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The Steiner-Lehmus angle-bisector theorem

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1. *Introduction*

In 1840 C. L. Lehmus sent the following problem to Charles Sturm: 'If two angle bisectors of a triangle have equal length, is the triangle necessarily isosceles?' The answer is 'yes', and indeed we have the *reverse-comparison theorem*: Of two unequal angles, the larger has the shorter bisector (see [1, 2]).

Sturm passed the problem on to other mathematicians, in particular to the great Swiss geometer Jakob Steiner, who provided a proof. In this paper we give several proofs and discuss the old query: 'Is there a direct proof?' before suggesting that this is no longer the right question to ask.

We go on to discuss *all* cases when an angle bisector (internal or external) of some angle is equal to one of another.

2. *The schizoid scissors – an indirect proof*

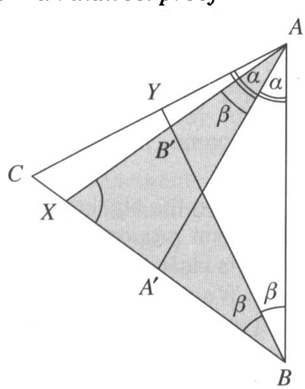

FIGURE 1: The Schizoid Scissors

The following proof is simplified from one in Coxeter and Greitzer's *Geometry revisited* [1]. We find our vivid title helps us to recall both the construction and the proof.

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Suppose that one of the bisected angles A and B of triangle ABC, say the one above, 2α at A, is strictly larger than the one below, 2β at B, as in Figure 1. Then we can cut off a proper part of size β from the angle bisector AA' towards the side AC. This yields the shaded 'scissors' of the figure, whose cutting edges are the equal angle bisectors. Then the blades above and below, namely the triangles $AA'X$ and $BB'X$ share an angle γ at X and have another angle β at A or B and so are similar. We easily reach two opposite conclusions!

On the one hand, the blade above is clearly smaller that the one below since AX is opposite the smaller angle 2β and BX opposite the larger one $\alpha + \beta$ in the triangle ABX that they span. (Note that this proves $AA' < BB' < BY$ giving the reverse-comparison theorem.)

On the other hand, the blade above is larger than the one below because $AA' > BB'$, the former being a complete bisector and the latter only a proper part of one. (Note that this is the first time we have used the hypothesis that the bisectors are equal.)

This contradiction shows the bisected angles cannot be different, and so proves the theorem. However, this synthetic proof is blatantly indirect. Before discussing the directness question, we give a simple algebraic proof.

3. *An algebraic proof*

Let the lengths of the three bisectors be y_a, y_b, y_c. Then it is not too hard to see that

$$y_a^2 = bc\left\{1 - \left(\frac{a}{b+c}\right)^2\right\}$$

and from this some tedious algebra tells us that

$$(a+c)^2(b+c)^2\left(y_a^2 - y_b^2\right) = c(b-a)(a+b+c)\left\{(a+b+c)\left(c^2+ab\right) + 2abc\right\}.$$

In this the factor

$$(a+b+c)\left(c^2+ab\right) + 2abc$$

cannot vanish, proving the theorem. Also if $b > a$, then $y_a > y_b$, proving the Comparison Theorem.

Coxeter and Greitzer mention the algebraic proof and say that 'Several allegedly direct proofs have been proposed, but each of them is really an indirect proof in disguise.' It is clear from these words that they regard this algebraic proof as indirect. We now restrict ourselves to the question of whether there can be a direct proof. First we show that:

4. *There cannot be a direct proof ...*

We define a process called *extraversion* ('turning inside out') of a triangle. Extraversion is a smooth process that transforms a triangle into its mirror image as in Figure 2, in which we have taken the edge AB which joins the two bisected angles (the *joining edge*) as base.

We start by moving A and B towards each other as hinted at by the bold arrows, then they pass through each other and continue until they form the reflected triangle. The numbers a and b smoothly vary but return to their initial positive values since they never pass through 0. However, c decreases uniformly, passing through zero and finishing at $-c$. In a similar way we can find what happens to the angles. When c passes through 0 so does C, and ends at $-C$, while A and B become their supplements.

