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DUALITY THEORY FOR EXPONENTIAL UTILITY-BASED HEDGING IN THE ALMGREN–CHRISS MODEL

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Abstract

In this paper we obtain a duality result for the exponential utility maximization problem where trading is subject to quadratic transaction costs and the investor is required to liquidate her position at the maturity date. As an application of the duality, we treat utility-based hedging in the Bachelier model. For European contingent claims with a quadratic payoff, we compute the optimal trading strategy explicitly.

Keywords: Exponential utility; duality; linear price impact; Bachelier model

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1. Introduction

In financial markets, trading moves prices against the trader: buying faster increases execution prices, and selling faster decreases them. This aspect of liquidity is known as market depth [5] or price impact. In this paper we consider the problem of optimal liquidation for the exponential utility function in the Almgren–Chriss model [1] with linear temporary impact for the underlying asset.

This problem goes back to Schied *et al.* [15], who considered a market model given by a Lévy process. They proved that the optimal trading strategy is deterministic and hence reduced the primal problem to a deterministic variational problem that can be solved explicitly. A similar phenomenon occurs in [2], where Bank and Voß consider an optimal liquidation problem with transient price impact in the Bachelier model. Namely, the fact that the utility function is exponential and the risky asset has independent increments allows us to reduce the primal hedging problem to a deterministic control problem.

For the case where the market model is not given by a process with independent increments, the exponential utility maximization problem in the Almgren–Chriss model is much more complicated and typically does not have an explicit solution. Gatheral and Schied [12] found a closed-form solution for the optimal trade execution strategy in the Almgren–Chriss framework assuming that the risky asset is given by the Black–Scholes model. However, the risk criterion they used was given by the expected value of the terminal wealth, and hence it is analytically simpler than the exponential utility maximization problem. In general, although the current paper is focused on the Almgren–Chriss model, there are other models for optimal liquidation problems. For instance, one common approach is via limit order books (see [4], [11], and the references therein).

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Our first result is Theorem 2.1, which provides a dual representation for the optimal portfolio and the corresponding value of the exponential utility maximization problem. Our duality result is obtained under quite general assumptions on the market model. As usual, for the case of exponential utility, by applying a change of measure one can reduce the problem of utility-based hedging of a European contingent claim to the standard utility maximization problem. This brings us to our second result.

Our second result (Theorem 3.1) deals with explicit computations for the case where the risky asset is given by a linear Brownian motion, i.e. the Bachelier model. We consider a European contingent claim with the payoff given by κS_T^2, where $\kappa > 0$ is a constant and S_T is the stock price at the maturity date. We apply the Girsanov theorem and the Itô isometry in order to derive a particularly convenient representation of the dual target functional which leads to deterministic variational problems. These problems can be solved explicitly and allow us both to construct the solution to the dual problem and to compute the primal optimal strategy. We show that the optimal strategy is given by a feedback form which we compute explicitly. For the case $\kappa = 0$, i.e. there is no option, Theorem 3.1 recovers the optimal portfolio found in [15] for the Bachelier model.

The problem of utility-based hedging for the Almgren–Chriss model in the Bachelier setup was studied recently by Ekren and Nadtochiy [9]: they apply the Hamilton–Jacobi–Bellman (HJB) methodology and obtain a representation of the value function and the optimal strategy. Still, they do not require the liquidation of the portfolio at the maturity date. Moreover, they assume that the payoff function is globally Lipschitz.

A natural question that for now remains open is whether Theorem 2.1 can be applied beyond the Bachelier model. In particular, it is not clear whether by applying this duality result one can recover the optimal portfolio from [15] for a general Lévy process (beyond Brownian motion).

The rest of the paper is organized as follows. In Section 2 we introduce the model and formulate a general duality result (Theorem 2.1). In Section 3 we consider the Bachelier model and we explicitly solve the problem of utility-based hedging for European contingent claims with a quadratic payoff (Theorem 3.1). In Section 4 we derive an auxiliary result from the field of deterministic variational analysis.

2. Preliminaries and the duality result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space equipped with the completed and right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. We do not make any assumptions on $\mathcal{F}_0$. Consider a simple financial market with a riskless savings account bearing zero interest (for simplicity) and with an RCLL (right-continuous with left limits) risky asset $S = (S_t)_{t\in[0,T]}$ which is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We assume the following growth condition.

Assumption 2.1. *There exists* $a > 0$ *such that* $\mathbb{E}_{\mathbb{P}}\big[\exp\big(a \sup_{0\le t\le T} S_t^2\big)\big] < \infty$.

Following [1], we model the investor's market impact in a temporary linear form, and thus, when at time t the investor turns over her position Φ_t at the rate $\phi_t = \dot{\Phi}_t$, the execution price is $S_t + (\Lambda/2)\phi_t$ for some constant $\Lambda > 0$. In our setup the investor has to liquidate her position, namely $\Phi_T = \Phi_0 + \int_0^T \phi_t \,\mathrm{d}t = 0$.

As a result, the profits and losses from trading are given by

$$V_T^{\Phi_0,\phi} := -\Phi_0 S_0 - \int_0^T \phi_t S_t \,\mathrm{d}t - \frac{\Lambda}{2}\int_0^T \phi_t^2 \,\mathrm{d}t, \tag{2.1}$$

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where Φ_0 is the initial number (deterministic) of shares. Observe that all the above integrals are defined pathwise. In particular, we do not assume that S is a semi-martingale.

Remark 2.1. Let us explain formula (2.1) in more detail. At time 0 the investor has Φ_0 stocks and the sum $-\Phi_0 S_0$ on her savings account. At time $t \in [0, T)$ the investor buys $\phi_t \, \mathrm{d}t$, an infinitesimal number of stocks, or more intuitively sells $-\phi_t \, \mathrm{d}t$ shares, so the (infinitesimal) change in the savings account is given by $-\phi_t(S_t + (\Lambda/2)\phi_t) \, \mathrm{d}t$. Since we liquidate the portfolio at the maturity date, the terminal portfolio value is equal to the terminal amount on the savings account and given by

$$-\Phi_0 S_0 - \int_0^T \phi_t \left(S_t + \frac{\Lambda}{2} \phi_t \right) \mathrm{d}t.$$

We arrive at the right-hand side of (2.1). For the case where S is a semi-martingale, by applying the integration by parts formula we get that the right-hand side of (2.1) is equal to

$$\int_0^T \Phi_t \, \mathrm{d}S_t - \frac{\Lambda}{2} \int_0^T \phi_t^2 \, \mathrm{d}t.$$

For a given Φ_0, the natural class of admissible strategies is

$$\mathcal{A}_{\Phi_0} := \left\{ \phi : \phi \text{ is } (\mathcal{F}_t)_{t \in [0,T]}\text{-optional with } \int_0^T \phi_t^2 \, \mathrm{d}t < \infty \text{ and } \Phi_0 + \int_0^T \phi_t \, \mathrm{d}t = 0 \right\}.$$

As usual, all the equalities and the inequalities are understood in the almost sure sense.

The investor's preferences are described by an exponential utility function $u(x) = -\exp(-\alpha x)$, $x \in \mathbb{R}$, with constant absolute risk aversion parameter $\alpha > 0$, and for a given Φ_0 her goal is to

$$\text{maximize } \mathbb{E}_{\mathbb{P}}\big[-\exp\big(-\alpha V_T^{\Phi_0,\phi}\big)\big] \text{ over } \phi \in \mathcal{A}_{\Phi_0}. \tag{2.2}$$

Next we introduce some notation. Let $\mathcal{Q}$ denote the set of all equivalent probability measures $\mathbb{Q} \sim \mathbb{P}$ with finite entropy

$$\mathbb{E}_{\mathbb{Q}}\left[\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] < \infty$$

relative to $\mathbb{P}$. For any $\mathbb{Q} \in \mathcal{Q}$, let $\mathcal{M}^{\mathbb{Q}}_{[0,T]}$ be the set of all square-integrable $\mathbb{Q}$-martingales $M = (M_t)_{0 \le t \le T}$. Moreover, let $\mathcal{M}^{\mathbb{Q}}_{[0,T)}$ be the set of all $\mathbb{Q}$-martingales $M = (M_t)_{0 \le t < T}$ that are defined on the half-open interval $[0, T)$ and satisfy

$$\|M\|_{L^2(\mathrm{d}t \otimes \mathbb{Q})} := \mathbb{E}_{\mathbb{Q}}\left[\int_0^T M_t^2 \, \mathrm{d}t\right] < \infty.$$

We arrive at the duality result.

