DECOMPOSITION OF METRIC SPACES WITH A 0-DIMENSIONAL SET OF NON-DEGENERATE ELEMENTS

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1. Introduction. Various conditions under which an upper semi-continuous (u.s.-c.) decomposition of E^3 yields E^3 as its decomposition space have been given by Armentrout (**1**; **2**; **5**), Bing (**7**; **8**), Lambert (**13**), McAuley (**14**), Smythe (**17**), and Wardwell (**18**). If the projection of the non-degenerate elements is 0-dimensional in the decomposition space, then "shrinking" or "Condition B" (**6**) has proven particularly useful.

In this paper we shall investigate monotone u.s.-c. decompositions of a locally compact connected metric space M, where the projection of the non-degenerate elements is 0-dimensional. We show in Theorem 1 that each open covering of the non-degenerate elements of a 0-dimensional decomposition has a locally finite refinement.

In § 5, we use Theorem 1 to investigate the following question which is similar to one raised by Bing (**11**, p. 19): Let G, G', and G'' be decompositions of M such that the non-degenerate elements of G are those of G' together with those of G''. If G' and G'' are shrinkable, does it follow that G is shrinkable? Example 1 gives us a negative answer for the above question and the one raised by Bing. However, we obtain an affirmative answer if we impose the additional hypothesis that whenever the limiting set of a convergent sequence of non-degenerate elements of G' intersects a non-degenerate element of G'', it is that element. We also show that G' is shrinkable if G is shrinkable.

A decomposition G is shrinkable at a set X if there exists an open set A containing X such that the decomposition, whose non-degenerate elements are those of G which are subsets of A, is shrinkable. Using the above definition we obtain the following two theorems.

A decomposition G of M is shrinkable if and only if G is shrinkable at each of its non-degenerate elements (Theorem 10).

If G is a decomposition of M and A is an open set such that G is shrinkable at each non-degenerate element of G which is a subset of A, then G yields the same decomposition space as the decomposition whose non-degenerate elements are those of G which are not subsets of A (Theorem 12).

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Section 6 contains restatements of the above theorems in terms of a 3-manifold, where the shrinking requirement is replaced by the requirement that the decomposition yield a 3-manifold.

Section 7 contains a short discussion on a few results which may be obtained when it is not required that the projection of the non-degenerate elements be 0-dimensional.

For presentations of a number of fundamental results on monotone u.s.-c. decompositions of separable metric spaces, see (**16**, Chapter V) and (**19**, Chapter 7). An extensive recent bibliography on u.s.-c. decompositions of E^3 can be found in (**6**).

2. Definitions and notation. Throughout this paper, M will denote a locally compact connected metric space. If G is a decomposition of M, then M/G denotes the associated decomposition space, P denotes the projection map of M onto M/G, and $H(G)$ denotes the set of all non-degenerate elements of G.

A decomposition G is monotone if each element of G is a compact continuum. We say that G is a 0-dimensional decomposition if $P(H(G))$ is a 0-dimensional subset of M/G.

Let G and G' be decompositions of M such that if g and g' are intersecting elements of $H(G)$ and $H(G')$, respectively, then $g = g'$. Let $G + G'$ denote the decomposition of M whose non-degenerate elements are those of G together with those of G'. Whenever we use $G + G'$, we shall assume that G and G' intersect as described above. Unless otherwise specified, G, G', and $G + G'$ will denote u.s.-c. decompositions of M which are monotone and 0-dimensional.

We say that K is an open covering of $H(G)$ in M whenever K is a collection of open subsets of M such that each element of $H(G)$ is a subset of some element of K. We shall also say that K covers $H(G)$ if K is a covering of $H(G)$. For each subset X of M, let XG denote $\bigcup\{g \in G | g \subset X\}$, and let $G(X)$ denote the decomposition of M such that $H(G(X)) = \{g \in H(G) | g \subset X\}$.

A collection J of subsets of M is null if for each $\epsilon > 0$ there are only a finite number of elements of J with diameter greater than ϵ. A collection J of subsets of M is locally null if for each point of M there is an open set A containing x such that the collection of all sets of J that intersect A is a null collection.

A collection J of subsets of M satisfies Property 1 if for each compact subset X of M, the closure of the set $\bigcup\{j \in J | j \text{ intersects } X\}$ is a compact set.

If J is a collection of subsets of M and f is a function of M into a metric space, then $f(J)$ will denote the set $\{f(j) | j \in J\}$.

We say that G satisfies Condition B if for each open set A containing $\bigcup H(G)$ and each positive number ϵ, there is a homeomorphism f from M onto M such that $\text{Diam}(f(g)) < \epsilon$ for each $g \in G$ and $f(x) = x$ for each $x \in (M - A)$.

We say that G satisfies Condition B* if for each open set A containing $\bigcup H(G)$, each positive number ϵ, and each homeomorphism h from M onto M, there is a homeomorphism f from M onto M such that $\operatorname{Diam}(f(g)) < \epsilon$ for each $g \in G$ and $f(x) = h(x)$ for each $x \in (M - A)$.

Armentrout (**4**; **6**) has shown that if M is E^3, then Conditions B and B* are equivalent. We show that they are equivalent for the decompositions of metric spaces considered in this paper. Thus, we shall say that a decomposition is shrinkable if it satisfies either Condition B or Condition B*.

A decomposition G is shrinkable at an element g of G if there exists an open set A containing g such that $\operatorname{Bd}(A)$ does not intersect any element of $H(G)$, and $G(A)$ is shrinkable.

An element g of G is pointlike if $M - g$ is homeomorphic to the complement of some point in M. We say that G is a pointlike decomposition if each element of G is pointlike.