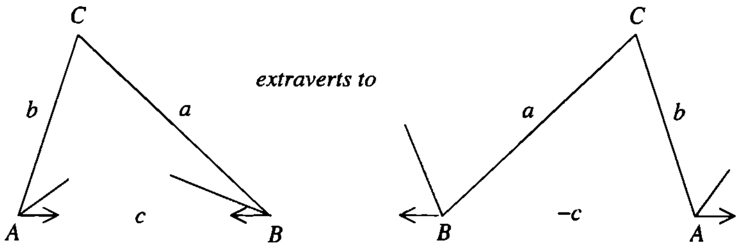

FIGURE 2: Extraverting the joining edge

Summary: c-extraversion replaces:

$$a, b, c \text{ by } a, b, -c \text{ and } A, B, C \text{ by } \pi - A, \pi - B, -C$$

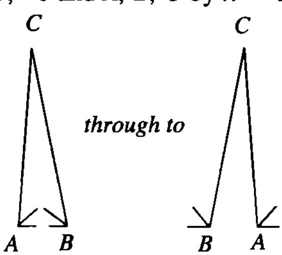

FIGURE 3: Snapshot as B passes through A so C passes through 0

Figure 3 is a snapshot of what happens as c passes from being small and positive to being small and negative. The internal bisectors at the ends A and B of the joining edge pass smoothly to external bisectors, while the bisector at C stays internal.

The direct proofs of various theorems about angle bisectors extravert to corresponding proofs of similar theorems in which some internal bisectors have been swapped with external ones. For instance, the proof that three internal bisectors meet (at an incentre) becomes a proof that one internal and two external bisectors meet (at an excentre). However we shall see later that no proof of the Steiner-Lehmus theorem can survive all such extraversions*. Under b-extraversion, our formula for $y_a^2 - y_b^2$ becomes the following:

$$(a+c)^2(c-b)^2\left(x_a^2 - y_b^2\right) = -c(a+b)(a-b+c)\left\{(a-b+c)\left(c^2-ab\right)-2abc\right\}$$

where x_a is the length of the external bisector segment for the angle at A. However, this does not prove that $a + b = 0$; now the sign of b has

* J.H.C. confesses to having made stronger assertions that now seem unjustified.

changed we can have:

$$(a - b + c)(c^2 - ab) - 2abc = 0.$$

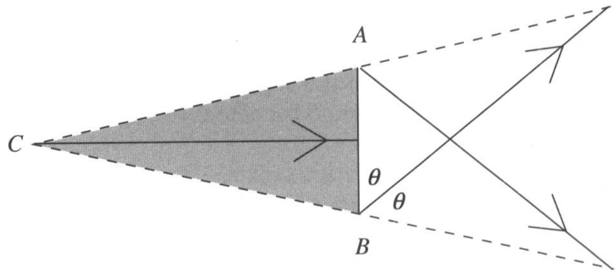

FIGURE 4: The inequilateral triangle

Indeed, the slanting external bisectors of the triangle in Figure 4 with sides 1, 1 and $-2\cos 2\theta$ are $\dfrac{-2\cos 2\theta}{\sin 3\theta}$ times as long as the vertical one. So if θ is the acute angle satisfying $\sin 3\theta + 2\cos 2\theta = 0$, namely $\sin^{-1}\dfrac{\sqrt{17} - 1}{4}$ $\approx 51.332°$, it has three bisectors (one internal and two external) of equal length, and so if Steiner-Lehmus survived extraversion, it would be equilateral. However, clearly it isn't – we call it the *inequilateral triangle* (it has angles of 77.336…, 77.336… and 25.328… degrees).

5. … *or can there*?

However, some proofs that don't survive extraversion have been considered direct. We have already remarked for instance that the algebraic proof might be considered direct. Here we consider some other plausibly direct proofs.

The schizoid scissors proof shows that both

$$above < below \text{ and } above > below,$$

where *above* and *below* are any two corresponding edges of the scissor blades. Everybody will agree that use of this blatant contradiction makes the proof indirect.

