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Asymptotics of Hele-Shaw flows with multiple point sources

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We study the asymptotic behaviour of a Hele-Shaw flow produced by the injection of fluid from a finite number of points at different speeds. We prove that, as time tends to infinity, the boundary of the fluid domain approaches the circle centred at the barycentre of the injection points with weights proportional to the injection rates. The distances from the barycentre to the boundary points are estimated from both above and below.

1. Introduction

Hele-Shaw flows are viscous fluid flows in an experimental device that consists of two closely placed parallel plates. Since the gap between two plates is sufficiently narrow, one can regard them as two-dimensional flows. One of the significant features of the Hele-Shaw flow is that the flow, despite being viscous fluid flow, is characterized as a two-dimensional potential flow with its potential being the pressure of the fluid [9, pp. 581–582].

We consider a Hele-Shaw flow produced by the injection of incompressible viscous fluid into the device from multiple points. Let the fluid initially occupy a bounded domain $\Omega(0) \subset \mathbb{C}$ and let $c_1, \dots, c_l \in \Omega(0)$ be the injection points. From each point c_j, more fluid is injected at the rate $\alpha_j > 0$ per unit time. The fluid domain at time $t > 0$ is denoted by $\Omega(t)$ and its boundary by $\partial\Omega(t)$. We write n for the unit outer normal vector to $\partial\Omega(t)$. To formulate the mathematical problem we now introduce a function T that is defined by $T(z) := \inf\{t \geqslant 0 \mid z \in \overline{\Omega(t)}\}$ for each $z \in \mathbb{C}$, i.e. $T(z)$ denotes the first time when the boundary $\partial\Omega(t)$ touches z. Let $p = p(z,t)$ be the pressure of the fluid at position $z = x + \mathrm{i}y \in \Omega(t)$ and time $t > 0$, where $\mathrm{i} = \sqrt{-1}$. By the theory of Hele-Shaw flows (see, for example, [7, 11]), p and T are assumed to satisfy the following equation and boundary conditions:

$$-\Delta p = \sum_{j=1}^{l} \alpha_j \delta_{c_j} \quad \text{for } z \in \Omega(t), \quad t > 0; \tag{1.1}$$

$$p = 0 \quad \text{for } z \in \partial\Omega(t), \quad t > 0; \tag{1.2}$$

$$\frac{\partial p}{\partial n}\frac{\partial T}{\partial n} = -1 \quad \text{for } z \in \partial\Omega(t), \quad t > 0, \tag{1.3}$$

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where $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian in $\mathbb{R}^2$ and δ_c is the Dirac measure at c. From (1.1) and (1.2), for each time $t > 0$ the function p can be represented by

$$p(z,t) = \sum_{j=1}^{l} \alpha_j G_{c_j,\Omega(t)}(z) \quad \text{for } z \in \Omega(t), \tag{1.4}$$

where $G_{c_j,\Omega(t)}$ is the Green's function of $\Omega(t)$ for the Laplacian under the homogeneous Dirichlet boundary condition with pole at c_j. By substituting (1.4) into (1.3), we obtain

$$\left(\sum_{j=1}^{l} \alpha_j \frac{\partial G_{c_j,\Omega(t)}}{\partial n}\right)\frac{\partial T}{\partial n} = -1 \quad \text{for } z \in \partial\Omega(t),\ t > 0. \tag{1.5}$$

Thus, the Hele-Shaw problem is characterized by finding a monotone increasing family of domains $\{\Omega(t)\}_{t>0}$ with smooth boundaries such that the corresponding function T is smooth and satisfies (1.5). We call such a family $\{\Omega(t)\}_{t>0}$ a classical solution of the Hele-Shaw problem [13, § 13].

The problem has been investigated by many researchers with different methods. Elliott and Janovský [2] adopted a variational inequality approach to the Hele-Shaw problem and proved the global-in-time existence and the uniqueness of a weak solution. Sakai [13, 14] developed the theory of quadrature domains [6] and applied it to the Hele-Shaw problem to obtain the existence and the uniqueness of a weak solution and its several properties. With this approach, Sakai [16] was able to obtain an estimate for the distances from a fixed point to the boundary points of $\Omega(t)$, which is stated as follows: let $\Omega(0) \subset D(c,r)$ and

$$t\sum_{j=1}^{l} \alpha_j + m(\Omega(0)) \geqslant 4\pi r^2,$$

where $D(c,r)$ denotes the disc of radius r with centre c and m two-dimensional Lebesgue measure. Then it holds that

$$\sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j + \frac{m(\Omega(0))}{\pi}} - r \leqslant |z-c| \leqslant \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j + \frac{m(\Omega(0))}{\pi}} + r \tag{1.6}$$

for all $z \in \partial\Omega(t)$, $t > 0$. As a matter of fact, Sakai proved this result as a more general estimate on quadrature domains, which we will define in the next section. By the estimate (1.6), we see that

$$\max_{z\in\partial\Omega(t)} |z-c| - \min_{z\in\partial\Omega(t)} |z-c| \leqslant 2r.$$

Another approach was taken by Escher and Simonett [3]. They converted the problem into a nonlinear evolution equation on a fixed domain and constructed a unique classical solution locally in time. Following this approach, in the case of a single injection point, Vondenhoff [18] recently proved the existence of a classical solution globally in time when the initial domain is sufficiently close to a disc centred at the injection point. Detailed information on the asymptotic behaviour of the Hele-Shaw flow was also obtained by means of spectral analysis. However, the method of spectral analysis [3, 18] seems to need non-trivial refinement in the case of multiple injection points in order to study the asymptotic behaviour, since it depends on the linearization of the evolution operator around an explicit solution.

We are interested in the shape of the interface $\partial\Omega(t)$ of the Hele-Shaw flow for sufficiently large time $t > 0$. To treat general domains we work within the framework of quadrature domains. In particular, we do not impose any restriction on the smoothness or the connectivity of the initial domain. We present a precise estimate for the asymptotic behaviour of the interface of the Hele-Shaw flow in the case when $l \geqslant 2$, in terms of the distances from a fixed point to the boundary points of $\Omega(t)$. To state our main theorem, we introduce the following important quantities:

$$w_l := \frac{\sum_{j=1}^{l} \alpha_j c_j}{\sum_{j=1}^{l} \alpha_j}, \tag{1.7}$$

$$r_0 := \inf\{r \geqslant 0 \mid \Omega(0) \subset D(c, r) \text{ for some } c \in \mathbb{C}\}, \tag{1.8}$$

$$\Lambda := \sqrt{\frac{\pi}{\sum_{j=1}^{l} \alpha_j}} \min_{\sigma \in S_l} \biggl(\sum_{k=2}^{l} \frac{\alpha_{\sigma(k)} \sum_{j=1}^{k-1} \alpha_{\sigma(j)}}{(\sum_{j=1}^{k} \alpha_{\sigma(j)})^2} \biggl| \frac{\sum_{j=1}^{k-1} \alpha_{\sigma(j)} c_{\sigma(j)}}{\sum_{j=1}^{k-1} \alpha_{\sigma(j)}} - c_{\sigma(k)} \biggr|^2 \biggr), \tag{1.9}$$

where the minimum is taken over the symmetric group S_l on the finite set $\{1, \dots, l\}$. Note that w_l is the barycentre of the injection points $c_1, \dots, c_l$ with weights proportional to the respective injection rates $\alpha_1, \dots, \alpha_l$, and r_0 is the smallest one among the radii of all discs containing $\Omega(0)$. The following is the main result in this paper.

THEOREM 1.1. *Let $\Omega(0)$, c_j and α_j be as in the above setting and define w_l, r_0 and Λ by (1.7), (1.8) and (1.9), respectively. Suppose that $\{\Omega(t)\}_{t>0}$ is a classical solution of the Hele-Shaw problem. Then there exist non-negative functions $\varepsilon_-(t)$ and $\varepsilon_+(t)$ such that the inequality*

$$\sqrt{\frac{t}{\pi} \sum_{j=1}^{l} \alpha_j} - \varepsilon_-(t) \leqslant |z - w_l| \leqslant \sqrt{\frac{t}{\pi} \sum_{j=1}^{l} \alpha_j} + \varepsilon_+(t) \tag{1.10}$$

holds for all $z \in \partial\Omega(t)$, $t > 0$, and they have the following asymptotic behaviour:

$$\left.\begin{aligned} \varepsilon_-(t) &= \Lambda t^{-1/2} + O(t^{-1}), \\ \varepsilon_+(t) &= \biggl(\Lambda + \frac{{r_0}^2}{2} \sqrt{\frac{\pi}{\sum_{j=1}^{l} \alpha_j}}\biggr) t^{-1/2} + O(t^{-1}) \end{aligned}\right\} \tag{1.11}$$

as $t \to \infty$.

By the estimate (1.10), we have

$$\max_{z \in \partial\Omega(t)} |z - w_l| - \min_{z \in \partial\Omega(t)} |z - w_l| \leqslant \varepsilon_+(t) + \varepsilon_-(t) = O(t^{-1/2}) \quad \text{as } t \to \infty.$$

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Therefore, for the Hele-Shaw flow with multiple injection points we see that the interface $\partial\Omega(t)$ of the fluid domain approaches the circle centred at the barycentre w_l as $t \to \infty$.

This paper is organized as follows. In § 2 we observe that a classical solution of the Hele-Shaw problem (1.5) satisfies an integral inequality for subharmonic functions. By the inequality, $\Omega(t)$ can be regarded as a quadrature domain of a positive measure, so we will be concerned with the shapes of quadrature domains in subsequent sections. The definition of quadrature domains and their elementary properties are also presented. In § 3 we introduce the notion of the Schwarz function and show relations between the Schwarz function and quadrature domains. We see that the problem of finding a certain quadrature domain can be reduced to the construction of a domain with the corresponding Schwarz function. Section 4 deals with the quadrature domain of the two Dirac measures. We construct rational mappings which map the unit disc onto the quadrature domains by means of the Schwarz functions. With the rational mappings we can estimate the distances from a certain point to the boundary points of each quadrature domain. The estimate is the key to the proof of theorem 1.1. In § 5 we complete the proof of theorem 1.1 by induction on l, after proving some technical lemmas.