Theorem 2.1. *Let Assumption 2.1 be in force. Then, for any $\Phi_0 \in \mathbb{R}$,*

$$\begin{aligned} &\max_{\phi \in \mathcal{A}_{\Phi_0}} \left\{ -\frac{1}{\alpha} \log \mathbb{E}_{\mathbb{P}}\big[\exp\big(-\alpha V_T^{\Phi_0,\phi}\big)\big] \right\} \\ &\quad = \inf_{\mathbb{Q} \in \mathcal{Q}} \inf_{M \in \mathcal{M}^{\mathbb{Q}}_{[0,T)}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\alpha} \log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) + \Phi_0(M_0 - S_0) + \frac{1}{2\Lambda} \int_0^T |M_t - S_t|^2 \, \mathrm{d}t \right] \right\}. \end{aligned} \tag{2.3}$$

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Furthermore, there is a unique minimizer $(\hat{\mathbb{Q}}, \hat{M} \in \mathcal{M}^{\hat{\mathbb{Q}}}_{[0,T)})$ *for the dual problem (right-hand side of* (2.3)*) and the process given by*

$$\hat{\phi}_t := \frac{\hat{M}_t - S_t}{\Lambda}, \quad t \in [0, T) \tag{2.4}$$

is the unique optimal portfolio ($\mathrm{d}t \otimes \mathbb{P}$ *a.s.) for the primal problem* (2.2).

Remark 2.2. Note that it is sufficient to define the optimal portfolio on the half-open interval $[0, T)$. We can just set $\phi_T := 0$.

Remark 2.3. Theorem 2.1 can be viewed as an extension of Proposition A.2 in [3] for the case where the investor liquidates her portfolio at the maturity date. The liquidation requirement adds additional difficulty to the dual representation. In particular, the maximization in the dual representation is over all equivalent probability measures and the corresponding martingales, in contrast to Proposition A.2 in [3] where the dual objects are just equivalent probability measures.

The duality result in [3] was used to solve the problem of exponential utility maximization in the Bachelier setting for the case where the investor can peek some time units into the future (frontrunner). Theorem 2.1 allows us to solve the same problem with the additional requirement that the portfolio has to be liquidated at the maturity date. Since the corresponding computations are not straightforward, we leave this problem for future research.

In the proof of the duality we assume that $\Lambda > 0$. However, if we formally take $\Lambda = 0$ in the right-hand side of (2.3) and use the convention $0/0 := 0$, we get the relation

$$\max_{\phi}\left\{-\log \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\alpha V_T^{\Phi_0,\phi}\right)\right]\right\} = \inf_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}\left[\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right],$$

where the infimum is taken over all martingale measures. This is (roughly speaking) the classical duality result for exponential hedging in the frictionless setup (see [7], [10]). Of course, in the frictionless setup there is no meaning to the initial number of shares Φ_0 and there is no real restriction in the requirement $\Phi_T = 0$.

We will prove Theorem 2.1 at the end of this section, after suitable preparations. We start by proving the superhedging theorem.

Lemma 2.1. *Let X be a random variable. Assume that there exists* $\alpha > 0$ *for which*

$$\mathbb{E}_{\mathbb{P}}[\exp(\alpha \max(-X, 0))] < \infty. \tag{2.5}$$

There exists $\phi \in \mathcal{A}_{\Phi_0}$ *such that* $V_T^{\Phi_0,\phi} \geq X$ *if and only if*

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \sup_{M\in\mathcal{M}^{\mathbb{Q}}_{[0,T]}} \mathbb{E}_{\mathbb{Q}}\left[X - \Phi_0(M_0 - S_0) - \frac{1}{2\Lambda}\int_0^T |M_t - S_t|^2\,\mathrm{d}t\right] \leq 0. \tag{2.6}$$

Proof. We start with the 'only if' part of the claim. Let $\phi \in \mathcal{A}_{\Phi_0}$ such that $V_T^{\Phi_0,\phi} \geq X$. Choose $\mathbb{Q} \in \mathcal{Q}$ and $M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}$. From (2.1) and the Cauchy–Schwarz inequality, it follows that

$$\sqrt{\int_0^T S_t^2\,\mathrm{d}t}\sqrt{\int_0^T \phi_t^2\,\mathrm{d}t} - \frac{\Lambda}{2}\int_0^T \phi_t^2\,\mathrm{d}t \geq X + \Phi_0 S_0,$$

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and hence, by exploiting the behaviour of the (random) parabola

$$x \to x\sqrt{\int_0^T S_t^2 \, \mathrm{d}t} - \frac{\Lambda}{2}x^2,$$

we get

$$\int_0^T \phi_t^2 \, \mathrm{d}t \le c\bigg(1 + \max(-X, 0) + \int_0^T S_t^2 \, \mathrm{d}t\bigg)$$

for some constant $c > 0$. This, together with Assumption 2.1, (2.5), and the well-known inequality $\mathrm{e}^x + y \log y \ge xy$, $x \in \mathbb{R}$, $y > 0$, yields

$$\mathbb{E}_{\mathbb{Q}}\bigg[\int_0^T \phi_t^2 \, \mathrm{d}t\bigg] < \infty.$$

Hence

$$\mathbb{E}_{\mathbb{Q}}\bigg[\Phi_0 M_0 + \int_0^T M_t \phi_t \, \mathrm{d}t\bigg] = 0,$$

so from (2.1) and the simple inequality

$$xy - \frac{\Lambda}{2}x^2 \le \frac{y^2}{2\Lambda}, \quad x, y \in \mathbb{R}$$

we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[X] &\le \mathbb{E}_{\mathbb{Q}}\bigg[\Phi_0(M_0 - S_0) + \int_0^T \phi_t(M_t - S_t) \, \mathrm{d}t - \frac{\Lambda}{2}\int_0^T \phi_t^2 \, \mathrm{d}t\bigg] \\ &\le \mathbb{E}_{\mathbb{Q}}\bigg[\Phi_0(M_0 - S_0) + \frac{1}{2\Lambda}\int_0^T |M_t - S_t|^2 \, \mathrm{d}t\bigg], \end{aligned}$$

and the result follows.

Next, we prove the 'if' part of the claim. Assume by contradiction that this part does not hold true. Namely, there exists X which satisfies (2.5)–(2.6) and there is no $\phi \in \mathcal{A}_{\Phi_0}$ such that $V_T^{\Phi_0,\phi} \ge X$.

Using the same arguments as in the proof of Proposition 3.5 in [14], it follows that the set

$$\Upsilon := \big(\big\{V_T^{\Phi_0,\phi} : \phi \in \mathcal{A}_{\Phi_0}\big\} - L_+^0(\mathbb{P})\big) \cap L^1(\mathbb{P})$$

is convex and closed in $L^1(\mathbb{P})$. Observe that from Assumption 2.1 and (2.5)–(2.6) (take $\mathbb{Q} = \mathbb{P}$ and $M \equiv 0$ in (2.6)) it follows that $X \in L^1(\mathbb{P})$. Since there is no $\phi \in \mathcal{A}_{\Phi_0}$ such that $V_T^{\Phi_0,\phi} \ge X$, we get $X \in L^1(\mathbb{P}) \setminus \Upsilon$.

Thus, by the Hahn–Banach separation theorem, we can find $Z \in L^\infty \setminus \{0\}$ such that

$$\mathbb{E}_{\mathbb{P}}[ZX] > \sup_{\upsilon \in \Upsilon} \mathbb{E}_{\mathbb{P}}[Z\upsilon].$$

Since

$$\big(V_T^{\Phi_0,\hat{\phi}} - L_+^0(\mathbb{P})\big) \cap L^1(\mathbb{P}) \subset \Upsilon \quad \text{for } \hat{\phi} \equiv -\frac{\Phi_0}{T},$$

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we must have $Z \geq 0$. Moreover, from (2.1) we have

$$V_T^{\Phi_0,\phi} \leq -\Phi_0 S_0 + \frac{1}{2\Lambda}\int_0^T S_t^2 \, \mathrm{d}t \quad \text{for all } \phi,$$

so from Assumption 2.1 it follows that there exists $\epsilon > 0$ such that

$$\mathbb{E}_{\mathbb{P}}[(\epsilon + Z)X] > \sup_{\upsilon \in \Upsilon} \mathbb{E}_{\mathbb{P}}[(\epsilon + Z)\upsilon].$$

We conclude that for the probability measure $\mathbb{Q}$ given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \frac{\epsilon + Z}{\epsilon + \mathbb{E}_{\mathbb{P}}[Z]}$$

we have $\mathbb{Q} \in \mathcal{Q}$ and

$$\mathbb{E}_{\mathbb{Q}}[X] > \sup_{\phi \in \mathcal{A}_{\Phi_0}} \mathbb{E}_{\mathbb{Q}}\big[V_T^{\Phi_0,\phi}\big], \tag{2.7}$$

where we set

$$\mathbb{E}_{\mathbb{Q}}\big[V_T^{\Phi_0,\phi}\big] := -\infty$$

if $V_T^{\Phi_0,\phi} \notin L^1(\mathbb{Q})$.