3. Decomposition and coverings. In this section we present several modifications of a given u.s.-c. decomposition such that the new decomposition is u.s.-c. Several theorems are presented on refinements of open coverings of 0-dimensional sets and decompositions. As indicated in § 2, the decompositions studied in this section are assumed to be 0-dimensional and monotone. Lemmas 1–4 seem to be well known and will be presented without proof.

LEMMA 1. *If X is a closed subset of M such that each element of G that intersects X is a subset of X, then $G(X)$ is u.s.-c.*

Remark. If A is an open subset of M whose boundary does not intersect any element of $H(G)$, then $\operatorname{Cl}(A)$ satisfies the hypothesis of Lemma 1, and $G(A)$ is the same decomposition as $G(\operatorname{Cl}(A))$.

LEMMA 2. *If f is a homeomorphism of M onto M and $f(G) = \{f(g) \mid g \in G\}$, then $f(G)$ is a 0-dimensional monotone, u.s.-c. decomposition of M.*

LEMMA 3. *The decomposition $G + G'$ is u.s.-c.*

LEMMA 4. *There exists an open cover K of M which satisfies Property* 1.

LEMMA 5. *If g is an element of $H(G)$ and A is an open set containing g, then there exists an open subset B of A containing g whose boundary does not intersect any element of $H(G)$.*

Proof. Note that $P(AG)$ is an open subset of M/G containing $P(g)$. Since $P(H(G))$ is 0-dimensional, there exists an open subset C of $P(AG)$ containing $P(g)$ whose boundary does not intersect $P(H(G))$. Then $P^{-1}(C)$ is the required set B.

LEMMA 6. *If K is an open covering of $H(G)$ in M, then there exists a countable collection K' of disjoint open sets which covers $H(G)$ and is a refinement of K.*

Proof. For each $g \in H(G)$, choose an element A of K which contains g. Apply Lemma 5 to obtain an open subset O_g of A containing g whose boundary

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does not intersect any element of $H(G)$. Since $\{O_g | g \in H(G)\}$ covers $H(G)$ in M, M is separable, and G is monotone, it follows that there is a countable subcollection $\{O_i\}$ that covers $H(G)$. Let $C_1 = O_1$, and for $i > 1$ let $C_i = O_i - \bigcup \mathrm{Cl}(O_j)$ $(1 \leqq j < i)$. Let $K' = \{C_i\}$. Then K' is a collection of disjoint open sets that refines K. For each element g of G, there is a first integer k such that $g \subset O_k$. For each $i < k$, it follows that $g \cap \mathrm{Cl}(O_i) = \emptyset$ and that $g \subset C_k$. Thus, K' is a disjoint collection of open sets which covers $H(G)$ in M.

LEMMA 7. *If A is an open set whose closure is compact and whose boundary does not intersect $\bigcup H(G)$, then there exists a countable collection K of disjoint open sets covering $H(G(A))$ such that each element B of K is a subset of A and there is an element g of $G(B)$ such that* $\mathrm{Diam}(B) < 2\,\mathrm{Diam}(g)$.

Proof. Let $J = \{g \in H(G(A)) | \mathrm{Diam}(g) \geqq 1\}$; then $\bigcup J$ is a closed subset of A, and hence is compact. For each element g of J use Lemma 5 to obtain an open subset O_g of $A \cap N(g, \mathrm{Diam}(g)/2)$ whose boundary does not intersect $\bigcup H(G)$. Since $\{O_g | g \in J\}$ is a covering of $\bigcup J$, there exists a finite subcover $O_{g_1}, \ldots, O_{g_n}$. Since each g_i is compact, it follows that there exist disjoint open sets $B_1, \ldots, B_n$ such that $g_i \subset B_i \subset O_{g_i}$ and $\mathrm{Bd}(B_i) \cap (\bigcup H(G)) = \emptyset$. Define open sets $C_1, \ldots, C_n$ as follows:

$$C_1 = O_{g_1} - \mathrm{Cl}(\bigcup B_j) \qquad (j = 2, 3, \ldots, n),$$
$$C_i = O_{g_i} - \mathrm{Cl}((\bigcup B_j) \cup (\bigcup C_k)) \qquad (1 \leqq k < i, j \neq i).$$

It now follows that $g_i \subset B_i \subset C_i \subset O_{g_i}$ and the set $K_1 = \{C_1, \ldots, C_n\}$ is a finite disjoint collection of open sets. Any point in $O_{g_1} \cup \ldots \cup O_{g_n}$ that is not in $\bigcup K_1$ is either a point of the boundary of some O_{g_i} or a point of the boundary of some B_i. Thus, K_1 covers J. Furthermore, $\mathrm{Diam}(C_i) \leqq \mathrm{Diam}(O_{g_i}) < 2\,\mathrm{Diam}(g_i)$. Proceeding inductively, assume that K_i has been chosen; then, in the above argument, replace A by

$$A_i = A - \mathrm{Cl}(\bigcup(\bigcup K_j)),$$

where j ranges from 1 to i, and J by $J_i = \{g \in H(G(A_i)) | \mathrm{Diam}(g) \geqq 1/i\}$ and obtain a finite disjoint collection K_{i+1} of open sets covering J_i. If we let $K = \bigcup_{i=1}^{\infty} K_i$, then K is a collection of disjoint open sets and for each $B \in K$, there is an element g of $G(B)$ such that $\mathrm{Diam}(B) \leqq 2\,\mathrm{Diam}(g)$. Since $\bigcup_{i=1}^{n} K_i$ covers all elements of $H(G(A))$ with diameter greater than $1/n$, it follows that K covers $H(G(A))$.