2. Weak formulation and quadrature domains

In equation (1.5), the smoothness of the boundary $\partial\Omega(t)$ and of the function T is required. This is a difficulty in dealing with equation (1.5). Following [13] we generalize the notion of classical solution so that it does not require any regularity of the boundary. Let $\{\Omega(t)\}_{t>0}$ be a classical solution of the Hele-Shaw problem. Then, for any subharmonic function s defined in $\Omega(t)$ which is integrable with respect to Lebesgue measure m, we see that

$$\begin{aligned}\int_{\Omega(t)\setminus\Omega(0)} s\,\mathrm{d}m &= \int_0^t \int_{\partial\Omega(\tau)} s\frac{1}{\partial T/\partial n}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \sum_{j=1}^{l}\alpha_j \int_0^t \int_{\partial\Omega(\tau)} s\biggl(-\frac{\partial G_{c_j,\Omega(\tau)}}{\partial n}\biggr)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\geqslant \sum_{j=1}^{l}\alpha_j \int_0^t s(c_j)\,\mathrm{d}\tau = t\sum_{j=1}^{l}\alpha_j s(c_j).\end{aligned}$$

Therefore, any classical solution $\{\Omega(t)\}_{t>0}$ satisfies, for each $t > 0$,

$$\int_{\Omega(0)} s\,\mathrm{d}m + t\sum_{j=1}^{l}\alpha_j s(c_j) \leqslant \int_{\Omega(t)} s\,\mathrm{d}m \tag{2.1}$$

for all integrable subharmonic functions s defined in $\Omega(t)$. In particular, since the constant functions $s = \pm 1$ are integrable and subharmonic in $\Omega(t)$, we have

$$m(\Omega(t)) = t\sum_{j=1}^{l}\alpha_j + m(\Omega(0)). \tag{2.2}$$

In general, for a given finite (positive Borel) measure ν with compact support, a bounded open set Ω is called a quadrature domain of ν for subharmonic functions if $\nu(\mathbb{C} \setminus \Omega) = 0$ and

$$\int s \, \mathrm{d}\nu \leqslant \int_{\Omega} s \, \mathrm{d}m$$

holds for all integrable subharmonic functions s defined in Ω. Quadrature domains for harmonic functions and for analytic functions are defined in the same way, but we take equality instead of inequality in these definitions. From (2.1), for a classical solution $\{\Omega(t)\}_{t>0}$ of the Hele-Shaw problem, each $\Omega(t)$ can be interpreted as a quadrature domain of the measure

$$\chi_{\Omega(0)} + t \sum_{j=1}^{l} \alpha_j \delta_{c_j}$$

for subharmonic functions, where $\chi_{\Omega(0)}$ denotes the characteristic function of $\Omega(0)$ and we regard it as the measure $\chi_{\Omega(0)} m$.

Here we summarize some elementary properties of quadrature domains [13, §§ 1–3].

(i) A quadrature domain for subharmonic functions is also one for harmonic functions. A quadrature domain for harmonic functions is also one for analytic functions.

(ii) For any finite measure ν which is singular with respect to m, there exists a quadrature domain of ν for subharmonic functions. For any finite measure ν being of the form $\nu = \chi_{\Omega} + \mu$, where Ω is a bounded domain and μ is a finite measure satisfying $\mu(\Omega) > 0$ and $\mu(\mathbb{C} \setminus \Omega) = 0$, there exists a quadrature domain of ν for subharmonic functions.

(iii) If a measure ν satisfies one of the conditions in (ii), then a quadrature domain of ν for subharmonic functions is uniquely determined up to a null set with respect to m. Moreover, the minimum quadrature domain $\Omega(\nu)$ exists, i.e. $\Omega(\nu) \subset \Omega$ holds for all quadrature domains Ω of ν for subharmonic functions.

(iv) If measures ν_1 and ν_2 satisfy one of the conditions in (ii) and $\nu_1 \leqslant \nu_2$, then $\Omega(\nu_1) \subset \Omega(\nu_2)$.

(v) For $\alpha > 0$ and $c \in \mathbb{C}$, a quadrature domain of the measure $\alpha\delta_c$ for subharmonic (also for harmonic and for analytic) functions is uniquely determined and is equal to $D(c, \sqrt{\alpha/\pi})$.

By the above properties of quadrature domains we see that, for each $t > 0$, there exists the minimum quadrature domain of the measure

$$\chi_{\Omega(0)} + t \sum_{j=1}^{l} \alpha_j \delta_{c_j}$$

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for subharmonic functions. Sakai [13] defined a weak solution of the Hele-Shaw problem as the family of the minimum quadrature domains

$$\left\{\Omega\left(\chi_{\Omega(0)} + t\sum_{j=1}^{l}\alpha_j\delta_{c_j}\right)\right\}_{t>0}.$$

There is another weak solution that is defined by using variational inequalities [2,4], but it was proved by Sakai [14] that these two weak solutions are equivalent. In the rest of the paper we work within the framework of quadrature domains and estimate them to prove theorem 1.1. One of the advantages of dealing with quadrature domains is that we do not need to be concerned about the smoothness of the free boundary $\partial\Omega(t)$ or topological changes of the domains $\{\Omega(t)\}_{t>0}$.

We conclude this section with an easy consequence of the above-mentioned properties of quadrature domains. The following proposition corresponds to the asymptotic behaviour of a Hele-Shaw flow with a single point source.

PROPOSITION 2.1. *Let $\Omega(0)$ be a bounded domain with $\Omega(0) \subset D(c,r)$, where $c \in \Omega(0)$ and $r > 0$, and let $\Omega(t)$ be a quadrature domain of the measure $\chi_{\Omega(0)} + t\delta_c$ for subharmonic functions. Then the inequality*

$$\sqrt{\frac{t}{\pi}} \leqslant |z-c| \leqslant \sqrt{r^2 + \frac{t}{\pi}} \tag{2.3}$$

holds for all $z \in \partial\Omega(t)$, $t > 0$.

Proof. Let $\Omega(t\delta_c)$, $\Omega(\chi_{\Omega(0)} + t\delta_c)$ and $\Omega(\chi_{D(c,r)} + t\delta_c)$ be the minimum quadrature domains of the measures $t\delta_c$, $\chi_{\Omega(0)} + t\delta_c$ and $\chi_{D(c,r)} + t\delta_c$ for subharmonic functions, respectively. Then we have

$$\Omega(t\delta_c) \subset \Omega(\chi_{\Omega(0)} + t\delta_c) \subset \Omega(\chi_{D(c,r)} + t\delta_c). \tag{2.4}$$

Note that $\Omega(t\delta_c) = D(c, \sqrt{t/\pi})$ by property (v) of quadrature domains. On the other hand, since

$$(\pi r^2 + t)s(c) \leqslant \int_{D(c,r)} s\,\mathrm{d}m + ts(c) \leqslant \int_{\Omega(\chi_{D(c,r)}+t\delta_c)} s\,\mathrm{d}m$$

holds for any integrable subharmonic function s defined in $\Omega(\chi_{D(c,r)} + t\delta_c)$, the domain $\Omega(\chi_{D(c,r)} + t\delta_c)$ is a quadrature domain of the measure $(\pi r^2 + t)\delta_c$ for subharmonic functions. Hence, we have $\Omega(\chi_{D(c,r)} + t\delta_c) = D(c, \sqrt{r^2 + t/\pi})$. Therefore, (2.4) is equivalent to the estimate

$$D\left(c, \sqrt{\frac{t}{\pi}}\right) \subset \Omega(\chi_{\Omega(0)} + t\delta_c) \subset D\left(c, \sqrt{r^2 + \frac{t}{\pi}}\right), \tag{2.5}$$

which shows (2.3) for $\Omega(t) = \Omega(\chi_{\Omega(0)} + t\delta_c)$.

Let $\Omega(t)$ be any quadrature domain of $\chi_{\Omega(0)} + t\delta_c$ for subharmonic functions. Then, by (2.5) we see that $D(c, \sqrt{t/\pi}) \subset \Omega(t)$. Let us prove

$$\Omega(t) \subset D(c, \sqrt{r^2 + t/\pi}).$$

Assume the contrary. Then there exists an open disc $D \neq \emptyset$ such that $D \in \Omega(t) \setminus D(c, \sqrt{r^2 + t/\pi})$. Hence, $m(\Omega(t) \setminus \Omega(\chi_{\Omega(0)} + t\delta_c)) \geqslant m(D) > 0$. This contradicts the uniqueness of quadrature domains for subharmonic functions. Therefore, (2.3) holds for any quadrature domain $\Omega(t)$. □

Note that $\sqrt{r^2 + t/\pi} - \sqrt{t/\pi} = (r^2\sqrt{\pi}/2)t^{-1/2} + O(t^{-3/2})$ as $t \to \infty$. Thus, for the Hele-Shaw problem with one injection point c, the boundary $\partial\Omega(t)$ of the fluid domain approaches the disc centred at c as $t \to \infty$. An essential fact that we have used in the proof of proposition 2.1 is that a quadrature domain of the measure $\alpha\delta_c$ is uniquely and explicitly determined as the disc $D(c, \sqrt{\alpha/\pi})$. This indicates that an explicit representation of the quadrature domains of the measure $t\sum_{j=1}^{l} \alpha_j \delta_{c_j}$ for subharmonic functions is the key to proving theorem 1.1.

3. The Schwarz function

To prove theorem 1.1, as a first step, we construct an explicit representation of the minimum quadrature domain of the measure $\pi(\alpha\delta_{\mathrm{i}} + \beta\delta_{-\mathrm{i}})$ for subharmonic functions, where $\alpha, \beta > 0$. It will be given as a univalent rational mapping from the unit disc onto the quadrature domain, and we estimate the distances from the barycentre $(\alpha - \beta)\mathrm{i}/(\alpha + \beta)$ to the boundary points of the quadrature domain. The construction of the rational mapping and its estimate will be discussed in the next section.

