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Conditions for the completeness of functional and algebraic equational reasoning†

JON G. RIECKE‡ and RAMESH SUBRAHMANYAM§

‡ *Bell Laboratories, Lucent Technologies, 700 Mountain Avenue, Murray Hill, New Jersey 07974, USA.*

§ *Department of Mathematics, Wesleyan University, Middletown, Connecticut 06459, USA.*

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We consider the following question: in the simply-typed λ-calculus with algebraic operations, is the set of equations valid in a particular model exactly those provable from (β), (η) and the set of algebraic equations, E, that are valid in the model? We find conditions for determining whether $\beta\eta E$-equational reasoning is complete. We demonstrate the utility of the results by presenting a number of simple corollaries for particular models.

1. Introduction

The two axioms of the λ-calculus,

$$(\beta) \qquad ((\lambda x.\ M)\ N) = M[x := N]$$
$$(\eta) \qquad (\lambda x.\ M\ x) = M, \quad \text{if } x \text{ not free in } M$$

lie at the heart of reasoning about functional programs: (β) explains function application syntactically, and (η) states that the meaning of functions can be based solely on their meaning under application. The (β) and (η) axioms turn out to be fundamental: not only are they sound, they also are **complete** for proving equations that hold in all models of the simply-typed λ-calculus (Friedman, 1975). In other words, an equation between simply-typed λ-terms is valid in all models if and only if it is provable from (β) and (η). These axioms can also be complete for *particular* models. Friedman, for example, shows that an equation is valid in the full set-theoretic model – *i.e.*, one with all total functions at function types, defined precisely in Section 2.2 below – over an infinite base type iff it is provable from (β) and (η) (Friedman, 1975). The completeness theorem holds for other

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§ Current address: Lucent Technologies, 2000 Naperville Road, PO Box 3033, Naperville, Illinois 60566-7033, USA.

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models as well, including ones based on continuous functions (Plotkin, 1982; Statman, 1982; Statman, 1985a; Riecke, 1995).

These theorems say nothing, however, about extensions of the λ-calculus involving constants. For instance, suppose we add the constants 0, *succ*, and $+$ to the simply-typed λ-calculus, where *succ* is a unary function and $+$ is a binary function. Suppose we interpret the constants in the standard way – with *succ* as the function that adds one, and $+$ as the function that adds its arguments – in the model $\mathscr{M}$ with the natural numbers as the base type, and all total functions at higher type. Then the equation

$$(succ\ (x+0)) = (succ\ x)$$

is valid in the model. Nevertheless, the fact is not provable from (β) and (η) alone: some form of algebraic reasoning must be used.

This paper explores the consequences of adding equations for algebraic constants to (β) and (η). The obvious algebraic equations are usually sound; our goal is to discover when such equational systems are complete for proving all equations in a particular model. We give conditions when the 'obvious' approach to achieving completeness works, but we also show when that approach fails to achieve completeness. Both positive and negative results have implications for reasoning about programs.

The 'obvious' approach to achieving completeness between terms with algebraic constants is simple: combine a complete set of algebraic equations with (β) and (η). Often the approach yields a completeness theorem. For example, consider reasoning about terms with 0, *succ*, and $+$ in the model mentioned above. The axioms of (β) and (η), together with the equations

$$\begin{array}{lrcl} (\mathrm{Z}) & x+0 & = & x \\ (\mathrm{C}) & x+y & = & y+x \\ (\mathrm{A}) & x+(y+z) & = & (x+y)+z \\ (\mathrm{S}) & x+(succ\ y) & = & succ\ (x+y) \end{array}$$

are sound and complete for proving all valid equations in the model. More precisely, we have the following theorem.

Theorem 1.1. Suppose M, N are simply-typed λ-terms involving only the constants 0, *succ*, and $+$. Then $M = N$ is valid in $\mathscr{M}$ iff M and N are provably equivalent using (β), (η), (Z), (C), (A), and (S). Moreover, since this set of equations forms a decidable equational theory, the set of valid equations in $\mathscr{M}$ is also decidable.

Theorem 1.1 follows as a corollary of Theorem 5.10 below, but relies crucially upon the fact that (Z), (C), (A), and (S) are complete for proving equations (containing variables) in the natural numbers with the operations of 0, *succ*, and $+$.

Theorem 1.1 may not be all that surprising. Indeed, we might suspect that if a set E of algebraic equations is complete for proving all valid algebraic equations in a particular model, the equational theory of E, (β), and (η) is complete for proving the valid equations. But this does not hold – the resulting equational system may be *incomplete*. Said slightly differently, reasoning with algebraic constants cannot always be decomposed neatly into reasoning about functions and reasoning about the algebraic constants. For instance,

Table 1. *Syntax and equational rules for the simply-typed λ-calculus.*

Variables	$x^\sigma : \sigma, \quad x^\sigma \in Var$	Constants	$c^\sigma : \sigma, \quad c^\sigma \in \Sigma$
Abstraction	$\dfrac{M : \nu}{(\lambda x^\tau.\, M) : (\tau \to \nu)}$	Application	$\dfrac{M : (\tau \to \nu) \qquad N : \tau}{(M\ N) : \nu}$

$$
\begin{array}{llcll}
(\beta) & ((\lambda x.\, M)\ N) & = & M[x := N] & \\
(\eta) & (\lambda x.\, M\ x) & = & M, & \text{if } x \notin FV(M) \\
(\mathit{refl}) & M & = & M &
\end{array}
$$

$$
(\mathit{symm})\ \frac{M = N}{N = M} \qquad (\mathit{trans})\ \frac{M = N \qquad N = P}{M = P}
$$

$$
(\mathit{cong})\ \frac{M = N \qquad P = Q}{(M\ P) = (N\ Q)} \qquad (\xi)\ \frac{M = N}{(\lambda x.\, M) = (\lambda x.\, N)}
$$

consider the simply-typed λ-calculus with constants 0, *iszero*, and *min*, and a model in which the base type is the natural numbers, *min* is interpreted as the minimum function, and *iszero* is interpreted as the function that returns one if its argument is zero, and zero otherwise. Suppose f is a unary function variable. Then the equation

$$(\mathit{min}\ (f\ 0)\ (\mathit{iszero}\ (f\ (\mathit{iszero}\ (f\ 0))))) = 0$$

is valid in the model, using a case analysis on whether $(f\ 0) = 0$ or $(f\ 0) \neq 0$. Nevertheless, the equation is not provable from (β), (η) and the algebraic equations that are valid in the model. Intuitively, the free function variable f acts as a barrier, so that algebraic reasoning can only happen under an f or by treating an expression $(f\ M)$ as a unit. One can turn this intuition into a proof; we give the details in Section 3.

This example shows that reasoning by cases, or some other method of extending equational reasoning to terms with free function variables, is essential in arriving at a general completeness theorem. In this paper we consider only the question of when the combination of algebraic equations E and (β) and (η) are complete for proving equations. Algebraic reasoning is a well-studied field replete with meta-theoretic results, and hence forms an important subcase to the full problem. Also, proving completeness for algebraic systems turns out to be a complex problem in and of itself, and we currently do not have a means of generalizing our methods.

Section 2 briefly reviews the syntax and definition of models for the simply-typed λ-calculus and extensions of the simply-typed λ-calculus to include algebraic constants and equations. Section 4 gives a generalization of the main lemma used in proving Statman's 1-Section Theorem (Statman, 1982; Statman, 1985a; Riecke, 1995). This theorem shows how to reduce checking the completeness of (β), (η) and algebraic equations E to checking the completeness of E. The theorem precisely characterizes when $\beta\eta E$-reasoning is complete, but it is a syntactic theorem that can be difficult to apply to specific models. Sections 5 and 6 give four general corollaries of Theorem 4.10 that are easier to apply. These corollaries are then used to deduce new completeness and decidability theorems. Section 7 concludes with a discussion of open problems.

2. Algebra and the simply-typed λ-calculus

We now briefly review the syntax and Henkin semantics of the simply-typed λ-calculus, and how algebras fit into the framework.

2.1. *Simply-typed λ-calculus*

Each term in the simply-typed λ-calculus comes with a **simple type** generated by the grammar

$$\sigma ::= \iota \mid (\sigma \to \sigma)$$

Parentheses are often dropped with the assumption that $\to$ associates to the right, for example, $(\iota \to \iota \to \iota)$ denotes $(\iota \to (\iota \to \iota))$. A **first-order type** is a type of the form $(\iota \to \ldots \to \iota)$; we often use T^n to stand for

$$(\underbrace{\iota \to \ldots \to \iota}_{n+1}).$$

Typically, the base type ι is chosen to be the natural numbers, as in PCF (Plotkin, 1977), but nothing in the syntax forces this choice; some of the theorems below are for models in which the base type is interpreted to be lists or sets of natural numbers.

To construct terms, assume that Σ is a **signature**, *i.e.*, a countable set of typed constants, and that Var is a countable set of typed variables. The set of simply-typed terms over Σ is given by the formation rules of Table 1. To simplify the notation, we often drop types from variables and constants when the context is clear, and drop parentheses with the assumption that application associates to the left. The usual definitions of free and bound variables apply to this set of terms, and terms are identified up to renaming of bound variables (Barendregt, 1981). The set of simply-typed terms containing constants Σ and free variables X is denoted $terms_{\Sigma,X}$.

The equational axioms and rules of the simply-typed λ-calculus also appear in Table 1; we write $(M =_{\beta\eta} N)$ if M and N are provably equivalent in this system. A Σ, λ**-theory** is a set of equations between terms in $terms_{\Sigma,X}$ of the same type, where X is the set of all variables, that is closed under the axioms and rules of Table 1. In the axioms, $M[x := N]$ denotes substitution of N for x in M, where the bound variables of M are renamed to avoid the capture of the free variables of N (Barendregt, 1981). We will also use the notation $M\theta$, where $\theta = [x_1 := N_1, \ldots, x_n := N_n]$ is a substitution, to denote a simultaneous substitution of N_i for x_i.

Two special forms of terms are useful in the following sections. If a term P has no subterms of the form $((\lambda x.\ M)\ N)$ or $(\lambda y.\ M\ y)$ with y not free in M, we say that P is in $\beta\eta$**-normal form**. Any term P is equivalent to a unique term Q in $\beta\eta$-normal form (Barendregt, 1981); we use $\beta\eta\text{-}nf(P)$ to denote the $\beta\eta$-normal form of P. **Long** $\beta\eta$**-normal forms** are slightly different: they have no subterms of the form $((\lambda x.\ M)\ N)$ (and hence are β-normal forms), but may have subterms of the form $(\lambda y.\ M\ y)$. Long $\beta\eta$-normal forms are terms of the form

$$\lambda x_1^{\sigma_1}.\ \ldots \lambda x_n^{\sigma_n}.\ \ell\ M_1 \ldots\ M_k$$

where $k, n \geqslant 0$, ℓ is a constant or variable of type $(\tau_1 \to \ldots \to \tau_k \to \iota)$, and $M_1, \ldots, M_k$ are long $\beta\eta$-normal forms. Any term is $\beta\eta$-equivalent to a unique long $\beta\eta$-normal form – see Jensen and Pietrzykowski (1976). These terms are useful because the body has type ι, making certain inductions simpler to perform.

2.2. *Models*

We use the notion of environment models to give meaning to simply-typed terms (Friedman, 1975; Meyer, 1982).

Definition 2.1. A **frame** is a pair $(\{\mathscr{M}^\sigma\}, \{ap^{\sigma,\tau}\})$, where each $\mathscr{M}^\sigma$ is a nonempty set, and each $ap^{\sigma,\tau}$ is an 'application' function with $ap^{\sigma,\tau} : \mathscr{M}^{\sigma\to\tau} \times \mathscr{M}^\sigma \to \mathscr{M}^\tau$. (We omit the types from ap when they are clear from context.) The functions $ap^{\sigma,\tau}$ must satisfy the extensionality property, *i.e.*, for any $f, g \in \mathscr{M}^{\sigma\to\tau}$, $f = g$ iff for all $d \in \mathscr{M}^\sigma$, $ap^{\sigma,\tau}(f,d) = ap^{\sigma,\tau}(g,d)$. An **environment** ρ is a type-respecting map from variables to $(\bigcup_\sigma \mathscr{M}^\sigma)$. A **Σ-model** $\mathscr{M}$ is a tuple $(\{\mathscr{M}^\sigma\}, \{ap^{\sigma,\tau}\}, \mathscr{I})$, where the first two components form a frame and $\mathscr{I}$ is a type-respecting map from Σ to $(\bigcup_\sigma \mathscr{M}^\sigma)$. Furthermore, there must be a well-defined meaning function $\mathscr{M}[\![\cdot]\!]$ satisfying the equations

$$\begin{array}{rcl} \mathscr{M}[\![x^\sigma]\!]\rho &=& \rho(x^\sigma) \\ \mathscr{M}[\![c^\sigma]\!]\rho &=& \mathscr{I}(c^\sigma) \\ \mathscr{M}[\![M\ N]\!]\rho &=& ap(\mathscr{M}[\![M]\!]\rho, \mathscr{M}[\![N]\!]\rho) \\ \mathscr{M}[\![\lambda x^\tau.\ M]\!]\rho &=& f, \text{where } ap(f,d) = \mathscr{M}[\![M]\!]\rho[x^\tau \mapsto d]. \end{array}$$

For example, the **full set-theoretic model** $\mathscr{S}$ over a base set X, defined by

$$\begin{array}{rcl} \mathscr{S}^\iota &=& X \\ \mathscr{S}^{\sigma\to\tau} &=& [\mathscr{S}^\sigma \to \mathscr{S}^\tau] \\ ap(f,d) &=& f(d), \end{array}$$

where $[A \to B]$ is the set of all total functions from σ to τ, is a model for the empty signature: the meanings of every λ-abstraction exist because the frame contains *all* functions. Another example is the **full continuous model** over a complete partial order (cpo) X, defined by

$$\begin{array}{rcl} \mathscr{C}^\iota &=& X \\ \mathscr{C}^{\sigma\to\tau} &=& [\mathscr{C}^\sigma \to_c \mathscr{C}^\tau] \\ ap(f,d) &=& f(d), \end{array}$$

where $[A \to_c B]$ is the cpo of continuous functions from cpo A to cpo B ordered pointwise. Full continuous models are important because they can be used to obtain models of programming languages that are **fully abstract**, *i.e.*, models in which equality coincides with operational notions of equivalence, *cf.*, Sazonov (1976), Plotkin (1977) and Cosmadakis (1989).

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An equation $M = N$ is **valid** in a model $\mathscr{M}$, written $\mathscr{M} \models M = N$, if for all environments ρ, $\mathscr{M}[\![M]\!]\rho = \mathscr{M}[\![N]\!]\rho$. Note that a model $\mathscr{M}$ generates a Σ, λ-theory, denoted $Th(\mathscr{M})$, namely the set $\{(M = N) \mid \mathscr{M} \models M = N\}$.

2.3. *Algebraic terms and equations*

Recall that a (single-sorted) **algebraic signature** is a pair (Σ_{alg}, r), where Σ_{alg} is a set of constants and r is a function mapping Σ_{alg} to the natural numbers. For any $c \in \Sigma_{alg}$, $r(c)$ is called the **arity** of c. In writing algebraic signatures, one usually leaves the function r to be deduced from the context. For example, the algebraic signature $(0, +)$ is an algebraic signature with two constants, the first of arity zero and the second of arity two. The set of **algebraic terms** over the algebraic signature Σ_{alg} is given by the grammar

$$A ::= x \mid c_0\ (\underbrace{A, \ldots, A}_{n_0}) \mid c_1\ (\underbrace{A, \ldots, A}_{n_1}) \mid \ldots,$$

where x ranges over an infinite set of variables and each c_i has arity n_i.