THEOREM 1. *Let K be an open covering of $H(G)$ in M. Then there exists a refinement K' of K such that*

(1) *K' is an open covering of $H(G)$ in M,*
(2) *K' is a disjoint countable collection,*
(3) *K' satisfies Property* 1, *and*
(4) *K' is a locally null collection.*

Proof. By applying Lemma 4 to M/G and then lifting the open covering by P^{-1}, we see that there exists an open covering C of $H(G)$ in M satisfying Property 1. We note that any refinement of C will also have Property 1. Hence, $C' = \{k \cap c | k \in K \text{ and } c \in C\}$ is a refinement of K satisfying Property 1 which covers $H(G)$.

Using Lemma 6, we obtain a countable refinement C'' of C' such that C'' is an open covering of $H(G)$ and the elements of C'' are disjoint.

If we let K' be a collection obtained by applying Lemma 7 to each element of C'', then K' is a refinement of K such that

(1) K' is an open covering of $H(G)$ in M,

(2) K' is a countable collection of disjoint open sets,

(3) K' satisfies Property 1, and

(4) if B is an element of K', then there exists an element g of $H(G(B))$ such that $\text{Diam}(B) < 2 \,\text{Diam}(g)$.

It also follows that K' is a locally null collection.

THEOREM 2. *Suppose that*

(1) $K = \{O_i\}$ *is a locally null collection of disjoint open subsets of* M,

(2) N *is a metric space,*

(3) $\{C_i\}$ *is a locally null collection of subsets of* N,

(4) f *is a continuous map of* $M - \cup K$ *into* N,

(5) f_i *is a continuous map of* $\text{Cl}(O_i)$ *into* $\text{Cl}(C_i)$,

(6) $f|\text{Bd}(O_i) = f_i|\text{Bd}(O_i)$, *and*

(7) $F(x) = f(x)$ *if* x *is an element of* $M - \cup K$,
$F(x) = f_i(x)$ *if* x *is an element of* O_i.

Then $F(x)$ *is a continuous map of* M *into* N.

Proof. Let $\{x_i\}$ be a sequence of points converging to a point x of M. Define a sequence $\{y_i\}$ by the following technique: (1) if $x_i \in M - \cup K$, let $y_i = x_i$; (2) if $x_i \in O_n$ and there are only finitely many j's so that $x_j \in O_n$, let y_i be an element of $\text{Bd}(O_n)$; (3) if $x_i \in O_n$ and there are infinitely many j's so that x_j is an element of O_n, let $y_i = x$.

We shall first show that $\{y_i\}$ converges to x. Let A be an open set containing x. Choose an open subset B of A containing x so that only finitely many elements of K have closures which intersect both B and $M - A$. There exists an m such that if $i > m$, then $x_i \in B$. If $i > m$, then the only y_n's that do not lie in A will be in the closure of an element of K that contains only a finite number of x_j's and intersects both B and $M - A$. Therefore, there are only a finite number of such y_n's.

If $x \in O_n$, then $\{F(x_i)\}$ converges to $F(x)$.

If $x \in M - \cup K$, then each y_i is an element of $M - \cup K$ and this implies that $F(y_i) = f(y_i)$. Thus, $\{F(y_i)\}$ converges to $F(x)$. Let A' denote an open subset of N that contains $F(x)$. There exists an open subset B' of A' and an integer k such that if $i > k$, then $\text{Cl}(C_i)$ does not intersect both B' and $M - A'$. For each O_n there exists an integer m_n such that if $i > m_n$ and x_i is an element of

$\mathrm{Cl}(O_n)$, then $f_n(x_i)$ is an element of A'. There exists an integer m such that if $i > m$, then $F(y_i)$ is an element of B'. Let $m' = \max(m, m_1, m_2, \ldots, m_k)$. Thus, if $i > m'$, it follows that $F(x_i) \in A'$, and hence that F is continuous.

LEMMA 8. *If X is a compact subset of M and K is a locally null collection of subsets of M, then there is an open set A containing X such that*

$$J = \{k \in K | k \cap A \neq \emptyset\}$$

is a null collection.

Proof. Since K is locally null, then for each point x of X we can choose an open set A_x containing x such that $K_x = \{k \in K | k \cap A_x \neq \emptyset\}$ is a null collection. There exists a finite collection $A_{x_1}, \ldots, A_{x_n}$ that covers X. If we let $A = \bigcup_{i=1}^n A_{x_i}$, then $J = \bigcup_{i=1}^n K_{x_i}$, and this is a null collection.

LEMMA 9. *If K is a locally null collection of subsets of M and*

$$P(K) = \{P(k) | \, k \in K\},$$

then $P(K)$ is a locally null collection of subsets of M/G.

Proof. Let x be an element of M/G. There exists an open set A containing x whose closure is compact. Now $P^{-1}(\mathrm{Cl}(A))$ is compact and by Lemma 8 there is an open set B containing it such that $C = \{k \in K | k \cap B \neq \emptyset\}$ is a null collection. Note that $P(C)$ contains $D = \{P(k) \in P(K) | \, P(k)$ intersects $A\}$. Assume, by way of contradiction, that D is not a null collection. Then there is a $\delta > 0$ such that there exist infinitely many elements $\{P(O_i)\}$ of D with diameter greater than 2δ, and, for each i, points x_i and y_i of $P(O_i)$ such that $x_i \in A$ and $\rho(x_i, y_i) > \delta$. Let x_i' and y_i' be points of $P^{-1}(x_i)$ and $P^{-1}(y_i)$, respectively, that are contained in O_i. We may assume, without loss of generality, that $\{x_i'\}$ converges to a point y of $P^{-1}(\mathrm{Cl}(A))$. For any $\epsilon > 0$, there are only finitely many elements of C that have a diameter larger than ϵ. This implies that $\{y_i'\}$ also converges to y. Since P is continuous, $\{P(x_i')\}$ and $\{P(y_i')\}$ both converge to $P(y)$, and this is a contradiction to the choice of $\{x_i\}$ and $\{y_i\}$. Thus, D is null, and since x was an arbitrary point of M/G, it follows that $P(K)$ is locally null.