But it is surely a positive statement that for any two lengths *above* and *below* we have either

$$above \leqslant below \text{ or } above \geqslant below.$$

Now if in the scissors proof we replace '<' and '>' by '⩽' and '⩾' and make a similar replacement of 'proper part' by 'part or whole', this modifies the proof to show that both

$$above \leqslant below \text{ and } above \geqslant below,$$

which is no longer a contradiction, but a seemingly direct proof that

$$above = below.$$

6. *The direct proof that was there all along*

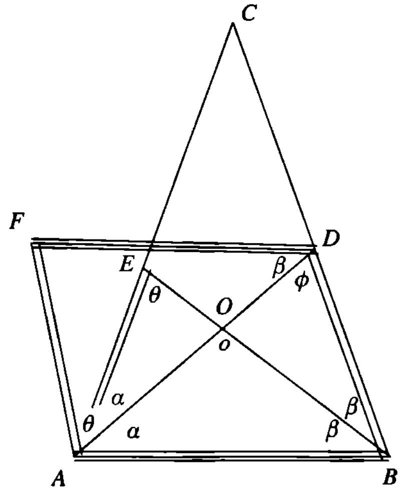

FIGURE 5: Hesse's construction

Just pOSSibly F. G. Hesse was one of the mathematicians that Sturm wrote to in 1840. In any case he produced the following proof by 1842. It uses the now generally forgotten fact (criterion OSS in the Appendix) that two triangles are congruent if they agree at two pairs of corresponding sides and a pair of corresponding non-included but obtuse angles.

In Figure 5 we picture Hesse's construction using some multiply-ruled lines. Letting *AD* and *BE* be the (1-ruled) equal angle bisectors in triangle *ABC*, Hesse constructs a 3, 2 and 1-ruled triangle *ADF* congruent to the similarly ruled triangle *EBA* with *B* and *F* on opposite sides of *AD*. The proof will show that the 3 and 2-ruled quadrilateral *ABDF* is a parallelogram.

We let *O* be the intersection of the two bisectors. Then the angle o at *O* of the triangle *OAB* is the supplement of $\alpha + \beta$. Now, because $2\alpha + 2\beta$ is less than 2 right angles, $\alpha + \beta$ is less than 1 right angle and so o is obtuse. Since it is the external angle of both the triangles *OAE* and *OBD* we have:

$$o = \alpha + \theta = \beta + \phi.$$

The forgotten fact now shows that *ABDF* is a parallelogram, since its (0-ruled!) diagonal *BF* divides it into two 3, 2, 0-ruled triangles that share this diagonal, and have two equal sides *AB* and *DF* and the obtuse angle o at corresponding vertices *A* and *D*.

Now by Hesse's construction *AE* equals *AF* which equals *BD* from the parallelogram so the triangles *ABE* and *BAD* are congruent, showing that $2\alpha = 2\beta$, and so *ABC* is isosceles.

We find it surprising that in Coxeter's long life (9 February, 1907 – 31 March, 2003) he does not seem to have commented on this proof, which in our opinion is the most 'direct' one.

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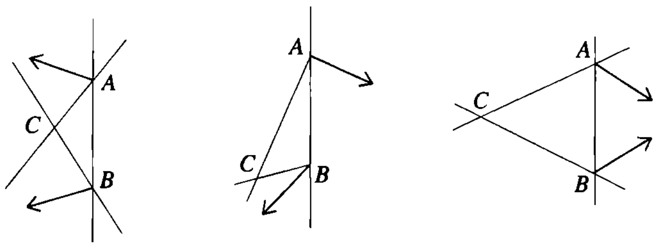
7. *External S-L theorems*?

In the usual discussions of the Steiner-Lehmus theorem it is often supposed tacitly that angle bisectors are internal. If they are both external, there are three possible cases (MAX, MID, MIN), distinguished by whether C is the maximal, middle, or minimal one of the three angles (or equivalently whether c is the maximal, middle, or minimal one of the three edges), illustrated in Figure 6. (The switch between these cases is when C equals A or B because then the external B or A bisector is parallel to the opposite side.)