To add algebraic terms to the simply-typed λ-calculus, we must first assign a type to the sort of the algebraic signature; the base type ι serves as the interpretation of the sort. Second, we must add the constants in some form to the simply-typed λ-calculus. The nullary constants all have type ι, for example, for the algebraic signature $(0, +)$, the nullary algebraic constant 0 is a constant of type ι. However, the function constants in an algebraic signature may be added in a variety of ways. The usual interpretation of the function constant $+$, for instance, is a function from $(\iota \times \iota)$ to ι; to interpret this directly in the λ-calculus would require us to add product types. While adding products to the λ-calculus is not difficult, there is another way to add algebraic constants without changing the type structure of the language. Recall that for any number n, the type T^n is defined to be

$$(\underbrace{\iota \to \ldots \to \iota}_{n} \to \iota).$$

Given a function constant c_i of arity n in the algebraic signature Σ_{alg}, we introduce a constant $c_i : T^n$ in the λ-calculus. The λ-calculus signature consisting of these newly introduced constants is denoted Σ. The algebraic terms over Σ_{alg} built by the grammar

$$A ::= x \mid c_0\ (\underbrace{A, \ldots, A}_{n_0}) \mid c_1\ (\underbrace{A, \ldots, A}_{n_1}) \mid \ldots$$

thus correspond to the λ-terms in the grammar

$$B ::= x^\iota \mid (c_0\ \underbrace{B\ \ldots B}_{n_0}) \mid (c_1\ \underbrace{B\ \ldots B}_{n_1}) \mid \ldots,$$

which are terms in the simply-typed λ-calculus over the signature Σ. In this paper, we identify Σ_{alg} with Σ, and identify the two sets of terms above. From now on, when we write algebraic terms we write terms generated by the second grammar above. We let $algterms_{\Sigma,X}$ be the set of algebraic terms over the signature Σ containing variables drawn

Table 2. *Algebraic provability from a set of equations E, where F ranges over constants in the algebraic signature.*

(*refl*)	$s = s$	(*axiom*)	$\dfrac{(s = t) \in E}{s = t}$
(*symm*)	$\dfrac{s = t}{t = s}$	(*trans*)	$\dfrac{s = t \qquad t = u}{s = u}$
(*cong*)	$\dfrac{s_1 = t_1 \dots s_n = t_n}{(F\ s_1 \dots s_n) = (F\ t_1 \dots t_n)}$	(*sub*)	$\dfrac{s = t}{s[x := u] = t[x := u]}$

from the set X of base type variables, and use the letters s, t, u to denote elements in this set.

An **algebraic equation** $t = t'$ is just an equation between algebraic terms t and t'. Since we identify algebraic terms with their λ-term counterparts by the correspondence above, these equations may be easily added to the equational theory of the λ-calculus by adding these equations to the axioms of Table 1. Given a set of algebraic equations E and algebraic terms t and t', we say that $t =_E t'$ if the equation is derived using the rules of Table 2. It is important to note that any algebraic equation provable in this system is provable in the simply-typed λ-calculus from the equations E and the axioms and rules of Table 1: all but (*sub*) appears in Table 1, and the (*sub*) rule can be derived from (ξ), (β) and (*trans*).

A **Σ-theory** is a set of equations between terms in the set $algterms_{\Sigma,X}$ that is closed under the rules of Table 2. We will use the symbol $\mathscr{E}$ to denote algebraic theories.

We also consider terms of base type that are built from algebraic constants Σ and a set of first-order variables FO with no λ-abstractions. Strictly speaking, such terms are *not* algebraic terms, since the first-order variables are not constants. The set of such terms is denoted $algterms_{\Sigma,FO,X}$, and the terms are called **extended algebraic terms**. If E is a set of algebraic equations over Σ, and $t_1, t_2 \in algterms_{\Sigma,FO,X}$, we write $t_1 =_{E+FO} t_2$ if the equation is derivable from the rules in Table 2 and the rule

$$(extCong) \qquad \frac{M_1 = N_1 \quad \dots \quad M_k = N_k}{(f\ M_1 \dots M_k) = (f\ N_1 \dots N_k)}$$

where f ranges over FO. An **extended Σ-theory** is a set of equations between terms in the set $algterms_{\Sigma,FO,X}$ that is closed under the rules of Table 2 and the rule (*extCong*) above.

Simply-typed terms containing the algebraic constants are called **mixed terms**. If M, N are mixed terms, we write $M =_{\beta\eta E} N$ if M and N are provably equal using the rules in Tables 1 and 2. On terms of base type, $\beta\eta$-reasoning and E reasoning can be separated, as in the following lemma.

Lemma 2.2. Let $M, N \in terms_{\Sigma,FO}$ be terms of type ι. Then we have that $M =_{\beta\eta E} N$ iff $\beta\eta\text{-}nf(M) =_{E+FO} \beta\eta\text{-}nf(N)$.

Proof. Note that $\beta\eta\text{-}nf(M)$ and $\beta\eta\text{-}nf(N)$ are (typable) terms in the grammar

$$S ::= (f\ S \dots S) \mid (c\ S \dots S)$$

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where f is drawn from FO and c is drawn from Σ. Hence, $\beta\eta\text{-}nf(M), \beta\eta\text{-}nf(N) \in algterms_{\Sigma,FO,X}$. Thus,

$$\begin{array}{rcl} M =_{\beta\eta E} N & \text{iff} & \beta\eta\text{-}nf(M) =_{\beta\eta E} \beta\eta\text{-}nf(N) \\ & \text{iff} & \beta\eta\text{-}nf(M) =_{E+FO} \beta\eta\text{-}nf(N) \end{array}$$

where the last line follows by invoking the fact that the simply-typed λ-calculus conservatively extends algebraic reasoning (Breazu-Tannen, 1987; Breazu-Tannen and Meyer, 1987). □

This fact will be used frequently in the proofs.

2.4. *Algebras in simply-typed models*

Recall that a **Σ-algebra** $\mathscr{A}$, where Σ is an algebraic signature, is a tuple $(A, \mathscr{K})$, where A is the (nonempty) **carrier set** of the algebra and $\mathscr{K}$ is a function with domain Σ such that for all $c^{T^n} \in \Sigma$,

$$\mathscr{K}(c^{T^n}) \in [\underbrace{A \to \ldots \to A}_{n} \to A]$$

(the usual set-theoretic function space). For instance, the $\{0, +\}$-algebra $(\mathbf{N}, \mathscr{K})$ is the algebra with the set of natural numbers as the carrier set, and where the number zero serves as the interpretation of the constant 0 and the usual addition function serves as the interpretation of the constant $+$. Given a Σ-algebra $(A, \mathscr{K})$, the meaning of an algebraic term is also standard, as given by the following definition.

Definition 2.3. An **algebraic environment** ρ is a map from variables of type ι to A. The **meaning** of a term $t \in algterms_{\Sigma,X}$ is defined by induction on the structure of t:

$$\begin{array}{rcl} \mathscr{A}[\![x^\iota]\!]\rho & = & \rho(x^\iota) \\ \mathscr{A}[\![c^\sigma\ t_1\ \ldots t_n]\!]\rho & = & (\mathscr{K}(c^\sigma)\ \mathscr{A}[\![t_1]\!]\rho \ldots \mathscr{A}[\![t_n]\!]\rho) \end{array}$$

We often write $\mathscr{A}[\![c^\sigma]\!]$ for $\mathscr{K}(c^\sigma)$.

As with environment models, an equation $t_1 = t_2$ is **valid** in an algebra $\mathscr{A}$, written $\mathscr{A} \models t_1 = t_2$, if for all environments ρ, $\mathscr{A}[\![t_1]\!]\rho = \mathscr{A}[\![t_2]\!]\rho$. The **theory of** $\mathscr{A}$, written $Th(\mathscr{A})$ with an abuse of notation, is the set $\{(s = t) \mid \mathscr{A} \models s = t\}$.

Algebras appear naturally in models of the simply-typed λ-calculus (with constants). Suppose Σ is an algebraic signature, and $\mathscr{M} = (\{\mathscr{M}^\sigma\}, \{ap^{\sigma,\tau}\}, \mathscr{I})$ is a Σ-model. Then the **Σ-algebra induced by** $\mathscr{M}$ is the algebra $(\mathscr{M}^\iota, \mathscr{K})$ with operations given by

$$(\mathscr{K}(c^\sigma)\ a_1 \ldots a_n) = ap(\ldots ap(ap(\mathscr{I}(c^\sigma), a_1), \ldots), a_n)$$

for every constant $c^\sigma \in \Sigma$. The other part of the Σ-model gives meaning to λ-definable functions over the algebra. Thus, when Σ is an algebraic signature, a Σ-model contains two parts: an algebra for giving meaning to algebraic terms, and an environment model for giving meaning to simply-typed terms. The meaning function then shows how to give meaning to terms involving both λ-calculus and algebraic constructions. Algebraic

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equations also have an interpretation in Σ-models: saying that a Σ-model validates an equation is exactly the same as saying that the algebra induced by the model validates the equation.

3. Incompleteness of combined theories

Since Σ-models 'contain' a Σ-algebra and an environment model, one might wonder whether reasoning about mixed-term equalities may be performed by combining a proof system for algebraic equational reasoning with a proof system for higher-order equational reasoning. There are instances in which the combination of algebraic equations and $\beta\eta$-equality, while complete for proving equations between algebraic and pure λ-terms, respectively, may not be complete for reasoning about mixed terms. Such an example is illuminating for describing when two complete theories *do* lead to a complete theory for reasoning about mixed terms.

Let $\mathscr{A} = (\mathbf{N}, \mathscr{K})$ be the $\{0, min, iszero\}$-algebra from the introduction. Let f be a variable of type $(\iota \to \iota)$. We claim that the equation

$$(min\ (f\ 0)\ (iszero\ (f\ (iszero\ (f\ 0))))) = 0$$

is valid in the full type hierarchy over $\mathscr{A}$, but is not provable using (β), (η) and the equational theory of $\mathscr{A}$. In rough outline, the proof proceeds by showing that algebraic reasoning cannot cross the 'boundaries' of the free variable f. More precisely, in each proof step, a subterm of the form $(f\ M)$ must be treated as one unit, or M must be proved equal to some other term M' by an algebraic step.

Instead of proving that the specific equation above is not provable, we develop some more general machinery that is also useful in the next section. Suppose Σ is an algebraic signature, E is a set of algebraic equations over Σ, X and Y are infinite sets of variables of type ι, and FO is a finite set of variables of first-order types; also assume that Σ, X, Y and FO are pairwise disjoint. We define a means of eliminating subterms beginning with symbols in FO as follows.

Definition 3.1. Let $Rep_{\Sigma,FO,E} : algterms_{\Sigma,FO,X} \to Y$ be a function such that

$$Rep_{\Sigma,FO,E}(s) = Rep_{\Sigma,FO,E}(s') \text{ iff } s =_{E+FO} s'.$$

Define $top_{\Sigma,FO,E}(s) : algterms_{\Sigma,FO,X} \to algterms_{\Sigma,X\cup Y}$ by

$$top_{\Sigma,FO,E}(s) = \begin{cases} x & \text{if } s = x \text{ and } x \in X \\ Rep_{\Sigma,FO,E}(f\ s_1 \ldots s_n) & \text{if } s = (f\ s_1 \ldots s_n),\ n \geqslant 0, \\ & \text{and } f \in FO \\ (c\ top_{\Sigma,FO,E}(s_1) \ldots top_{\Sigma,FO,E}(s_n)) & \text{if } s = (c\ s_1 \ldots s_n),\ n \geqslant 0, \\ & \text{and } c \in \Sigma. \end{cases}$$

The function $top_{\Sigma,FO,E}$ prunes its argument s by replacing each maximal subterm containing a symbol from FO at its head by the variable in Y. The result is an algebraic term built from the symbols in Σ and the variables in $X \cup Y$. In the rest of this section the subscripts on Rep and top are omitted, and should be taken to be Σ, FO, E.

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We want to show that a proof of $t =_{E+FO} t'$ yields a proof of $top(t) =_E top(t')$. Before we can prove this statement, we need a technical lemma.

Lemma 3.2. Suppose $s, t \in algterms_{\Sigma,FO,X}$ and $x \in X$. Let $\theta : Y \to Y$ be a map such that if $Rep(u) = y$, then $\theta(y) = Rep(u[x := t])$. Then we have that $top(s[x := t]) = top(s)\theta[x := top(t)]$.

Proof. To simplify notation, let $\theta' = \theta[x := top(t)]$. We proceed by induction on s.

— $s = x$: Then $top(x[x := t]) = top(t) = x[x := top(t)] = top(x)\theta'$.
— $s = x' \in X$ and $x' \neq x$: Trivial.
— $s = (f\ s_1\ \ldots\ s_n)$, where $f \in FO$: Then

$$\begin{aligned} top(s[x := t]) &= Rep(s[x := t]) = y \\ top(s)\theta' &= Rep(s)\theta' = \theta(Rep(s)) = \theta(y') \end{aligned}$$

for some $y, y' \in Y$. Suppose u is the term such that $y' = Rep(u)$ and $\theta(y') = Rep(u[x := t])$. By the properties of Rep, it follows that $u =_{E+FO} s$. Thus, since $u[x := t] =_{E+FO} s[x := t]$,

$$Rep(u[x := t]) = Rep(s[x := t])$$

and hence $\theta(y') = y$, which proves the claim.
— $s = (c\ s_1\ \ldots\ s_n)$ where $c \in \Sigma$: Follows easily by the induction hypothesis.

This completes the induction and hence the proof. □

Lemma 3.3. Suppose E is a set of algebraic equations over the algebraic signature Σ. Let $t, t' \in algterms_{\Sigma,FO,X}$ such that $t =_{E+FO} t'$. Then $top(t) =_E top(t')$.

Proof. By induction on the derivation of $t =_{E+FO} t'$. The base case, when $(t = t') \in E$ follows because top is the identity on $algterms_{\Sigma,X}$. The other base case, namely when t and t' are identical, is trivial. Consider the induction case. There are five cases to check; we only illustrate the cases where (*extCong*), (*cong*) and (*sub*) are the last rules in the derivation.

When (*extCong*) is the last rule, we have $t = (f\ t_1 \ldots t_n)$, $t' = (f\ t'_1 \ldots t'_n)$, and $f \in FO$. Then $top(t) = Rep(t)$ and $top(t) = Rep(t')$; since $t =_{E+FO} t'$, we have $Rep(t) = Rep(t')$.

When (*cong*) is the last rule, $t = (c\ t_1 \ldots t_n)$, $t' = (c\ t'_1 \ldots t'_n)$, and $c \in \Sigma$, so

$$\begin{aligned} top(t) &=_E (c\ top(t_1) \ldots top(t_n)) \\ &=_E (c\ top(t'_1) \ldots top(t'_n)) \\ &=_E top(t') \end{aligned}$$

where the second line follows by the induction hypothesis.

When (*sub*) is the last rule, we have $t = s[x := u]$ and $t' = s'[x := u]$. By the induction

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hypothesis, $top(s) =_E top(s')$. Applying Lemma 3.2,

$$\begin{aligned} top(t) &=_E top(s[x := u]) \\ &=_E top(s)\theta[x := top(u)] \\ &=_E top(s')\theta[x := top(u)] \\ &=_E top(s'[x := u]) = top(t') \end{aligned}$$

where the second line follows by Lemma 3.2, which defines θ, and the third line follows from rule (*sub*). This completes the induction step and hence the proof. □

Theorem 3.4. Let $f^{\iota\to\iota}$ be a variable. Then the equation

$$(min\ (f\ 0)\ (iszero\ (f\ (iszero\ (f\ 0))))) = 0$$

is valid in the full type hierarchy over $\mathscr{A}$, but is not provable using (β), (η) and $\mathscr{E} = Th(\mathscr{A})$.