Next, fix $n \in \mathbb{N}$ and introduce the set

$$\mathcal{B}_n := \big\{\phi \in L^2(\mathrm{d}t \otimes \mathbb{Q}) : \|\phi\|_{L^2(\mathrm{d}t\otimes\mathbb{Q})} \leq n\big\}.$$

We argue that for any $n \in \mathbb{N}$

$$\begin{aligned}
\sup_{\phi \in \mathcal{A}_{\Phi_0}} \mathbb{E}_{\mathbb{Q}}\big[V_T^{\Phi_0,\phi}\big] &\geq \sup_{\phi \in \mathcal{A}_{\Phi_0} \cap \mathcal{B}_n} \mathbb{E}_{\mathbb{Q}}\big[V_T^{\Phi_0,\phi}\big] \\
&= \sup_{\phi \in \mathcal{B}_n} \inf_{M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}} \mathbb{E}_{\mathbb{Q}}\bigg[V_T^{\Phi_0,\phi} + M_T\bigg(\Phi_0 + \int_0^T \phi_t \, \mathrm{d}t\bigg)\bigg] \\
&= \inf_{M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}} \sup_{\phi \in \mathcal{B}_n} \mathbb{E}_{\mathbb{Q}}\bigg[V_T^{\Phi_0,\phi} + M_T\bigg(\Phi_0 + \int_0^T \phi_t \, \mathrm{d}t\bigg)\bigg].
\end{aligned} \tag{2.8}$$

Indeed, the inequality is obvious. The first equality follows from the fact that if $\phi \in \mathcal{B}_n \setminus \mathcal{A}_{\Phi_0}$, then

$$\inf_{M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}} \mathbb{E}_{\mathbb{Q}}\bigg[M_T\bigg(\Phi_0 + \int_0^T \phi_t \, \mathrm{d}t\bigg)\bigg] = -\infty.$$

For the last equality in (2.8) we apply a minimax theorem. Consider the vector space $\mathcal{M}^{\mathbb{Q}}_{[0,T]}$ with the $L^2(\mathrm{d}t \otimes \mathbb{Q})$ norm and the set $\mathcal{B}_n$ with the weak topology which corresponds to $L^2(\mathrm{d}t \otimes \mathbb{Q})$. Then both of these sets are convex subsets of topological vector spaces and the latter set is even compact. Moreover,

$$(\phi, M) \to \mathbb{E}_{\mathbb{Q}}\bigg[V_T^{\Phi_0,\phi} + M_T\bigg(\Phi_0 + \int_0^T \phi_t \, \mathrm{d}t\bigg)\bigg]$$

is upper semi-continuous and concave in ϕ and convex (indeed affine) in M. We can thus apply Theorem 4.2 in [16] to obtain the second equality.

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Next, choose $M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}$ and introduce the process ϕ (which depends on n and M):

$$\phi_t := \frac{n(M_t - S_t)}{\max\big(\Lambda n,\, \|M - S\|_{L^2(\mathrm{d}t\otimes\mathbb{Q})}\big)}, \qquad t \in [0, T].$$

Observe that $\phi \in \mathcal{B}_n$ and simple computations give

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\bigg[V_T^{\Phi_0,\phi} + M_T\bigg(\Phi_0 + \int_0^T \phi_t \,\mathrm{d}t\bigg)\bigg] \\
&\quad = \mathbb{E}_{\mathbb{Q}}\bigg[\Phi_0(M_0 - S_0) + \int_0^T \phi_t(M_t - S_t)\,\mathrm{d}t - \frac{\Lambda}{2}\int_0^T \phi_t^2 \,\mathrm{d}t\bigg] \\
&\quad = \mathbb{E}_{\mathbb{Q}}[\Phi_0(M_0 - S_0)] + G_n\big(\|M - S\|_{L^2(\mathrm{d}t\otimes\mathbb{Q})}\big),
\end{aligned}$$

where

$$G_n(x) = \begin{cases} \dfrac{x^2}{2\Lambda} & \text{if } x < \Lambda n, \\ nx - \dfrac{\Lambda}{2}n^2 & \text{otherwise.} \end{cases}$$

Since $n \in \mathbb{N}$ and $M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}$ were arbitrary, from (2.7)–(2.8) we conclude that there exists a sequence of martingales $M^n \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}$, $n \in \mathbb{N}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X] > \sup_{n\in\mathbb{N}}\big\{\mathbb{E}_{\mathbb{Q}}\big[\Phi_0\big(M_0^n - S_0\big)\big] + G_n\big(\|M^n - S\|_{L^2(\mathrm{d}t\otimes\mathbb{Q})}\big)\big\}. \tag{2.9}$$

From the fact that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is bounded we have $\mathbb{E}_{\mathbb{Q}}[X] < \infty$, so $\sup_{n\in\mathbb{N}} \|M^n - S\|_{L^2(\mathrm{d}t\otimes\mathbb{Q})} < \infty$. Thus from (2.9) we obtain that for any $k > (1/\Lambda)\sup_{n\in\mathbb{N}} \|M^n - S\|_{L^2(\mathrm{d}t\otimes\mathbb{Q})}$,

$$\mathbb{E}_{\mathbb{Q}}[X] > \mathbb{E}_{\mathbb{Q}}\bigg[\Phi_0\big(M_0^k - S_0\big) + \frac{1}{2\Lambda}\int_0^T |M_t^k - S_t|^2 \,\mathrm{d}t\bigg],$$

which is a contradiction to (2.6). This completes the proof. □

Lemma 2.2. *There exists a unique minimizer*

$$\big(\hat{\mathbb{Q}}, \hat{M} \in \mathcal{M}^{\hat{\mathbb{Q}}}_{[0,T)}\big)$$

for the optimization problem given by the right-hand side of (2.3).

Proof. Let C denote the set of all pairs (Z, Y) such that $Z > 0$ is a random variable satisfying $\mathbb{E}_{\mathbb{P}}[Z] = 1$, $\mathbb{E}_{\mathbb{P}}[Z \log Z] < \infty$, and $Y = (Y_t)_{0\le t<T}$ is a $\mathbb{P}$-martingale satisfying

$$\mathbb{E}_{\mathbb{P}}\bigg[\int_0^T \frac{Y_t^2}{\mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{F}_t]}\,\mathrm{d}t\bigg] < \infty.$$

Note that the function $(z, y) \to y^2/z$ is convex on $\mathbb{R}_{++} \times \mathbb{R}$, so C is a convex set. Define a map $\Psi : C \to \mathbb{R}$ by

$$\Psi(Z, Y) := \mathbb{E}_{\mathbb{P}}\bigg[\frac{1}{\alpha}Z \log Z + \Phi_0(Y_0 - S_0 Z) + \frac{Z}{2\Lambda}\int_0^T \bigg(\frac{Y_t}{\mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{F}_t]} - S_t\bigg)^2 \,\mathrm{d}t\bigg].$$

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Observe that there is a bijection

$$(Y, Z) \in C \leftrightarrow \left(\mathbb{Q} \in \mathcal{Q}, M \in \mathcal{M}^{\mathbb{Q}}_{[0,T)}\right)$$

given by

$$Z := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \quad \text{and} \quad M_t := \frac{Y_t}{\mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{F}_t]}, \qquad t \in [0, T),$$

and for this bijection we have

$$\Psi(Z, Y) = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\alpha}\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) + \Phi_0(M_0 - S_0) + \frac{1}{2\Lambda}\int_0^T |M_t - S_t|^2 \,\mathrm{d}t\right].$$

Thus, in order to prove the lemma, it is sufficient to show that there exists a unique minimizer for $\Psi : C \to \mathbb{R}$. Note that the convexity of the map $(z, y) \to y^2/z$ implies the convexity of Ψ. From the strict convexity of the functions $z \to z\log z$ and the map $y \to y^2$, it follows that Ψ is strictly convex, and hence the uniqueness of a minimizer is immediate. It remains to prove the existence of a minimizer.