Case MAX: Theorem true by scissor proof

Case MID: Theorem false

Case MIN: Theorem true; needs new proof

FIGURE 6: The three cases

7.1 *Case MAX: The backward external S-L theorem*

In this case, the two given bisector segments point backward (from the joining edge AB into the half-plane containing C). The theorem and scissors proof remain valid, provided that all inequalities are reversed. We suppose $\alpha > \beta$ as in Figure 7, so that we can choose X between A' and C to make the angle $A'AX$ equal β. Then for the two similar triangles AXA' and BXB', we reach two opposite conclusions (as in the internal case, but now with reversed inequalities).

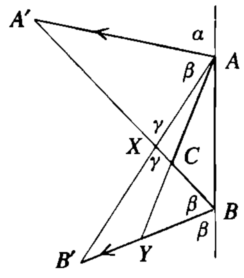


FIGURE 7: Case MAX:Backward Bisectors

In the triangle AXB, AX is opposite the larger angle $\pi - 2\beta$ while BX is opposite the smaller one $\pi - \alpha - \beta$ so AXA' is bigger. On the other hand, $AA' = BY < BB'$ so BXB' is bigger. The comparison theorem in this case is that the larger exterior angle has the longer bisector.

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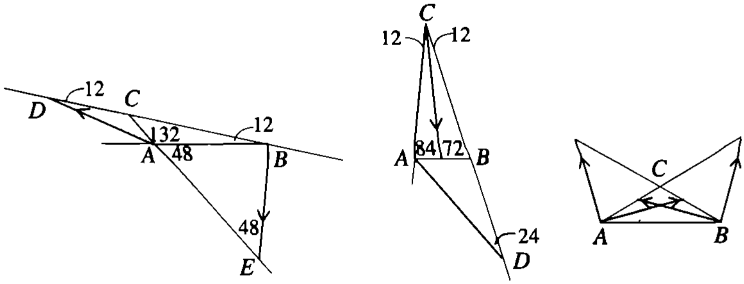
7.2 *Case MID: The non-theorem*

When C is between the two other angles, one external bisector is forward and the other backward. In this case, the theorem fails, a famous counterexample having been found by Oene Bottema.



(a) Bottema's integral triangle

(b) The extra integral triangle

(c) The 30-30-120 Triangle with four equal bisectors

FIGURE 8: Integral triangles

There is in fact just a 1-parameter family of counter-examples – '*Bottema's variable triangles*'. The equation that governs these is

$$(a + b - c)(c^2 + ab) - 2abc = 0,$$

in which the left-hand side is found by c-extraverting the displayed formula from Section 3. In trigonometric form this becomes

$$\sin \tfrac{5}{4}(A + B) \cos \tfrac{1}{4}(A - B) = \sin \tfrac{1}{4}(A + B) \cos \tfrac{3}{4}(A - B)$$

or equivalently

$$\sin^2\left(\frac{A - B}{4}\right) = \cos^2\left(\frac{A + B}{4}\right)\left[4 \sin^2\left(\frac{A + B}{4}\right) - 1\right].$$

The last form shows that for each value of C (or equivalently $A + B$), there is a unique value of $|A - B|$ (or equivalently the pair $\{A, B\}$). This means that if $4 \sin^2\left(\tfrac{1}{4}(A + B)\right) \geqslant 1$, that is $C \leqslant 60°$, there is a unique triangle for which the theorem fails. (Really this triangle is only unique up to interchange of A and B, but we shall abuse the word 'unique' in this sense whenever convenient.)

One obvious solution to the first trigonometric form of the equation has $\tfrac{5}{4}(A + B) = 180°$ and $\tfrac{3}{4}(A - B) = 90°$, yielding $A = 132°$, $B = 12°$ and $C = 36°$. This is Bottema's triangle, or more specifically *Bottema's integral triangle*, because its angles are integral in degrees. In it, the external bisectors at A and B have the same length as the joining edge, as is evident from the indicated isosceles triangles in Figure 8(a).

