\|u\|^2_{L^{2p'}(0,T;L^{2s'})}\\ &\leqslant C\|u\|^{2\theta}_{L^{1+\frac{2}{s}}(0,T;L^\infty)}\|u\|^{2(1-\theta)}_{L^\infty(0,T;L^2)}\\ &\leqslant C\|\nabla u\|^{2\theta}_{L^p(0,T;L^s)}\\ &\leqslant C, \end{aligned} \tag{3.15}$$

where $\theta=1/s$. By (3.15) and lemma 2.1, we see that (3.13) is valid for $s>3$.

Using similar method, one deduces

$$\lim_{\epsilon\downarrow 0} I_2=0. \tag{3.16}$$

Equation (3.9) thus follows from (3.10), (3.13) and (3.16). The proof of lemma 3.2 is completed. □

Lemma 3.3. *Under the assumptions listed in theorem 1.2, we have*

$$\lim_{\epsilon\downarrow 0}\int_0^T\int(\rho^\gamma)^\epsilon \mathrm{div}\, U\psi=\int_0^T\psi'\int\frac{\rho^\gamma}{\gamma-1}. \tag{3.17}$$

Proof. Mollifying the mass equation and expressing the resultant equation as

$$\partial_t\rho^\epsilon+u\cdot\nabla\rho^\epsilon=\mathrm{div}(\rho^\epsilon u-(\rho u)^\epsilon)-\rho^\epsilon\mathrm{div}\, u. \tag{3.18}$$

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By this, we calculate

$$\begin{aligned}
&\int_0^T\int(\rho^\gamma)^\epsilon \operatorname{div} U\psi \\
&=\int_0^T\int(\rho^\epsilon)^\gamma \operatorname{div} u\psi-\int_0^T\int\left((\rho^\epsilon)^\gamma-(\rho^\gamma)^\epsilon\right)\operatorname{div} u\psi+\int_0^T\int(\rho^\gamma)^\epsilon \operatorname{div}(U-u)\psi \\
&=-\int_0^T\int(\rho^\epsilon)^{\gamma-1}\left(\partial_t\rho^\epsilon+u\cdot\nabla\rho^\epsilon\right)\psi \\
&\quad+\int_0^T\int(\rho^\epsilon)^{\gamma-1}\operatorname{div}(\rho^\epsilon u-(\rho u)^\epsilon)\psi-\int_0^T\int\left((\rho^\epsilon)^\gamma-(\rho^\gamma)^\epsilon\right)\operatorname{div} u\psi \\
&\quad+\int_0^T\int(\rho^\gamma)^\epsilon \operatorname{div}(U-u)\psi.
\end{aligned}\tag{3.19}$$

Using (3.18) once more, one has

$$\begin{aligned}
&\int_0^T\int(\rho^\epsilon)^{\gamma-1}\left(\partial_t\rho^\epsilon+u\cdot\nabla\rho^\epsilon\right)\psi \\
&=\frac{1}{\gamma-1}\int_0^T\int\rho^\epsilon\left(\partial_t(\rho^\epsilon)^{\gamma-1}+u\cdot\nabla(\rho^\epsilon)^{\gamma-1}\right)\psi \\
&=-\frac{1}{\gamma-1}\int_0^T\psi'\int(\rho^\epsilon)^\gamma-\frac{1}{\gamma-1}\int_0^T\psi\int(\rho^\epsilon)^{\gamma-1}(\partial_t\rho^\epsilon+\operatorname{div}(\rho^\epsilon u)) \\
&=-\frac{1}{\gamma-1}\int_0^T\psi'\int(\rho^\epsilon)^\gamma-\frac{1}{\gamma-1}\int_0^T\psi\int(\rho^\epsilon)^{\gamma-1}\operatorname{div}(\rho^\epsilon u-(\rho u)^\epsilon).
\end{aligned}$$

Substituting it back into (3.19) yields

$$\begin{aligned}
\int_0^T\int(\rho^\gamma)^\epsilon \operatorname{div} U\psi&=\frac{1}{\gamma-1}\int_0^T\psi'\int(\rho^\epsilon)^\gamma \\
&\quad+\frac{\gamma}{\gamma-1}\int_0^T\int(\rho^\epsilon)^{\gamma-1}\operatorname{div}(\rho^\epsilon u-(\rho u)^\epsilon)\psi \\
&\quad-\int_0^T\int\left((\rho^\epsilon)^\gamma-(\rho^\gamma)^\epsilon\right)\operatorname{div} u\psi+\int_0^T\int(\rho^\gamma)^\epsilon \operatorname{div}(U-u)\psi \\
&=\frac{1}{\gamma-1}\int_0^T\psi'\int(\rho^\epsilon)^\gamma+J_1+J_2+J_3.
\end{aligned}$$

We need to control the terms appeared on the right-hand side of (3.20).