Let us introduce the notion of the Schwarz function. The Schwarz function $S = S(z)$ of a curve Γ is defined as a holomorphic function on a neighbourhood of Γ which satisfies

$$S(z) = \bar{z} \quad \text{for } z \in \Gamma,$$

where $\bar{z}$ is the complex conjugate of z. Note that the Schwarz function of Γ is uniquely determined for a given curve Γ by its analyticity.

Let us explain how the Schwarz function relates to quadrature domains (see [1, ch. 14] and [17, ch. 3]). Let $\Omega \subset \mathbb{C}$ be a bounded domain with smooth boundary and let f be a holomorphic function in a neighbourhood of $\bar{\Omega}$, where $\bar{\Omega}$ denotes the closure of Ω. By the analyticity of f and Stokes's theorem, we see that

$$\int_{\Omega} f \, \mathrm{d}m = \frac{1}{2\mathrm{i}} \int_{\partial\Omega} f(z)\bar{z} \, \mathrm{d}z,$$

where $\partial\Omega$ is positively oriented. Now assume that there exists the Schwarz function S of $\partial\Omega$ and that it can be extended to a holomorphic function in $\Omega \setminus \{c_1, \dots, c_l\}$ such that $c_j \in \Omega$ is a simple pole with residue $t\alpha_j/\pi$ for $j = 1, \dots, l$. Then we have

$$\int_{\partial\Omega} f(z)\bar{z} \, \mathrm{d}z = \int_{\partial\Omega} f(z)S(z) \, \mathrm{d}z = 2\mathrm{i}t \sum_{j=1}^{l} \alpha_j f(c_j).$$

Thus,

$$\int_{\Omega} f \, \mathrm{d}m = t \sum_{j=1}^{l} \alpha_j f(c_j) \tag{3.1}$$

holds for all holomorphic functions f defined in a neighbourhood of $\bar{\Omega}$. From (3.1), Ω is expected to be a quadrature domain of the measure $t\sum_{j=1}^{l}\alpha_j\delta_{c_j}$ for subharmonic functions.

To obtain such a candidate for the quadrature domain, we therefore find a domain Ω such that the Schwarz function of $\partial\Omega$ has simple poles at $c_1,\dots,c_l\in\Omega$ with respective residues $t\alpha_1/\pi,\dots,t\alpha_l/\pi$. As we will see later, the domain Ω that we found is, in fact, a quadrature domain of the measure $t\sum_{j=1}^{l}\alpha_j\delta_{c_j}$ for subharmonic functions. In order to find such a domain Ω, we assume that Ω can be represented as the image of the unit disc $D(0,1)$ by a rational function φ, i.e. $\Omega=\varphi(D(0,1))$, where φ is holomorphic and injective in a neighbourhood of $\overline{D(0,1)}$. Then the Schwarz function of $\partial\Omega$ is given by

$$S(z):=\overline{\varphi\left(\frac{1}{\overline{\varphi^{-1}(z)}}\right)}\quad\text{for } z \text{ in a neighbourhood of } \partial\Omega. \tag{3.2}$$

Moreover, if φ has only the simple poles at $w_1,\dots,w_l\in(\mathbb{C}\cup\{\infty\})\setminus\overline{D(0,1)}$, then S can be meromorphically extended into Ω with simple poles at $\varphi(1/\overline{w_1}),\dots,\varphi(1/\overline{w_l})$. Hence, our task is to choose a rational function φ appropriately so that $\varphi(1/\overline{w_j})=c_j$ and that the residue of the corresponding function S at c_j is $t\alpha_j/\pi$.

However, in general it is quite difficult to construct such a rational function φ. In particular, for $l\geqslant 3$, there are infinitely many possibilities of the disposition of $c_1,\dots,c_l$. As we will see later, in the case when $l=2$, by using translation, rotation and dilation we only have to consider the case where $c_1=\mathrm{i}$ and $c_2=-\mathrm{i}$.

4. Quadrature domains of two point masses

In this section we deal with quadrature domains of the measure $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$. Note that the measure $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$ corresponds to a Hele-Shaw flow with two injection points. When the injection rates are the same, i.e. $\alpha=\beta$, Richardson [12] showed that the interface of the Hele-Shaw flow is a curve formed by inverting an ellipse with respect to the unit circle. Such a curve is called an elliptic lemniscate of Booth, which is named after the Reverend James Booth. Here we are also concerned with the case $\alpha\neq\beta$.

In chapter 3 of [17], the rational function $\varphi_0(w):=2Rw/(w^2+R^2)$, $R>1$, is used to construct such a quadrature domain. To treat the case where $\alpha\neq\beta$ we introduce a new rational function φ defined by

$$\varphi(w)=\varphi_{a,R,\eta}(w):=\frac{aR(w-\mathrm{i}\eta)}{w^2+R^2}+\eta R. \tag{4.1}$$

Here, the function $\varphi=\varphi_{a,R,\eta}$ is parametrized by $a>0$, $R>1$ and $\eta\in\mathbb{R}$. For given $\alpha,\beta>0$, we choose a, R and η appropriately so that the domain $\Omega(a,R,\eta):=\varphi_{a,R,\eta}(D(0,1))$ is a quadrature domain of the measure $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$.

4.1. Construction of a rational mapping

LEMMA 4.1. *Let α and β be positive numbers such that $\alpha+\beta$ is sufficiently large. Then, by taking some $a>0$, $R>1$ and $\eta\in\mathbb{R}$ and defining a rational function φ by (4.1), the Schwarz function S of $\partial\Omega(a,R,\eta)$, where $\Omega(a,R,\eta):=\varphi(D(0,1))$,* *is meromorphic in a neighbourhood of* $\overline{\Omega(a,R,\eta)}$ *having only simple poles at* i, $-$i *with residues* α *and* β*, respectively.*

Proof. STEP 1. We determine $a > 0$, $R > 1$ and $\eta \in \mathbb{R}$ after some observations. For the time being let us assume that φ is holomorphic and injective in the disc $D(0,2)$. Then the Schwarz function S is given by (3.2). Since φ has two simple poles at $\pm iR$, the function S is meromorphic in $\Omega(a,R,\eta)$ with only two simple poles at

$$\varphi\left(\frac{1}{\mp iR}\right) = \frac{iaR^2(\pm 1 - \eta R)}{R^4 - 1} + i\eta R.$$

Hence, we take $a > 0$ to be $(R^4-1)/R^2$ so that the poles of S are at $\pm i$. Let us calculate the residues of S at $\pm i$. Now we have a parametric representation of S as follows:

$$z = \varphi(w), \qquad S(z) = \tilde{S}(w), \qquad w \in D(0,2),$$

where

$$\tilde{S}(w) = \overline{\varphi\left(\frac{1}{\bar{w}}\right)} = \frac{R^4-1}{R}\frac{w + i\eta w^2}{w^2R^2 + 1}.$$

By some computations we see that if $\tilde{S}$ has a simple pole at w_0 with residue ρ, then S has a simple pole at $\varphi(w_0)$ with residue $\rho\varphi'(w_0)$. In the present case, $w_0 = \pm iR^{-1}$, $\rho = (R^4-1)(R \mp \eta)/(2R^4)$ and

$$\begin{aligned} \varphi(w_0) &= \pm i, \\ \varphi'(w_0) &= \frac{R(R^4 \mp 2\eta R + 1)}{R^4 - 1}, \\ \rho\varphi'(w_0) &= \frac{1}{2R^3}(R^5 + R + 2\eta^2 R \mp \eta R^4 \mp \eta \mp 2\eta R^2). \end{aligned}$$

Therefore, S has simple poles at $\pm i$ with residues

$$\begin{aligned} \rho_1 &:= \frac{1}{2R^3}(R^5 + R + 2\eta^2 R - \eta R^4 - \eta - 2\eta R^2), \\ \rho_2 &:= \frac{1}{2R^3}(R^5 + R + 2\eta^2 R + \eta R^4 + \eta + 2\eta R^2), \end{aligned}$$

respectively.

STEP 2. By the above observations, for given $\alpha, \beta > 0$ we choose $R > 2$ and $\eta \in \mathbb{R}$ appropriately so that $\rho_1 = \alpha$, $\rho_2 = \beta$ and so that φ is injective in the disc $D(0,2)$. This means that we have to solve the following system of equations:

$$\alpha + \beta = \rho_1 + \rho_2 = \frac{1}{R^2}(R^4 + 1 + 2\eta^2); \tag{4.2}$$

$$\beta - \alpha = \rho_2 - \rho_1 = \frac{\eta}{R^3}(R^2+1)^2 \tag{4.3}$$

for R and η. By (4.3),

$$\eta = \frac{(\beta-\alpha)R^3}{(R^2+1)^2}. \tag{4.4}$$

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Substituting (4.4) into (4.2), we see that

$$(\alpha+\beta)R^2 = R^4 + 1 + \frac{2(\alpha-\beta)^2 R^6}{(R^2+1)^4}. \tag{4.5}$$

Let us solve the algebraic equation (4.5) for R, and then substitute a solution R into (4.4) to obtain η. Dividing both sides of equation (4.5) by R^2 gives us

$$\alpha+\beta = (R+R^{-1})^2 - 2 + \frac{2(\alpha-\beta)^2}{(R+R^{-1})^4}.$$

Putting $x := (R+R^{-1})^2$ into the above equation, we have the cubic equation

$$x^3 - (2+\alpha+\beta)x^2 + 2(\alpha-\beta)^2 = 0. \tag{4.6}$$

Moreover, we set $\xi := x - (\alpha+\beta+2)/3$ and substitute it into (4.6), obtaining

$$\xi^3 + 3p\xi + 2q = 0, \tag{4.7}$$

where

$$p = -\frac{(\alpha+\beta+2)^2}{9} \quad \text{and} \quad q = \frac{-(\alpha+\beta+2)^3 + 27(\alpha-\beta)^2}{27}.$$