Proof. The equation is valid in the full type hierarchy over $\mathscr{A}$; a case analysis based on whether $(f\ 0) = 0$ or $(f\ 0) \neq 0$ can be used to establish that it is valid in the model. This equation is not, however, provable. Suppose, by way of contradiction, that

$$(min\ (f\ 0)\ (iszero\ (f\ (iszero\ (f\ 0))))) =_{\beta\eta\mathscr{E}} 0.$$

By Lemma 2.2,

$$(min\ (f\ 0)\ (iszero\ (f\ (iszero\ (f\ 0))))) =_{\mathscr{E}+\{f\}} 0.$$

Note that $(f\ 0) = (f\ (iszero\ (f\ 0)))$ is not valid in the full type hierarchy over $\mathscr{A}$, so $(f\ 0) \neq_{\mathscr{E}+\{f\}} (f\ (iszero\ (f\ 0)))$. Thus,

$$y_1 = Rep(f\ 0) \neq Rep(f\ (iszero\ (f\ 0))) = y_2.$$

By Lemma 3.3, if $(min\ (f\ 0)\ (iszero\ (f\ (iszero\ (f\ 0))))) =_{\mathscr{E}+\{f\}} 0$, then

$$(min\ y_1\ (iszero\ y_2)) =_{\mathscr{E}} 0,$$

which is clearly not the case. We have reached a contradiction, so the equation is not provable. □

4. A necessary and sufficient condition for completeness

The goal of this section is to prove the following theorem:

> Suppose Σ is an algebraic signature, X is an infinite set of variables of type ι, and $f^{\iota\to\iota\to\iota}$ is a variable. Let E be a Σ,λ-theory, and E_1 be the restriction of E to $algterms_{\Sigma,X}$. Then (β), (η) and E_1 prove all equations in E iff E_1 and $(extCong)$ prove all equations in
>
> $$\{(t_1 = t_2) \in E \mid t_1, t_2 \in algterms_{\Sigma,\{f^{\iota\to\iota\to\iota}\},X}\}.$$

Note that the theorem concerns *theories*, not models, which emphasizes the syntactic nature of the proof. At the end of the section, we use the theorem to deduce a fact about models.

4.1. *Outline of the proof*

We generalize techniques in Statman (1982b) and Breazu-Tannen (1988) to the case when algebraic equations are present. In broad outline, the proof has two main steps:

1. If M, N are closed terms and $M \neq_{\beta\eta E} N$, there is a sequence of arguments, driving both terms to base type and having only first-order free variables, that distinguishes the terms. We refer to this as the *first main lemma*, and state it precisely as Lemma 4.5.
2. There exists a set of terms, involving only free variables $f^{\iota\to\iota\to\iota}$ and x^{ι}, such that for any closed second-order terms M, N with $M \neq_{\beta\eta E} N$, there is a sequence of arguments drawn from the set and driving both terms to base type, that distinguishes the terms. We refer to this as the *second main lemma*, and state it precisely as Lemma 4.7.

The proof of the theorem is simple given these two statements.

We need some notation before beginning the proofs. Fix some algebraic signature Σ. First, we use the notation $C[t_1, \ldots, t_k]$ as shorthand for a substitution $C[x_1 := t_1, \ldots, x_k := t_k]$ with $C \in algterms_{\Sigma,X}$ and $x_1, \ldots, x_k \in X$. Second, some of the proofs go by induction on the structure of Φ-normal forms.

Definition 4.1. Algebraic Φ-normal forms and **regular Φ-normal forms** – together called the set of **Φ-normal forms** – are subsets of terms over an algebraic signature Σ. The sets are defined by simultaneous induction:

— $(\lambda\vec{x}.\ y\ (M_1\ \vec{x}) \ldots (M_k\ \vec{x}))$, where $\vec{x}$ is not free in any of the M_i's, is a regular Φ-normal form if $k \geqslant 0$, each M_i is a Φ-normal form, and y is a variable of type $(\sigma_1 \to \ldots \to \sigma_k \to \iota)$; and

— $\lambda\vec{x}.\ C[(M_1\ \vec{x}), \ldots, (M_k\ \vec{x})]$, where $\vec{x}$ is not free in any of the M_i's, is an algebraic Φ-normal form if each M_i is a regular Φ-normal form and $C \in algterms_{\Sigma,X}$ is a non-trivial term (that is, it is not a variable).

Note that when M is a *closed* Φ-normal form, each of the constituent terms of M are also closed Φ-normal forms. This is the reason we use Φ-normal forms and *not* $\beta\eta$-normal forms in induction proofs; the $\beta\eta$-normal form of a closed term may not be composed of closed terms. For example, in the $\beta\eta$-normal form $(\lambda x.\ \lambda f.\ f\ (f\ x))$, the subterm $(f\ x)$ is not closed.

It is not hard to see that any term is $\beta\eta$-equivalent to a term in Φ-normal form.

Lemma 4.2. For any term M, there is a Φ-normal form N such that $M =_{\beta\eta} N$.

Proof. Without loss of generality, assume that M is in long-$\beta\eta$-normal form; it is thus a λ-abstraction whose body is of base type, with the form

$$M =_{\beta\eta} \lambda\vec{x}.\ u\ M_1 \ldots\ M_n.$$

Each M_j is in long-$\beta\eta$-normal form and u is either a variable or a constant drawn from Σ. We proceed by induction on the structure of M.

If u is a variable, by the induction hypothesis, $M_1, \ldots, M_n$ – which are strictly smaller in size than M – are $\beta\eta$-equivalent to Φ-normal forms $N_1, \ldots, N_n$. The term

$$\lambda\vec{x}.\ u\ ((\lambda\vec{x}.\ N_1)\ \vec{x}) \ldots ((\lambda\vec{x}.\ N_n)\ \vec{x})$$

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is in Φ-normal form and is $\beta\eta$-equivalent to M. If u is a constant, there is an algebraic term C such that

$$(u\ M_1 \ldots\ M_n) = C[P_1, \ldots, P_k]$$

where each P_i is a long $\beta\eta$-normal form that begins with a variable (they cannot be λ-abstractions, because each P_i has type ι). Since $P_1, \ldots, P_k$ are smaller than M, by the induction hypothesis there are terms Q_i in Φ-normal form such that $P_i =_{\beta\eta} Q_i$. Moreover, since each P_i is of base type and begins with a variable, each Q_i must begin with a variable. Thus, the terms Q_i are regular Φ-normal forms, so the term

$$\lambda\vec{x}.\ C[((\lambda\vec{x}.\ Q_1)\ \vec{x}), \ldots, ((\lambda\vec{x}.\ Q_k)\ \vec{x})]$$

is in Φ-normal form and is $\beta\eta$-equivalent to M. □

4.2. *First main lemma*

We now turn to the proof of the first main lemma. Fix some set FO of first-order variables disjoint from Σ and with an infinite number of variables of each first-order type. We begin with a preliminary lemma.

Lemma 4.3. Suppose $M, N \in algterms_{\Sigma,FO,X}$ and $M =_{E+FO} N$. If $f \in FO$ and $M = (f\ M_1 \ldots M_m)$ and $N = (f\ N_1 \ldots N_m)$, then for all $1 \leqslant i \leqslant m$, $M_i =_{E+FO} N_i$.

Proof. Let $1 \leqslant i \leqslant m$. It is not hard to show that there is a sequence

$$(f\ M_1 \ldots M_m) = T_0 =_{E+FO} T_1 =_{E+FO} \ldots =_{E+FO} T_n = (f\ N_1 \ldots N_m),$$

where each $T_j =_{E+FO} T_{j+1}$ has the form $D[s\theta] =_{E+FO} D[t\theta]$ with $(s = t) \in E$ or $(t = s) \in E$ and $D[x]$ is a term with one occurrence of x (see Breazu-Tannen (1987, Proposition 2.17)). The goal of the proof is to transform the sequence into a proof of $M_i =_{E+FO} N_i$.

Consider any term Q. The term $\overline{Q}$ is obtained by replacing every maximal subterm Q' of Q satisfying the property

$$Q' = (f\ P_1 \ldots P_m) \text{ and, for every } 1 \leqslant j \leqslant m,\ M_j =_{E+FO} P_j \text{ or } N_j =_{E+FO} P_j \qquad (1)$$

by the term P_i. We claim that $\overline{T_j} =_{E+FO} \overline{T_{j+1}}$, and thus

$$\overline{T_0} =_{E+FO} \overline{T_1} =_{E+FO} \ldots =_{E+FO} \overline{T_n}.$$

Since $\overline{T_0} = M_i$ and $\overline{T_n} = N_i$, this is enough to prove that $M_i =_{E+FO} N_i$.

To prove the claim, suppose the proof step is $D[s\theta] =_{E+FO} D[t\theta]$, where $(s = t) \in E$ or $(t = s) \in E$ and $D[x]$ has one occurrence of x. There are two cases:

- Suppose $s\theta$ is contained in no subterm of $D[s\theta]$ satisfying Property 1. Thus, $t\theta$ is also not contained in any subterm of $D[t\theta]$ satisfying Property 1. Then $\overline{D[s\theta]} = \overline{D}[s\overline{\theta}]$, where $\overline{\theta}(x_i) = \overline{\theta(x_i)}$. Thus $\overline{D[t\theta]} = \overline{D}[t\overline{\theta}]$, and therefore $\overline{D[s\theta]} =_{E+FO} \overline{D[t\theta]}$.
- Suppose $s\theta$ is contained in a subterm of $D[s\theta]$ satisfying Property 1, and let the maximal such term be $u = (f\ P_1\ \ldots\ P_m)$. Let $u' = (f\ P_1'\ \ldots\ P_m')$ be the term at the corresponding position in $D[t\theta]$. Note that u' also satisfies Property 1, and so is

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replaced by P_i' when the transformation $\overline{(\cdot)}$ is applied to it. It is clear that $P_i =_{E+FO} P_i'$. Therefore, $\overline{D[s\theta]} =_{E+FO} \overline{D[t\theta]}$.

This completes the proof of the claim and hence the lemma. □

Suppose we have a set of equations such that for any equation $M = N$ in the set there is a sequence of arguments $\vec{V} \in terms_{\Sigma,FO}$ such that $(M\ \vec{V})$ and $(N\ \vec{V})$ are of type ι, and $(M\ \vec{V}) \neq_{\beta\eta E} (N\ \vec{V})$. The next lemma says that one may obtain a *single* list of arguments that achieves the same purpose for all the equations.

Lemma 4.4. Suppose $T \subseteq terms_{\Sigma,\varnothing}$ is a finite set of (closed) terms of type $\sigma = (\sigma_1 \to \ldots \to \sigma_n \to \iota)$ in regular Φ-normal form. Suppose for each $M, N \in T$ with $M \neq_{\beta\eta E} N$, there are terms $V_1, \ldots, V_n \in terms_{\Sigma,FO}$ such that

$$(M\ V_1 \ldots V_n) \neq_{\beta\eta E} (N\ V_1 \ldots V_n).$$

Then there exist $U_1, \ldots U_n \in terms_{\Sigma,FO}$ such that:

1 for all $M \in T$, $\beta\eta$-*nf*$(M\ \vec{U})$ has a variable at the head of an application; and
2 for all $M, N \in T$ if $M \neq_{\beta\eta E} N$, then $(M\ U_1 \ldots U_n) \neq_{\beta\eta E} (N\ U_1 \ldots U_n)$.

Proof. Let $(M_1, N_1), \ldots, (M_k, N_k)$ be the (distinct) pairs of terms in T with $M_i \neq_{\beta\eta E} N_i$. Let

$$(V_{1,1}, \ldots, V_{1,n}), \ldots, (V_{k,1}, \ldots, V_{k,n})$$

be vectors such that

$$(M_i\ V_{i,1} \ldots V_{i,n}) \neq_{\beta\eta E} (N_i\ V_{i,1} \ldots V_{i,n}).$$

Let $f_1, \ldots, f_n \in FO$ be distinct variables and

$$U_j = \lambda y_1^{\sigma_1}. \ldots \lambda y_n^{\sigma_n}. f_j\ (V_{1,j}\ y_1 \ldots y_n) \ldots (V_{k,j}\ y_1 \ldots y_n).$$

We claim that this choice of $U_1, \ldots, U_n$ satisfies the condition desired. Since each $M \in T$ is a closed regular Φ-normal form, it follows easily that each $\beta\eta$-*nf*$(M\ \vec{U})$ has a variable at the head of an application, namely one of the f_j's. To see the second part, note that

$$U_j[f_j := \lambda\vec{x}.\ x_i] =_{\beta\eta} V_{i,j}.$$

Since $(M_i\ V_{i,1} \ldots V_{i,n}) \neq_{\beta\eta E} (N_i\ V_{i,1} \ldots V_{i,n})$, and f_j does not appear in any term other than U_j, it follows that $(M_i\ \vec{U}) \neq_{\beta\eta E} (N_i\ \vec{U})$. □

Lemma 4.5. Suppose $M, N \in terms_{\Sigma,\varnothing}$ are closed terms and $M \neq_{\beta\eta E} N$. Then there exist terms $V_1, \ldots, V_m$ in $terms_{\Sigma,FO}$ such that $(M\ V_1 \ldots V_m) \neq_{\beta\eta E} (N\ V_1 \ldots V_m)$ at base type.

Proof. Without loss of generality, we may assume that M, N are in Φ-normal form. We proceed by induction on the sum of the sizes of M and N.

1 Both M and N are in algebraic Φ-normal form. Then for some $k, l, n \geqslant 0$ and terms C and D, both of which begin with a constant from Σ,

$$\begin{aligned} M &= \lambda x_1^{\sigma_1}. \ldots \lambda x_n^{\sigma_n}. C[(M_1\ x_1 \ldots x_n), \ldots, (M_k\ x_1 \ldots x_n)] \\ N &= \lambda x_1^{\sigma_1}. \ldots \lambda x_n^{\sigma_n}. D[(N_1\ x_1 \ldots x_n), \ldots, (N_l\ x_1 \ldots x_n)] \end{aligned}$$

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where $T = \{M_1, \dots, M_k, N_1, \dots, N_l\}$ are regular Φ-normal forms. By the induction hypothesis, for every pair $P, Q \in T$ such that $P \neq_{\beta\eta E} Q$, there exists a vector $\vec{V} \in terms_{\Sigma,FO}$ such that $(P\ \vec{V}) \neq_{\beta\eta E} (Q\ \vec{V})$. Thus, by Lemma 4.4, there exists a vector $\vec{U} \in terms_{\Sigma,FO}$ such that for every pair $P, Q \in T$ with $P \neq_{\beta\eta E} Q$, $(P\ \vec{U}) \neq_{\beta\eta E} (Q\ \vec{U})$, and $\beta\eta\text{-}nf(P\ \vec{U})$ and $\beta\eta\text{-}nf(Q\ \vec{U})$ have a variable at the head. Let $M' = \beta\eta\text{-}nf(M\ \vec{x})$, $N' = \beta\eta\text{-}nf(N\ \vec{x})$, $M'' = \beta\eta\text{-}nf(M\ \vec{U})$, and $N'' = \beta\eta\text{-}nf(N\ \vec{U})$. Therefore, the pair of terms

$$(top_{\Sigma,\{x_1,\dots,x_n\},E}(M'), top_{\Sigma,\{x_1,\dots,x_n\},E}(N'))$$

is the same as the pair

$$(top_{\Sigma,FO,E}(M''), top_{\Sigma,FO,E}(N''))$$

up to variable renaming. Since $M \neq_{\beta\eta E} N$,

$$top_{\Sigma,\{x_1,\dots,x_n\},E}(M') \neq_{\beta\eta E} top_{\Sigma,\{x_1,\dots,x_n\},E}(N'),$$

and hence $top_{\Sigma,FO,E}(M'') \neq_{\beta\eta E} top_{\Sigma,FO,E}(N'')$. By Lemmas 2.2 and 3.3, we have that $(M\ \vec{U}) \neq_{\beta\eta E} (N\ \vec{U})$.

2 Exactly one of M and N is in algebraic Φ-normal form. Without loss of generality, assume that M is in algebraic Φ-normal form and N is in regular Φ-normal form. For some $k, n \geqslant 0$ and algebraic term C,

$$M \quad = \quad \lambda x_1^{\sigma_1}.\ \dots \lambda x_n^{\sigma_n}.\ C[(M_1\ x_1 \dots x_n), \dots, (M_k\ x_1 \dots x_n)]$$

where the M_i's and N are regular Φ-normal forms. The case goes similarly to the previous case, taking $T = \{M_1, \dots, M_k, N\}$.