Thanks to (1.12), one easily checks

$$\lim_{\epsilon\downarrow 0}\int_0^T\psi'\int(\rho^\epsilon)^\gamma=\int_0^T\psi'\int\rho^\gamma.\tag{3.20}$$

Let us discuss J_1 in three cases.

- Case I: $\frac{3}{2} < s \leqslant \frac{9}{5}$.

Choose a number $\underline{s} \in (\frac{3}{2}, s)$ be close to s and satisfy $\underline{s}/(2\underline{s}-3) < p$. If we select $\underline{p} \in (\underline{s}/(2\underline{s}-3)), p)$, then we see that $\underline{p}$, $\underline{s}$ still satisfy case I of (1.13). Utilizing (1.12), (1.13) and lemma 2.2, we deduce

$$\begin{aligned} |J_1| &= \left| \int_0^T \int (\rho^\epsilon)^{\gamma-1} \mathrm{div}(\rho^\epsilon u - (\rho u)^\epsilon) \right| \\ &\leqslant C \|\mathrm{div}(\rho^\epsilon u - (\rho u)^\epsilon)\|_{L^{\underline{p}}(0,T;L^{\underline{s}})} \\ &\leqslant C \|\nabla u\|_{L^{\underline{p}}(0,T;L^{\underline{s}})} \\ &\leqslant C \|\nabla u\|_{L^p(0,T;L^s)}, \end{aligned} \tag{3.21}$$

and additionally, $\mathrm{div}(\rho^\epsilon u - (\rho u)^\epsilon) \to 0$ in $L^{\underline{p}}(0, T; L^{\underline{s}})$. Therefore,

$$\lim_{\epsilon \downarrow 0} J_1 = 0. \tag{3.22}$$

- Case II: $\frac{9}{5} < s \leqslant 3$.

Choose a number $\underline{s} \in (\frac{9}{5}, s)$ be close to s and satisfy $(5\underline{s}/(5\underline{s}-6)) < p$. If we select $\underline{p} \in ((5\underline{s}/(5\underline{s}-6)), p)$, then we see that $\underline{p}$, $\underline{s}$ satisfy case II of (1.13). The same deduction as case I ensures that (3.22) is still valid.

- Case III: $3 < s < \infty$.

Choose a number $\underline{s} \in (3, s)$ be close to s and satisfy $1 + 2/\underline{s} < p$. If we select $\underline{p} \in (1 + 2/\underline{s}, p)$, then we see that $\underline{p}$, $\underline{s}$ satisfy case III of (1.13). Using the same argument as case I, we conclude (3.22).

Thanks to (1.12) and (1.13), we use similar method as in J_1 and obtain

$$\begin{aligned} \lim_{\epsilon \downarrow 0} |J_3| &= \lim_{\epsilon \downarrow 0} \left| \int_0^T \int (\rho^\gamma)^\epsilon \mathrm{div}(U - u) \right| \\ &\leqslant C \lim_{\epsilon \downarrow 0} \|\mathrm{div}(U - u)\|_{L^{\underline{p}}(0,T;L^{\underline{s}})} \\ &= 0, \end{aligned} \tag{3.23}$$

for some $\underline{s} < s$ and $\underline{p} < p$.