According to Cardano's method for solving cubic equations, if $q^2 + p^3 \leqslant 0$, the solutions of (4.7) can be represented as follows:

$$\xi_1 := 2\sqrt[3]{r}\cos\frac{\theta}{3}, \qquad \xi_2 := 2\sqrt[3]{r}\cos\frac{\theta+2\pi}{3}, \qquad \xi_3 := 2\sqrt[3]{r}\cos\frac{\theta+4\pi}{3},$$

where $r > 0$ and $-\pi < \theta \leqslant \pi$ are defined by $\mathrm{e}^{\mathrm{i}\theta} = -q + \sqrt{q^2+p^3}$. Observe that, for α, $\beta > 0$,

$$\begin{aligned} 27^2(q^2+p^3) &= (-(\alpha+\beta+2)^3 + 27(\alpha-\beta)^2)^2 - (\alpha+\beta+2)^6 \\ &= 27(\alpha-\beta)^2(-2(\alpha+\beta+2)^3 + 27(\alpha-\beta)^2) \\ &\leqslant 27(\alpha-\beta)^2 \max_{\lambda\geqslant 0}(-2(\lambda+2)^3 + 27\lambda^2) \\ &= 0 \end{aligned}$$

by simple computations. We also have

$$r = |-q+\sqrt{q^2+p^3}| = \sqrt{q^2-(q^2+p^3)} = \sqrt{-p^3} = \frac{(\alpha+\beta+2)^3}{27}$$

and

$$\cos\theta = -\frac{q}{r} = \frac{(\alpha+\beta+2)^3 - 27(\alpha-\beta)^2}{(\alpha+\beta+2)^3} = 1 - \frac{27(\alpha-\beta)^2}{(\alpha+\beta+2)^3},$$

so that

$$0 \leqslant 1 - \cos\frac{\theta}{3} \leqslant 1 - \cos\theta = \frac{27(\alpha-\beta)^2}{(\alpha+\beta+2)^3}.$$

Thus, the solution ξ_1 of (4.7) satisfies

$$0 \leqslant \frac{2(\alpha+\beta+2)}{3} - \xi_1 \leqslant \frac{18(\alpha-\beta)^2}{(\alpha+\beta+2)^2}$$

and, consequently, $x_1 := \xi_1 + (\alpha+\beta+2)/3$, a solution of (4.6), satisfies

$$0 \leqslant \alpha+\beta-(x_1-2) \leqslant \frac{18(\alpha-\beta)^2}{(\alpha+\beta+2)^2} \leqslant 18. \tag{4.8}$$

When $\alpha+\beta>0$ is sufficiently large, the solution x_1 is a large positive number and satisfies

$$0 \leqslant \sqrt{\alpha+\beta} - \sqrt{x_1-2} \leqslant \frac{9}{\sqrt{x_1-2}} \leqslant \frac{9}{\sqrt{\alpha+\beta-18}}.$$

Now we define $R>2$ by

$$R := \frac{\sqrt{x_1}+\sqrt{x_1-4}}{2}. \tag{4.9}$$

Then R satisfies equation (4.5) and the following estimate:

$$\begin{aligned} 0 &\leqslant \sqrt{\alpha+\beta} - R \\ &= (\sqrt{\alpha+\beta}-\sqrt{x_1-2}) + \left(\sqrt{x_1-2} - \frac{\sqrt{x_1}+\sqrt{x_1-4}}{2}\right) \\ &\leqslant (\sqrt{\alpha+\beta}-\sqrt{x_1-2}) + \frac{1}{2}\left(-\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_1-4}}\right) \\ &\leqslant \frac{9}{\sqrt{\alpha+\beta-18}} + \frac{1}{2\sqrt{\alpha+\beta-20}}. \end{aligned} \tag{4.10}$$

In particular,

$$R = \sqrt{\alpha+\beta} + O\left(\frac{1}{\sqrt{\alpha+\beta}}\right) \quad \text{as } \alpha+\beta\to\infty. \tag{4.11}$$

By using R defined by (4.9), we define η by (4.4). Then R and η solve the system of equations (4.2), (4.3). Note that

$$\begin{aligned} \eta &= \frac{(\beta-\alpha)R}{(R+R^{-1})^2} \\ &= \frac{(\beta-\alpha)R}{x_1} = \frac{(\beta-\alpha)(\sqrt{\alpha+\beta}+O(1/\sqrt{\alpha+\beta}))}{\alpha+\beta+O(1)} \\ &= (\beta-\alpha)\left\{\frac{1}{\sqrt{\alpha+\beta}} + O((\alpha+\beta)^{-3/2})\right\} \quad \text{as } \alpha+\beta\to\infty. \end{aligned} \tag{4.12}$$

STEP 3. We are now in a position to prove the injectivity of φ in $D(0,2)$ when $\alpha+\beta$ is sufficiently large. Suppose that $\varphi(w_1)=\varphi(w_2)$ with w_1, $w_2 \in D(0,2)$. Then

$$(w_1-\mathrm{i}\eta)(w_2^2+R^2) = (w_2-\mathrm{i}\eta)(w_1^2+R^2),$$

and hence

$$(w_1-w_2)(R^2+\mathrm{i}\eta(w_1+w_2)-w_1w_2) = 0. \tag{4.13}$$

However, by (4.11) and (4.12),

$$|R^2 + \mathrm{i}\eta(w_1 + w_2) - w_1 w_2| \geqslant R^2 - |\mathrm{i}\eta(w_1 + w_2) - w_1 w_2| = \alpha + \beta + O(\sqrt{\alpha+\beta}) \quad \text{as } \alpha + \beta \to \infty.$$

Therefore, from (4.13) it follows that $w_1 = w_2$ if $\alpha + \beta$ is sufficiently large. Consequently, φ is injective in $D(0,2)$. Thus, when $\alpha+\beta$ is sufficiently large, by choosing a, $\eta \in \mathbb{R}$ and $R > 2$ as above, we have shown that the Schwarz function S of $\partial\Omega(a,R,\eta)$ satisfies the desired conditions. □

Remark 4.2. We make some observations on the shapes of the domains $\Omega(a,R,\eta)$ constructed in lemma 4.1. From $\Omega(a,R,\eta) = \varphi(D(0,1))$ we can see that $\Omega(a,R,\eta)$ is a simply connected bounded domain with analytic boundary. We also observe that $\Omega(a,R,\eta)$ is symmetric with respect to the imaginary axis, i.e. if $X + \mathrm{i}Y \in \Omega(a,R,\eta)$, then $-X + \mathrm{i}Y \in \Omega(a,R,\eta)$. Indeed, substituting $w = x + \mathrm{i}y$ into (4.1) yields

$$\begin{aligned}\varphi(x + \mathrm{i}y) &= \frac{aR(x + \mathrm{i}y - \mathrm{i}\eta)}{(x+\mathrm{i}y)^2 + R^2} + \mathrm{i}\eta R \\ &= \frac{aR(x + \mathrm{i}(y-\eta))}{x^2 - y^2 + R^2 + 2\mathrm{i}xy} + \mathrm{i}\eta R \\ &= \frac{aR\{x(x^2 - y^2 + R^2 + 2y(y-\eta)) + \mathrm{i}((x^2 - y^2 + R^2)(y-\eta) - 2x^2y)\}}{(x^2-y^2+R^2)^2 + 4x^2y^2} + \mathrm{i}\eta R,\end{aligned}$$

so that

$$\left.\begin{aligned}\operatorname{Re}\varphi(x+\mathrm{i}y) &= \frac{aRx(x^2 - y^2 + R^2 + 2y(y-\eta))}{(x^2-y^2+R^2)^2 + 4x^2y^2}, \\ \operatorname{Im}\varphi(x+\mathrm{i}y) &= \frac{aR((x^2-y^2+R^2)(y-\eta) - 2x^2y)}{(x^2-y^2+R^2)^2 + 4x^2y^2} + \eta R.\end{aligned}\right\} \tag{4.14}$$

For any $X + \mathrm{i}Y \in \Omega(a,R,\eta)$, there exists $x + \mathrm{i}y \in D(0,1)$ such that $\varphi(x+\mathrm{i}y) = X + \mathrm{i}Y$. Then, by (4.14) we see that $\varphi(-x+\mathrm{i}y) = -X + \mathrm{i}Y$, which implies that $-X + \mathrm{i}Y \in \Omega(a,R,\eta)$.

By virtue of lemma 4.1 we see that the domain $\Omega(a,R,\eta)$ satisfies

$$\int_{\Omega(a,R,\eta)} f \, \mathrm{d}m = \pi\alpha f(\mathrm{i}) + \pi\beta f(-\mathrm{i}) \tag{4.15}$$

for all holomorphic functions f defined in a neighbourhood of $\overline{\Omega(a,R,\eta)}$. Now we confirm that the domain $\Omega(a,R,\eta)$ is indeed a quadrature domain for subharmonic functions.

Lemma 4.3. *Let α and β be positive numbers such that $\alpha + \beta$ is sufficiently large. Then the domain $\Omega(a,R,\eta)$ constructed in lemma 4.1 is a unique quadrature domain of the measure $\pi(\alpha\delta_{\mathrm{i}} + \beta\delta_{-\mathrm{i}})$ for subharmonic functions.*

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Proof. First we prove that $\Omega(a,R,\eta)$ is a quadrature domain of the measure $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$ for harmonic functions. Let h be a harmonic function defined in a neighbourhood of $\overline{\Omega(a,R,\eta)}$. Choosing $\varepsilon>0$ so small that h is harmonic in the simply connected domain $\varphi(D(0,1+\varepsilon))$, we see that there exists a holomorphic function f defined in $\varphi(D(0,1+\varepsilon))$ whose real part is h. Then the function f satisfies (4.15). Taking the real part of each side of (4.15), we have

$$\int_{\Omega(a,R,\eta)} h\,\mathrm{d}m = \pi\alpha h(\mathrm{i}) + \pi\beta h(-\mathrm{i}). \tag{4.16}$$

According to the approximation theorem by Sakai [13, lemma 7.3], any integrable harmonic function h defined in $\Omega(a,R,\eta)$ can be approximated in $L^1(\Omega(a,R,\eta))$ by some linear combinations of $\mathrm{Re}(1/(\cdot-\zeta))$, $\mathrm{Im}(1/(\cdot-\zeta))$ and $\log|\cdot-\zeta|$ with $\zeta\in\mathbb{C}\setminus\Omega(a,R,\eta)$. Hence, we only have to check that equality (4.16) holds for $\mathrm{Re}(1/(\cdot-\zeta))$, $\mathrm{Im}(1/(\cdot-\zeta))$ and $\log|\cdot-\zeta|$ for $\zeta\in\mathbb{C}\setminus\Omega(a,R,\eta)$. When $\zeta\in\mathbb{C}\setminus\overline{\Omega(a,R,\eta)}$, it follows from the argument in the previous paragraph. So we will prove it for $\zeta\in\partial\Omega(a,R,\eta)$.