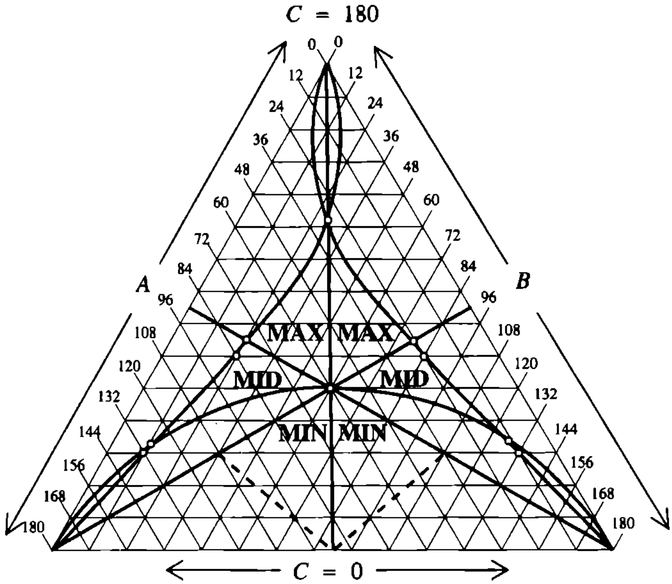


FIGURE 9: The triangle of triangles

7.3 The triangle of triangles

In our 'triangle of triangles', Figure 9, each point corresponds to a triple of numbers A, B and C that add to 180, and so to a shape of triangle. In the figure, A is constant on downward sloping lines, B on upward sloping ones, and C on horizontals. Isosceles triangles lie along the medians.

The approximately circular arc in the lower part of the figure contains Bottema's variable triangles and the lowest marked points on it are Bottema's integral triangle (and its A-B reflection). Above the Bottema curve the comparison theorem is direct (the larger bisector bisects the larger angle). Below it the comparison theorem is reversing (as in the internal case).

By a- and b-extraverting the equation of the Bottema curve we obtain the equations

$$(a-b+c)(c^2-ab)-2abc=0 \text{ and } (a-b-c)(c^2-ab)+2abc=0,$$

for the two other curves of the figure, corresponding to triangles for which an internal bisector of either A or B is equal to an external bisector of the other.

Working upwards on either of these extraverted curves, we find that after narrowly missing the Bottema integral triangle it crosses the Bottema curve at a marked point corresponding to a triangle for which the external bisector at one of A and B is equal to both bisectors at the other. (The squares of the sides of such a triangle are proportional to 1, σ^5 and σ, where $\sigma = \frac{1}{2}(\sqrt{5}-1)$.)

It then passes through a marked point corresponding to another triangle with integral angles, the '*extra integral triangle*'. This is analogous to Bottema's, with bisected angles of 24° and 84°. Again the equal bisectors have the same length as the joining edge, see Figure 8(b).

The next marked point (on a median) is our inequilateral triangle, with three equal bisectors, and the last one (on both extraverted curves and the vertical median) corresponds to the 30, 30, 120 triangle, which has four equal bisectors (Figure 8(c)).

7.4 *Case MIN: The forward external S-L theorem*

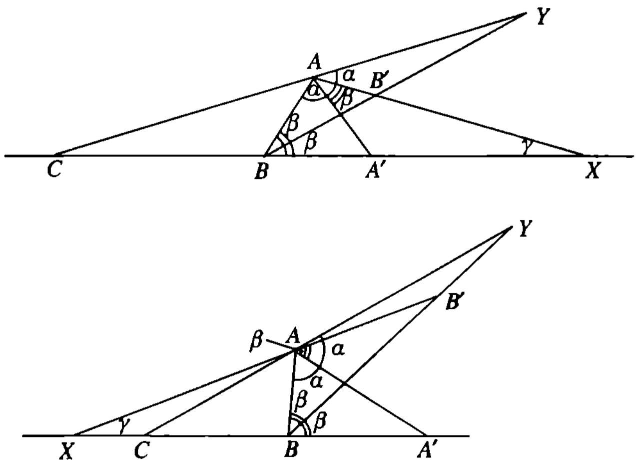


FIGURE 10: The MIN case diagram changes as X passes infinity

In this case, the usual scissor diagram takes two forms according to where the point X lies on the line BC. In the upper part of Figure 10 (where X is to the right of B) the proof for the internal case continues to work without changing a word. In particular the comparison theorem is reversing.

However, in the bottom part, X has 'passed infinity' to reappear at the left of C. Now the scissor proof argument fails because it gives $AA' > BB'$ and $BY > BB'$ which do not contradict the equality $AA' = BY$.