Next to estimate J_2. By (1.12) and (1.13), as well as the fact

$$|(\rho^\epsilon)^\gamma - \rho^\gamma| \leqslant C(\overline{\rho}, \underline{\rho}, \gamma) |\rho^\epsilon - \rho|,$$

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we deduce

$$\begin{aligned}\lim_{\epsilon\downarrow 0}|J_2| &= \lim_{\epsilon\downarrow 0}\left|\int_0^T\int ((\rho^\epsilon)^\gamma-(\rho^\gamma)^\epsilon)\,\mathrm{div}\,u\right|\\ &\leqslant \lim_{\epsilon\downarrow 0}\int_0^T\int (|(\rho^\epsilon)^\gamma-\rho^\gamma|+|\rho^\gamma-(\rho^\gamma)^\epsilon|)\,|\mathrm{div}\,u|\\ &\leqslant C\lim_{\epsilon\downarrow 0}\int_0^T\int (|\rho^\epsilon-\rho|+|\rho^\gamma-(\rho^\gamma)^\epsilon|)\,|\mathrm{div}\,u|\\ &\leqslant \lim_{\epsilon\downarrow 0}(\|\rho^\epsilon-\rho\|_{L^{p'}(0,T;L^{s'})}+\|(\rho^\gamma)^\epsilon-\rho^\gamma\|_{L^{p'}(0,T;L^{s'})})\|\nabla u\|_{L^p(0,T;L^s)}\\ &\leqslant C\lim_{\epsilon\downarrow 0}(\|\rho^\epsilon-\rho\|_{L^{p'}(0,T;L^{s'})}+\|(\rho^\gamma)^\epsilon-\rho^\gamma\|_{L^{p'}(0,T;L^{s'})})\\ &= 0.\end{aligned} \tag{3.24}$$

As a result of (3.20)–(3.24), we get the required (3.17). The proof of lemma 3.3 is finished. □

LEMMA 3.4. *Under the assumptions listed in theorem 1.2, we have*

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\int_0^T\int (\mu\nabla u^\epsilon:\nabla U\psi+(\mu+\lambda)\mathrm{div}\,u^\epsilon\mathrm{div}\,U\psi)\\ &\quad=\int_0^T\psi\int\left(\mu|\nabla u|^2+(\mu+\lambda)|\mathrm{div}\,u|^2\right).\end{aligned} \tag{3.25}$$

Proof. Define the smooth approximate sequence $\{u^n\}$ such that

$$\nabla u^n\to\nabla u \quad \text{in} \quad L^2(0,T;L^2). \tag{3.26}$$

By this, we write

$$\begin{aligned}\mu\int_0^T\int\nabla u^\epsilon:\nabla U\psi &= \mu\int_0^T\int\nabla(u^\epsilon-(u^n)^\epsilon):\nabla U\psi\\ &\quad+\mu\int_0^T\int\nabla(u^n)^\epsilon:\nabla(U-u)\psi+\mu\int_0^T\int\nabla(u^n)^\epsilon:\nabla u\psi\\ &= K_1+K_2+K_3.\end{aligned} \tag{3.27}$$

The $K_i\,(i=1\sim 3)$ are estimated as follows:

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As in lemma 3.3, we may choose exponents $\underline{s}$, $\underline{p}$ such that $\underline{s} < s$, $\underline{p} < p$ and (1.13) holds true. Then, we deduce from lemma 2.3, for fixed n,

$$\begin{aligned}|K_2| &\leqslant \left|\mu \int_0^T \int \nabla(u^n)^\epsilon : \nabla(U-u)\psi\right| \\ &\leqslant \|\nabla(u^n)^\epsilon\|_{L^{\underline{p}'}(0,T;L^{\underline{s}'})}\|\nabla(U-u)\|_{L^{\underline{p}}(0,T;L^{\underline{s}})} \\ &\leqslant C(n)\|\nabla(U-u)\|_{L^{\underline{p}}(0,T;L^{\underline{s}})} \\ &\leqslant C(n)\|\nabla u\|_{L^{\underline{p}}(0,T;L^{\underline{s}})} \\ &\leqslant C(n)\|\nabla u\|_{L^{p}(0,T;L^{s})} \\ &\leqslant C(n),\end{aligned}$$

and furthermore, $\nabla(U-u) \to 0$ in $L^{\underline{p}}(0,T;L^{\underline{s}})$ as ϵ goes to zero. Hence,

$$\lim_{\epsilon\downarrow 0} K_2 = 0. \tag{3.28}$$

Next, observe from (1.4), (1.12) and lemma 2.3 that

$$\|\nabla U\|_{L^2(0,T:L^2)} \leqslant \|\nabla(U-u)\|_{L^2(0,T:L^2)} + \|\nabla u\|_{L^2(0,T:L^2)} \leqslant C\|\nabla u\|_{L^2(0,T:L^2)} \leqslant C.$$