3 Both M and N are in regular Φ-normal form. There are two subcases:

(a) The head variables are unequal. Then for some $k, l, n \geqslant 0$ and $i \neq j$,

$$\begin{aligned} M &= \lambda x_1^{\sigma_1}.\ \dots \lambda x_n^{\sigma_n}.\ x_i\ (M_1\ x_1 \dots x_n)\ \dots\ (M_k\ x_1 \dots x_n) \\ N &= \lambda x_1^{\sigma_1}.\ \dots \lambda x_n^{\sigma_n}.\ x_j\ (N_1\ x_1 \dots x_n)\ \dots\ (N_l\ x_1 \dots x_n). \end{aligned}$$

Choose the sequence $\vec{V}$ such that $V_i = \lambda\vec{y}.\ z_1$ and $V_j = \lambda\vec{y}.\ z_2$, where $z_1^\iota, z_2^\iota \in FO$, and $z_1^\iota \neq z_2^\iota$; the other terms in the sequence may be chosen arbitrarily (but with the appropriate type). Then $(M\vec{V}) =_{\beta\eta E} z_1 \neq_{\beta\eta E} z_2 =_{\beta\eta E} (N\vec{V})$.

(b) The head variables are equal. For some $k, n \geqslant 1$,

$$\begin{aligned} M &= \lambda x_1^{\sigma_1}.\ \dots \lambda x_n^{\sigma_n}.\ x_i\ (M_1\ x_1 \dots x_n)\ \dots\ (M_k\ x_1 \dots x_n) \\ N &= \lambda x_1^{\sigma_1}.\ \dots \lambda x_n^{\sigma_n}.\ x_i\ (N_1\ x_1 \dots x_n)\ \dots\ (N_k\ x_1 \dots x_n). \end{aligned}$$

Since $M \neq_{\beta\eta E} N$, it must be the case that $M_j \neq_{\beta\eta E} N_j$ for some $1 \leqslant j \leqslant k$. By the induction hypothesis, there exist terms $U_1, \dots, U_n, \dots, U_m \in terms_{\Sigma,FO}$ such that $(M_j\ U_1 \dots U_m) \neq_{\beta\eta E} (N_j\ U_1 \dots U_m)$, and both $(M_j\ U_1 \dots U_m)$ and $(N_j\ U_1 \dots U_m)$ are of type ι. Choose fresh variables $h^{\iota\to\iota\to\iota} \in FO$ and $y_1, \dots, y_k$, that is, variables not appearing free in any of the terms $U_1, \dots, U_m$. For any $1 \leqslant p \leqslant n$, define

$$V_p = \begin{cases} \lambda y_1.\ \dots \lambda y_k.\ h\ (y_j\ U_{n+1} \dots U_m)\ (U_i\ y_1 \dots y_k) & \text{if } p = i \\ U_p & \text{otherwise.} \end{cases}$$

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We just need to verify that $(M\ V_1 \dots V_n) \neq_{\beta\eta E} (N\ V_1 \dots V_n)$. First, we do a little calculation:

$$\begin{aligned}(M\ V_1 \dots V_n) &=_{\beta\eta E} V_i\,(M_1\ V_1 \dots V_n) \dots (M_k\ V_1 \dots V_n)\\ &=_{\beta\eta E} h\,(M_j\ V_1 \dots V_n\ U_{n+1} \dots U_m)\,(U_i\,(M_1\ V_1 \dots V_n) \dots\\ &\qquad\qquad (M_k\ V_1 \dots V_n))\end{aligned}$$

Similarly,

$$\begin{aligned}(N\ V_1 \dots V_n) &=_{\beta\eta E} h\,(N_j\ V_1 \dots V_n\ U_{n+1} \dots U_m)\,(U_i\,(N_1\ V_1 \dots V_n) \dots\\ &\qquad\qquad (N_k\ V_1 \dots V_n)).\end{aligned}$$

By way of contradiction, assume that $(M\ V_1 \dots V_n) =_{\beta\eta E} (N\ V_1 \dots V_n)$. It follows from Lemmas 2.2 and 4.3 that

$$(M_j\ V_1 \dots V_n\ U_{n+1} \dots U_m) =_{\beta\eta E} (N_j\ V_1 \dots V_n\ U_{n+1} \dots U_m).$$

Let $H = \lambda u^\iota.\ \lambda v^\iota.\ v$; substituting H for variable h, we obtain

$$(M_j\ V_1 \dots V_n\ U_{n+1} \dots U_m)[h := H] =_{\beta\eta E} (N_j\ V_1 \dots V_n\ U_{n+1} \dots U_m)[h := H]. \quad (2)$$

But because we chose h to be fresh with respect to the terms $U_1, \dots, U_m$, h only occurs in the term V_i. Calculating,

$$\begin{aligned}V_i[h := H] &= \lambda y_1.\ \dots \lambda y_k.\ H\ (y_j\ U_{n+1} \dots U_m)\ (U_i\ y_1 \dots y_k)\\ &=_{\beta\eta} \lambda y_1.\ \dots \lambda y_k.\ (\lambda u.\ \lambda v.\ v)\ (y_j\ U_{n+1} \dots U_m)\ (U_i\ y_1 \dots y_k)\\ &=_{\beta\eta} \lambda y_1.\ \dots \lambda y_k.\ U_i\ y_1 \dots y_k\\ &=_{\beta\eta} U_i\end{aligned}$$

Thus,

$$\begin{aligned}(M_j\ V_1 \dots V_n\ U_{n+1} \dots U_m)[h := H] &=_{\beta\eta E} (M_j\ U_1 \dots U_n\ U_{n+1} \dots U_m)\\ (N_j\ V_1 \dots V_n\ U_{n+1} \dots U_m)[h := H] &=_{\beta\eta E} (N_j\ U_1 \dots U_n\ U_{n+1} \dots U_m).\end{aligned}$$

It follows from Equation (2) that

$$(M_j\ U_1 \dots U_n\ U_{n+1} \dots U_m) =_{\beta\eta E} (N_j\ U_1 \dots U_n\ U_{n+1} \dots U_m).$$

This contradicts our original choice of $U_1, \dots, U_m$, so we must have

$$(M\ V_1 \dots V_n) \neq_{\beta\eta E} (N\ V_1 \dots V_n).$$

This completes the induction and hence the proof. □

4.3. *Second main lemma*

Moving to the second main lemma, pick variables $f^{\iota\to\iota\to\iota}$ and x^ι. Define

$$V_i^{T^k} = \lambda y_1^\iota.\ \dots \lambda y_k^\iota.\ f\ \bar{i}\ (f\ y_1\ (f\ y_2\ (\dots(f\ y_k\ x)\dots)))$$

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where

$$\begin{aligned} \overline{0} &= x \\ \overline{n+1} &= (f\ \overline{n}\ x). \end{aligned}$$

Lemma 4.6. Let E be a consistent set of algebraic equations over signature Σ. For any $m \neq n$, $\overline{m} \neq_{\beta\eta E} \overline{n}$.

Proof. By Lemma 4.3, it is enough to show that for any $n > 0$, $\overline{0} \neq_{\beta\eta E} \overline{n}$. Suppose, by way of contradiction, that $\overline{0} = x =_{\beta\eta E} \overline{n} = (f\ \overline{n-1}\ x)$. Then

$$(\lambda f.\ x)\ (\lambda x.\ \lambda z.\ y) =_{\beta\eta E} (\lambda f.\ (f\ \overline{n-1}\ x))\ (\lambda x.\ \lambda z.\ y),$$

so $x =_{\beta\eta E} y$. By Lemma 2.2, $x =_E y$, a contradiction. Thus, $\overline{0} \neq_{\beta\eta E} \overline{n}$. □

Lemma 4.7. Suppose $M, N \in terms_{\Sigma,\varnothing}$, $M \neq_{\beta\eta E} N$, and the type of M and N is $\sigma = (\sigma_1 \to \ldots \to \sigma_n \to \iota)$, where each σ_i is a first-order type. Then $(M\ V_1^{\sigma_1} \ldots V_n^{\sigma_n}) \neq_{\beta\eta E} (N\ V_1^{\sigma_1} \ldots V_n^{\sigma_n})$.

Proof. We use induction on the sum of the sizes of M and N.

1 Both M and N are in algebraic Φ-normal form. Then for some $k, l, n \geqslant 0$, and terms C and D with C, D both beginning with a constant from Σ,

$$\begin{aligned} M &= \lambda x_1^{\sigma_1}.\ \ldots \lambda x_n^{\sigma_n}.\ C[(M_1\ x_1 \ldots x_n), \ldots, (M_k\ x_1 \ldots x_n)] \\ N &= \lambda x_1^{\sigma_1}.\ \ldots \lambda x_n^{\sigma_n}.\ D[(N_1\ x_1 \ldots x_n), \ldots, (N_l\ x_1 \ldots x_n)] \end{aligned}$$

where $M_1, \ldots, M_k, N_1, \ldots, N_l$ are regular Φ-normal forms. Let $M' = \beta\eta\text{-}nf(M\ \vec{x})$ and $N' = \beta\eta\text{-}nf(N\ \vec{x})$, $M'' = \beta\eta\text{-}nf(M\ \vec{V})$ and $N'' = \beta\eta\text{-}nf(N\ \vec{V})$. By the induction hypothesis, for every pair of terms $P, Q \in \{M_1, \ldots, M_k, N_1, \ldots, N_l\} = T$ such that $P \neq_{\beta\eta E} Q$, it must be that $(P\ \vec{V}) \neq_{\beta\eta E} (Q\ \vec{V})$. Note also that since all $P, Q \in T$ are regular Φ-normal forms, $\beta\eta\text{-}nf(P\ \vec{V})$ and $\beta\eta\text{-}nf(Q\ \vec{V})$ both begin with the function variable f. Therefore, the pair of terms

$$(top_{\Sigma,\{x_1,\ldots,x_n\},E}(M'), top_{\Sigma,\{x_1,\ldots,x_n\},E}(N'))$$

is the same as the pair

$$(top_{\Sigma,\{f\},E}(M''), top_{\Sigma,\{f\},E}(N''))$$

up to variable renaming. Since $M \neq_{\beta\eta E} N$,

$$top_{\Sigma,\{x_1,\ldots,x_n\},E}(M') \neq_{\beta\eta E} top_{\Sigma,\{x_1,\ldots,x_n\},E}(N'),$$

and hence $top_{\Sigma,\{f\},E}(M'') \neq_{\beta\eta E} top_{\Sigma,\{f\},E}(N'')$. Then, from Lemmas 2.2 and 3.3, $(M\ \vec{V}) \neq_{\beta\eta E} (N\ \vec{V})$.

2 Exactly one of M and N is in algebraic Φ-normal form. Without loss of generality, assume that M is in algebraic Φ-normal form and N is in regular Φ-normal form. For some $k, n \geqslant 0$ and algebraic term C,

$$M = \lambda x_1^{\sigma_1}.\ \ldots \lambda x_n^{\sigma_n}.\ C[(M_1\ x_1 \ldots x_n), \ldots, (M_k\ x_1 \ldots x_n)]$$

where the M_i's and N are regular Φ-normal forms. The case goes similarly to the previous case, taking $T = \{M_1, \ldots, M_k, N\}$.

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3 Both M and N are in regular Φ-normal form. There are two subcases:

(a) The head variables are unequal. Then for some $k, l, n \geqslant 0$ and $i \neq j$,

$$\begin{aligned} M &= \lambda x_1^{\sigma_1}. \ldots \lambda x_n^{\sigma_n}. x_i\,(M_1\,x_1 \ldots x_n)\ \ldots\ (M_k\,x_1 \ldots x_n) \\ N &= \lambda x_1^{\sigma_1}. \ldots \lambda x_n^{\sigma_n}. x_j\,(N_1\,x_1 \ldots x_n)\ \ldots\ (N_l\,x_1 \ldots x_n). \end{aligned}$$

Note that $\beta\eta\text{-}nf(M\ \vec{V}) = (f\ \bar{i}\ t_1)$, and, similarly, that $\beta\eta\text{-}nf(N\ \vec{V}) = (f\ \bar{j}\ t_2)$. By Lemma 4.6 $\bar{i} \neq_E \bar{j}$. By Lemma 4.3, $(M\ \vec{V}) \neq_{\beta\eta E} (N\ \vec{V})$.

(b) The head variables are equal. For some $k, n \geqslant 1$ and $i \neq j$,

$$\begin{aligned} M &= \lambda x_1^{\sigma_1}. \ldots \lambda x_n^{\sigma_n}. x_i\,(M_1\,x_1 \ldots x_n)\ \ldots\ (M_k\,x_1 \ldots x_n) \\ N &= \lambda x_1^{\sigma_1}. \ldots \lambda x_n^{\sigma_n}. x_i\,(N_1\,x_1 \ldots x_n)\ \ldots\ (N_k\,x_1 \ldots x_n). \end{aligned}$$

Notice that k must be greater than 0, and that for some j, $M_j \neq_{\beta\eta E} N_j$. Thus, by the induction hypothesis, $(M_j\ \vec{V}) \neq_{\beta\eta E} (N_j\ \vec{V})$. Now doing some calculation,

$$\begin{aligned} (M\ \vec{V}) &=_{\beta\eta E} (f\ \bar{i}\,(f\ (M_1\ \vec{V})\,(\ldots(f\ (M_{k-1}\ \vec{V})\,(f\ (M_k\ \vec{V})\ x))\ldots))) \\ (N\ \vec{V}) &=_{\beta\eta E} (f\ \bar{i}\,(f\ (N_1\ \vec{V})\,(\ldots(f\ (N_{k-1}\ \vec{V})\,(f\ (N_k\ \vec{V})\ x))\ldots))). \end{aligned}$$

Suppose, by way of contradiction, that $(M\ \vec{V}) =_{\beta\eta E} (N\ \vec{V})$. Using Lemma 4.3 repeatedly, we then get $(M_j\ \vec{V}) =_{\beta\eta E} (N_j\ \vec{V})$, which is a contradiction. Thus, $(M\ \vec{V}) \neq_{\beta\eta E} (N\ \vec{V})$.

This completes the induction and hence the proof. □

4.4. *Putting it together*

Theorem 4.8. Suppose Σ is an algebraic signature, X is an infinite set of variables of type ι, and $f^{\iota\to\iota\to\iota}$ is a variable. Let E be a Σ,λ-theory, and E_1 be the restriction of E to $algterms_{\Sigma,X}$. Then (β), (η) and E_1 prove all equations in E iff E_1 and $(extCong)$ prove all equations in

$$E_2 = \{(t_1 = t_2) \in E \mid t_1, t_2 \in algterms_{\Sigma,\{f^{\iota\to\iota\to\iota}\},X}\}.$$

Proof. ($\Rightarrow$) Let $t_1, t_2 \in algterms_{\Sigma,\{f^{\iota\to\iota\to\iota}\},X}$, and suppose $(t_1 = t_2) \in E_2$. Regard $t_1 = t_2$ as an 'algebraic' equation over the algebraic signature $\Sigma \cup \{f\}$, f being a symbol of arity 2. Since $t_1 =_{\beta\eta E} t_2$, by Lemma 2.2, $t_1 =_{E_1} t_2$.