By the smoothness of $\partial\Omega(a,R,\eta)$, we can take an open right-spherical cone V_ζ with vertex ζ, angle $0<2\psi<\pi$ and height ρ which lies in $\mathbb{C}\setminus\overline{\Omega(a,R,\eta)}$. Then, by taking a sequence $\{\zeta_j\}$ on the axis of V_ζ such that $\zeta_j\to\zeta$, we have

$$\int_{\Omega(a,R,\eta)} \left|\mathrm{Re}\left(\frac{1}{z-\zeta_j}\right)-\mathrm{Re}\left(\frac{1}{z-\zeta}\right)\right|\mathrm{d}m(z)\to 0 \quad \text{as } j\to\infty. \tag{4.17}$$

To verify this we may assume $\zeta_j\in D(\zeta,\frac{1}{2}\rho)$. Then, from $|z-\zeta|\leqslant|z-\zeta_j|+|\zeta_j-\zeta|$, it follows that

$$\frac{1}{|z-\zeta_j|}\leqslant\frac{1}{|z-\zeta|}+\frac{|\zeta_j-\zeta|}{|z-\zeta||z-\zeta_j|}\leqslant\frac{1}{|z-\zeta|}\left(1+\frac{1}{\sin\psi}\right) \quad \text{for } z\in\Omega(a,R,\eta). \tag{4.18}$$

Hence, by Lebesgue's convergence theorem, (4.17) holds. By (4.17) we see that (4.16) holds for $h(z)=\mathrm{Re}(1/(z-\zeta))$ with $\zeta\in\partial\Omega(a,R,\eta)$. The same argument shows that (4.16) holds also for $\mathrm{Im}(1/(z-\zeta))$ and $\log|z-\zeta|$ with $\zeta\in\partial\Omega(t,\alpha)$. Here we have used the fact that

$$-\log|z-\zeta_j|\leqslant-\log|z-\zeta|+\log\left(1+\frac{1}{\sin\psi}\right) \quad \text{for } z\in\Omega(a,R,\eta),$$

which is deduced from (4.18).

By now we have shown that $\Omega(a,R,\eta)$ is a quadrature domain of $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$ for harmonic functions. Next we show that $\Omega(a,R,\eta)$ is, in fact, a unique quadrature domain for subharmonic functions. As has been mentioned in the previous section, there exists the minimum quadrature domain of $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$ for subharmonic functions. We denote it by Ω_0 and show that $\Omega(a,R,\eta)=\Omega_0$. Since Ω_0 is also a quadrature domain for harmonic functions, it suffices to show the uniqueness of quadrature domains of $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$ for harmonic functions.

The following fact is due to [13] (see also [17, proposition 4.8 and theorem 4.9] for the proof). Let ν be a finite measure whose support is a compact subset of a line L in $\mathbb{C}$. If there exists a quadrature domain Ω of the measure ν for harmonic functions which satisfies the following conditions, then Ω is a unique quadrature domain of ν for harmonic functions:

(i) $\mathbb{C} \setminus \bar{\Omega}$ is connected;

(ii) the interior of $\bar{\Omega}$ is identical with Ω;

(iii) Ω is symmetric with respect to L;

(iv) the support of ν is contained in $L \cap \Omega$.

Since the domain $\Omega(a, R, \eta)$ and the measure $\pi(\alpha\delta_{\mathrm{i}} + \beta\delta_{-\mathrm{i}})$ satisfy conditions (i)–(iv) with L being the imaginary axis, we see that $\Omega(a, R, \eta)$ is a unique quadrature domain of $\pi(\alpha\delta_{\mathrm{i}} + \beta\delta_{-\mathrm{i}})$ for harmonic functions. Therefore, $\Omega(a, R, \eta) = \Omega_0$ and hence $\Omega(a, R, \eta)$ is a unique quadrature domain of $\pi(\alpha\delta_{\mathrm{i}} + \beta\delta_{-\mathrm{i}})$ for subharmonic functions. □

4.2. Estimates of quadrature domains

By lemma 4.3 we see that a unique quadrature domain $\Omega(\alpha, \beta)$ of the measure $\pi(\alpha\delta_{\mathrm{i}} + \beta\delta_{-\mathrm{i}})$ for subharmonic functions is represented by

$$\Omega(\alpha, \beta) = \varphi_{a,R,\eta}(D(0, 1)).$$

On the other hand, $a > 0$, $R > 1$ and $\eta \in \mathbb{R}$ are estimated in the proof of lemma 4.1. In the following theorem, we proceed to the calculation of the distance from the point $(\alpha-\beta)\mathrm{i}/(\alpha+\beta)$ to a boundary point $z \in \partial\Omega(\alpha, \beta)$ and obtain the asymptotics of the quadrature domain $\Omega(\alpha, \beta)$ when $(\alpha + \beta)\min\{\alpha, \beta\} \to \infty$. Note that

$$\sqrt{(\alpha + \beta)\min\{\alpha, \beta\}} \leqslant \alpha + \beta.$$

Hence, $(\alpha + \beta)\min\{\alpha, \beta\} \to \infty$ implies $\alpha + \beta \to \infty$.

Theorem 4.4. *For $\alpha, \beta > 0$ such that $\alpha + \beta$ is sufficiently large, let $\Omega(\alpha, \beta)$ be a unique quadrature domain of the measure $\pi(\alpha\delta_{\mathrm{i}} + \beta\delta_{-\mathrm{i}})$ for subharmonic functions. Then, as $(\alpha + \beta)\min\{\alpha, \beta\} \to \infty$,*

$$\min_{z\in\partial\Omega(\alpha,\beta)} \left| z - \frac{\alpha - \beta}{\alpha + \beta}\mathrm{i} \right| = \sqrt{\alpha + \beta - 2} + \frac{(\alpha - \beta)^2}{(\alpha + \beta)^{5/2}} + (\alpha - \beta)^2 O((\alpha + \beta)^{-7/2}), \tag{4.19}$$

$$\max_{z\in\partial\Omega(\alpha,\beta)} \left| z - \frac{\alpha - \beta}{\alpha + \beta}\mathrm{i} \right| = \sqrt{\alpha + \beta + 2} - \frac{(\alpha - \beta)^2}{(\alpha + \beta)^{5/2}} + \frac{8\alpha\beta|\alpha - \beta|}{(\alpha + \beta)^4} + (\alpha - \beta)^2 O((\alpha + \beta)^{-7/2}) + (\alpha - \beta)O((\alpha + \beta)^{-3}). \tag{4.20}$$

Proof. Step 1. In view of the representation $\partial\Omega(\alpha, \beta) = \varphi(\partial D(0, 1))$, where $\varphi = \varphi_{a,R,\eta}$ with $a > 0$, $R > 1$ and $\eta \in \mathbb{R}$ as defined in the proof of lemma 4.1, it is sufficient to calculate the minimum and the maximum of the function

$$d(w) := \left| \varphi(w) - \frac{\alpha - \beta}{\alpha + \beta}\mathrm{i} \right| \quad \text{for } w \in \partial D(0, 1),$$

which is the distance from the point $\mathrm{i}(\alpha-\beta)/(\alpha+\beta)$ to a boundary point $\varphi(w) \in \partial\Omega(\alpha,\beta)$. For simplicity we set $K := (\alpha-\beta)/(\alpha+\beta)$. Here we have

$$\begin{aligned}
d(w)^2 &= (\varphi(w) - K\mathrm{i})\overline{(\varphi(w) - K\mathrm{i})} \\
&= \left(\frac{R^4-1}{R}\frac{w-\mathrm{i}\eta}{w^2+R^2} + \mathrm{i}(\eta R - K)\right)\left(\frac{R^4-1}{R}\frac{\bar{w}+\mathrm{i}\eta}{\bar{w}^2+R^2} - \mathrm{i}(\eta R - K)\right) \\
&= \left(\frac{R^4-1}{R}\right)^2 \frac{1+\mathrm{i}\eta(w-\bar{w})+\eta^2}{R^4+R^2(w^2+\bar{w}^2)+1} + (\eta R - K)^2 \\
&\quad - \frac{R^4-1}{R}\frac{w-\mathrm{i}\eta}{w^2+R^2}(\eta R - K)\mathrm{i} + \frac{R^4-1}{R}\frac{\bar{w}+\mathrm{i}\eta}{\bar{w}^2+R^2}(\eta R - K)\mathrm{i} \\
&= \left(\frac{R^4-1}{R}\right)^2 \frac{1+\mathrm{i}\eta(w-\bar{w})+\eta^2}{R^4+R^2(w^2+\bar{w}^2)+1} + (\eta R - K)^2 \\
&\quad + \frac{R^4-1}{R}(\eta R - K)\frac{\mathrm{i}(w-\bar{w}) - \eta(w^2+\bar{w}^2) + \mathrm{i}R^2(\bar{w}-w) - 2\eta R^2}{|w^2+R^2|^2}.
\end{aligned}$$

Substituting $w = \mathrm{e}^{\mathrm{i}\psi}$ into the above equality, we see that

$$\begin{aligned}
d(\mathrm{e}^{\mathrm{i}\psi})^2 &= \left(\frac{R^4-1}{R}\right)^2 \frac{1-2\eta\sin\psi+\eta^2}{R^4+2R^2\cos 2\psi+1} + (\eta R - K)^2 \\
&\quad + \frac{R^4-1}{R}(\eta R - K)\frac{-2\sin\psi - 2\eta\cos 2\psi + 2R^2\sin\psi - 2\eta R^2}{R^4+2R^2\cos 2\psi+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{R^2(R^4+2R^2\cos 2\psi+1)}{R^4-1}(d(\mathrm{e}^{\mathrm{i}\psi})^2 - (\eta R - K)^2) \\
&\quad = R^4 - \eta^2R^4 - \eta^2 + 2\eta\sin\psi - 1 \\
&\qquad + 2KR((1-R^2)\sin\psi + \eta\cos 2\psi + \eta R^2) - 2\eta R^2(\sin\psi + \eta\cos 2\psi) \\
&\quad = (4\eta^2R^2 - 4K\eta R)\sin^2\psi - 2(R^2-1)(\eta+KR)\sin\psi \\
&\qquad - \eta^2(R^2+1)^2 + (R^2-1+2K\eta R)(R^2+1). \qquad (4.21)
\end{aligned}$$

To estimate the minimum and the maximum of $d(w)$, we introduce a function F defined by

$$F(\lambda) := \frac{A\lambda^2 + B\lambda + C}{(R^2+1)^2 - 4R^2\lambda^2} \quad \text{for } -1 \leqslant \lambda \leqslant 1,$$

where

$$\begin{aligned}
A &:= 4\eta^2R^2 - 4K\eta R, \\
B &:= -2(R^2-1)(\eta+KR), \\
C &:= -\eta^2(R^2+1)^2 + (R^2-1+2K\eta R)(R^2+1).
\end{aligned}$$

By (4.21) we see that

$$F(\sin\psi) = \frac{R^2}{R^4-1}(d(\mathrm{e}^{\mathrm{i}\psi})^2 - (\eta R - K)^2). \tag{4.22}$$

In view of relation (4.22), we first find points where the function F attains its minimum or maximum instead of the function d. Then we calculate the minimum and the maximum of $d(w)$.