This together with (3.26) provide us

$$\begin{aligned}|K_1| &\leqslant \left|\mu \int_0^T \int \nabla(u^\epsilon - (u^n)^\epsilon) : \nabla U\psi\right| \\ &\leqslant C\|\nabla(u^\epsilon - (u^n)^\epsilon)\|_{L^2(0,T:L^2)}\|\nabla U\|_{L^2(0,T:L^2)} \\ &\leqslant C\|\nabla(u-u^n)\|_{L^2(0,T:L^2)} \quad \text{(uniformly in } \epsilon).\end{aligned} \tag{3.29}$$

Thus,

$$\lim_{n\uparrow\infty} K_1 = 0. \tag{3.30}$$

Finally, taking limits in ϵ and n in subsequent yields

$$K_3 = \mu \int_0^T \int \nabla(u^n)^\epsilon : \nabla u\psi \to \mu \int_0^T \int \nabla u^n : \nabla u\psi \to \mu \int_0^T \int |\nabla u|^2\psi. \tag{3.31}$$

In terms of (3.28)–(3.31), we conclude

$$\lim_{\epsilon\downarrow 0} \mu \int_0^T \int \nabla u^\epsilon : \nabla U\psi = \mu \int_0^T \int |\nabla u|^2\psi. \tag{3.32}$$

Similar argument runs that

$$(\mu+\lambda)\lim_{\epsilon\downarrow 0} \int_0^T \int \operatorname{div} u^\epsilon \operatorname{div} U\psi = (\mu+\lambda)\int_0^T \int |\operatorname{div} u|^2\psi. \tag{3.33}$$

With the aid of (3.32) and (3.33), we get (3.25). The proof of lemma 3.4 is completed. □

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Proof of (1.14). As a result of lemmas 3.1–3.4, we pass ϵ-limit in (3.4) and deduce

$$\int_0^T \left[\psi' \int \left(\frac{1}{2}\rho|u|^2 + \frac{\rho^\gamma}{\gamma - 1}\right) - \psi \int \left(\mu|\nabla u|^2 + (\mu + \lambda)|\mathrm{div}\, u|^2\right)\right] = 0.$$

This together with (1.6) give birth to the desired (1.14). The proof of theorem 1.2 is completed. □

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References

1 I. Akramov, T. Debiec, J. Skipper and E. Wiedemann. Energy conservation for the compressible Euler and Navier–Stokes equations with vacuum, arXiv: 1808.05029.

2 C. Bardos and E. S. Titi. Onsager's conjecture for the incompressible Euler equations in bounded domains. *Arch. Ration. Mech. Anal.* **228** (2018), 197–207.

3 C. Bardos and E. S. Titi. Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit. *Commun. Math. Phys.* **370** (2019), 291–310.

4 L-C. Berselli. On a regularity criterion for the solutjions to the 3D Navier–Stokes equations. *Differ. Integral Equ.* **15** (2002), 1129–1137.

5 L-C. Berselli and E. Chiodaroli. On the energy equality for the 3D Navier–Stokes equations. *Nonlinear Anal.* **192** (2020), 111704.

6 T. Buckmaster, C. De Lellis, L. Szekelyhidi, Jr.V. Vicol. Onsager's conjecture for admissible weak solutions. *Commun. Pure Appl. Math.* **72** (2019), 229–274.

7 R.-M. Chen and C. Yu. Onsager's energy conservation for inhomogeneous Euler equations. *J. Math. Pures Appl.* **131** (2019), 1–16.

8 R.-M. Chen, Z Liang, D. Wang and R. Xu. Energy equality in compressible fluids with physical boundaries. *SIAM J. Math. Anal.* **52** (2020), 1363–1385.

9 A. Cheskidov and X Luo. On the energy equality for Navier–Stokes equations in weak-in-time Onsager spaces. *Nonlinearity* **33** (2020), 1388–1403.

10 A. Cheskidov, P. Constantin, S. Friedlander and R. Shvydkoy. Energy conservation and Onsagers conjecture for the Euler equations. *Nonlinearity* **21** (2008), 1233–1252.

11 P. Constantin, E. Wiedemann and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Commun. Math. Phys.* **165** (1994), 207–209.