($\Leftarrow$) Suppose E_1 and $(extCong)$ prove all equations in E_2. It is obvious that if (β), (η) and E_1 prove an equation, that equation is also in E; this follows merely from the fact that E is a Σ,λ-theory. So suppose $M \neq_{\beta\eta E_1} N$. Without loss of generality, we may assume that M, N are closed terms. By Lemma 4.5, there are terms $U_1, \ldots, U_m$ in $terms_{\Sigma,FO}$ such that $(M\ U_1 \ldots U_m) \neq_{\beta\eta E_1} (N\ U_1 \ldots U_m)$ at base type. Let $y_1, \ldots, y_k$ be the free (and necessarily first-order) variables of $U_1, \ldots, U_m$. Then

$$M' = (\lambda y_1. \ldots \lambda y_k. M\ U_1 \ldots U_m) \neq_{\beta\eta E_1} (\lambda y_1. \ldots \lambda y_k. N\ U_1 \ldots U_m) = N'.$$

Now, by Lemma 4.7, there is a sequence of terms $V_1, \ldots, V_n$ in $terms_{\Sigma,f,x}$ such that $(M'\ V_1 \ldots V_n) \neq_{\beta\eta E_1} (N'\ V_1 \ldots V_n)$ at base type. By Lemma 2.2,

$$\beta\eta\text{-}nf(M'\ V_1 \ldots V_n) \neq_{E_1+\{f\}} \beta\eta\text{-}nf(N'\ V_1 \ldots V_n).$$

By hypothesis, this equation is not in E_2, and hence not in E. Therefore, $(M = N)$ must not be in E either. □

It may be helpful to consider a slight restatement of the above theorem (suggested by a referee) to get a sense of the types involved. Suppose E is a Σ, λ-theory, and

$$\begin{aligned} E_1 &= \{(t_1 = t_2) \in E \mid t_1, t_2 \text{ are closed of first-order type}\} \\ E_2 &= \{(t_1 = t_2) \in E \mid t_1, t_2 \text{ are closed of } (\iota \to \iota \to \iota) \to \iota \to \iota\}. \end{aligned}$$

Essentially the same proof as above shows that

Theorem 4.9. E_1, (β) and (η) prove all the equations of E iff E_1, (β) and (η) prove all the equations of E_2.

One might be tempted to say that E_2 alone determines E, but this is not true. For a counterexample, consider the language with base type constants Ω, *true*, *false* (all of type ι), and *if* $: \iota \to \iota \to \iota \to \iota$. Consider the Σ, λ-theory generated by the fully abstract model of sequential PCF over the booleans. The subtheory consisting of equations between closed terms of order no more than three is decidable (Sieber, 1992). The full theory is undecidable (Loader, 1997). Therefore, the full theory is not generated from the subtheory: there are two ways of generating a theory from the subtheory, one through the use of $\beta\eta$ (which will be a decidable theory), and the other to the full theory.

4.5. *Application to models*

Theorem 4.10. Suppose Σ is an algebraic signature, X is an infinite set of variables of type ι, and $f^{\iota\to\iota\to\iota}$ is a variable. Suppose $\mathscr{M}$ is a Σ-model and E_1 is a set of algebraic equations over signature Σ. Then (β), (η) and E_1 completely axiomatize the equations of $\mathscr{M}$ iff E_1 and $(extCong)$ prove all equations between terms in $algterms_{\Sigma,\{f^{\iota\to\iota\to\iota}\},X}$ that are valid in $\mathscr{M}$.

Proof. The proof is immediate from Theorem 4.8, taking $E = Th(\mathscr{M})$. □

Theorem 4.10 is difficult to apply directly. Suppose we are given a Σ-model $\mathscr{M}$ and a set E of algebraic equations over the signature Σ. To use the theorem, we need to consider all extended algebraic terms $algterms_{\Sigma,\{f\},X}$, and prove that E and $(extCong)$ prove all extended algebraic equations that are valid in $\mathscr{M}$. It would be simpler to apply a theorem that depended only on properties of the *induced algebra* of $\mathscr{M}$, and some minimal property on the rest of the model. The minimal property on the model that we use in the next two sections is given by the following definition.

Definition 4.11. A model $\mathscr{M}$ has **enough first-order functions** if every binary, finite partial function over the base type ι can be extended to some element in the model. More precisely, if $g : \mathscr{M}^\iota \times \mathscr{M}^\iota \rightharpoonup \mathscr{M}^\iota$ is a finite partial function, there is an $f \in \mathscr{M}^{\iota\to\iota\to\iota}$ such that for all $(d, e) \in dom(g)$, $ap(ap(f, d), e) = g(d, e)$.

The goal in the next two sections is to find suitable conditions on the algebra.

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5. Disjunctively closed algebras

A Σ-algebra $\mathscr{A}$ is **disjunctively closed** if for every finite set of algebraic equations E, none of which is valid in $\mathscr{A}$, there is a *single* $\mathscr{A}$-environment σ such that for every equation $s = t \in E$, $\mathscr{A}[\![s]\!]\sigma \neq \mathscr{A}[\![t]\!]\sigma$. Equivalently, $\mathscr{A}$ is disjunctively closed if for every valid disjunction $(E_1 \vee \ldots \vee E_n)$ of equations, where the variables are universally quantified over the entire formula, at least one of the equations E_i is valid in $\mathscr{A}$.

The property of disjunctive closedness may sound *ad hoc* at first glance, but the theory of the simply-typed λ-calculus *without* algebraic equations already satisfies the following analogous property.

Theorem 5.1. If $\beta\eta$-reasoning is complete for a model $\mathscr{M}$ over the empty signature, and if the equations $(M_1 = N_1), \ldots, (M_k = N_k)$ are not valid in $\mathscr{M}$, there is a single $\mathscr{M}$-environment ρ such that $\mathscr{M}[\![M_i]\!]\rho \neq \mathscr{M}[\![N_i]\!]\rho$ for all i.

Proof. We use the restrictions of Lemmas 4.4, 4.5, and 4.7 to the simply-typed λ-calculus without constants from Statman (1982). We begin by transforming each M_i and N_i into closed terms of the same type σ: we can apply each side of each equation $(M_i = N_i)$ to fresh variables $\vec{x}_i$ to get base type terms, and then λ-abstract each M_i and N_i over *all* the free variables of $M_1, \ldots, M_k, N_1, \ldots, N_k$ and $\vec{x}_1, \ldots, \vec{x}_k$. Call the results of this transformation M_i' and N_i'. By the analogue of Lemma 4.5, there are vectors $\vec{V}_i$ of terms in $terms_{\varnothing,FO}$ for each equation with $(M_i' \, \vec{V}_i) \neq_{\beta\eta} (N_i' \, \vec{V}_i)$. By the analogue of Lemma 4.4, we can collapse these vectors into a single distinguishing vector $\vec{V}$ of terms in $terms_{\varnothing,FO}$. Let $x_1, \ldots, x_k$ be the free variables of $\vec{V}$, and consider the terms

$$\begin{array}{rcl} M_i'' & = & (\lambda x_1. \ \ldots \lambda x_k. \ M_i' \, \vec{V}) \\ N_i'' & = & (\lambda x_1. \ \ldots \lambda x_k. \ N_i' \, \vec{V}) \end{array}$$

By the analogue of Lemma 4.7, there is a single vector $\vec{U}$ of terms in $terms_{\varnothing,\{f^{\iota\to\iota\to\iota},x^{\iota}\}}$ for all i, $(M_i'' \, \vec{U}) \neq_{\beta\eta E} (N_i'' \, \vec{U})$. By completeness, there is an environment ρ' – in particular, a choice for f and x – such that $\mathscr{M}[\![(M_i'' \, \vec{U})]\!]\rho' \neq \mathscr{M}[\![(N_i'' \, \vec{U})]\!]\rho'$. It is then not hard to turn this into a distinguishing environment ρ for the original terms (by using the meanings of $\vec{V}$ and $\vec{U}$). □

Not every algebra is disjunctively closed. Trivially, any finite algebra with at least two elements cannot be disjunctively closed: if the algebra has $n > 1$ elements, the distinct variables $x_1, \ldots, x_{n+1}$ are pairwise distinguishable, but there is no single environment that distinguishes all of them. Consequently, if we consider the set of equations $\{x_i = x_j \mid i \neq j\}$, none of the equations is valid, and yet no single environment can invalidate all the equations. Another example is the $\{iszero\}$-algebra $(\mathbf{N}, \mathscr{K})$, where $\mathscr{K}(iszero)$ is the unary function that returns zero on a non-zero argument, and one on zero. This algebra is not disjunctively closed because four terms – x, y, $iszero(x)$, and $iszero(y)$ – are pairwise unequal but not simultaneously distinguishable. Nevertheless, some interesting algebras are disjunctively closed.

Theorem 5.2. The following algebras are disjunctively closed.

— The $\{0, 1, +, *\}$-algebra $\mathscr{N} = (\mathbf{N}, \mathscr{K})$ with the evident interpretation.

— The $\{0, succ, pred\}$-algebra $(\mathbf{Z}, \mathscr{K})$ with the evident interpretation.
— The $\{nil, append, \langle 0\rangle, \langle 1\rangle, \dots\}$-algebra $(\mathbf{NatList}, \mathscr{K})$, where the carrier is the lists over the naturals, $\langle i\rangle$ is interpreted as the singleton list constructor, and *append* is interpreted as the list appending function.
— The $\{\varnothing, \cup, \cap\}$-algebra $(\mathbf{Set}, \mathscr{K})$, where the carrier consists of sets over some unspecified infinite type.

Proof. We prove the disjunctive closedness of the first algebra $\mathscr{N}$ here, using a simplification of the proof of Benedikt *et al.* (1998, Lemma 4), and leave the others to the reader. Let $\Sigma = \{0, 1, +, *\}$ and $E = Th(\mathscr{N})$. Suppose $E_0 = \{t_1 = t'_1, \dots, t_m = t'_m\}$ is a set of equations such that $t_i, t'_i \in algterms_{\Sigma,X}$, and, for all i, $t_i \neq_E t'_i$. We have to find a single distinguishing environment for all the equations. Assume that the variables appearing in $t_1, t'_1, \dots, t_m, t'_m$ are $x_1, \dots, x_n$. We find an environment ρ by induction on n. For $n = 0$, any environment works. Now suppose $n = (k+1)$. Notice that, for each $j \leqslant m$, the jth equation in E_0 can be written in the form

$$(\sum_{i\in I_j} p_{i,j}(x_{k+1}) * v_{i,j}) \quad = \quad 0.$$

In this notation, each I_j is a finite set, each $p_{i,j}(x_{k+1})$ is a polynomial over x_{k+1}, each $v_{i,j}$ is a product $(x_1^{e_1} * \dots * x_k^{e_k})$ with $e_l \geqslant 0$, and for any $i, l \in I_j$ such that $i \neq l$, $v_{i,j}$ and $v_{l,j}$ are different products of $x_1, \dots, x_k$. Now choose a value r for x_{k+1} such that all the non-zero polynomials among the polynomials $p_{i,j}(x_{k+1})$ evaluate to a non-zero value; this is possible because there are only a finite number of roots for each polynomial. Evaluate the polynomials, and substitute the value in each of the above equations, leaving a set of equations over the variables $x_1, \dots, x_k$; note that this does not result in any of the equations becoming valid. By the induction hypothesis, there is an environment ρ that distinguishes all of these equations. Then the environment $\rho[x_{k+1} \mapsto r]$ distinguishes all of the equations in E_0. □

5.1. *Completeness for disjunctively closed algebras*

We show that $\beta\eta E$ is complete for reasoning about the equality of mixed terms in a Σ-model $\mathscr{M}$ that has enough first-order functions, and whose induced algebra is disjunctively closed and is axiomatized by E. Using Theorem 4.10, it is enough to consider terms in $algterms_{\Sigma,\{f^{\iota\to\iota\to\iota}\},X}$. So suppose M and N are such terms with $M \neq_{E+\{f\}} N$; we need to find a distinguishing environment in the model, namely, an environment ρ such that $\mathscr{M}[\![M]\!]\rho \neq \mathscr{M}[\![N]\!]\rho$. The proof carves M and N up into subterms that are purely algebraic, and then uses disjunctive closedness of the algebra to find a distinguishing environment for certain equations between the subterms.

We begin with some notation. Fix an algebraic signature Σ, a set X of variables of type ι, and a disjunctively closed Σ-algebra $\mathscr{B}$. Denote the equational theory of the algebra by E. Let $f^{\iota\to\iota\to\iota}$ be a variable. We say that a term in $algterms_{\Sigma,\{f\},X}$ is **Σ-top** if it has a symbol from $\Sigma \cup X$ at its root (non-Σ-top terms must thus begin with an f). When given a Σ-top term, we can traverse each branch downwards and stop when we encounter an occurrence of the symbol f: the subterm at each such point is a non-Σ-top term. In this

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manner we can decompose the given term into a purely algebraic Σ-term and a set of non-Σ-top terms.

Lemma 5.3. If t is Σ-top, there is a unique non-trivial Σ-term $C[x_1, \ldots, x_n]$ and non-Σ-top terms $s_1, \ldots, s_n$ such that $t = C[s_1, \ldots, s_n]$.

Proof. The proof is by structural induction on t. □

Our goal is to decompose a term – or, more generally, a set of terms – over Σ and f into a set of algebraic terms over Σ. Recall, from Section 3, the function $top_{\Sigma,\{f\},E}$ from terms to terms, and the function $Rep_{\Sigma,\{f\},E}$ from $algterms_{\Sigma,\{f\},X}$ to Y, where Y is a set of variables of type ι disjoint from $X \cup \{f\}$. Recall also that $Rep_{\Sigma,\{f\},E}(t) = Rep_{\Sigma,\{f\},E}(s)$ iff $t =_{E+\{f\}} s$. We define a variant of *top* that applies *top* to the term t and then replaces each occurrence of a variable $x \in X$ in it by $Rep_{\Sigma,\{f\},E}(x)$, yielding a term in $algterms_{\Sigma,Y}$. To that end, define

$$topp(t) = \theta(top_{\Sigma,\{f\},E}(t)),$$

where θ is a substitution such that $dom(\theta) = X$ and for all $x \in X$, $\theta(x) = Rep_{\Sigma,\{f\},E}(x)$.

Lemma 5.4. If $topp(t) =_E topp(t')$, then $t =_{E+\{f\}} t'$.

Proof. Let $y_1, \ldots, y_n$ be the free variables in $topp(t)$ and $topp(t')$. Then for some $t_1, \ldots, t_n$,

$$\begin{array}{rcl} t &=& topp(t)[y_1 \mapsto t_1, \ldots, y_n \mapsto t_n] \\ t' &=& topp(t')[y_1 \mapsto t_1, \ldots, y_n \mapsto t_n] \end{array}$$

Therefore, $topp(t) =_E topp(t')$ implies $t =_{E+\{f\}} t'$. □

The operation *topp* is just what we need to decompose a set of algebraic terms over Σ and f into a set over Σ. Recall that a Σ' **extension** of a Σ-algebra, where $\Sigma \subseteq \Sigma'$, has the same carrier as the Σ-algebra $\mathscr{A}$, and interprets the additional symbols in Σ' as well.

Lemma 5.5. Suppose $\mathscr{B}$ is a disjunctively closed Σ-algebra, $\mathscr{E} = Th(\mathscr{B})$ and $t, t' \in algterms_{\Sigma,\{f\},X}$, and $t \neq_{\mathscr{E}+\{f\}} t'$. Then there is a finite, partial, binary function g over the carrier of $\mathscr{B}$ such that for any $\Sigma \cup \{f\}$-algebra $\mathscr{A}$ extending $\mathscr{B}$ with $\mathscr{A}[\![f]\!] \restriction dom(g) = g$, there is an environment σ_X such that $\mathscr{A}[\![t]\!]\sigma_X \neq \mathscr{A}[\![t']\!]\sigma_X$.

Proof. Let $S = \{topp(u) \mid u \text{ is a subterm of } t \text{ or } t'\}$. Since $\mathscr{B}$ is disjunctively closed, there is a $\mathscr{B}$-environment σ_Y such that for every pair of terms $s, s' \in S$, if $s \neq_{\mathscr{E}+\{f\}} s'$, then $\mathscr{B}[\![s]\!]\sigma_Y \neq \mathscr{B}[\![s']\!]\sigma_Y$. Since $t \neq_{\mathscr{E}+\{f\}} t'$, by Lemma 5.4, $topp(t) \neq_{\mathscr{E}+\{f\}} topp(t')$. It follows that

$$\mathscr{B}[\![topp(t)]\!]\sigma_Y \neq \mathscr{B}[\![topp(t')]\!]\sigma_Y.$$

Define g as follows. Suppose $(f\ s\ s')$ is a subterm of t or t', and $u = topp(s)$, $u' = topp(s')$. Let

$$\begin{array}{rcl} a &=& \mathscr{B}[\![u]\!]\sigma_Y \\ b &=& \mathscr{B}[\![u']\!]\sigma_Y. \end{array}$$

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Then define

$$g\ a\ b = \sigma_Y(Rep_{\Sigma,\{f\},E}(f\ s\ s')).$$

Note that this function is well defined.