STEP 2. In the case when $\alpha = \beta$, by (4.4) and (4.8) we have $\eta = 0$ and $(R+R^{-1})^2 = x_1 = \alpha + \beta + 2$. Since

$$F(\lambda) = \frac{(R^2-1)(R^2+1)}{(R^2+1)^2 - 4R^2\lambda^2},$$

F attains its minimum at $\lambda = 0$ and its maximum at $\lambda = \pm 1$. Consequently, by (4.22), we see that $d(\mathrm{e}^{\mathrm{i}\psi})$ attains its minimum when $\sin\psi = 0$ and its maximum when $\sin\psi = \pm 1$. Therefore, we deduce that

$$\min_{|w|=1} d(w) = d(\pm 1) = \sqrt{\frac{R^4-1}{R^2}F(0)} = \sqrt{\frac{(R^2-1)^2}{R^2}} = \sqrt{x_1 - 4} = \sqrt{\alpha+\beta-2},$$

$$\max_{|w|=1} d(w) = d(\pm i) = \sqrt{\frac{R^4-1}{R^2}F(\pm 1)} = \sqrt{\frac{(R^2+1)^2}{R^2}} = \sqrt{x_1} = \sqrt{\alpha+\beta+2},$$

which satisfy (4.19) and (4.20).

STEP 3. To treat the case where $\alpha \neq \beta$ we prove that $R > |\eta|$. By (4.2), (4.11) and (4.12) we see that

$$\begin{aligned}\left(R + \frac{1}{R}\right)^2 &= \alpha + \beta + 2 - \frac{2\eta^2}{R^2} \\ &= \alpha + \beta + 2 - \frac{2(\beta-\alpha)^2}{(\alpha+\beta)^2} + (\beta-\alpha)^2 O((\alpha+\beta)^{-3}) \quad \text{as } \alpha+\beta \to \infty.\end{aligned} \tag{4.23}$$

Hence, from (4.11), (4.12) and (4.23) it follows that

$$\begin{aligned}R - |\eta| &= \frac{R}{(R+R^{-1})^2}((R+R^{-1})^2 - |\alpha-\beta|) \\ &= \frac{R}{(R+R^{-1})^2}\left\{\alpha + \beta + \frac{8\alpha\beta}{(\alpha+\beta)^2} + O((\alpha+\beta)^{-1}) - |\alpha-\beta|\right\} \\ &= \frac{2R}{(R+R^{-1})^2}\left\{\min\{\alpha,\beta\} + \frac{4\alpha\beta}{(\alpha+\beta)^2} + O((\alpha+\beta)^{-1})\right\} \quad \text{as } \alpha+\beta \to \infty.\end{aligned} \tag{4.24}$$

Therefore, we have $R > |\eta|$ when $(\alpha+\beta)\min\{\alpha,\beta\}$ is sufficiently large.

STEP 4. Let us consider the case $\alpha \neq \beta$. Here we may assume that $\alpha < \beta$; otherwise we change the roles of α and β. We first claim that $B > 0$. To prove this, we note that $\eta > 0$ and

$$K = \frac{\alpha - \beta}{\alpha + \beta} = -\frac{\eta(R^2+1)^2}{R(R^4+2\eta^2+1)} = -\frac{\eta}{R} - \frac{2\eta(R^2-\eta^2)}{R(R^4+2\eta^2+1)} \tag{4.25}$$

by (4.2) and (4.3). Combining (4.24) with (4.25), we see that

$$B = -2(R^2-1)(\eta + KR) = \frac{4\eta(R^2-1)(R^2-\eta^2)}{R^4+2\eta^2+1} > 0. \tag{4.26}$$

In order to find a maximum point and a minimum point of F, we differentiate F and obtain

$$\begin{aligned} F'(\lambda)((R^2+1)^2 - 4R^2\lambda^2)^2 &= 4R^2B\lambda^2 + 2((R^2+1)^2A + 4R^2C)\lambda + (R^2+1)^2B \\ &= 4R^2B(\lambda - \lambda_+)(\lambda - \lambda_-), \end{aligned} \tag{4.27}$$

where

$$\lambda_\pm = \frac{-(R^2+1)^2A - 4R^2C \pm \sqrt{((R^2+1)^2A + 4R^2C)^2 - 4R^2(R^2+1)^2B^2}}{4R^2B},$$

respectively. For sufficiently large $(\alpha+\beta)\min\{\alpha,\beta\}$, let us prove that $\lambda_- \leqslant -1 \leqslant \lambda_+ \leqslant 0$. By (4.24) and (4.25), we have

$$\begin{aligned} (R^2+1)^2A + 4R^2C &= 4(R^2+1)(R^2-1)(R^2 + K\eta R) \\ &= 4(R^2+1)(R^2-1)\left(R^2 - \eta^2 - \frac{2\eta^2(R^2-\eta^2)}{R^4+2\eta^2+1}\right) \\ &= \frac{4(R^2+1)(R^2-1)(R^4+1)(R^2-\eta^2)}{R^4+2\eta^2+1} \\ &> 0. \end{aligned} \tag{4.28}$$

Then (4.26) and (4.28) show that

$$\begin{aligned} &((R^2+1)^2A + 4R^2C)^2 - 4R^2(R^2+1)^2B^2 \\ &\qquad = \frac{16(R^2+1)^2(R^2-1)^2(R^2-\eta^2)^2}{(R^4+2\eta^2+1)^2}((R^4+1)^2 - 4\eta^2R^2) > 0, \end{aligned} \tag{4.29}$$

which implies that $\lambda_+, \lambda_- \in \mathbb{R}$. From (4.24), (4.26), (4.28) and (4.29) it follows that

$$\lambda_- < -\frac{(R^2+1)^2A + 4R^2C}{4R^2B} = -\frac{(R^2+1)(R^4+1)}{4\eta R^2} < -\frac{R^3}{4} < -1$$

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and

$$
\begin{aligned}
0 > \lambda_+ &= \frac{R^2+1}{4\eta R^2}\big(-(R^4+1) + \sqrt{(R^4+1)^2 - 4\eta^2 R^2}\big) \\
&= \frac{R^2+1}{4\eta R^2}\bigg(-\frac{2\eta^2 R^2}{R^4+1} + O(\eta^4 R^{-8})\bigg) \\
&= -\frac{\eta(R^2+1)}{2(R^4+1)} + (\beta-\alpha)^3 O((\alpha+\beta)^{-11/2}) \\
&= (\beta-\alpha)\bigg\{-\frac{1}{2(\alpha+\beta)^{3/2}} + O((\alpha+\beta)^{-5/2})\bigg\} \quad \text{as } \alpha+\beta\to\infty. \qquad (4.30)
\end{aligned}
$$

Hence, we have $\lambda_- < -1 < \lambda_+ < 0$ when $(\alpha+\beta)\min\{\alpha,\beta\}$ is sufficiently large. Consequently, by (4.27), $F'(\lambda) \leqslant 0$ if $-1 \leqslant \lambda \leqslant \lambda_+$, and $F'(\lambda) \geqslant 0$ if $\lambda_+ \leqslant \lambda \leqslant 1$. Therefore, F attains its minimum at $\lambda = \lambda_+$ and its maximum at $\lambda = 1$ or -1. By the definition of F and (4.26), it is easy to see that $F(-1) \leqslant F(1)$, so that F attains its maximum at $\lambda = 1$. Hence, by (4.22), we see that $d(\mathrm{e}^{\mathrm{i}\psi})$ attains its minimum when $\sin\psi = \lambda_+$ and its maximum when $\sin\psi = 1$.

Let us calculate the maximum of $d(w)$. Since

$$
\varphi(\mathrm{i}) = \frac{(R^4-1)(1-\eta)\mathrm{i}}{R(R^2-1)} + \mathrm{i}\eta R = \bigg(R + \frac{1}{R} - \frac{\eta}{R}\bigg)\mathrm{i},
$$

it follows from (4.25) that

$$
\begin{aligned}
\max_{|w|=1} d(w) = d(\mathrm{i}) &= |\varphi(\mathrm{i}) - K\mathrm{i}| \\
&= R + \frac{1}{R} - \frac{\eta}{R} - K \\
&= R + \frac{1}{R} + \frac{2\eta(R^2-\eta^2)}{R(R^4+2\eta^2+1)}.
\end{aligned}
$$

Here, by (4.11), (4.12) and (4.23),

$$
\begin{aligned}
R + \frac{1}{R} &= \sqrt{\alpha+\beta+2} - \frac{(\beta-\alpha)^2}{(\alpha+\beta)^{5/2}} + (\beta-\alpha)^2 O((\alpha+\beta)^{-7/2}), \\
\frac{2\eta(R^2-\eta^2)}{R(R^4+2\eta^2+1)} &= \frac{8\alpha\beta(\beta-\alpha)}{(\alpha+\beta)^4} + (\beta-\alpha) O((\alpha+\beta)^{-3})
\end{aligned}
$$

as $\alpha+\beta\to\infty$. Therefore,

$$
\begin{aligned}
\max_{|w|=1} d(w) = \sqrt{\alpha+\beta+2} &- \frac{(\beta-\alpha)^2}{(\alpha+\beta)^{5/2}} + \frac{8\alpha\beta(\beta-\alpha)}{(\alpha+\beta)^4} \\
&+ (\beta-\alpha)^2 O((\alpha+\beta)^{-7/2}) + (\beta-\alpha) O((\alpha+\beta)^{-3})
\end{aligned}
$$

as $\alpha+\beta\to\infty$, which proves (4.20).