Let $(Z^n, Y^n) \in C$, $n \in \mathbb{N}$ be a sequence such that

$$\lim_{n\to\infty} \Psi(Z^n, Y^n) = \inf_{(Z,Y)\in C} \Psi(Z, Y). \tag{2.10}$$

Assumption 2.1 and (2.10) yield that without loss of generality we can assume that

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[Z^n \log Z^n] < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left[\int_0^T \frac{|Y_t^n|^2}{\mathbb{E}_{\mathbb{P}}[Z^n \mid \mathcal{F}_t]}\,\mathrm{d}t\right] < \infty.$$

Thus the de la Vallée–Poussin criterion ensures that Z^n, $n \in \mathbb{N}$ are uniformly integrable. Let us argue that for any $s < T$ the random variables Y_s^n, $n \in \mathbb{N}$ are uniformly integrable

Fix $s < T$. From the Jensen inequality and fact that Y^n is a martingale, it follows that for any given n the function

$$t \to \mathbb{E}_{\mathbb{P}}\left[\frac{|Y_t^n|^2}{\mathbb{E}_{\mathbb{P}}[Z^n \mid \mathcal{F}_t]}\right]$$

is non-decreasing, and hence

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left[\frac{|Y_s^n|^2}{\mathbb{E}_{\mathbb{P}}[Z^n \mid \mathcal{F}_s]}\right] < \infty.$$

This, together with the inequality $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[Z^n \log Z^n] < \infty$, gives $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[g|Y_s^n|] < \infty$, where

$$g(y) := \inf_{z>0}\left\{\frac{y^2}{z} + z\log z\right\}, \qquad y > 0.$$

For a given $y > 0$, the function $z \to y^2/z + z\log z$ is convex and attains its minimum at the unique $z = z(y)$ which satisfies $1 + \log z = y^2/z^2$. Obviously $\lim_{y\to\infty} z(y) = \infty$ and $y = z(y)\sqrt{1 + \log(z(y))}$. Thus

$$\lim_{y\to\infty} \frac{g(y)}{y} \geq \lim_{z\to\infty} \frac{z\log z}{z\sqrt{1+\log z}} = \infty.$$

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Hence, from the de la Vallée–Poussin criterion, we conclude that Y_s^n, $n \in \mathbb{N}$ are uniformly integrable.

Next, define the sequence of random variables $(H_n)_{n\in\mathbb{N}} \in L^1(\mathrm{d}t \otimes \mathbb{P}, [0, 2T] \times \Omega)$ by

$$H^n(t, \omega) := \begin{cases} Y_t^n(\omega) & \text{if } t < T, \\ Z^n(\omega) & \text{otherwise.} \end{cases}$$

Observe that the relations

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[Z^n \log Z^n] < \infty, \quad \sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left[\int_0^T \frac{|Y_t^n|^2}{\mathbb{E}_{\mathbb{P}}[Z^n \mid \mathcal{F}_t]} \,\mathrm{d}t\right] < \infty, \quad \lim_{y\to\infty} \frac{g(y)}{y} = \infty$$

yield that $(H^n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^1(\mathrm{d}t \otimes \mathbb{P}, [0, 2T] \times \Omega)$. Hence, from the well-known Komlós argument (see Theorem 1.3 in [6]), there exists a sequence $(\hat{H}^n) \in \mathrm{conv}\,(H^n, H^{n+1} \cdots)$, $n \in \mathbb{N}$ such that $(\hat{H}^n)_{n\in\mathbb{N}}$ converge in probability ($\mathrm{d}t \otimes \mathbb{P}$) to some $\hat{H} \in L^1(\mathrm{d}t \otimes \mathbb{P}, [0, 2T] \times \Omega)$. From the bounded convergence theorem, $\arctan(\hat{H}^n) \to \arctan(H)$ in $L^1(\mathrm{d}t \otimes \mathbb{P}, [0, 2T] \times \Omega)$. Thus, from Fubini's theorem, we obtain that there exists a dense set $\mathcal{I} \subset [0, 2T]$ such that for any $t \in \mathcal{I}$, $(\hat{H}_t^n)_{n\in\mathbb{N}}$ converge in probability to $\hat{H}_t$. Choose a countable subset $\mathcal{J} \subset \mathcal{I} \cap [0, T)$ of the form $\mathcal{J} = \{t_1 < t_2 < \cdots\}$ such that $\lim_{n\to\infty} t_n = T$.

We conclude that there exist convex combinations (the same combinations as for H^n) $(\hat{Z}^n, \hat{Y}^n) \in \mathrm{conv}\,((Z^n, Y^n), (Z^{n+1}, Y^{n+1}) \cdots)$, $n \in \mathbb{N}$, such that the sequence $(\hat{Z}^n)_{n\in\mathbb{N}}$ converges in probability to some $\hat{Z}$, and for any $m \in \mathbb{N}$ the sequence $(\hat{Y}_{t_m}^n)_{n\in\mathbb{N}}$ converges in probability to some U_m. From the uniform integrability of the sequences $(Z^n)_{n\in\mathbb{N}}$ and $(Y_{t_m}^n)_{n\in\mathbb{N}}$, $m \in \mathbb{N}$, we conclude that

$$\hat{Z}^n \to \hat{Z} \quad \text{in } L^1(\mathbb{P}), \tag{2.11}$$

and for any $m \in \mathbb{N}$

$$\hat{Y}_{t_m}^n \to U_m \quad \text{in } L^1(\mathbb{P}). \tag{2.12}$$

Notice that (2.11) implies $\mathbb{E}_{\mathbb{P}}[\hat{Z}] = 1$. Moreover, the function $x \to x \log x$, $x > 0$ is bounded from below, so from the Fatou lemma and the convexity of the function $x \to x \log x$ we get $\mathbb{E}_{\mathbb{P}}[\hat{Z} \log \hat{Z}] \le \sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[Z^n \log Z^n] < \infty$.

Next, define the process $\hat{Y} = (\hat{Y}_t)_{0\le t<T}$ by

$$\hat{Y}_t := \sum_{m=1}^{\infty} \mathbf{1}_{\{t\in[t_{m-1},t_m)\}} \mathbb{E}_{\mathbb{P}}[U_m \mid \mathcal{F}_t],$$

where we set $t_0 := 0$. Clearly, for any n the process $\hat{Y}^n = \hat{Y}^n_{[0,T)}$ is a martingale (convex combination of martingales), so from (2.12) we obtain that $\hat{Y} = \hat{Y}_{[0,T)}$ is a martingale and

$$\hat{Y}_t^n \to \hat{Y}_t \quad \text{in } L^1(\mathbb{P}) \quad \text{for all } t \in [0, T]. \tag{2.13}$$

From (2.11) we get

$$\mathbb{E}_{\mathbb{P}}[\hat{Z}^n \mid \mathcal{F}_t] \to \mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t] \quad \text{in } L^1(\mathbb{P}) \quad \text{for all } t \in [0, T]. \tag{2.14}$$

By combining the Fatou lemma, the convexity of Ψ, (2.10), and (2.13)–(2.14), we obtain

$$\Psi(\hat{Z}, \hat{Y}) \le \lim_{n\to\infty} \Psi(\hat{Z}^n, \hat{Y}^n) = \inf_{(Z,Y)\in C} \Psi(Z, Y).$$

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A priori it might happen that $dt \otimes \mathbb{P}(\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t] = 0) > 0$, so we need to be careful with the definition of $\Psi(\hat{Z}, \hat{Y})$. From (2.13)–(2.14) it follows that we have the convergence in probability $\mathbb{E}_{\mathbb{P}}[\hat{Z}^n \mid \mathcal{F}_.] \to \mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_.]$ and $\hat{Y}^n \to \hat{Y}$ with respect to the product measure $dt \otimes \mathbb{P}$. Hence, by taking a subsequence (which for simplicity we still denote by n), we can assume that $\mathbb{E}_{\mathbb{P}}[\hat{Z}^n \mid \mathcal{F}_.] \to \mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_.]$ and $\hat{Y}^n \to \hat{Y}$ $dt \otimes \mathbb{P}$ a.s. Since $\lim_{n\to\infty} \Psi(\hat{Z}^n, \hat{Y}^n) < \infty$, from the Fatou lemma it follows that

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left(\liminf_{n\to\infty} \frac{|\hat{Y}_t^n|^2}{\mathbb{E}_{\mathbb{P}}[\hat{Z}^n \mid \mathcal{F}_t]}\right) dt\right] < \infty.$$

In particular,

$$\liminf_{n\to\infty} \frac{|\hat{Y}_t^n|^2}{\mathbb{E}_{\mathbb{P}}[\hat{Z}^n \mid \mathcal{F}_t]} < \infty \quad dt \otimes \mathbb{P} \text{ a.s.}$$

This together with the above convergence of the sequences $(\mathbb{E}_{\mathbb{P}}[\hat{Z}^n \mid \mathcal{F}_.])_{n\in\mathbb{N}}$ and $(\hat{Y}^n)_{n\in\mathbb{N}}$ yields the implication $\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t] = 0 \Rightarrow \hat{Y}_t = 0$ $dt \otimes \mathbb{P}$ a.s. Thus we set

$$\frac{\hat{Y}_t}{\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t]} := 0$$

if $\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t] = 0$.