12 R. J. DiPerna and P.-L. Lions. Ordinary differential equations transport theory and Sobolev spaces. *Invent. Math.* **98** (1989), 511–547.

13 T. D. Drivas and G. Eyink. An Onsager singularity theorem for turbulent solutions of compressible Euler equations. *Commun. Math. Phys.* **359** (2018), 733–763.

14 T. D. Drivas and H. Q. Nguyen. Onsager's conjecture and anomalous dissipation on domains with boundary. *SIAM J. Math. Anal.* **50** (2018), 4785–4811.

15 J. Duchon and R. Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations. *Nonlinearity* **13** (2000), 249–255.

16 L. C. Evans, Ed. Partial differential equations. In *Graduate Studies in Mathematics*, vol. 19, 2nd edn, (Providence, RI: American Mathematical Society, 2010).

17 E. Feireisl, A. Novotnśy and H. Petzeltovśa. On the existence of globally defined weak solutions to the Navier–Stokes equations. *J. Math. Fluid Mech.* **3** (2001), 358–392.

18 E. Feireisl, P. Gwiazda, A. Swierczewska-Gwiazda and E. Wiedemann. Regularity and energy conservation for the compressible Euler equations. *Arch. Ration. Mech. Anal.* **223** (2017), 1375–1395.

19 P. Isett. A proof of Onsager's conjecture. *Ann. Math.* 188 (2018), 871–963.

20 A. N. Kolmogoroff. Dissipation of energy in the locally isotropic turbulence. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **32** (1941), 16–18.

https://doi.org/10.1017/prm.2020.87 Published online by Cambridge University Press

21 O. A. Ladyžzenskaja, V. A. Solonnikov and N. N. Ural'ceva, Eds. Linear and quasilinear equations of parabolic type, translated from the Russian by S. Smith. *Translations of Mathematical Monographs*, vol. 23, (Providence, RI: American Mathematical Society, 1968).

22 T. M. Leslie and R. Shvydkoy: The energy balance relation for weak solutions of the density-dependent Navier–Stokes equations, *J. Differ. Equ.* **261** (2016), 3719–3733.

23 T. M. Leslie and R. Shvydkoy: Conditions implying energy equality for weak solutions of the Navier–Stokes equations, *SIAM J. Math. Anal.* **50** (2018), 870–890.

24 J.-L. Lions. Sur la réegularitśe et l'unicitśe des solutions turbulentes des śequations de Navier–Stokes. *Rend. Sem. Mat. Univ. Padova* **30** (1960), 16–23.

25 P.-L. Lions. *Mathematical topics in fluid mechanics. Incompressible models. Oxford Lecture Series in Mathematics and its Applications*, vol. 1, (New York: 3 Oxford Science Publications, The Clarendon Press, Oxford University Press, 1996).

26 P.-L. Lions. Mathematical topics in fluid mechanics. *Compressible models. Oxford Lecture Series in Mathematics and its Applications*, vol. **2**, (New York: 10 Oxford University Press, Oxford Science Publications. The Clarendon Press, 1998).

27 Q.-H. Nguyen, P.-T. Nguyen and B Tang. Energy equalities for compressible Navier–Stokes equations. *Nonlinearity* **32** (2019), 4206–4231.

28 L. Onsager. Statistical Hydrodynamics. *Nuovo Cimento (Supplemento)* **6** (1949), 279–287.

29 G. Prodi. Un teorema di unicità per le equazioni di Navier–Stokes. *Ann. Mat. Pura Appl.* **48** (1959), 173–182.

30 J. Serrin. The initial value problem for the Navier–Stokes equations. In *Proc. Symp. 1963 Nonlinear Problems, Madison, WI*, pp. 69–98 (University of Wisconsin Press, Madison, WI, 1962).

31 M. Shinbrot. The energy equation for the Navier–Stokes system. *SIAM J. Math. Anal.* **5** (1974), 948–954.

32 C. Yu. Energy conservation for the weak solutions of the compressible Navier–Stokes equations. *Arch. Ration. Mech. Anal.* **225** (2017), 1073–1087.

33 C. Yu. A new proof to the energy conservation for the Navier–Stokes equations, arXiv: 1604.05697v1.

34 C. Yu. The energy conservation for the Navier–Stokes equations in bounded domains. arXiv:1802.07661v1.