Bull. Aust. Math. Soc. **108** (2023), 511–512
doi:10.1017/S0004972723000473

EXTRACTING FEATURES FROM EIGENFUNCTIONS: HIGHER CHEEGER CONSTANTS AND SPARSE EIGENBASIS APPROXIMATION

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(Received 5 May 2023; first published online 22 August 2023)

2020 *Mathematics subject classification*: primary 58J50; secondary 37C60, 62H30.

Keywords and phrases: higher Cheeger constants, nonautonomous dynamical systems, dynamic Laplacian, Lagrangian coherent sets, feature identification, sparse eigenbasis approximation.

This thesis investigates links between the eigenvalues and eigenfunctions of the Laplace–Beltrami operator, and the higher Cheeger constants of smooth Riemannian manifolds, possibly with boundary. The higher Cheeger constants give a loose description of the major geometric features of a manifold. We obtain a new lower bound on the negative Laplace–Beltrami eigenvalues in terms of the corresponding higher Cheeger constant.

The level sets of Laplace–Beltrami eigenfunctions sometimes reveal sets with small Cheeger ratio, representing well-separated features of the manifold. Some manifolds have their major features entwined across several eigenfunctions, and no single eigenfunction contains all the major features. In this case, there may exist carefully chosen linear combinations of the eigenfunctions, each with large values on a single feature and small values elsewhere. We can then apply a soft-thresholding operator to these linear combinations to obtain new functions, each supported on a single feature. We show that the Cheeger ratios of the level sets of these functions also give an upper bound on the Laplace–Beltrami eigenvalues. We extend these level set results to nonautonomous dynamical systems and show that the dynamic Laplacian eigenfunctions reveal sets with small dynamic Cheeger ratios.

In a later chapter, we propose a numerical method for identifying features represented in eigenvectors arising from spectral clustering methods when those features are not cleanly represented in a single eigenvector. This method provides explicit candidates for the soft-thresholded linear combinations of eigenfunctions mentioned above. Many data clustering techniques produce collections of orthogonal vectors

Thesis submitted to the University of New South Wales in June 2022; degree approved on 7 November 2022; primary supervisor Gary Froyland, secondary supervisor Mark Holzer.

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https://doi.org/10.1017/S0004972723000473 Published online by Cambridge University Press

(for example, eigenvectors) which contain connectivity information about the dataset. This connectivity information must be disentangled by some secondary procedure. We propose a method for finding an approximate sparse basis for the space spanned by the leading eigenvectors, by applying thresholding to linear combinations of eigenvectors. Our procedure is natural, robust and efficient, and it provides soft-thresholded linear combinations of the inputted eigenfunctions. We develop a new Weyl-inspired eigengap heuristic and heuristics based on the sparse basis vectors, suggesting how many eigenvectors to pass to our method.

Some of this material has been published in [2], or is in prepration as [1].

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https://doi.org/10.1017/S0004972723000473 Published online by Cambridge University Press