Finally, in order to complete the proof it remains to show that $\hat{Z} > 0$ a.s. To this end, define the function $f : [0, 1] \to \mathbb{R}$ by $f(\alpha) := \Psi(\alpha + (1-\alpha)\hat{Z}, \hat{Y})$, $\alpha \in [0, 1]$. From the convexity of Ψ it follows that f is convex. The inequality $\Psi(\hat{Z}, \hat{Y}) \le \inf_{(Z,Y)\in C} \Psi(Z, Y)$ yields that the right-hand derivative $f'(0+) \ge 0$. Moreover, from the monotone (derivative of a convex function) convergence theorem it follows that we can interchange derivative and expectation. Thus

$$\begin{aligned} 0 &\le f'(0+) \\ &= \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\alpha}(1-\hat{Z}) \log \hat{Z} - \Phi_0 S_0 (1-\hat{Z})\right] \\ &\quad + \frac{1}{2\Lambda}\mathbb{E}_{\mathbb{P}}\left[(1-\hat{Z}) \int_0^T S_t^2 \, dt + \int_0^T \mathbf{1}_{\{\mathbb{E}_{\mathbb{P}}[\hat{Z}|\mathcal{F}_t]>0\}} \left(\frac{\hat{Y}_t^2}{\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t]} - \frac{\hat{Y}_t^2}{\mathbb{E}_{\mathbb{P}}^2[\hat{Z} \mid \mathcal{F}_t]}\right) dt\right]. \end{aligned}$$

We conclude that $\mathbb{E}_{\mathbb{P}}[\log \hat{Z}] > -\infty$ and complete the proof. □

Now we have all the ingredients for the proof of Theorem 2.1.

Proof. Let $\big(\hat{\mathbb{Q}} \in \mathcal{Q}, \hat{M} \in \mathcal{M}_{[0,T]}^{\hat{\mathbb{Q}}}\big)$ be the minimizer from Lemma 2.2. Denote

$$D := \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{\alpha}\log\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right) + \Phi_0(\hat{M}_0 - S_0) + \frac{1}{2\Lambda}\int_0^T |\hat{M}_t - S_t|^2 \, dt\right].$$

Let us show that there exists $\hat{\phi} \in \mathcal{A}_{\Phi_0}$ such that

$$V_T^{\Phi_0,\hat{\phi}} \ge D - \frac{1}{\alpha}\log\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right). \tag{2.15}$$

We apply Lemma 2.1 for

$$X := D - \frac{1}{\alpha}\log\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right).$$

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Clearly X satisfies (2.5), so we need to show that for any $\mathbb{Q} \in \mathcal{Q}$ and $M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\alpha}\log\left(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right) + \Phi_0(M_0 - S_0) + \frac{1}{2\Lambda}\int_0^T |M_t - S_t|^2\, dt\right] \geq D. \tag{2.16}$$

Choose $\mathbb{Q} \in \mathcal{Q}$ and $M \in \mathcal{M}^{\mathbb{Q}}_{[0,T]}$. Define $(Z, Y), (\hat{Z}, \hat{Y}) \in C$ by

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}, \quad \hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}, \quad Y_t = M_t \mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{F}_t] \quad \text{and} \quad \hat{Y}_t = \hat{M}_t \mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t], \quad t < T.$$

Define the convex function $h : [0, 1] \to \mathbb{R}_+$ by

$$h(\alpha) := \Psi(\alpha Z + (1-\alpha)\hat{Z}, \alpha Y + (1-\alpha)\hat{Y}), \quad \alpha \in [0, 1].$$

The function h attains its minimum at $\alpha = 0$, so $h'(0+) \geq 0$. Again, the monotone convergence theorem allows us to interchange derivative and expectation. Thus

$$\begin{aligned}
0 &\leq h'(0+) \\
&= \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\alpha}(Z - \hat{Z})\log \hat{Z} + \Phi_0(Y_0 - \hat{Y}_0) - \Phi_0 S_0(Z_0 - \hat{Z}_0)\right] \\
&\quad + \frac{1}{2\Lambda}\mathbb{E}_{\mathbb{P}}\left[\int_0^T ((Z - \hat{Z})S_t^2 - 2(Y_t - \hat{Y}_t)S_t)\, dt\right] \\
&\quad + \frac{1}{2\Lambda}\mathbb{E}_{\mathbb{P}}\left[\frac{1}{2\Lambda}\int_0^T \frac{2\hat{Y}_t(Y_t - \hat{Y}_t)\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t] - \mathbb{E}_{\mathbb{P}}[Z - \hat{Z} \mid \mathcal{F}_t]\hat{Y}_t^2}{\mathbb{E}^2_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t]}\right].
\end{aligned} \tag{2.17}$$

Observe that for any $t < T$

$$2Y_t\hat{Y}_t\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t] \leq \mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{F}_t]\hat{Y}_t^2 + \frac{Y_t^2\mathbb{E}^2_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t]}{\mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{F}_t]}.$$

This together with (2.17) gives

$$\begin{aligned}
0 &\leq \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\alpha}(Z - \hat{Z})\log \hat{Z} + \Phi_0(Y_0 - \hat{Y}_0) - \Phi_0 S_0(Z_0 - \hat{Z}_0)\right] \\
&\quad + \frac{1}{2\Lambda}\mathbb{E}_{\mathbb{P}}\left[\int_0^T ((Z - \hat{Z})S_t^2 - 2(Y_t - \hat{Y}_t)S_t)\, dt\right] \\
&\quad + \frac{1}{2\Lambda}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left(\frac{Y_t^2}{\mathbb{E}_{\mathbb{P}}[Z \mid \mathcal{F}_t]} - \frac{\hat{Y}_t^2}{\mathbb{E}_{\mathbb{P}}[\hat{Z} \mid \mathcal{F}_t]}\right) dt\right],
\end{aligned}$$

which is exactly (2.16). We conclude that (2.15) holds true, and thus

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\alpha V_T^{\Phi_0,\hat{\phi}}}\right] \leq e^{-\alpha D}. \tag{2.18}$$

We arrive at the final step of the proof. Choose $\phi \in \mathcal{A}_{\Phi_0}$. Without loss of generality, assume that

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\alpha V_T^{\Phi_0,\phi}}\right] < \infty,$$

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and hence arguments similar to those in the proof of Lemma 2.1 yield

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[\int_0^T \phi_t^2 \, \mathrm{d}t\right] < \infty.$$

Let us argue that for any $\gamma > 0$

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{-\alpha V_T^{\Phi_0,\phi}}\right] \\
&\quad \geq \alpha\gamma \mathbb{E}_{\hat{\mathbb{Q}}}\left[\Phi_0 S_0 + \int_0^T S_t \phi_t \, \mathrm{d}t + \frac{\Lambda}{2}\int_0^T \phi_t^2 \, \mathrm{d}t\right] - \mathbb{E}_{\mathbb{P}}\left[\gamma \hat{Z}(\log(\gamma \hat{Z}) - 1)\right] \\
&\quad = \alpha\gamma \mathbb{E}_{\hat{\mathbb{Q}}}\left[\Phi_0(S_0 - \hat{M}_0) + \int_0^T (S_t - \hat{M}_t)\phi_t \, \mathrm{d}t + \frac{\Lambda}{2}\int_0^T \phi_t^2 \, \mathrm{d}t\right] - \gamma(\log\gamma - 1) - \gamma \mathbb{E}_{\hat{\mathbb{Q}}}[\log \hat{Z}] \\
&\quad \geq \alpha\gamma \mathbb{E}_{\hat{\mathbb{Q}}}\left[\Phi_0(S_0 - \hat{M}_0) - \frac{1}{2\Lambda}\int_0^T |\hat{M}_t - S_t|^2 \, \mathrm{d}t\right] - \gamma(\log\gamma - 1) - \gamma \mathbb{E}_{\hat{\mathbb{Q}}}[\log \hat{Z}]. \qquad (2.19)
\end{aligned}$$

Indeed, the first inequality follows from the simple inequality $\mathrm{e}^x \geq xy - y(\log y - 1)$, $x \in \mathbb{R}$, $y > 0$. The equality is due to

$$\mathbb{E}_{\hat{\hat{\mathbb{Q}}}}\left[\Phi_0 \hat{M}_0 + \int_0^T \hat{M}_t \phi_t \, \mathrm{d}t\right] = 0;$$

for this we need the bound $\mathbb{E}_{\hat{\mathbb{Q}}}\left[\int_0^T \phi_t^2 \, \mathrm{d}t\right] < \infty$. The last inequality follows from the maximization of the quadratic pattern in ϕ.