The boundary between the two cases is given by $2C = |A - B|$, the dashed line in the triangle of triangles. The scissor proof works only below this line, however the theorem continues to hold on and above this dashed line by the following continuity argument.

The triangle of triangles is divided into six triangular parts by its medians; inside any one of these the bisector lengths change continuously. So unless a path between two points in the same part crosses the Bottema curve, the triangles they represent must have the same comparison status (reversed or not). Since the Bottema curve lies entirely in the MID parts, this implies the comparison status is reversing everywhere in the MIN parts and direct in the MAX parts.

8. *Conclusion*

Our friend Richard Parker says that a significant mathematical assertion can be regarded as a definition of the one word you do not know in terms of the ones you do. Parker's principle suggests that when a proof of the Steiner-Lehmus theorem is described as direct, this merely tells us how the author is using the term 'direct'. More than 170 years of discussion has taught us only that there is no agreed meaning to this term. The directness question is therefore outmoded and we should ask instead whether and where proofs involve inequalities as the extraversion argument suggests they will.

Hesse's proof does so (o is obtuse), as does the algebraic proof (a, b, c are positive), the scissors proofs (blatantly); so too do all proofs in Sherri Gardner's recent collection [3].

Our first proof makes no attempt to be direct: we call it the *strictly-schizoid proof* and, whether direct or not, the second proof certainly remains schizoid, so we call it the *still-schizoid proof*. Since it seems that every proof must involve inequalities, we are inclined to disquote George Orwell:

all proofs are inequal, but some are more inequal than others.

Appendix: Some forgotten facts

Students are warned not to be ASSes by using the ASS condition in which the angle is not included between the two sides, since this can fail as in the first part of Figure 11, which shows that there two such triangles can be different. The forgotten fact used by Hesse is that a deduction is still pOSSible when the given angle is obtuse (the OSS criterion).

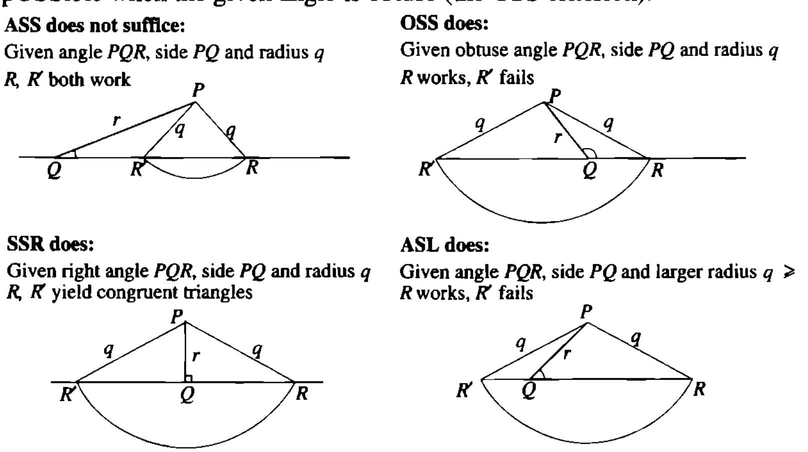


FIGURE 11: When do two sides and a non-included angle imply congruence?

The SSR case of two sides and a non-included right angle is still taught, so we call it the 'claSSRoom' criterion. It is interesting that this criterion with a right angle does not specialize either the acute ASS case (being valid) or the obtuse OSS one (since it permits two triangles, but these are congruent).

Figure 11 finishes with a short and simple condition that covers all valid cases* of ASS (two sides and a non-included angle), see also [4]. This is the ASL criterion, standing for 'Angle, Side, Longer (or equal) side', meaning that the given angle is opposite the longer of the two sides. Here 'longer' is to be interpreted so as to include equality. Our name for this is the 'ULTRASLICK' criterion, in which the middle-sized 'I' hints at the inclusive interpretation of 'longer'.

To include SAS along with these, we should use the ABLE condition – that (the given) Angle Belongs to a Longest Edge (of the two given ones) – note the inclusive sense. This covers *absolutely all* cases in which one is ABLE to deduce congruence from two sides and an angle!

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* Like the referee we hope that ASS can be brought back from 'the outer darkness with weeping and gnashing of teeth' to which it is usually relegated.

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