Let $\mathscr{A}$ be any algebra extending $\mathscr{B}$ to the signature $\Sigma \cup \{f\}$ with $\mathscr{A}[\![f]\!] \restriction dom(g) = g$. Let σ_X be any $\mathscr{A}$-environment such that for all $x \in X$, $\sigma_X(x) = \sigma_Y(Rep_{\Sigma,\{f\},E}(x))$. We claim that for any term $s \in S$, $\mathscr{A}[\![s]\!]\sigma_X = \mathscr{B}[\![topp(s)]\!]\sigma_Y$. From this it follows that

$$\mathscr{A}[\![t]\!]\sigma_X \neq \mathscr{A}[\![t']\!]\sigma_X.$$

The proof of the claim goes by induction on the structure of s.

— $s = x$. Then $topp(s) = y$, where $Rep_{\Sigma,\{f\},E}(x) = y$. Then

$$\mathscr{A}[\![s]\!]\sigma_X = \sigma_X(x) = \sigma_Y(Rep_{\Sigma,\{f\},E}(x)) = \sigma_Y(y) = \mathscr{B}[\![y]\!]\sigma_Y = \mathscr{B}[\![topp(s)]\!]\sigma_Y.$$

— $s = (f\ s_1\ s_2)$ and $topp(s) = y$, where $y = Rep_{\Sigma,\{f\},E}(s)$. By the induction hypothesis,

$$\begin{aligned}\mathscr{A}[\![s]\!]\sigma_X &= (\mathscr{A}[\![f]\!]\ \mathscr{A}[\![s_1]\!]\sigma_X\ \mathscr{A}[\![s_2]\!]\sigma_X)\\ &= (\mathscr{A}[\![f]\!]\ \mathscr{B}[\![topp(s_1)]\!]\sigma_Y\ \mathscr{B}[\![topp(s_2)]\!]\sigma_Y)\\ &= \sigma_Y(Rep_{\Sigma,\{f\},E}(f\ s_1\ s_2))\\ &= \mathscr{B}[\![topp(s)]\!]\sigma_Y.\end{aligned}$$

— $s = (c\ s_1\ \dots\ s_n)$, where $c \in \Sigma$. By the induction hypothesis,

$$\begin{aligned}\mathscr{A}[\![s]\!]\sigma_X &= (\mathscr{A}[\![c]\!]\ \mathscr{A}[\![s_1]\!]\sigma_X \dots \mathscr{A}[\![s_n]\!]\sigma_X)\\ &= (\mathscr{A}[\![c]\!]\ \mathscr{B}[\![topp(s_1)]\!]\sigma_Y \dots \mathscr{B}[\![topp(s_n)]\!]\sigma_Y)\\ &= \mathscr{B}[\![topp(s)]\!]\sigma_Y.\end{aligned}$$

This completes the proof of the claim, and hence the lemma. □

We now have enough facts to prove the following theorem.

Theorem 5.6. Let $\mathscr{M}$ be a model, $\mathscr{B}$ be the Σ-algebra induced by $\mathscr{M}$, and E be a complete axiomatization of equations valid in $\mathscr{B}$. Suppose $\mathscr{B}$ is disjunctively closed, and $\mathscr{M}$ has enough first-order functions. Then (β), (η) and E completely axiomatize equality between mixed terms in $\mathscr{M}$.

Proof. Suppose $M \neq_{\beta\eta E} N$ for closed terms M and N. Then by Lemmas 4.5 and 4.7, there exists a term $V \in terms_{\Sigma,\{x^\iota, f^{\iota\to\iota\to\iota}\}}$ (actually, V contains no symbols from Σ, but this fact is not important) such that

$$(V\ M) \neq_{\beta\eta E} (V\ N)$$

at type ι. Thus,

$$\beta\eta\text{-}nf(V\ M) \neq_E \beta\eta\text{-}nf(V\ N).$$

Since the $\beta\eta$-normal forms of $(V\ M)$ and $(V\ N)$ are in $algterms_{\Sigma,\{f\},X}$, by Lemma 5.5 there is a $\Sigma \cup \{f\}$-extension $\mathscr{A}$ of $\mathscr{B}$ such that the currying of the interpretation of f in $\mathscr{A}$ is in

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$\mathscr{M}^{\iota\to\iota\to\iota}$ and

$$\mathscr{A} \not\models \beta\eta\text{-}nf(V\ M) = \beta\eta\text{-}nf(V\ N).$$

Thus $\mathscr{M} \not\models M = N$. □

Corollary 5.7. The full type hierarchy over any of the algebras from Theorem 5.2 is completely axiomatized by (β), (η) and the equations of the algebra.

5.2. *Continuous models over flat algebras*

Continuous type hierarchies over algebras form an important class of models: they can be used to give semantics to programming languages with recursion. In this section we establish the fact that disjunctive closedness of a flat algebra $\mathscr{B}$ is a sufficient condition for completeness of $\beta\eta E$-reasoning for proving valid equations in the full continuous type hierarchy over $\mathscr{B}$.

Two technical problems arise in proving such a theorem for full continuous models. First, it seems difficult to apply our techniques when the base type has some arbitrary partial order structure. We can, however, use our techniques in models whose induced algebra is a **flat algebra**, *i.e.*, an algebra with a flat poset (with a distinguished least element $\bot$) as the carrier, and whose algebraic operations are monotone (and thus continuous) functions. Second, we cannot directly apply Theorem 5.6, because the 'has enough first-order functions' condition does not hold in the full continuous type hierarchy. For instance, any non-monotonic, partial function cannot be extended to an element in the model. The proof must be careful in constructing a monotone, partial function.

We need the following two lemmas.

Lemma 5.8. Let $\mathscr{B}$ be a disjunctively closed, flat Σ-algebra, and E be a finite set of equations between terms in $algterms_{\Sigma,X}$ such that no equation in E is valid in $\mathscr{B}$. Then there is an environment σ such that:

1 If $(s = t) \in E$, then $\mathscr{B}[\![s]\!]\sigma \neq \mathscr{B}[\![t]\!]\sigma$.
2 For all x, $\sigma(x) \neq \bot$.

Proof. Let

$$E_1 \quad = \quad \{(s = x) \mid (s = t) \text{ or } (t = s) \in E,\ x \in FV(s), \text{ and } \mathscr{B} \not\models (s = x)\}$$

and let $E_2 = E \cup E_1$. Note that E_2 is finite. By disjunctive closedness of $\mathscr{B}$, there is an environment σ_0 that distinguishes all equations in E_2. We turn the environment σ_0 into another environment σ as follows. Define

$$R = \{\mathscr{B}[\![s]\!]\sigma_0 \mid (s = t) \text{ or } (t = s) \in E_2\} \cup \{\bot\}$$

and pick b to be an element of $\mathscr{B}$ that is not in R; such a b must exist because the carrier of $\mathscr{B}$ must be infinite and R is finite. Then let

$$\sigma(x) = \begin{cases} \sigma_0(x) & \text{if } \sigma_0(x) \neq \bot \\ b & \text{if } \sigma_0(x) = \bot. \end{cases}$$

We claim that σ – which maps all variables to non-$\bot$ elements – distinguishes all equations

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in E_2, and hence all equations in E. Suppose, by way of contradiction, that $\mathscr{B}[\![s]\!]\sigma = \mathscr{B}[\![t]\!]\sigma$ with $(s = t) \in E_2$. Since $\mathscr{B}[\![s]\!]\sigma_0 \neq \mathscr{B}[\![t]\!]\sigma_0$,

$$\begin{aligned} &\mathscr{B}[\![s]\!]\sigma_0 = \bot \text{ and } \mathscr{B}[\![t]\!]\sigma_0 = \mathscr{B}[\![t]\!]\sigma \\ \text{or}\quad &\mathscr{B}[\![t]\!]\sigma_0 = \bot \text{ and } \mathscr{B}[\![s]\!]\sigma_0 = \mathscr{B}[\![s]\!]\sigma. \end{aligned}$$

because for any term $t \in S$, $\mathscr{B}[\![t]\!]\sigma_0 \sqsubseteq \mathscr{B}[\![t]\!]\sigma$. Without loss of generality, assume the former. Since $\mathscr{B}[\![s]\!]\sigma_0 \neq \mathscr{B}[\![s]\!]\sigma$, for some $x \in FV(s)$, it must be that $\sigma_0(x) = \bot$. Therefore, $\mathscr{B}[\![s]\!]\sigma_0 = \mathscr{B}[\![x]\!]\sigma_0$. So $(s = x) \notin E_1$ and thus $\mathscr{B} \models (s = x)$. By the definition of $\sigma(x)$, we have $\sigma(x) = b \neq \mathscr{B}[\![t]\!]\sigma_0 = \mathscr{B}[\![t]\!]\sigma$. Since $\mathscr{B} \models (s = x)$, we contradict $\mathscr{B}[\![s]\!]\sigma = \mathscr{B}[\![t]\!]\sigma$. □

Lemma 5.9. Suppose $\mathscr{B}$ is a flat, disjunctively closed Σ-algebra, $t, t' \in algterms_{\Sigma,\{f\},X}$, and $t \neq_{E+\{f\}} t'$. Then there is a finite, partial, and monotone binary function g over the carrier of $\mathscr{B}$ such that:

1 $\bot$ is not in the domain or range of g; and
2 For any $\Sigma \cup \{f\}$-algebra $\mathscr{A}$ extending the Σ-algebra $\mathscr{B}$ with $\mathscr{A}[\![f]\!] \upharpoonright dom(g) = g$, there is an environment σ_X such that $\mathscr{A}[\![t]\!]\sigma_X \neq \mathscr{A}[\![t']\!]\sigma_X$.

Proof. The proof is exactly as in Lemma 5.5, using Lemma 5.8 to turn environments into environments that map all variables to non-$\bot$ values. □

Using this lemma, we can prove the following theorem in exactly the same way as we proved Theorem 5.6.

Theorem 5.10. If a Σ-algebra with an underlying flat cpo structure is disjunctively closed, the continuous type hierarchy over it is completely axiomatized by (β), (η) and the equational theory of the algebra.

In the rest of this section we focus on flat algebras whose operations preserve as well as reflect $\bot$. More precisely, given a Σ-algebra $\mathscr{A}$, the **lifted algebra** $\mathscr{A}_\bot$ is constructed by adding a least element $\bot$, defining $\sqsubseteq$ on elements of $\mathscr{A}_\bot$ by

$$a \sqsubseteq a' \text{ iff } a = \bot \text{ or } a = a',$$

and defining, for any $c \in \Sigma$

$$\begin{aligned} (\mathscr{A}_\bot[\![c]\!]\ a_1 \dots a_n) &= \bot \text{ if any } a_i = \bot \\ (\mathscr{A}_\bot[\![c]\!]\ a_1 \dots a_n) &= (\mathscr{A}[\![c]\!]\ a_1 \dots a_n) \text{ if no } a_i = \bot. \end{aligned}$$

The following lemma gives a simple test for the disjunctive closedness of lifted algebras.

Lemma 5.11. $\mathscr{A}_\bot$ is disjunctively closed iff $\mathscr{A}$ is disjunctively closed and every equation valid in $\mathscr{A}$ has the same free variables occurring on both sides.

Proof. ($\Leftarrow$) Let E be a nonempty set of equations none of which is valid in $\mathscr{A}_\bot$. We claim that none of the equations are valid in $\mathscr{A}$. From this we can extract an environment σ in $\mathscr{A}$ by disjunctive closedness that distinguishes all the equations in E; this same σ invalidates in $\mathscr{A}_\bot$ every equation in the set E. To prove the claim, suppose $(s = t) \in E$. We know that there is an $\mathscr{A}_\bot$-environment σ such that $\mathscr{A}_\bot[\![s]\!]\sigma \neq \mathscr{A}_\bot[\![t]\!]\sigma$. Since every equation valid in $\mathscr{A}$ has the same variables occurring on both sides, $FV(s) = FV(t)$. Thus,

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for all $x \in FV(s)$, we have $\sigma(x) \neq \bot$ (if it were $\bot$, both $\mathscr{A}_\bot[\![s]\!]\sigma$ and $\mathscr{A}_\bot[\![t]\!]\sigma$ would be $\bot$). But then σ is an $\mathscr{A}$-environment, so $\mathscr{A} \not\models s = t$, proving the claim.

($\Rightarrow$) Let $\mathscr{E}$ be the equational theory of $\mathscr{A}$. First, we show that $\mathscr{A}$ is disjunctively closed. Let E_0 be a set of equations, none of which is valid in $\mathscr{A}$. Then none of these equations can be valid in $\mathscr{A}_\bot$ either. Let

$$E_1 \quad = \quad \{(s = x) \mid (s = t) \text{ or } (t = s) \in \mathscr{E},\ x \in FV(s), \text{ and } \mathscr{A}_\bot \not\models (s = x)\},$$

and let $E_2 = E_0 \cup E_1$. Note that E_2 is finite. By disjunctive closedness of $\mathscr{A}_\bot$, there is an environment σ_0 that distinguishes all equations in E_2. As in the proof of Lemma 5.8 above, there is an environment σ distinguishing all equations in E_2 – and hence in E – that does not assign any variable to $\bot$. Thus, $\mathscr{A}$ is disjunctively closed.

To show that every equation in $\mathscr{E}$ has the same variables occurring on both sides, it suffices to show (up to symmetry) that if $s = t$ is an equation with a variable x free in s but not in t, then $\mathscr{A} \not\models s = t$. Let $s = t$ be such an equation. Suppose $\mathscr{A} \models s = x$. If $\mathscr{A} \models s = t$, then $\mathscr{A} \models x = t$, making $\mathscr{A}$ the trivial (one-element) algebra. However, this would make $\mathscr{A}_\bot$ a two-element algebra, contradicting its disjunctive closedness. Thus, $\mathscr{A} \not\models s = t$.

Next, suppose $\mathscr{A} \not\models s = x$. Let x' be a fresh variable. It is clear that $\mathscr{A}_\bot \not\models s = x$, $\mathscr{A}_\bot \not\models s = s[x := x']$ and $\mathscr{A}_\bot \not\models s[x := x'] = x'$. By disjunctive closedness of $\mathscr{A}_\bot$, there is an environment σ that invalidates all three equations. Since x is a free variable of s, it is clear that $\sigma(x), \sigma(x') \neq \bot$. Furthermore, $\sigma(y) \neq \bot$ for any other variable y free in s, since otherwise σ cannot distinguish $s = s[x := x']$. Let σ' be any environment that maps all variables to non-$\bot$ values, and agrees with σ on $FV(s) \cup \{x'\}$. Since $\mathscr{A}_\bot[\![s]\!]\sigma' \neq \mathscr{A}_\bot[\![s[x := x']]\!]\sigma'$, we can conclude that $\mathscr{A}[\![s]\!]\sigma' \neq \mathscr{A}[\![s[x := x']]\!]\sigma'$. If $\mathscr{A} \models s = t$, noting that x does not occur in t, we would have $\mathscr{A} \models s[x := x'] = t$, contradicting $\mathscr{A}[\![s]\!]\sigma' \neq \mathscr{A}[\![s[x := x']]\!]\sigma'$. Therefore, it must be the case that $\mathscr{A} \not\models s = t$. □

Corollary 5.12. The following algebras are disjunctively closed, but their liftings are not:

— The $\{0, 1, +, *\}$-algebra $\mathscr{N} = (\mathbf{N}, \mathscr{K})$ with the evident interpretation.

— The $\{\emptyset, \cup, \cap\}$-algebra $(\mathbf{Set}, \mathscr{K})$, where the carrier consists of sets over some unspecified infinite type.

Proof. The equation $(x * 0) = 0$ is valid in the first; the equation $(x \cap \emptyset) = \emptyset$ is valid in the second. □

Corollary 5.13. The full continuous type hierarchy over the result of lifting the following algebras is completely axiomatized by (β), (η) and the equational theory of the (unlifted) algebra:

— The $\{0, 1, +\}$-algebra $\mathscr{N} = (\mathbf{N}, \mathscr{K})$ with the evident interpretation.