Optimizing (2.19) in $\gamma > 0$, we arrive at

$$\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{-\alpha V_T^{\Phi_0,\phi}}\right] \geq \mathrm{e}^{-\alpha D}. \qquad (2.20)$$

Since $\phi \in \mathcal{A}_{\Phi_0}$ was arbitrary, from (2.18), (2.20) and the fact that $\big(\hat{\mathbb{Q}} \in \mathcal{Q}, \hat{M} \in \mathcal{M}_{[0,T]}^{\hat{\mathbb{Q}}}\big)$ is the minimizer from Lemma 2.2, we obtain (2.3). Moreover, note that there is an equality in (2.19) if and only if

$$\phi = \frac{\hat{M} - S}{\Lambda} \quad \mathrm{d}t \otimes \mathbb{P} \text{ a.s.}$$

This yields (2.4) and completes the proof. □

3. Explicit computations in the Bachelier model

In this section we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a one-dimensional Wiener process $W = (W_t)_{t\in[0,T]}$ and the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural augmented filtration generated by W. The risky asset S is given by

$$S_t = S_0 + \sigma W_t + \mu t, \qquad t \in [0, T], \qquad (3.1)$$

where $S_0 \in \mathbb{R}$ is the initial asset price, $\sigma > 0$ is the constant volatility, and $\mu \in \mathbb{R}$ is the constant drift.

Consider a European contingent claim with the quadratic payoff $\mathcal{X} = \kappa S_T^2$, where $\kappa \in (0, 1/(2\alpha\sigma^2 T))$ is a constant. We say that $\hat{\phi} \in \mathcal{A}_{\Phi_0}$ is a utility-based optimal hedging strategy if

$$\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\alpha\left(\mathcal{X} - V_T^{\Phi_0,\hat{\phi}}\right)}\right] = \inf_{\phi\in\mathcal{A}_{\Phi_0}} \mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\alpha\left(\mathcal{X} - V_T^{\Phi_0,\phi}\right)}\right].$$

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Theorem 3.1. *Let $\rho := \alpha\sigma^2/\Lambda$ be the risk–liquidity ratio. The utility-based optimal hedging strategy $\hat{\phi}_t$, $t \in [0, T)$ is unique and given by the feedback form*

$$\hat{\phi}_t = \frac{\left(2\kappa S_t + \mu/\left(\alpha\sigma^2\right)\right)\tanh\left(\sqrt{\rho}(T-t)/2\right) - \left(\coth\left(\sqrt{\rho}(T-t)\right) - 2\Lambda\sqrt{\rho}\kappa\right)\hat{\Phi}_t}{\frac{1}{\sqrt{\rho}} - 4\kappa\Lambda\tanh\left(\sqrt{\rho}(T-t)/2\right)}, \tag{3.2}$$

where

$$\hat{\Phi}_t := \Phi_0 + \int_0^t \hat{\phi}_s\,\mathrm{d}s, \quad t \in [0, T].$$

Our feedback description (3.2) can be interpreted as follows. From the simple inequality $\tanh(z) < z$, for all $z > 0$ and the assumption $\kappa \in (0, 1/(2\alpha\sigma^2 T))$, it follows that the denominator in (3.2) is positive and $\coth(\sqrt{\rho}(T-t)) - 2\Lambda\sqrt{\rho}\kappa > 0$. Thus the optimal trading strategy is a mean reverting strategy towards the process

$$\frac{\left(2\kappa S_t + \mu/\left(\alpha\sigma^2\right)\right)\tanh\left(\sqrt{\rho}(T-t)/2\right)}{\coth\left(\sqrt{\rho}(T-t)\right) - 2\Lambda\sqrt{\rho}\kappa}, \quad t \in [0, T].$$

This process can be viewed as a tradeoff between the optimal trading strategy in the frictionless case $2\kappa S_t + \mu/\left(\alpha\sigma^2\right)$, $t \in [0, T]$ and the liquidation requirement.

Next, we prove Theorem 3.1.

Proof. First, from the assumption $\kappa \in (0, 1/(2\alpha\sigma^2 T))$, it follows that $\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\alpha\mathcal{X}}\right] < \infty$. Thus we define the probability measure $\tilde{\mathbb{P}}$ by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} := \frac{\mathrm{e}^{\alpha\mathcal{X}}}{\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\alpha\mathcal{X}}\right]}.$$

Observe that for any probability measure $\mathbb{Q} \sim \mathbb{P}$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\tilde{\mathbb{P}}}\right)\right] = \mathbb{E}_{\mathbb{Q}}\left[\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) - \alpha\mathcal{X}\right] + \alpha\log\left(\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\alpha\mathcal{X}}\right]\right).$$

From Hölder's inequality and Assumption 2.1 it follows that there exists $b > 0$ such that

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left(b\sup_{0\le t\le T} S_t^2\right)\right] < \infty.$$

Hence, using Theorem 2.1 for the probability measure $\tilde{\mathbb{P}}$, we obtain

$$\min_{\phi\in\mathcal{A}_{\Phi_0}}\left\{\frac{1}{\alpha}\log\mathbb{E}_{\mathbb{P}}\left[\exp\left(\alpha\left(\mathcal{X} - V_T^{\Phi_0,\phi}\right)\right)\right]\right\}$$
$$= \sup_{\mathbb{Q}\in\mathcal{Q}}\sup_{M\in\mathcal{M}^{\mathbb{Q}}_{[0,T)}} \mathbb{E}_{\mathbb{Q}}\left[\mathcal{X} - \frac{1}{\alpha}\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) - \Phi_0(M_0 - S_0) - \frac{1}{2\Lambda}\int_0^T |M_t - S_t|^2\,\mathrm{d}t\right]. \tag{3.3}$$

Moreover, there exists a unique maximizer $\left(\hat{\mathbb{Q}} \in \mathcal{Q}, \hat{M} \in \mathcal{M}^{\hat{\mathbb{Q}}}_{[0,T)}\right)$ for the right-hand side of (3.3), and the process given by (2.4) is the unique utility-based optimal hedging strategy.

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Observe that by the Markov property of Brownian motion, in order to prove Theorem 3.1 it is sufficient to establish (3.2) for $t=0$. Thus, in view of (2.4), it remains to establish that

$$\frac{\hat{M}_0 - S_0}{\Lambda} = \frac{\big(2\kappa S_0 + \mu/(\alpha\sigma^2)\big)\tanh\big(\sqrt{\rho}T/2\big) - \big(\coth\big(\sqrt{\rho}T\big) - 2\Lambda\sqrt{\rho}\kappa\big)\Phi_0}{\frac{1}{\sqrt{\rho}} - 4\kappa\Lambda\tanh\big(\sqrt{\rho}T/2\big)}. \tag{3.4}$$

To this end, let $\mathbb{Q} \in \mathcal{Q}$ and $M \in \mathcal{M}^{\mathbb{Q}}_{[0,T)}$. From the Girsanov theorem it follows that there exists a progressively measurable process $\theta \in L^2(\mathrm{d}t \otimes \mathbb{Q})$ such that

$$\mathbb{E}_{\mathbb{Q}}[\log(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})] = \mathbb{E}_{\mathbb{Q}}\bigg[\int_0^T \theta_s^2\,\mathrm{d}s\bigg]/2 < \infty \quad \text{and} \quad W_t^{\mathbb{Q}} := W_t - \int_0^t \theta_s\,\mathrm{d}s, \quad t \in [0, T]$$

is a $\mathbb{Q}$-Brownian motion. By applying the martingale representation theorem, there exists a process $\gamma = (\gamma_t)_{0\le t<T}$ such that

$$M_t = M_0 + \sigma \int_0^t \gamma_s\,\mathrm{d}W_s^{\mathbb{Q}}, \quad \mathrm{d}t \otimes \mathbb{P}\ \text{a.s.} \tag{3.5}$$