4. Shrinking decompositions. Bing (**7**; **8**) used the shrinking of a decomposition of E^3 to show that the decomposition space was E^3. He also used the non-shrinking of a decomposition of E^3 to show (see **9**; **10**; **11**) that the decomposition space was not E^3. McAuley (**14**; **15**) extended Bing's shrinking process. In Theorem 4 of this section we use a proof similar to that of Bing (**8**, Theorem 1). Theorem 5 is similar to a lemma used by Bing (**9**, p. 497). Armentrout (**4**; **6**) has shown the equivalence of Conditions B and B* for monotone, 0-dimensional, u.s.-c. decompositions of E^n. In Theorem 3 of this section, we show the equivalence of Conditions B and B* for the decompositions of metric spaces considered in this paper. In Theorem 5, together with Theorem 4, we give another condition that is equivalent to Condition B.

THEOREM 3. *Conditions* B *and* B* *are equivalent.*

Proof. We observe that Condition B* implies Condition B by letting the homeomorphism h be the identity map of M onto M.

We shall now show that Condition B implies Condition B*. To this end, let A denote an open set containing $\bigcup H(G)$, ϵ a positive number, and h a homeomorphism of M onto M. By Theorem 1, there exists a locally null collection of disjoint open subsets $O_1, O_2, \ldots$ of A such that K covers $H(G)$ and each $\mathrm{Cl}(O_i)$ is compact. Since each homeomorphism of M forms a natural decomposition of M into points, it follows from Lemma 9 that $h(K)$ is a locally null collection of disjoint open subsets of $h(A)$.

We now wish to define, for each i, a homeomorphism h_i of $\mathrm{Cl}(O_i)$ onto $h(\mathrm{Cl}(O_i))$ such that $h|\mathrm{Bd}(O_i) = h_i|\mathrm{Bd}(O_i)$ and $\mathrm{Diam}(h_i(g)) < \epsilon$ for $g \in H(G(O_i))$. There exists a positive number δ_i such that if $X \subset \mathrm{Cl}(O_i)$ and $\mathrm{Diam}(X) < \delta_i$, then $\mathrm{Diam}(h(X)) < \epsilon$. There exists an open covering J' of $\bigcup H(G)$ such that the closure of each open set is a subset of an element of K. Let J be a refinement of J' satisfying the conclusion of Theorem 1. By Condition B, there exists a homeomorphism f_i of M onto M such that $f_i(x) = x$ for $x \in (M - \bigcup J)$ and $\mathrm{Diam}(f_i(g)) < \delta_i$ for $g \in G$. Let h_i denote hf_i restricted to $\mathrm{Cl}(O_i)$. Then h_i is a homeomorphism of $\mathrm{Cl}(O_i)$ onto $h(\mathrm{Cl}(O_i))$ with the desired properties.

With such a function defined for each O_i, let

$$f(x) = \begin{cases} h(x) & \text{if } x \in (M - \bigcup K), \\ h_i(x) & \text{if } x \in O_i. \end{cases}$$

Clearly, f is one-to-one and maps M onto itself. By applying Theorem 2 twice, we conclude that f and f^{-1} are continuous, and thus f is a homeomorphism which satisfies the requirements of Condition B*.

THEOREM 4. *If G is shrinkable and A is an open set containing $\bigcup H(G)$, then there is a homeomorphism F of M/G onto M such that*

$$P|(M - A) = F^{-1}|(M - A).$$

Proof. First we shall define a sequence $\{K_i\}$ of open coverings of $H(G)$ and a sequence of homeomorphisms $\{f_i\}$ of M onto M. Let K_0 denote an open covering of $H(G)$ such that each element is a subset of A and has a compact closure. Let f_0 be the identity map. Assume, inductively, that f_i and K_i have been chosen. Since $\bigcup K_i$ is an open set containing $\bigcup H(G)$, Condition B* may be applied with $\epsilon = 1/(i+1)$ and $h = f_i$ to obtain a homeomorphism f_{i+1} such that

(1) $f_{i+1}|(M - \bigcup K_i) = f_i|(M - \bigcup K_i)$, and

(2) $\mathrm{Diam}(f_{i+1}(g)) < 1/(i+1)$ for $g \in G$.

We shall now choose K_{i+1}. For each $g \in H(g)$, let O_g be an open set containing g such that

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(3) O_g is a subset of $P^{-1}(N(P(g), 1/(i+1)))$,

(4) $\mathrm{Cl}(O_g)$ is a subset of some element of K_i, and

(5) $\mathrm{Diam}(f_{i+1}(O_g)) < 2\,\mathrm{Diam}(f_{i+1}(g))$.

Apply Theorem 1 to $\{O_g | g \in H(G)\}$ and obtain a disjoint collection K_{i+1} of open sets that cover $H(G)$. Statements (2) and (5) imply that f_{i+1} maps each element of K_{i+1} onto a set of diameter less than $2/(i+1)$.