— The $\{nil, append, \langle 0 \rangle, \langle 1 \rangle, \dots\}$-algebra $(\mathbf{NatList}, \mathscr{K})$, where the carrier is the lists over the naturals, $\langle i \rangle$ is interpreted as the singleton list constructor, and *append* is interpreted as the list appending function.

— The $\{\cup, \cap\}$-algebra $(\mathbf{Set}, \mathscr{K})$, where the carrier consists of sets over some unspecified infinite type.

6. Finitely collapsing algebras

Disjunctive closedness is sufficient, but not necessary, for completeness of the combined theories. The $\{0, succ, pred\}$-algebra $(\mathbf{N}, \mathscr{K})$ with the evident interpretation provides one counterexample: the terms x, 0, and $(succ\ (pred\ x))$ are all pairwise unequal in the model, but any environment that distinguishes $(succ\ (pred\ x))$ from x must assign zero to x. Clearly such an environment does not distinguish x and 0. Nevertheless, the full type hierarchy over this algebra is completely axiomatized by (β), (η) and the equational theory of this algebra. What makes the algebra special is its limited signature, consisting only of unary or nullary symbols. We call such a signature a **unary signature**. We prove a general completeness theorem for unary signatures in this section:

> Let Σ be a unary signature. Let $\mathscr{M}$ be any Σ-model with enough first-order functions, with induced Σ-algebra $\mathscr{B} = (B, \mathscr{K})$, where B is infinite. Suppose, for every $g^{\iota\to\iota} \in \Sigma$ and every element $a \in B$, we have $\mathscr{B}[\![g]\!]^{-1}(a)$ is finite. Then the equations (β), (η) and the equational theory of $\mathscr{B}$ completely axiomatize equality between mixed terms in $\mathscr{M}$.

Clearly, the $\{0, succ, pred\}$-algebra above satisfies the condition. This allows us to conclude that, for any complete equational axiomatization E of the algebraic equations valid in the algebra, the full type hierarchy over this algebra is completely axiomatized by (β), (η) and E.

The technical details of this theorem require some definitions. If A is a set and g is a function from A to A, we say g is **finitely collapsing** if $g^{-1}(e)$ is always finite for every element $e \in A$. Equivalently, g is finitely collapsing if $g(S)$ is infinite for every infinite $S \subseteq A$. In this section we only consider unary-signature algebras in which the carrier set is infinite and every operator is finitely collapsing. Such algebras are called **finitely collapsing unary algebras**. In such algebras, the following lemma shows that defined operators are finitely collapsing too.

Lemma 6.1. Finitely collapsing unary functions over a set A are closed under composition.

Finally, to simplify the presentation, we extend the definition of meaning function of an algebra $\mathscr{A} = (A, \mathscr{K})$ to $algterms_{\Sigma,\{f^{\iota\to\iota\to\iota}\},X}$. Suppose σ is a function from variables of type ι to elements of A, and from $f^{\iota\to\iota\to\iota}$ to an element of the set-theoretic function space $[A \to A \to A]$. If $t \in algterms_{\Sigma,\{f\},X}$, we define $\mathscr{A}[\![t]\!]\sigma$ in the evident way, using σ to interpret f. More precisely, the definition of the meaning function is

$$\begin{aligned}
\mathscr{A}[\![x^\iota]\!]\sigma &= \sigma(x^\iota) \\
\mathscr{A}[\![f\ t_1\ t_2]\!]\sigma &= (\sigma(f)\ \mathscr{A}[\![t_1]\!]\sigma\ \mathscr{A}[\![t_2]\!]\sigma) \\
\mathscr{A}[\![c^\sigma\ t_1\ \dots t_n]\!]\sigma &= (\mathscr{K}(c^\sigma)\ \mathscr{A}[\![t_1]\!]\sigma \dots \mathscr{A}[\![t_n]\!]\sigma),
\end{aligned}$$

which applies to all terms in $algterms_{\Sigma,\{f\},X}$.

Our goal is to prove the following result:

> Let Σ be a unary signature, and $\mathscr{A}$ be a unary, finitely collapsing algebra. Let $\mathscr{E} = Th(\mathscr{A})$, and suppose $t_1, t_2 \in algterms_{\Sigma,\{f\},X}$, where X is a set of variables of

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type ι. If $t_1 \neq_{\mathscr{E}+\{f\}} t_2$, there is an environment σ with domain $X \cup \{f\}$ such that $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$.

The strategy is to construct the environment σ by induction on the structure of t_1 and t_2. The statement must be strengthened, however. To gain some insight, consider the case when $t_1 = C_1[f\ s_1\ s_2]$, $t_2 = C_2[f\ s_3\ s_4]$, $C_1[x], C_2[x] \in algterms_{\Sigma,X}$, and $C_1[x] =_E C_2[x]$. Without loss of generality, we may assume that $C_1[x] = C_2[x]$ (we call this $C[x]$), as well as that $s_1 \neq_{E+\{f\}} s_3$. By the induction hypothesis, there is an environment σ such that $\mathscr{A}[\![s_1]\!]\sigma \neq \mathscr{A}[\![s_3]\!]\sigma$. Since $C[x]$ may be a trivial term, to conclude that $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$ we need to guarantee the following condition.

Condition 1. $\sigma(f)$ is a 'pairing function' that maps distinct inputs to different outputs.

With this assumption, $a = \mathscr{A}[\![f\ s_1\ s_2]\!]\sigma \neq \mathscr{A}[\![f\ s_3\ s_4]\!]\sigma = b$. However, $\mathscr{A}[\![C[x]]\!][x \mapsto a]$ might be equal to $\mathscr{A}[\![C[x]]\!][x \mapsto b]$; to avoid this, we need to add one more condition. Let $Def(t)$ be the set of unary functions defined in t; for instance,

$$Def(g_1(g_2(f\ x\ g_3(y)))) = \{h_1, h_2, h_3, h_4\}$$

where $h_1(a) = \mathscr{A}[\![g_1(g_2(x))]\!][x \mapsto a]$, $h_2(a) = \mathscr{A}[\![g_3(x)]\!][x \mapsto a]$, $h_3(a) = \mathscr{A}[\![g_1(x)]\!][x \mapsto a]$ and $h_4(a) = \mathscr{A}[\![g_2(x)]\!][x \mapsto a]$. Then we get the following condition.

Condition 2. For any $h \in Def(t_1) \cup Def(t_2)$, and $a, b \in range(\sigma(f))$, $h(a) = h(b)$ iff $a = b$.

Thus, continuing our argument, knowing that a, b satisfy this property because they belong to $range(\sigma(f))$, we are assured that $\mathscr{A}[\![C[x]]\!][x \mapsto a]$ is not equal to $\mathscr{A}[\![C[x]]\!][x \mapsto b]$.

To formalize the argument, we need a few definitions and lemmas. Fix a unary signature Σ, a finitely collapsing Σ-algebra $\mathscr{A} = (A, \mathscr{K})$ with A an infinite set, an infinite set of variables X all of type ι, and a variable $f^{\iota \to \iota \to \iota}$. We use the notion of 'partial environments' for $X \cup \{f\}$. A **partial environment** assigns values to some subset of X, and assigns a partial binary function to f. A partial binary function $g \in [A \to A \to A]$ is a **partial finite pairing function** if the domain is finite, and for every two ordered pairs such that $(a, b) \neq (c, d)$ and $(g\ a\ b)$ and $(g\ c\ d)$ are both defined, $(g\ a\ b) \neq (g\ c\ d)$. If g is a partial finite pairing function, we write

$$dom(g) = \{a \in A \mid \text{there is a } b \in A \text{ such that } (g\ a\ b) \text{ or } (g\ b\ a) \text{ is defined}\}.$$

A partial environment σ is said to be a **PFPF-environment** if $\sigma(f)$ is a partial finite pairing function. We use the symbols $\sigma, \sigma_0, \ldots, \sigma_n, \ldots$ to range over PFPF-environments. A PFPF-environment σ' **extends** a PFPF-environment σ if:

— $dom(\sigma) \subseteq dom(\sigma')$,
— for all variables $x \in X$ such that $x \in dom(\sigma)$, $\sigma(x) = \sigma'(x)$, and
— for the distinguished binary function variable f, $\sigma(f) \subseteq \sigma'(f)$.

A partial environment enables us to 'expand' the constructed environment at induction steps. The meaning function $[\![\cdot]\!]$ acting on $algterms_{\Sigma,\{f\}}$ can be defined exactly as total environments. The reader should note that the function is partial; that is, the meaning of certain terms may not be defined because of the partiality of the environment.

Let F be a finite set of finitely collapsing functions over A. We say $S \subseteq A$ is F**-rigid** if for every $g \in F$, $g \restriction S$ is injective. For an F-rigid set S, we say that an element $d \in A - S$ is F**-compliant with** S if $S \cup \{d\}$ is F-rigid.

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Lemma 6.2. Suppose F is a finite set of finitely collapsing unary functions over A. For every finite F-rigid set S, there are cofinitely many elements F-compliant with S.

Proof. Let S be a finite F-rigid subset of A. Consider the set

$$U = \{b \in A \mid g(b) = g(s) \text{ for some } g \in F, s \in S\}.$$

Note that for any given $g \in F$ and $s \in S$, there are only finitely many b's such that $g(b) = g(s)$; this follows from the fact that g is finitely collapsing. Thus, because S and F are finite, U must be finite. Thus, $A - U$ is infinite. Clearly, for any $d \in A - U$, $S \cup \{d\}$ is F-rigid. □

Lemma 6.3. Let F be a finite set of finitely collapsing unary functions over A, and let K be a finite subset of A. Let σ be a PFPF-environment such that $\mathit{range}(\sigma(f)) \cap K = \emptyset$ and $\mathit{range}(\sigma(f)) \cup K$ is F-rigid. Let $t \in \mathit{algterms}_{\Sigma,\{f\},X}$ such that $\mathscr{A}[\![t]\!]\sigma$ is undefined. Then there is a PFPF-environment σ' extending σ such that $\mathit{range}(\sigma'(f)) \cup K$ is F-rigid, $\mathit{range}(\sigma'(f)) \cap K = \emptyset$, $\mathscr{A}[\![t]\!]\sigma'$ is defined, and $\mathscr{A}[\![t]\!]\sigma' \notin \mathit{dom}(\sigma'(f))$.

Proof. The proof is by induction on the structure of t.

- t is algebraic. Then t has at most one free variable, say x. Choose $\sigma' = \sigma[x \mapsto a]$, where a is such that $\mathscr{A}[\![t]\!][x \mapsto a] \notin \mathit{dom}(\sigma(f))$. There must be such an a because $\mathscr{A}$ is finitely collapsing. Note that $\mathit{range}(\sigma'(f)) = \mathit{range}(\sigma(f))$. Clearly, $\mathit{range}(\sigma'(f)) \cup K$ is F-rigid and $\mathit{range}(\sigma'(f)) \cap K = \emptyset$, by hypothesis. Furthermore, σ' is a PFPF-environment extending σ.
- $t = C[f\ s_1\ s_2]$, where $C[x]$ is an algebraic term. First, we define a new PFPF-environment σ_1 by considering two cases:
  - If both $\mathscr{A}[\![s_1]\!]\sigma$ and $\mathscr{A}[\![s_2]\!]\sigma$ are defined (call them a and b), then since $\mathscr{A}[\![t]\!]\sigma$ is undefined we can infer that $(\sigma(f)\ a\ b)$ is undefined. Let $\sigma_1 = \sigma$.
  - If one of $\mathscr{A}[\![s_1]\!]\sigma$ or $\mathscr{A}[\![s_2]\!]\sigma$ is undefined, then we may without loss of generality assume that $\mathscr{A}[\![s_1]\!]\sigma$ is undefined. By the induction hypothesis, there is a PFPF-environment σ_0 extending σ such that $a = \mathscr{A}[\![s_1]\!]\sigma_0 \notin \mathit{dom}(\sigma_0(f))$, $\mathit{range}(\sigma_0(f)) \cap K = \emptyset$ and the set $\mathit{range}(\sigma_0(f)) \cup K$ is F-rigid. If $\mathscr{A}[\![s_2]\!]\sigma_0$ is defined, call it b and let $\sigma_1 = \sigma_0$; otherwise, by the induction hypothesis, there is a PFPF-environment σ_1 extending σ_0 such that $b = \mathscr{A}[\![s_2]\!]\sigma_1 \notin \mathit{dom}(\sigma_1(f))$, $\mathit{range}(\sigma_1(f)) \cap K = \emptyset$ and $\mathit{range}(\sigma_1(f)) \cup K$ is F-rigid. In either case, $(\sigma_1(f)\ a\ b)$ is undefined.

  Let h be the function

  $$h(c) = \mathscr{A}[\![C[x]]\!][x \mapsto c].$$

  Recall that $C[x]$ is an algebraic term (*i.e.*, contains no occurrences of f) with one hole; thus, h is a finitely collapsing unary function by Lemma 6.1. Now, choose d such that $h(d) \notin \mathit{dom}(\sigma_1(f)) \cup \{a,b\}$, $d \notin \mathit{range}(\sigma_1(f)) \cup K$, and d is F-compliant with $\mathit{range}(\sigma_1(f)) \cup K$ – there must be such a d by Lemma 6.2, because h is finitely collapsing, and $\mathit{range}(\sigma_1(f)) \cup K$ and $\mathit{dom}(\sigma_1(f)) \cup \{a,b\}$ are finite. Let

  $$\sigma' = \sigma_1[f \mapsto (\sigma_1(f) \cup \{(a,(b,d))\})].$$

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Then $range(\sigma'(f)) = range(\sigma_1(f)) \cup \{d\}$. Clearly, σ' extends σ, $range(\sigma'(f)) \cup K$ is F-rigid, $range(\sigma'(f)) \cap K = \emptyset$, $\mathscr{A}[\![t]\!]\sigma' = h(d)$ is defined, and $h(d) \notin dom(\sigma'(f))$.

This completes the induction and hence the proof. □

Lemma 6.4. Let F be a finite set of finitely collapsing unary functions over A. Suppose σ is a PFPF-environment, $\mathscr{A}[\![f\ s_1\ s_2]\!]\sigma$ is undefined and $range(\sigma(f))$ is F-rigid. Then there is a PFPF-environment σ' extending σ such that $range(\sigma'(f))$ is F-rigid, and $\mathscr{A}[\![s_1]\!]\sigma'$, $\mathscr{A}[\![s_2]\!]\sigma'$ are defined but $\mathscr{A}[\![f\ s_1\ s_2]\!]\sigma'$ is undefined.

Proof. The proof is similar to the proof of Lemma 6.3. □

We can now finally state the main lemma.

Lemma 6.5. Suppose $t_1, t_2 \in algterms_{\Sigma,\{f\},X}$. Let $\mathscr{E} = Th(\mathscr{A})$ and $F = Def(t_1) \cup Def(t_2)$. Suppose $t_1 \neq_{\mathscr{E}+\{f\}} t_2$. Then there exists a PFPF-environment σ such that

— $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$; and
— $range(\sigma(f))$ is F-rigid.

Proof. The proof is by induction on the sum of the sizes of t_1 and t_2. There are four main cases:

1 t_1 and t_2 are algebraic terms. Since $t_1 \neq_{\mathscr{E}+\{f\}} t_2$, there is a PFPF-environment σ, which assigns to f the empty partial function, such that $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$. Clearly, $range(\sigma(f)) = \emptyset$ is F-rigid.