Moreover, by applying the martingale representation theorem for θ_t, $t \in [0, T]$, we conclude that there exist a deterministic function a_t, $t \in [0, T]$ and a jointly measurable process $\beta_{t,s}$, $0 \le s \le t \le T$ such that $\beta_{t,s}$ is $\mathcal{F}_{t\wedge s}$ measurable and

$$\theta_t = a_t + \int_0^t \beta_{t,s}\,\mathrm{d}W_s^{\mathbb{Q}}, \quad \mathrm{d}t \otimes \mathbb{P}\ \text{a.s.} \tag{3.6}$$

Set

$$\nu_t := \mu t + \sigma \int_0^t a_s\,\mathrm{d}s, \quad t \in [0, T] \quad \text{and} \quad l_{t,s} := \int_s^t \beta_{u,s}\,\mathrm{d}u, \quad 0 \le s \le t \le T.$$

From Fubini's theorem, (3.1), and (3.6),

$$S_t = S_0 + \nu_t + \sigma \int_0^t (1 + l_{t,s})\,\mathrm{d}W_s^{\mathbb{Q}}, \quad t \in [0, T]. \tag{3.7}$$

Given the probability measure $\mathbb{Q}$, we are looking for a martingale $\tilde{M} \in \mathcal{M}^{\mathbb{Q}}_{[0,T)}$ which maximizes the right-hand side of (3.3). By combining (3.5), (3.7) and applying the Itô isometry and Fubini's theorem, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\bigg[\Phi_0(M_0 - S_0) + \frac{1}{2\Lambda}\int_0^T |M_t - S_t|^2\,\mathrm{d}t\bigg] \\
&\quad = \Phi_0(M_0 - S_0) + \frac{1}{2\Lambda}\int_0^T (M_0 - S_0 - \nu_t)^2\,\mathrm{d}t + \frac{\sigma^2}{2\Lambda}\mathbb{E}_{\mathbb{Q}}\bigg[\int_0^T \int_s^T (\gamma_s - 1 - l_{t,s})^2\,\mathrm{d}t\,\mathrm{d}s\bigg].
\end{aligned}$$

Given a and β, we are looking for $\hat{M}_0$ and $\hat{\gamma}$ which minimize the above right-hand side. Observe that the right-hand side is a quadratic function in M_0 and γ_s, $s \in [0, T)$. Hence we obtain that the minimizer is unique and given by

$$\hat{M}_0 = S_0 + \frac{1}{T}\int_0^T \nu_t\,\mathrm{d}t - \frac{\Phi_0\Lambda}{T} \tag{3.8}$$

and

$$\hat{\gamma}_s = 1 + \frac{1}{T-s}\int_s^T l_{t,s}\,\mathrm{d}t, \quad s < T. \tag{3.9}$$

Finally, we compute the optimal ν. From the Itô isometry, Fubini's theorem, and (3.6)–(3.7), we have

$$\mathbb{E}_{\mathbb{Q}}[S_T^2] = (S_0 + \nu_T)^2 + \sigma^2 \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \int_s^T (1 + l_{t,s})^2\,\mathrm{d}t\,\mathrm{d}s\right]$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] = \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \theta_s^2\,\mathrm{d}s\right] = \frac{1}{2}\left(\int_0^T a_t^2\,\mathrm{d}t + \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \int_s^T \beta_{t,s}^2\,\mathrm{d}t\,\mathrm{d}s\right]\right).$$

These equalities together with (3.8)–(3.9) give

$$\begin{aligned}
&\sup_{M \in \mathcal{M}^{\mathbb{Q}}_{[0,T)}} \mathbb{E}_{\mathbb{Q}}\left[\mathcal{X} - \frac{1}{\alpha}\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) - \Phi_0(M_0 - S_0) - \frac{1}{2\Lambda}\int_0^T |M_t - S_t|^2\,\mathrm{d}t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathcal{X} - \frac{1}{\alpha}\log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) - \Phi_0(\hat{M}_0 - S_0) - \frac{1}{2\Lambda}\int_0^T |\hat{M}_t - S_t|^2\,\mathrm{d}t\right] \\
&= I + \mathbb{E}_{\mathbb{Q}}\left[\int_0^T J_s\,\mathrm{d}s\right],
\end{aligned} \tag{3.10}$$

where

$$I = \kappa(S_0 + \nu_T)^2 - \frac{1}{2\alpha}\int_0^T a_t^2\,\mathrm{d}t + \frac{1}{2\Lambda}\left(\frac{1}{T}\left(\Phi_0\Lambda - \int_0^T \nu_t\,\mathrm{d}t\right)^2 - \int_0^T \nu_t^2\,\mathrm{d}t\right)$$

and

$$J_s = \kappa\sigma^2\int_s^T (1 + l_{t,s})^2\,\mathrm{d}t - \frac{1}{2\alpha}\int_s^T \beta_{t,s}^2\,\mathrm{d}t + \frac{1}{2\Lambda}\left(\frac{1}{T-s}\left(\int_s^T l_{t,s}\,\mathrm{d}t\right)^2 - \int_s^T l_{t,s}^2\,\mathrm{d}t\right).$$

From Proposition 4.1 we conclude that the optimal ν satisfies (4.2). Hence from (3.8) we obtain (3.4) and complete the proof. □

Remark 3.1. By applying (3.10) we can also compute the the right-hand side of (3.3). This requires computing the maximal I and for any $s \in [0, T]$ computing the maximal J_s. Observe that the latter is a deterministic variational problem where the control is $l_{\cdot,s}$, $\cdot \in [s, T]$. Computing both I and J_s, $s \in [0, T]$ can be done by computing the value which corresponds to the optimization problem given by (4.1) (for J_s replace T with $T - s$, S_0, σ with 1, and μ, Φ_0 with 0). The computations are quite cumbersome and hence omitted.

Remark 3.2. Note that the quadratic structure of the payoff $\mathcal{X}$ used in (3.10) is essential in reducing the dual problem to a deterministic control problem. This is due to the Itô isometry. Although for a general payoff the dual representation does not allow us to obtain an explicit solution, it can still be used for utility-based hedging problems. For instance, the recent paper [8] applies Theorem 2.1 and, for a general European contingent claim in the Bachelier model, computes the scaling limit of the corresponding utility indifference prices for a vanishing price impact which is inversely proportional to the risk aversion.

4. Auxiliary result

The following result deals with a purely deterministic setup.

Proposition 4.1. *Let* Γ *be the space of all continuous functions* $\delta : [0, T] \to \mathbb{R}$ *that are differentiable almost everywhere (with respect to the Lebesgue measure) and satisfy* $\delta(0) = 0$*. Then the maximizer* $\hat{\delta} \in \Gamma$ *of the optimization problem*

$$\max_{\delta\in\Gamma}\left\{\kappa(S_0+\delta_T)^2-\frac{1}{2\alpha\sigma^2}\int_0^T(\dot{\delta}_t-\mu)^2\,\mathrm{d}t+\frac{1}{2\Lambda}\left(\frac{1}{T}\left(\Phi_0\Lambda-\int_0^T\delta_t\,\mathrm{d}t\right)^2-\int_0^T\delta_t^2\,\mathrm{d}t\right)\right\} \tag{4.1}$$

is unique and satisfies

$$\frac{1}{T}\int_0^T\hat{\delta}_t\,\mathrm{d}t=\frac{\big(2\kappa S_0+\mu/(\alpha\sigma^2)\big)\tanh\big(\sqrt{\rho}T/2\big)-\big(\coth\big(\sqrt{\rho}T\big)-2\Lambda\sqrt{\rho}\kappa\big)\Phi_0}{\frac{1}{\sqrt{\rho}\Lambda}-4\kappa\tanh\big(\sqrt{\rho}T/2\big)}+\frac{\Phi_0\Lambda}{T}. \tag{4.2}$$

Proof. The proof will be done in two steps. First we will solve the optimization problem (4.1) for the case where δ_T and $\int_0^T\delta_t\,\mathrm{d}t$ are given. Then we will find the optimal δ_T and $\int_0^T\delta_t\,\mathrm{d}t$.