Since M is connected and since statement (1) implies that

$$f_i|(M - \cup K_i) = f_j|(M - \cup K_i)$$

for each $j > i$, it follows that either $f_i(x) = f_j(x)$ or they lie in elements of $f_i(K_i)$ whose boundaries intersect. Thus, it follows that

(6) $\rho(f_i(x), f_j(x)) < 4/i$ for $j > i$ and $x \in M$.

Now we define a function F' from M into M by

(7) $F'(x) = \lim_{i\to\infty} f_i(x)$,

and a function F from M/G into M by

(8) $F(g) = F'(x)$, where $x \in g \in G$.

We need to show that these two functions are well-defined. Consider any $x \in M$. The sequence $\{f_i(x)\}$ lies in a compact set. Hence, some subsequence of $\{f_i(x)\}$ converges to a point y of M, and it follows from (6) that $\{f_i(x)\}$ converges to y. Thus, $F'(x)$ is well-defined. If x and z are points of an element of G, then (2) implies that $F'(x) = F'(z)$. Therefore, $F(g)$ is well-defined.

Let $\{x_i\}$ be a sequence of points converging to a point x of M, and let ϵ be a positive number. It follows from (6) and (7) that there exists an integer j such that $\rho(f_j(y), F'(y))$ is less than $\epsilon/3$ for each point y of M. Since f_j is continuous, there exists an integer n such that $\rho(f_j(x_i), f_j(x))$ is less than $\epsilon/3$ for $i > n$. Thus, if $i > n$, then $\rho(F'(x), F'(x_i)) < \epsilon$, and thus F' is continuous. If C is an open set in $F'(M)$, then $F'^{-1}(C)$ is open. Since $P(F'^{-1}(C)) = F^{-1}(C)$, it follows that $F^{-1}(C)$ is an open subset of M/G. Therefore, F is continuous.

We have shown in the preceding three paragraphs that F is a continuous function carrying M/G into M. It remains for us to show that F is a one-to-one mapping of M/G onto M and that F^{-1} is continuous.

Let x be a point of M. For each integer i there exists a point x_i such that $f_i(x_i) = x$. The sequence $\{x_i\}$ lies in a compact set. Thus, we can assume, without loss of generality, that $\{x_i\}$ converges to a point y of M. Then $\{F'(x_i)\}$ converges to $F'(y)$. Statements (6) and (7) imply that $\rho(F'(x_i), f_i(x_i)) \leqq 4/i$, and hence that $\{F'(x_i)\}$ converges to x and that $F'(y) = x$. Let g be the element of G which contains y. Now $F(g) = F'(y) = x$, and thus F is a continuous map of M/G onto M.

If g and g' are elements of G and $g \neq g'$, then $P(g) \neq P(g')$. There exists an integer i such that $\rho(P(g), P(g')) > 9/i$. Statement (3) implies that g and g' do not lie in elements of K_i whose boundaries intersect. Thus, $F(g) \neq F(g')$, and it follows that F is one-to-one.

Let D be an open subset of M/G and let x be an element of D. There exists an integer n such that $N(x, 6/n)$ is a subset of D. Let J be the union of $N(x, 2/n)$

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and all elements of $P(K_n)$ which intersect $N(x, 2/n)$. Let J' be the union of J and all elements of $P(K_n)$ whose closures intersect $\mathrm{Cl}(J)$. It follows from (3) that J' is a subset of $N(x, 6/n)$. Notice that $F(J')$ contains $f_n(P^{-1}(J))$ which is an open set containing $F(x)$. Since x was an arbitrary point of D, it follows that $F(D)$ is an open set. Thus F^{-1} is continuous.

We have shown in the preceding paragraphs that F is a homeomorphism of M/G onto M. Since f_i restricted to $M - A$ is the identity map, it follows that F' restricted to $M - A$ is the identity map. We can also conclude that P and F^{-1} are equal when restricted to $M - A$.

THEOREM 5. *If for each open subset A of M containing $\bigcup H(G)$ there exists a homeomorphism f of* $\mathrm{Cl}(A)$ *onto* $P(\mathrm{Cl}(A))$ *such that* $f|\mathrm{Bd}(A) = P|\mathrm{Bd}(A)$, *then G is shrinkable.*

Proof. Suppose that ϵ is a positive number and that A is an open set containing $\bigcup H(G)$. If we let $K_0 = \{A\}$, then, by Theorem 1, there exists a locally null refinement K of K_0 of disjoint open sets $\{O_i\}$ covering $H(G)$ such that $\mathrm{Cl}(O_i)$ is compact for each i. Since $\bigcup K$ is an open set containing $\bigcup H(G)$, it follows from the hypothesis that there exists a homeomorphism f of $\mathrm{Cl}(\bigcup K)$ onto $P(\mathrm{Cl}(\bigcup K))$ such that $f|\mathrm{Bd}(\bigcup K) = P|\mathrm{Bd}(\bigcup K)$.