2 $t_1 = C_1[x]$ and $t_2 = C_2[f\ s_1\ s_2]$, where $C_1[x]$ and $C_2[x]$ are algebraic terms. Let a' be any element of $\mathscr{A}$, and let $\sigma_0 = [x \mapsto a']$ and $a = \mathscr{A}[\![t_1]\!]\sigma_0$. Note that σ_0 is a PFPF-environment, and $range(\sigma_0(f)) = \emptyset$ is F-rigid. By Lemma 6.4, there is a σ_1 extending σ_0 such that $range(\sigma_1(f))$ is F-rigid, $\mathscr{A}[\![s_1]\!]\sigma_1 = a_1$, and $\mathscr{A}[\![s_2]\!]\sigma_1 = a_2$, and $(\sigma_1(f)\ a_1\ a_2)$ is not defined. Pick $b \notin range(\sigma_1(f))$ such that $\mathscr{A}[\![C_2[x]]\!][x \mapsto b] \neq a$ and $range(\sigma_1(f)) \cup \{b\}$ is F-rigid; we know such a b must exist by Lemma 6.2, because $range(\sigma_1(f))$ is finite and F-rigid, and

$$h(d) = \mathscr{A}[\![C_2[x]]\!][x \mapsto d]$$

is a finitely collapsing unary function (Lemma 6.1). Let σ be the extension of σ_1 with

$$\sigma(f) = \sigma_1(f) \cup \{(a_1, (a_2, b))\}.$$

Clearly, $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$ and $range(\sigma(f)) \cup \{b\}$ is F-rigid.

3 $t_1 = C_1[f\ s_1\ s_2]$, $t_2 = C_2[f\ s_3\ s_4]$ and $C_1[x] \neq_{\mathscr{E}} C_2[x]$, where $C_1[x]$ and $C_2[x]$ are algebraic terms. Let

$$h_1(d) = \mathscr{A}[\![C_1[x]]\!][x \mapsto d] \quad \text{and} \quad h_2(d) = \mathscr{A}[\![C_2[x]]\!][x \mapsto d],$$

which are finitely collapsing unary functions by Lemma 6.1. Choose some c such that $h_1(c) \neq h_2(c)$. Repeatedly applying Lemma 6.3, taking $K = \{c\}$, we obtain a PFPF-environment σ_1 such that $\mathscr{A}[\![s_i]\!]\sigma_1 = a_i$, for $i = 1, \ldots, 4$, $c \notin range(\sigma_1(f))$ and $range(\sigma_1(f)) \cup \{c\}$ is F-rigid. At least one of $\mathscr{A}[\![t_1]\!]\sigma_1$ or $\mathscr{A}[\![t_2]\!]\sigma_1$ is undefined. Up to symmetry, there are two cases:

— Both $\mathscr{A}[\![t_1]\!]\sigma_1$ and $\mathscr{A}[\![t_2]\!]\sigma_1$ are undefined. There are two subcases:

  – $(a_1, a_2) \neq (a_3, a_4)$. By Lemma 6.2, we can choose $a' \in A$ that is F-compliant with $range(\sigma_1(f))$. Since h_1, h_2 are finitely collapsing unary functions, there are cofinitely many b' such that $h_1(a') \neq h_2(b')$. By Lemma 6.2, the set of elements F-compliant with $range(\sigma_1(f)) \cup \{a'\}$ is cofinite. Therefore, we can pick $b' \neq a'$ such that the set $range(\sigma_1(f)) \cup \{a', b'\}$ is F-rigid and $h_1(a') \neq h_2(b')$. Let σ be the extension of σ_1 with

$$\sigma(f) = \sigma_1(f) \cup \{(a_1, (a_2, a')), (a_3, (a_4, b'))\}.$$

  Clearly, $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$ and $range(\sigma(f))$ is F-rigid.

  – $(a_1, a_2) = (a_3, a_4)$. Let σ be the extension of σ_1 with

$$\sigma(f) = \sigma_1(f) \cup \{(a_1, (a_2, c))\}.$$

  Since $c \notin range(\sigma_1(f))$, $\sigma(f)$ is a PFPF-environment. Note that $range(\sigma(f))$ is the same as $range(\sigma_1(f)) \cup \{c\}$. Clearly, $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$ and $range(\sigma(f))$ is F-rigid.

— $\mathscr{A}[\![t_1]\!]\sigma_1 = a$ and $\mathscr{A}[\![t_2]\!]\sigma_1$ is undefined. By Lemma 6.2, there are cofinitely many elements F-compliant with $range(\sigma_1(f))$. Since h_2 defines a finitely collapsing function, there are cofinitely many b' such that $h_2(b') \neq a$. Thus, we can pick b' such that b' is F-compliant with $range(\sigma_1(f))$, $b' \notin range(\sigma_1(f))$ and $h_2(b') \neq a$. Let σ be the extension of σ_1 with argument f, and

$$\sigma(f) = \sigma_1(f) \cup \{(a_3, (a_4, b'))\}.$$

Clearly, $range(\sigma(f))$ is rigid and $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$.

4 $t_1 = C_1[f\ s_1\ s_2]$, $t_2 = C_2[f\ s_3\ s_4]$ and $C_1[x] =_{\mathscr{E}} C_2[x]$, where $C_1[x]$ and $C_2[x]$ are algebraic terms. Let

$$h(d) = \mathscr{A}[\![C_1[x]]\!][x \mapsto d].$$

Then either $s_1 \neq_{\mathscr{E}+\{f\}} s_3$ or $s_2 \neq_{\mathscr{E}+\{f\}} s_4$. Without loss of generality, assume the former. By the induction hypothesis, there is a σ_1 such that $a_1 = \mathscr{A}[\![s_1]\!]\sigma_1 \neq \mathscr{A}[\![s_3]\!]\sigma_1 = a_3$, and $range(\sigma_1(f))$ is F-rigid. If $\mathscr{A}[\![t_1]\!]\sigma_1$ and $\mathscr{A}[\![t_2]\!]\sigma_1$ are both defined, then $\mathscr{A}[\![f\ s_1\ s_2]\!]\sigma_1 = a' \neq b' = \mathscr{A}[\![f\ s_3\ s_4]\!]\sigma_1$, since $\sigma_1(f)$ is a PFPF-environment. Furthermore, since $range(\sigma_1(f))$ is F-rigid, $a', b' \in range(\sigma_1(f))$ and $h \in Def(t_1) \cup Def(t_2)$, we can conclude that $h(a') \neq h(b')$. Thus, $\mathscr{A}[\![t_1]\!]\sigma_1 \neq \mathscr{A}[\![t_2]\!]\sigma_1$.
However, if either or both $\mathscr{A}[\![t_1]\!]\sigma_1, \mathscr{A}[\![t_2]\!]\sigma_1$ are undefined, by Lemma 6.3 there is a PFPF-environment σ_2 extending σ_1 such that $range(\sigma_2(f))$ is F-rigid, $\mathscr{A}[\![s_i]\!]\sigma_2$ are defined, for $i = 1, \ldots, 4$, and at least one of $\mathscr{A}[\![t_1]\!]\sigma_2, \mathscr{A}[\![t_2]\!]\sigma_2$ is undefined. Without loss of generality, assume that $\mathscr{A}[\![t_2]\!]\sigma_2$ is undefined. Let $\mathscr{A}[\![s_i]\!]\sigma_2 = a_i$, for $i = 1, \ldots, 4$. There are two cases:

— $\mathscr{A}[\![t_1]\!]\sigma_2 = a$: By Lemma 6.2, we can choose b' satisfying the following properties: b' is F-compliant with $range(\sigma_2(f))$, $b' \notin range(\sigma_2(f))$ and $h(b') \neq a$. Let σ extend σ_2 such that

$$\sigma(f) = \sigma_2(f) \cup \{(a_3, (a_4, b'))\}.$$

Note that $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$ and $range(\sigma(f))$ is F-rigid.

— $\mathscr{A}[\![t_1]\!]\sigma_2$ is undefined: Then $(\sigma_2(f)\ a_1\ a_2)$ is undefined. We can choose a', b' satisfying the following properties: $a' \neq b'$, $a', b' \notin range(\sigma_2(f))$, $range(\sigma_2(f)) \cup \{a', b'\}$ is F-rigid, and $h(a') \neq h(b')$. Define

$$\sigma(f) = \sigma_2(f) \cup \{(a_1, (a_2, a')), (a_3, (a_4, b'))\}.$$

Clearly, $range(\sigma(f))$ is F-rigid and $\mathscr{A}[\![t_1]\!]\sigma \neq \mathscr{A}[\![t_2]\!]\sigma$.

This concludes the case analysis and hence the proof. □

Theorem 6.6. Let Σ be a unary signature. Let $\mathscr{M}$ be a Σ-model that has enough first-order functions and the induced algebra $\mathscr{B}$. Suppose $\mathscr{B}$ is finitely collapsing. Then the theory consisting of (β), (η) and $Th(\mathscr{B})$ completely axiomatizes equality of mixed terms in $\mathscr{M}$.

Proof. Suppose $t_1 = t_2$, an equation between closed terms, is not provable from the equations (β), (η) and $\mathscr{E} = Th(\mathscr{B})$. By Lemmas 4.5 and 4.7, there is a term V such that

— $(V\ t_1)$ and $(V\ t_2)$ are of base type.

— $\beta\eta\text{-}nf(V\ t_1)$ and $\beta\eta\text{-}nf(V\ t_2)$ are in $algterms_{\Sigma,\{f\},X}$, where X is a set of variables of type ι.

— $\beta\eta\text{-}nf(V\ t_1) \neq_{\mathscr{E}+\{f\}} \beta\eta\text{-}nf(V\ t_2)$.

By Lemma 6.5, there is a PFPF-environment σ' that distinguishes $\beta\eta\text{-}nf(V\ t_1)$ and $\beta\eta\text{-}nf(V\ t_2)$. There is a total extension f^+, in $\mathscr{M}$, to the partial function $\sigma'(f)$. We can extend σ' to a total environment by setting $\sigma(f) = f^+$. Thus, $\mathscr{M} \not\models t_1 = t_2$. □

Corollary 6.7. The full type hierarchy over the $\{0, succ, pred\}$-algebra $\mathscr{N} = (\mathbf{N}, \mathscr{K})$, with the evident interpretation, is completely, axiomatized by (β), (η) and $Th(\mathscr{N})$.

The condition of being finitely collapsing is not a necessary condition for the completeness theorem stated in Theorem 6.6. Consider the algebra $\mathscr{A}$ with carrier the set of natural numbers and whose signature Σ contains exactly one symbol f, where f on even numbers is zero, and f on odd numbers adds two. The subalgebra $\mathscr{B}$ of $\mathscr{A}$ consisting of the odd natural numbers is finitely collapsing. We show in the following theorem that the full type hierarchy over $\mathscr{A}$ is completely axiomatized by (β), (η) and $Th(\mathscr{B})$. Since $\mathscr{B}$ satisfies no non-trivial algebraic equation over Σ, $Th(\mathscr{A}) = Th(\mathscr{B})$. It follows that $(\beta), (\eta)$ and $Th(\mathscr{A})$ completely axiomatize equality of mixed terms in the full type hierarchy.

Theorem 6.8. Let Σ be a unary signature. Let $\mathscr{M}$ be the full type hierarchy over Σ-algebra $\mathscr{A}$. Suppose $\mathscr{A}$ contains a subalgebra $\mathscr{B}$ that is finitely collapsing. Then the equations (β), (η) and $Th(\mathscr{B})$ completely axiomatize equality between mixed terms in $\mathscr{M}$.

Proof. Let M and N be closed, mixed terms where $\mathscr{M} \models M = N$. Let $\mathscr{M}'$ be the full type hierarchy over $\mathscr{B}$. Let $\mathscr{R}$ be the logical relation over $\mathscr{M}$ and $\mathscr{M}'$ induced by the partial function from A to B with domain $B \subseteq A$ that is the identity on this domain. This logical relation relates the meanings of the symbols in Σ, since $\mathscr{B}$ is a subalgebra of $\mathscr{A}$. As Friedman has shown in Friedman (1975), $\mathscr{R}$ is a partial function at all types (in his phrase, a *partial homomorphism*). Combining this fact with the fundamental theorem of logical relations (Statman, 1985b; Mitchell, 1996), it follows that for closed terms M and

N, $\mathscr{M} \models M = N$ implies $\mathscr{M}' \models M = N$. Since $\mathscr{B}$ is finitely collapsing, by Theorem 6.6, (β), (η) and $Th(\mathscr{B})$ prove $M = N$. □

7. Discussion

We have considered the problem of equational reasoning about mixed terms in Σ-models, using $\beta\eta$ and the Σ-algebraic equations valid in the model. The main theorem, Theorem 4.10, which states that the equational theory of mixed terms in a model is completely determined by the structure of the first-order types, is difficult to apply. We have proposed two sufficient conditions, and given a few corollaries.

Our theorems can be used in combination with a theorem of Breazu-Tannen (Breazu-Tannen, 1988) for deducing decidability of various theories.

Theorem 7.1 (Breazu-Tannen). Suppose E is a set of equations over the signature Σ and $t_1, t_2 \in terms_{\Sigma,X}$. Then there exist finite sets of algebraic equations $S_1, \ldots, S_n$, effectively computable from $t_1 = t_2$, such that $t_1 =_{\beta\eta E} t_2$ iff for some i every equation in S_i is provable using E.

In particular, if E is decidable, then so is $\beta\eta E$. To see how to apply this theorem, consider the $\{0, succ, +, *\}$-algebra $(\mathbf{N}, \mathscr{K})$ with the evident interpretation. In Section 5 we proved that this algebra is disjunctively closed. By our completeness theorem for disjunctively closed algebras, it follows that the full type hierarchy over this algebra is completely axiomatized by (β), (η) and the equational theory of this algebra. An axiomatization of this theory is given by the equations

$$\begin{array}{rcl} x + y &=& y + x \\ x + (y + z) &=& (x + y) + z \\ x * (y + z) &=& x * y + x * z \\ 0 + x &=& x \end{array} \qquad \begin{array}{rcl} x * y &=& y * x \\ x * (y * z) &=& (x * y) * z \\ 0 * x &=& 0 \\ 1 * x &=& x \end{array}$$

Henkin (1977) shows that this is a complete axiomatization of validities in this algebra. Note that every term in the theory is provable equal to a multinomial, and two multinomials are provably equal iff up to rearrangements of summands they are identical. Therefore, this is a decidable theory. Invoking Theorem 7.1, it follows that equality of mixed terms in the full type hierarchy is decidable. In a similar fashion, one may show the following corollary.

Corollary 7.2. The equational theories of the full type hierarchies over the following algebras are decidable:

— The $\{succ, pred\}$-algebra $(\mathbf{N}, \mathscr{K})$ with the evident interpretation.
— The $\{0, succ, pred\}$-algebra $(\mathbf{Z}, \mathscr{K})$ with the evident interpretation.
— The $\{\varnothing, \cup, \cap\}$-algebra $(\mathbf{Set}, \mathscr{K})$, where the carrier consists of sets over some unspecified infinite type.

The full continuous type hierarchies over the following flat, continuous algebras have decidable equational theories:

— The lifted $\{0, 1, +\}$-algebra $\mathscr{N}_\perp$, where $\mathscr{N} = (\mathbf{N}, \mathscr{K})$ with the evident interpretation.

— The lifted $\{\cup, \cap\}$-algebra $\mathscr{S}_\perp$, where $\mathscr{S} = (\mathbf{Set}, \mathscr{K})$, where the carrier consists of sets over some unspecified infinite type.

Some of the technical results above raise further questions. First, can one generalize Lemma 4.7 to $f^{\iota \to \iota}$ instead of $f^{\iota \to \iota \to \iota}$? Simpson (1995) gives a counterexample for the case when the algebraic signature is empty, but one can ask the question for specific, non-empty signatures. Second, Theorem 4.9 essentially shows that if an equational theory of closed, first-order terms is sufficient for proving facts about closed, second-order terms, one can lift the first-order reasoning to all types. One can then ask whether this holds at other orders.

We have left a number of other, more important problems open as well. For instance, are there other easily characterized settings in which completeness holds? That is, are there other corollaries of Theorem 4.10 along the lines of Theorems 6.6 and 5.6? Also, can one extend the results to richer type systems? It appears that the extension to languages with products is easy. For sums, on the other hand, even though there is a complete axiomatization of the simply-typed λ-calculus with coproducts (Dougherty and Subrahmanyam, 1995), extending our results to this calculus is likely to be challenging. Likewise, we believe that results along our lines for type systems with second-order polymorphism (Girard, 1971; Reynolds, 1974) would be interesting and challenging as well. Finally, can one extend the results to *call-by-value* languages? One approach might be to take Moggi's computational λ-calculus (Moggi, 1991) as the starting point rather than (β) and (η).

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