Next we turn to the calculation of the minimum of $d(w)$. By (4.22), we see that

$$\begin{aligned}\Big(\min_{|w|=1} d(w)\Big)^2 &= \Big(R^2 - \frac{1}{R^2}\Big)F(\lambda_+) + (\eta R - K)^2 \\ &= \Big(R^2 - \frac{1}{R^2}\Big)F(0) + (\eta R - K)^2 + \Big(R^2 - \frac{1}{R^2}\Big)(F(\lambda_+) - F(0)) \\ &= d(1)^2 + \Big(R^2 - \frac{1}{R^2}\Big)(F(\lambda_+) - F(0)). \qquad (4.31)\end{aligned}$$

Observe that, by (4.11), (4.12), (4.26), (4.28) and (4.30),

$$\begin{aligned}&F(\lambda_+) - F(0) \\ &\quad= \frac{A\lambda_+^2 + B\lambda_+ + C}{(R^2+1)^2 - 4R^2\lambda_+^2} - \frac{C}{(R^2+1)^2} \\ &\quad= \frac{((R^2+1)^2A + 4R^2C)\lambda_+^2 + (R^2+1)^2 B\lambda_+}{(R^2+1)^2((R^2+1)^2 - 4R^2\lambda_+^2)} \\ &\quad= \frac{(\beta-\alpha)^2\{16\alpha\beta(\alpha+\beta) + O((\alpha+\beta)^2)\}\{1/(4(\alpha+\beta)^3) + O((\alpha+\beta)^{-4})\}}{(\alpha+\beta)^4 + O((\alpha+\beta)^3)} \\ &\qquad - \frac{(\beta-\alpha)^2\{16\alpha\beta/\sqrt{\alpha+\beta} + O(\sqrt{\alpha+\beta})\} \times\{1/(2(\alpha+\beta)^{3/2}) + O((\alpha+\beta)^{-5/2})\}}{(\alpha+\beta)^4 + O((\alpha+\beta)^3)} \\ &\quad= (\beta-\alpha)^2\Big\{-\frac{4\alpha\beta}{(\alpha+\beta)^6} + O((\alpha+\beta)^{-5})\Big\} \quad \text{as } \alpha+\beta\to\infty,\end{aligned}$$

and hence

$$\begin{aligned}\Big(R^2 - \frac{1}{R^2}\Big)(F(\lambda_+) - F(0)) &= (\beta-\alpha)^2\Big\{-\frac{4\alpha\beta}{(\alpha+\beta)^5} + O((\alpha+\beta)^{-4})\Big\} \\ &= (\beta-\alpha)^2 O((\alpha+\beta)^{-3}) \quad \text{as } \alpha+\beta\to\infty. \qquad (4.32)\end{aligned}$$

On the other hand, since

$$\varphi(1) = \frac{(R^4-1)(1-\mathrm{i}\eta)}{R(R^2+1)} + \mathrm{i}\eta R = R - \frac{1}{R} + \frac{\eta}{R}\mathrm{i},$$

it follows from (4.11), (4.12), (4.23) and (4.25) that

$$\begin{aligned}d(1)^2 = |\varphi(1) - K\mathrm{i}|^2 &= \Big(R - \frac{1}{R}\Big)^2 + \Big(\frac{\eta}{R} - K\Big)^2 \\ &= \Big(R + \frac{1}{R}\Big)^2 - 4 + \frac{4\eta^2}{R^2}\Big(1 + \frac{R^2-\eta^2}{R^4 + 2\eta^2 + 1}\Big)^2 \\ &= \alpha+\beta-2 + \frac{2(\beta-\alpha)^2}{(\alpha+\beta)^2} + (\beta-\alpha)^2 O((\alpha+\beta)^{-3}) \quad \text{as } \alpha+\beta\to\infty. \qquad (4.33)\end{aligned}$$

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Therefore, we deduce from (4.31), (4.32) and (4.33) that

$$\begin{aligned}\min_{|w|=1} d(w) &= \sqrt{\alpha+\beta-2+\frac{2(\beta-\alpha)^2}{(\alpha+\beta)^2}+(\beta-\alpha)^2 O((\alpha+\beta)^{-3})} \\ &= \sqrt{\alpha+\beta-2}+\frac{(\beta-\alpha)^2}{(\alpha+\beta)^{5/2}}+(\beta-\alpha)^2 O((\alpha+\beta)^{-7/2})\end{aligned}$$

as $\alpha+\beta\to\infty$, which proves (4.19). □

By an argument similar to the proof of theorem 4.4, we estimate the distance from the point $-\mathrm{i}$ to a boundary point of the quadrature domain $\Omega(\alpha,\beta)$ and show that the quadrature domain $\Omega(\alpha,\beta)$ approaches the disc centred at $-\mathrm{i}$ when $\alpha>0$ is fixed and $\beta\to\infty$.

Theorem 4.5. *Suppose that α is a fixed positive number. For sufficiently large $\beta>0$, let $\Omega(\alpha,\beta)$ be a unique quadrature domain of the measure $\pi(\alpha\delta_{\mathrm{i}}+\beta\delta_{-\mathrm{i}})$ for subharmonic functions. Then, as $\beta\to\infty$,*

$$\min_{z\in\partial\Omega(\alpha,\beta)}|z+\mathrm{i}| = \sqrt{\beta}+\frac{\alpha}{2\sqrt{\beta}}-\frac{2\alpha}{\beta}+\left(4\alpha-\frac{\alpha^2}{8}\right)\beta^{-3/2}+O(\beta^{-2}), \tag{4.34}$$

$$\max_{z\in\partial\Omega(\alpha,\beta)}|z+\mathrm{i}| = \sqrt{\beta}+\frac{\alpha}{2\sqrt{\beta}}+\frac{2\alpha}{\beta}+\left(4\alpha-\frac{\alpha^2}{8}\right)\beta^{-3/2}+O(\beta^{-2}). \tag{4.35}$$

Proof. We shall calculate the minimum and the maximum of the function

$$\tilde{d}(w) := |\varphi(w)+\mathrm{i}| \quad \text{for } w\in\partial D(0,1)$$

in an analogous way to the proof of lemma 4.4. We define a function $\tilde{F}$ by

$$\tilde{F}(\lambda) := \frac{\tilde{A}\lambda^2+\tilde{B}\lambda+\tilde{C}}{(R^2+1)^2-4R^2\lambda^2} \quad \text{for } -1\leqslant\lambda\leqslant 1,$$

where

$$\begin{aligned}\tilde{A} &:= 4\eta^2R^2+4\eta R, \\ \tilde{B} &:= 2(R^2-1)(R-\eta), \\ \tilde{C} &:= -\eta^2(R^2+1)^2+(R^2-1-2\eta R)(R^2+1).\end{aligned}$$

Note that $\tilde{B}>0$ by (4.24). After some computations we see that

$$\tilde{F}(\sin\psi) = \frac{R^2}{R^4-1}(\tilde{d}(\mathrm{e}^{\mathrm{i}\psi})^2-(\eta R+1)^2).$$

Let us find a minimum point and a maximum point of $\tilde{F}$. By differentiating $\tilde{F}$, we obtain

$$\begin{aligned}&\tilde{F}'(\lambda)((R^2+1)^2-4R^2\lambda^2)^2 \\ &\qquad = 4R^2\tilde{B}\lambda^2+2((R^2+1)^2\tilde{A}+4R^2\tilde{C})\lambda+(R^2+1)^2\tilde{B}.\end{aligned} \tag{4.36}$$

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However, we have

$$((R^2+1)^2\tilde{A}+4R^2\tilde{C})^2-4R^2(R^2+1)^2\tilde{B}^2=0.$$

This implies that $\tilde{F}'\geqslant 0$. Consequently, $\tilde{F}$ attains its minimum at $\lambda=-1$ and maximum at $\lambda=1$.

In order to calculate the minimum and the maximum of $d(w)$, we note that

$$R=\sqrt{\beta}+O\left(\frac{1}{\sqrt{\beta}}\right),\qquad \eta=\sqrt{\beta}+O\left(\frac{1}{\sqrt{\beta}}\right)\quad\text{as }\beta\to\infty \tag{4.37}$$

by (4.11) and (4.12). In view of (4.24), we also have

$$\begin{aligned}R-\eta&=\frac{2R}{(R+R^{-1})^2}\{\alpha+O((\alpha+\beta)^{-1})\}\\&=\frac{2\alpha}{\sqrt{\beta}}+O(\beta^{-3/2})\quad\text{as }\beta\to\infty.\end{aligned}\tag{4.38}$$

Therefore, from (4.23), (4.37) and (4.38) it follows that

$$\begin{aligned}\min_{|w|=1}d(w)&=|\varphi(-\mathrm{i})+\mathrm{i}|=R+\frac{1}{R}+\frac{\eta}{R}-1\\&=\sqrt{\alpha+\beta+\frac{2(R^2-\eta^2)}{R^2}}+\frac{\eta-R}{R}\\&=\sqrt{\alpha+\beta+\frac{8\alpha}{\beta}+O(\beta^{-2})}-\frac{2\alpha}{\beta}+O(\beta^{-2})\\&=\sqrt{\beta}+\frac{\alpha}{2\sqrt{\beta}}-\frac{2\alpha}{\beta}+\left(4\alpha-\frac{\alpha^2}{8}\right)\beta^{-3/2}+O(\beta^{-2})\quad\text{as }\beta\to\infty\end{aligned}$$

and

$$\begin{aligned}\max_{|w|=1}d(w)&=|\varphi(\mathrm{i})+\mathrm{i}|=R+\frac{1}{R}-\frac{\eta}{R}+1\\&=\sqrt{\alpha+\beta+\frac{2(R^2-\eta^2)}{R^2}}+\frac{R-\eta}{R}\\&=\sqrt{\alpha+\beta+\frac{8\alpha}{\beta}+O(\beta^{-2})}+\frac{2\alpha}{\beta}+O(\beta^{-2})\\&=\sqrt{\beta}+\frac{\alpha}{2\sqrt{\beta}}+\frac{2\alpha}{\beta}+\left(4\alpha-\frac{\alpha^2}{8}\right)\beta^{-3/2}+O(\beta^{-2})\quad\text{as }\beta\to\infty.\end{aligned}$$

□

5. Application to the Hele-Shaw problem

In this section we apply the results of the previous section to the Hele-Shaw problem. We give an estimate for quadrature domains of a linear combination of the Dirac measures by applying theorem 4.4. Theorem 1.1 is obtained as a consequence of the estimate combined with theorem 4.5. In what follows we write $\Omega(\nu)$ for the minimum quadrature domain of the measure ν for subharmonic functions. First we prove the following two lemmas.