We now wish to define, for each i, a homeomorphism F_i of $\mathrm{Cl}(O_i)$ onto $\mathrm{Cl}(f^{-1}P(O_i))$ such that $F_i|\mathrm{Bd}(O_i)$ is the identity map, and $\mathrm{Diam}(F_i(g)) < \epsilon$ for $g \in G(O_i)$. There is a positive number δ_i such that f^{-1} maps each subset of $P(\mathrm{Cl}(O_i))$ with diameter less than δ_i onto a set with diameter less than ϵ. For each $g \in H(G)$, let O_g be an open set containing g such that $\mathrm{Diam}(P(O_g)) < \delta_i$ and $\mathrm{Cl}(O_g)$ is a subset of an element of K. By Theorem 1, there exists a refinement J of $\{O_g | \, g \in H(G)\}$ which is a disjoint collection of open sets covering $H(G)$. Since $\bigcup J$ is an open set containing $\bigcup H(G)$, it follows from the hypothesis that there exists a homeomorphism h of $\mathrm{Cl}(\bigcup J)$ onto $P(\mathrm{Cl}(\bigcup J))$ such that $h|\mathrm{Bd}(\bigcup J) = P|\mathrm{Bd}(\bigcup J)$. We observe that if C is an element of J which is a subset of O_i, then $\mathrm{Diam}(f^{-1}h(C)) < \epsilon$. Let

$$F_i(x) = \begin{cases} f^{-1}P(x) & \text{if } x \in (\mathrm{Cl}(O_i) - \bigcup J), \\ f^{-1}h(x) & \text{if } x \in (O_i \cap (\bigcup J)). \end{cases}$$

Thus, $\mathrm{Diam}(F_i(g)) < \epsilon$ for each $g \in G(O_i)$. Since P and h are equal on the $\mathrm{Bd}(\bigcup J)$, it follows that F_i is the required homeomorphism.

With such a function defined for each O_i, let

$$F(x) = \begin{cases} x & \text{if } x \in (M - \bigcup K) \\ F_i(x) & \text{if } x \in O_i. \end{cases}$$

It follows, by applying Theorem 2 twice, that $F(x)$ is a homeomorphism of M onto M. Since K covers $H(G)$, it also follows that $\mathrm{Diam}(F(g)) < \epsilon$ for each $g \in H(G)$. Since $M - A$ is a subset of $M - \bigcup K$, it follows that F is the identity map when restricted to $M - A$. Thus, F is a homeomorphism which satisfies the requirements of Condition B.

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Remark. We note that Theorems 4 and 5 could be combined into the following form: G is shrinkable if and only if for each open set A containing $\bigcup H(G)$, there exists a homeomorphism f of $\mathrm{Cl}(A)$ onto $P(\mathrm{Cl}(A))$ such that $f|\mathrm{Bd}A = P|\mathrm{Bd}A$.

5. Adding decompositions. Bing (**11**, p. 19) raised the following question: "Let G_i ($i = 1, 2$) be a pointlike decomposition of E^3 such that each G_i yields E^3 and if a non-degenerate element of G_1 intersects a non-degenerate element of G_2, then the elements are the same. Let G be the decomposition of E^3 whose non-degenerate elements are the non-degenerate elements of G_1 and G_2. Does G yield E^3?".

Example 1 of this section provides a negative answer to Bing's question. Theorem 6 is also related to Bing's question and is an affirmative answer obtained by imposing an additional condition on the decompositions G_1 and G_2.

Lambert (**13**) used a result of Armentrout (**2**, Theorem 1) to obtain theorems similar to Theorems 7 and 9 of this section. Lambert had the additional hypothesis that G was a pointlike decomposition of S^3 such that $\mathrm{Cl}(P(\bigcup H(G)))$ was a compact 0-dimensional set.

Example 1. There exist pointlike decompositions G and G' of E^3 such that each decomposition yields E^3 and no non-degenerate element of G intersects a non-degenerate element of G', but $G + G'$ does not yield E^3.

In Bing's paper on pointlike decompositions of E^3 (see **10**), an example of a toroidal decomposition of E^3 is given so that the decomposition space is not E^3. There are two planes such that each non-degenerate element of the decomposition lies in one of them. Let G and G' denote decompositions of E^3 such that $H(G)$ is the set of non-degenerate elements in one plane and $H(G')$ those in the other plane. Thus, $G + G'$ is the decomposition described. It follows from a result of Dyer and Hamstrom (**12**) that G yields E^3 if $H(G)$ lies in a plane. Thus, G and G' yield E^3, but $G + G'$ does not yield E^3.

For the theorems in this section, the decompositions G and G' will be as defined in § 2.

THEOREM 6. *If G and G' are shrinkable, and, for any sequence of elements of $H(G')$ converging to a set X, either $X \in G$ or X does not intersect any non-degenerate element of G, then $G + G'$ is shrinkable.*

Proof. Assume that ϵ is a positive number and that A is an open set containing $\bigcup H(G + G')$. It follows that A also contains $\bigcup H(G)$ and $\bigcup H(G')$. Since G' is shrinkable, there exists a homeomorphism f of M onto M such that $f|(M - A)$ is the identity map and $\mathrm{Diam}(f(g')) \leqq \epsilon/2$ for each $g' \in G'$.

Let $J = \{g \in G | \mathrm{Diam}(f(g)) \geqq \epsilon\}$. Since $f(G)$ is u.s.-c., it follows that $\bigcup f(J)$ and $\bigcup J$ are closed subsets of M. It follows from the hypothesis that, for each $g \in J$, there exists an open subset O_g of A containing g such that O_g does not intersect any non-degenerate element of G'. If $g \in H(G)$ and g is not an element of J, then, since g is compact and $\bigcup J$ is closed, it follows that there exists an open subset O_g of A containing g such that O_g does not intersect any element of J. It follows from Theorem 1 that there exists a refinement K of $\{O_g | g \in H(G)\}$ such that K is a disjoint collection of open sets covering $H(G)$. It follows that $\bigcup K$ is an open set containing $\bigcup H(G)$, and for each $g \in J$ the component of $\bigcup K$ containing g does not intersect any element of $H(G')$. Let A' denote the union of all components of $\bigcup K$ that intersect $\bigcup J$. It follows from Condition B* that there is a homeomorphism h of M onto M such that $h|(M - \bigcup K) = f|(M - \bigcup K)$ and $\operatorname{Diam}(h(g)) < \epsilon$ for each $g \in G$. Let

$$F(x) = \begin{cases} h(x) & \text{if } x \in A', \\ f(x) & \text{if } x \in (M - A'). \end{cases}$$