Thus, for any x, y, let $\Gamma_{x,y}\subset\Gamma$ be the set of all functions $\delta\in\Gamma$ that satisfy $\delta_T = x$ and $\int_0^T\delta_t\,\mathrm{d}t = y$. Consider the minimization problem

$$\min_{\delta\in\Gamma_{x,y}}\int_0^T H(\dot{\delta}_t,\delta_t)\,\mathrm{d}t,$$

where

$$H(u,v) := \frac{1}{2\alpha\sigma^2}(u-\mu)^2+\frac{1}{2\Lambda}v^2\quad\text{for } u, v\in\mathbb{R}.$$

This optimization problem is convex, and thus it has a unique solution which has to satisfy the Euler–Lagrange equation (for details see [13])

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial\dot{\delta}_t}=\lambda+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial\delta_t}$$

for some constant $\lambda > 0$ (Lagrange multiplier due to the constraint $\int_0^T\delta_t\,\mathrm{d}t = y$). Thus the optimizer solves the ODE $\ddot{\delta}_t - \rho\delta \equiv$ const. (recall the risk–liquidity ratio $\rho = \alpha\sigma^2/\Lambda$). From the standard theory it follows that

$$\delta_t = c_1\sinh\big(\sqrt{\rho}t\big)+c_2\sinh\big(\sqrt{\rho}(T-t)\big)+c_3 \tag{4.3}$$

for some constants c_1, c_2, c_3. From the three constraints $\delta_0 = 0$, $\delta_T = x$, and $\int_0^T\delta_t\,\mathrm{d}t = y$ we obtain

$$c_1=\frac{x-c_3}{\sinh\big(\sqrt{\rho}T\big)},\quad c_2=-\frac{c_3}{\sinh\big(\sqrt{\rho}T\big)},\quad c_3=\frac{\sqrt{\rho}y-x\tanh\big(\sqrt{\rho}T/2\big)}{\sqrt{\rho}T-2\tanh\big(\sqrt{\rho}T/2\big)}. \tag{4.4}$$

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We argue that

$$
\begin{aligned}
&\rho \int_0^T \delta_t^2 \,\mathrm{d}t + \int_0^T \dot{\delta}_t^2 \,\mathrm{d}t \\
&\quad = \rho \int_0^T ((\delta_t - c_3) + c_3)^2 \,\mathrm{d}t + \int_0^T \dot{\delta}_t^2 \,\mathrm{d}t \\
&\quad = \frac{\sqrt{\rho}}{2}\big(c_1^2 + c_2^2\big) \sinh\big(2\sqrt{\rho}T\big) - 2c_1c_2\sqrt{\rho}\sinh\big(\sqrt{\rho}T\big) - \rho c_3^2 T + 2\rho c_3 y \\
&\quad = \sqrt{\rho}x^2 \coth\big(\sqrt{\rho}T\big) + 2\sqrt{\rho}c_1c_2 \sinh\big(\sqrt{\rho}T\big)\big(\cosh\big(\sqrt{\rho}T\big) - 1\big) - \rho c_3^2 T + 2\rho c_3 y \\
&\quad = \sqrt{\rho}x^2 \coth\big(\sqrt{\rho}T\big) + \big(2\sqrt{\rho}\tanh\big(\sqrt{\rho}T/2\big) - \rho T\big)c_3^2 + 2\big(\rho y - \sqrt{\rho}\tanh\big(\sqrt{\rho}T/2\big)x\big)c_3 \\
&\quad = \sqrt{\rho}\left(x^2 \coth\big(\sqrt{\rho}T\big) + \frac{\big(x\tanh\big(\sqrt{\rho}T/2\big) - \sqrt{\rho}y\big)^2}{\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)}\right).
\end{aligned}
\tag{4.5}
$$

Indeed, the first equality is obvious. The second equality follows from (4.3) and simple computations. The third equality is due to

$$c_1 - c_2 = \frac{x}{\sinh\big(\sqrt{\rho}T\big)}.$$

The fourth equality is due to

$$c_1c_2 = \frac{c_3^2 - xc_3}{\sinh^2\big(\sqrt{\rho}T\big)}.$$

The last equality follows from substituting c_3.

By combining (4.5) and the simple equality

$$\frac{1}{2\alpha\sigma^2}\int_0^T (\dot{\delta}_t - \mu)^2 \,\mathrm{d}t + \frac{1}{2\Lambda}\int_0^T \dot{\delta}_t^2 \,\mathrm{d}t = \frac{\mu^2 T}{2\alpha\sigma^2} - \frac{\mu x}{\alpha\sigma^2} + \frac{1}{2\alpha\sigma^2}\left(\rho \int_0^T \delta_t^2 \,\mathrm{d}t + \int_0^T \dot{\delta}_t^2 \,\mathrm{d}t\right),$$

we obtain

$$
\begin{aligned}
&\min_{\delta \in \Gamma_{x,y}} \int_0^T H(\dot{\delta}_t, \delta_t)\,\mathrm{d}t \\
&\quad = \frac{\mu^2 T}{2\alpha\sigma^2} - \frac{\mu x}{\alpha\sigma^2} + \frac{1}{2\Lambda\sqrt{\rho}}\left(x^2 \coth\big(\sqrt{\rho}T\big) + \frac{\big(x\tanh\big(\sqrt{\rho}T/2\big) - \sqrt{\rho}y\big)^2}{\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)}\right).
\end{aligned}
\tag{4.6}
$$

We arrive at the final step of the proof. In view of (4.6), the optimization problem (4.1) is reduced to finding $x := \delta_T$ and $y := \int_0^T \delta_t \,\mathrm{d}t$, which maximize the quadratic form

$$Ax^2 + By^2 + 2Cxy + \eta x + \theta y,$$

where

$$A := \kappa - \frac{1}{2\Lambda\sqrt{\rho}}\left(\coth\big(\sqrt{\rho}T\big) + \frac{\tanh^2\big(\sqrt{\rho}T/2\big)}{\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)}\right)$$
$$= -\frac{\sqrt{\rho}T\coth\big(\sqrt{\rho}T\big) + 4\Lambda\sqrt{\rho}\kappa\tanh\big(\sqrt{\rho}T/2\big) - 1 - 2\Lambda\rho T\kappa}{2\Lambda\sqrt{\rho}\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)},$$
$$B := \frac{1}{2\Lambda T} - \frac{\sqrt{\rho}}{2\Lambda\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)} = -\frac{\tanh\big(\sqrt{\rho}T/2\big)}{\Lambda T\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)},$$
$$C := \frac{\tanh\big(\sqrt{\rho}T/2\big)}{2\Lambda\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)}, \quad \eta := 2\kappa S_0 + \frac{\mu}{\alpha\sigma^2} \quad \text{and} \quad \theta := -\frac{\Phi_0}{T}.$$

Simple computations give

$$AB - C^2 = -\frac{\kappa\tanh\big(\sqrt{\rho}T/2\big)}{\Lambda T\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)}$$
$$+ \frac{\coth\big(\sqrt{\rho}T\big)\tanh\big(\sqrt{\rho}T/2\big)}{2\sqrt{\rho}\Lambda^2 T\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)} - \frac{\tanh^2\big(\sqrt{\rho}T/2\big)}{4\sqrt{\rho}\Lambda^2 T\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)}$$
$$= \frac{1/(4\sqrt{\rho}\Lambda) - \kappa\tanh\big(\sqrt{\rho}T/2\big)}{\Lambda T\big(\sqrt{\rho}T - 2\tanh\big(\sqrt{\rho}T/2\big)\big)}.$$

From the inequality $z > \tanh(z)z$, $z > 0$ and the assumption $\kappa \in (0, 1/(2\alpha\sigma^2 T))$, we obtain $B < 0$ and $AB - C^2 > 0$. Thus the above quadratic form has a unique maximizer

$$(\bar{x}, \bar{y}) := \frac{1}{2(AB - C^2)}(C\theta - B\eta, C\eta - A\theta).$$

We conclude that the optimization problem (4.1) has a unique solution which is given by (4.3)–(4.4) for $(x, y) := (\bar{x}, \bar{y})$. Moreover, direct computations yield

$$\frac{\bar{y}}{T} = \frac{\big(2\kappa S_0 + \mu/\big(\alpha\sigma^2\big)\big)\tanh\big(\sqrt{\rho}T/2\big) - \big(\coth\big(\sqrt{\rho}T\big) - 2\Lambda\sqrt{\rho}\kappa\big)\Phi_0}{\frac{1}{\sqrt{\rho}\Lambda} - 4\kappa\tanh\big(\sqrt{\rho}T/2\big)} + \frac{\Phi_0\Lambda}{T},$$

and (4.2) follows. □

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