Since $h|\operatorname{Bd}(A') = f|\operatorname{Bd}(A')$, it follows that F is a homeomorphism of M onto M. If $g \in (G + G')$ and g is a subset of A', then $g \in G$ and

$$\operatorname{Diam}(F(g)) = \operatorname{Diam}(h(g)) < \epsilon.$$

If $g \in (G + G')$ and g is not a subset of A', then $g \notin J$ and

$$\operatorname{Diam}(F(g)) = \operatorname{Diam}(f(g)) < \epsilon.$$

Since $f|(M - A)$ is the identity map and $F|(M - A) = f|(M - A)$, it follows that F is the homeomorphism required in Condition B.

THEOREM 7. *If G is shrinkable and $H(G) \supset H(G')$, then G' is shrinkable.*

Proof. Let ϵ be a positive number and let A be an open set containing $\bigcup H(G')$. Let $J = \{g' \in G' | \operatorname{Diam}(g') \geqq \epsilon\}$. Since $\bigcup J$ is closed, it follows that $\{A, (M - \bigcup J)\}$ is an open covering of $H(G)$. By Theorem 1, there exists a refinement K of $\{A, (M - \bigcup J)\}$ such that K is a disjoint collection of open sets covering $H(G)$. It follows that the union A' of all components of $\bigcup K$ which intersect $\bigcup J$ is a subset of A. Since G is shrinkable, there exists a homeomorphism f of M onto M such that $f|(M - \bigcup K)$ is the identity map and $\operatorname{Diam}(f(g)) < \epsilon$ for each $g \in G$. Let

$$F(x) = \begin{cases} f(x) & \text{if } x \in A', \\ x & \text{if } x \in (M - A'). \end{cases}$$

Since $f|\operatorname{Bd}(A')$ is the identity map, it follows that F is a homeomorphism of M onto M. If $g' \in G'$, then either g' is a subset of A' and

$$\operatorname{Diam}(F(g')) = \operatorname{Diam}(f(g')) < \epsilon$$

or g' is neither a subset of A' nor an element of J and

$$\operatorname{Diam}(F(g')) = \operatorname{Diam}(g') < \epsilon.$$

Thus, G' satisfies Condition B.

THEOREM 8. *If $H(G) \supset H(G')$, g is an element of G and of G', and G is shrinkable at g, then G' is shrinkable at g.*

Proof. Let A be an open set containing g such that $\mathrm{Bd}(A)$ does not intersect any non-degenerate element of G and $G(A)$ satisfies Condition B. Then $G(A)$ and $G'(A)$ satisfy the hypothesis of Theorem 7.

THEOREM 9. *If $g \in G$, $g \in G'$, $H(G) \supset H(G')$, and G' is not shrinkable at g, then G is not shrinkable at g.*

Proof. This theorem is a corollary to Theorem 8.

THEOREM 10. *The decomposition G is shrinkable if and only if G is shrinkable at each element of $H(G)$.*

Proof. If G is shrinkable, then for each $g \in G$, M is an open set containing g and $G(M)$ is shrinkable.

Let ϵ be a positive number and A an open set containing $\bigcup H(G)$. For each $g \in H(G)$, there exists an open set O_g containing g such that $G(O_g)$ is shrinkable. Use Theorem 1 to obtain a refinement K of $\{A \cap O_g | \, g \in H(G)\}$ that is a locally null disjoint collection of open sets $O_1, O_2, \ldots$ covering $H(G)$. Theorem 7 implies that $G(O_i)$ is shrinkable for each i. Since O_i is an open set containing $\bigcup H(G(O_i))$, Condition B implies that there exists a homeomorphism f_i of M onto M such that $f_i|(M - O_i)$ is the identity map and $\mathrm{Diam}(f_i(g)) < \epsilon$ for each $g \in G(O_i)$.

With such a function defined for each O_i, let

$$F(x) = \begin{cases} x & \text{if } x \in (M - \bigcup K), \\ f_i(x) & \text{if } x \in O_i. \end{cases}$$

Theorem 2 implies that F is a homeomorphism of M onto M. Since K covers $H(G)$ and $\bigcup K \subset A$, it follows that Condition B is satisfied.

THEOREM 11. *If A is an open subset of M such that G is shrinkable at each element of $H(G(A))$ and* $\mathrm{Bd}(A)$ *does not intersect any element of $H(G)$, then there is a homeomorphism f of* $\mathrm{Cl}(A)$ *onto* $P(\mathrm{Cl}(A))$ *such that f and P are equal when restricted to* $\mathrm{Bd}(A)$.

Proof. This theorem is a direct result of Theorems 4, 8, and 10.

THEOREM 12. *If A is an open subset of M such that for each element g of G that intersects A, g is a subset of A and G is shrinkable at g, then M/G is homeomorphic to $M/G(M - A)$.*

Proof. Let P' denote the projection map associated with $G(M - A)$. It follows from Theorem 20 (see § 7) that $P'(G)$ is a u.s.-c. decomposition of $M/G(M - A)$ and that M/G is homeomorphic to $(M/G(M - A))/P'(G)$. Since P' is a homeomorphism on the open set A and each non-degenerate element of $P'(G)$ is a subset of $P'(A)$, it follows that $P'(G)$ is shrinkable at each of its non-degenerate elements, and hence that $P'(G)$ is shrinkable. Thus, $M/G(M - A)$ is homeomorphic to $(M/G(M - A))/P'(G)$ and our conclusion follows.

THEOREM 13. *If $g \in G$ and G is shrinkable at g, then g is pointlike.*

Proof. Since G is shrinkable at g, there is an open set A containing g such that $G(A)$ is shrinkable. Since $M - g$ is an open set satisfying the hypothesis of Theorem 12 relative to the decomposition $G(A)$, it follows that

$$M = M/G(A) = M/(G(A))(g) = M/\{g\};$$

that is, the decomposition, whose only non-degenerate element is g, yields M as its decomposition space. Thus, $M - g$ is homeomorphic to the complement of a point in M.

6. Decompositions of 3-manifolds. If the space M is a 3-manifold, then several of the theorems presented earlier can be modified by using results of Armentrout (**5**) stated below as Theorems A_1 and A_2. Theorems which are similar to Theorems A_1 and A_2 also appear in other papers by Armentrout (**1**; **2**; **3**). Since Armentrout's results do not depend on the conditions used in the preceding sections of this paper, we shall state each theorem in a form that is independent of such conditions.

THEOREM A_1. *If M is a 3-manifold, G is a cellular u.s.-c. decomposition of M, and the associated decomposition space M/G is a 3-manifold, then M/G is homeomorphic to M.*

THEOREM A_2. *Suppose that M is a 3-manifold, G is a cellular u.s.-c. decomposition of M, and M/G is a 3-manifold. Suppose that A is an open subset of M whose boundary does not intersect any element of $H(G)$. Then there exists a homeomorphism h of $\mathrm{Cl}(A)$ onto $P(\mathrm{Cl}(A))$ such that h and P are equal on $\mathrm{Bd}(A)$.*

Although it is not required in Theorems A_1 and A_2 that G be 0-dimensional, we shall continue to impose this requirement in the following theorems. Throughout this section, G, G', and $G + G'$ will denote cellular, 0-dimensional, and u.s.-c. decompositions of a 3-manifold M.

THEOREM 14. *The decomposition G is shrinkable if and only if M/G is homeomorphic to M.*

Proof. If G is shrinkable, then Theorem 4 implies that M/G is homeomorphic to M. If M/G is homeomorphic to M, then Theorems A_2 and 5 imply that G is shrinkable.

THEOREM 15. *If $M = M/G = M/G'$ and for any sequence of elements of $H(G')$ converging to a set X, either X is an element of G or X does not intersect any element of $H(G)$, then $M/(G + G')$ is homeomorphic to M.*

Proof. This theorem is a result of Theorems 6 and 14.

THEOREM 16. *If $M/G = M$ and $H(G) \supset H(G')$, then M/G' is homeomorphic* to M.

Proof. This theorem is a result of Theorems 7 and 14.

THEOREM 17. *Suppose that g is an element of G. The decomposition G is shrinkable at g if there exists an open set containing g whose projection is homeomorphic to* E^3.

Proof. Apply Theorem 14 to the open set containing g.

THEOREM 18. *If A is an open subset of M such that each element of* $H(G)$ *that intersects A lies in an open subset of A whose projection is homeomorphic to* E^3, *then* $M/G(M - A)$ *is homeomorphic to* M/G.

Proof. This theorem follows from Theorems 12 and 17.

7. Decompositions which are not 0-dimensional. In this section we present two theorems that are related to the type of question studied in § 5. The first theorem is a direct result of Theorem A_1. In this section we do not place any restrictions on the decompositions considered other than those stated in each theorem.

THEOREM 19. *Let M be a* 3*-manifold. Let G and G′ be cellular u.s.-c. decompositions of M such that* $M/G = M/G' = M$ *and* $\mathrm{Cl}(\bigcup H(G))$ *does not intersect* $\mathrm{Cl}(\bigcup H(G'))$. *Let* $G + G'$ *be the decomposition of M such that*

$$H(G + G') = H(G) \cup H(G').$$

Then, $M/(G + G')$ *is homeomorphic to M.*

Proof. If g is any element of $G + G'$, then there exists an open set A containing g which intersects only one of $\mathrm{Cl}(\bigcup H(G))$ and $\mathrm{Cl}(\bigcup H(G'))$. Thus, the projection of A in $M/(G + G')$ is a 3-manifold. It follows from Theorem A_1 that $M/(G + G')$ is homeomorphic to M.

THEOREM 20. *If M is a metric space, G and G′ are u.s.-c. decompositions of M such that* $H(G) \supset H(G')$, *and P′ is the projection map of M onto* M/G', *then* $P'(G)$ *is a u.s.-c. decomposition of* M/G' *and* $M/G = (M/G')/P'(G)$.

Proof. Let $P'(g)$ be an element of $P'(G)$ and A an open subset of M/G' containing $P'(g)$. Since $P'^{-1}(A)$ is an open subset of M containing g, it follows that $C = P'((P'^{-1}(A))G)$ is an open subset of A containing $P'(g)$. If $P'(y)$ is any element of $P'(G)$ which intersects C, then $P'(y)$ is a subset of C. Thus, $P'(G)$ is a u.s.-c. decomposition of M/G'.

The fact that $(M/G')/P'(G)$ is homeomorphic to M/G follows directly from the definition of decomposition spaces.

Remark. In Theorem 20, the set $\bigcup H(P'(G))$ is homeomorphic to $\bigcup(H(G) - H(G'))$. However, Example 1 illustrates the fact that their respective embeddings might be different even when $M = M/G$.

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