Lemma 5.1. *Let β_1, β_2 and κ be positive numbers and $c_1, c_2 \in \mathbb{C}$. Then*

$$\Omega(\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2}) = \{\kappa z \in \mathbb{C} \mid z \in \Omega(\beta_1\delta_{c_1} + \beta_2\delta_{c_2})\}$$

holds.

Proof. First we prove that the domain

$$\kappa\Omega := \{\kappa z \in \mathbb{C} \mid z \in \Omega(\beta_1\delta_{c_1} + \beta_2\delta_{c_2})\}$$

is a quadrature domain of the measure $\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2}$ for subharmonic functions. For each subharmonic and integrable function f defined in $\kappa\Omega$, we set $g(z) := f(\kappa z)$. Then g is subharmonic and integrable in $\Omega(\beta_1\delta_{c_1} + \beta_2\delta_{c_2})$. Hence, we have

$$\beta_1 g(c_1) + \beta_2 g(c_2) \leqslant \int_{\Omega(\beta_1\delta_{c_1}+\beta_2\delta_{c_2})} g \, \mathrm{d}m,$$

so that

$$\beta_1 f(\kappa c_1) + \beta_2 f(\kappa c_2) \leqslant \frac{1}{\kappa^2}\int_{\kappa\Omega} f \, \mathrm{d}m.$$

Therefore, $\kappa\Omega$ is a quadrature domain of $\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2}$ for subharmonic functions, and hence $\Omega(\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2}) \subset \kappa\Omega$ holds.

Now assume $\Omega(\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2}) \subsetneq \kappa\Omega$. Then we have

$$\{\kappa^{-1}z \in \mathbb{C} \mid z \in \Omega(\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2})\} \subsetneq \Omega(\beta_1\delta_{c_1} + \beta_2\delta_{c_2}).$$

However, an argument similar to the previous paragraph shows that

$$\{\kappa^{-1}z \in \mathbb{C} \mid z \in \Omega(\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2})\}$$

is a quadrature domain of the measure $\beta_1\delta_{c_1} + \beta_2\delta_{c_2}$ for subharmonic functions. This contradicts the assumption that $\Omega(\beta_1\delta_{c_1} + \beta_2\delta_{c_2})$ is the minimum quadrature domain. □

The next lemma shows that minimum quadrature domains possesses the semigroup property. Gustafsson and Sakai [5] have already proved this property for more general measures, but it is established for saturated (or maximum) quadrature domains (see [5, theorem 2.2] for details). On the other hand, Sakai [13] proved the property for the minimum quadrature domains. We improve the result [13, proposition 3.10] and give a detailed proof for completeness.

Lemma 5.2. *Let μ and ν be finite measures with compact support such that there exist the bounded minimum quadrature domains $\Omega(\mu)$, $\Omega(\mu+\nu)$ and $\Omega(\chi_{\Omega(\mu)} + \nu)$ of the measures μ, $\mu+\nu$ and $\chi_{\Omega(\mu)} + \nu$ for subharmonic functions, respectively. In addition, we assume that ν is of the form*

$$\nu = f + \sum_{j=1}^{l} \alpha_j \delta_{c_j},$$

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where $f \in L^\infty(\mathbb{C})$, $\alpha_j > 0$ and $c_j \in \mathbb{C}$. Then it holds that

$$\Omega(\mu + \nu) = \Omega(\chi_{\Omega(\mu)} + \nu).$$

Proof. Since the inequality

$$\int s \, \mathrm{d}\mu + \int s \, \mathrm{d}\nu \leqslant \int_{\Omega(\mu)} s \, \mathrm{d}m + \int s \, \mathrm{d}\nu \leqslant \int_{\Omega(\chi_{\Omega(\mu)}+\nu)} s \, \mathrm{d}m$$

holds for all integrable subharmonic functions s defined in $\Omega(\chi_{\Omega(\mu)}+\nu)$, we see that $\Omega(\chi_{\Omega(\mu)} + \nu)$ is a quadrature domain of $\mu + \nu$ for subharmonic functions. Hence, $\Omega(\mu + \nu) \subset \Omega(\chi_{\Omega(\mu)} + \nu)$ and $\chi_{\Omega(\chi_{\Omega(\mu)}+\nu)} = \chi_{\Omega(\mu+\nu)}$ hold. Therefore, we need to prove $\Omega(\mu + \nu) \supset \Omega(\chi_{\Omega(\mu)} + \nu)$.

First we prove that $\Omega(\chi_{\Omega(\mu)} + \nu) = \{z \in \mathbb{C} | 0 < u(z) \leqslant \infty\}$, where

$$u(z) := \int_{\Omega(\chi_{\Omega(\mu)}+\nu)} \log|z - \zeta| \, \mathrm{d}m(\zeta) - \biggl(\int_{\Omega(\mu)} \log|z - \zeta| \, \mathrm{d}m(\zeta) + \int \log|z - \zeta| \, \mathrm{d}\nu(\zeta) \biggr). \tag{5.1}$$

Since $s(\zeta) = \log|z-\zeta|$ with $z \in \mathbb{C}$ and $s(\zeta) = -\log|z-\zeta|$ with $z \in \mathbb{C} \setminus \Omega(\chi_{\Omega(\mu)}+\nu)$ are integrable subharmonic functions in $\Omega(\chi_{\Omega(\mu)} + \nu)$, we see that $u(z) \geqslant 0$ in $\mathbb{C}$ and $u(z) = 0$ in $\mathbb{C} \setminus \Omega(\chi_{\Omega(\mu)} + \nu)$. Hence,

$$\Omega(\chi_{\Omega(\mu)} + \nu) \supset \{z \in \mathbb{C} \mid 0 < u(z) \leqslant \infty\}.$$

Let us assume $\Omega(\chi_{\Omega(\mu)} + \nu) \supsetneq \{z \in \mathbb{C} \mid 0 < u(z) \leqslant \infty\}$ and derive a contradiction. Then, by taking $z_0 \in \Omega(\chi_{\Omega(\mu)} + \nu) \setminus \{z \in \mathbb{C} \mid 0 < u(z) \leqslant \infty\}$, we will deduce that $\Omega(\chi_{\Omega(\mu)} + \nu) \setminus \{z_0\}$ is also a quadrature domain of $\chi_{\Omega(\mu)} + \nu$. To verify this, we take $z \in \mathbb{C} \setminus (\Omega(\chi_{\Omega(\mu)} + \nu) \setminus \{z_0\})$. Then $u(z) = 0$, and hence u attains its minimum there. On the other hand, by the form

$$\nu = f + \sum_{j=1}^{l} \alpha_j \delta_{c_j},$$

we see that $z \in \mathbb{C} \setminus \{c_1, \dots, c_l\}$ and u is of C^1 in a neighbourhood of z. Consequently, $\nabla u(z) = 0$. Therefore, by (5.1),

$$\int_{\Omega(\mu)} s \, \mathrm{d}m + \int s \, \mathrm{d}\nu \leqslant \int_{\Omega(\chi_{\Omega(\mu)}+\nu)\setminus\{z_0\}} s \, \mathrm{d}m \tag{5.2}$$

holds for functions such as

(i)

$$\begin{aligned} s(\zeta) &= \pm \log|z - \zeta|, \\ s(\zeta) &= \pm \frac{\partial(\log|z - \zeta|)}{\partial z_x}, \\ s(\zeta) &= \pm \frac{\partial(\log|z - \zeta|)}{\partial z_y} \quad \text{with } z \in \mathbb{C} \setminus (\Omega(\chi_{\Omega(\mu)} + \nu) \setminus \{z_0\}). \end{aligned}$$

Moreover, since $u(z) \geqslant 0$ in $\mathbb{C}$, (5.2) holds for

(ii) $s(\zeta) = \log|z-\zeta| \quad \text{with } z \in \mathbb{C}$.

According to Sakai [15, lemma 5.1], the family of all the linear combinations of the functions $s(\zeta)$ such as (i) and (ii) with positive coefficients is dense in the space of all integrable subharmonic functions in $\Omega(\chi_{\Omega(\mu)}+\nu) \setminus \{z_0\}$ with respect to the L^1 norm. Therefore, (5.2) holds for any integrable subharmonic functions s defined in $\Omega(\chi_{\Omega(\mu)}+\nu) \setminus \{z_0\}$, i.e. $\Omega(\chi_{\Omega(\mu)}+\nu) \setminus \{z_0\}$ is a quadrature domain of $\chi_{\Omega(\mu)}+\nu$. This is a contradiction. Therefore, $\Omega(\chi_{\Omega(\mu)}+\nu) = \{z \in \mathbb{C} \mid 0 < u(z) \leqslant \infty\}$ is verified.

Next let us assume $\Omega(\mu+\nu) \subsetneq \Omega(\chi_{\Omega(\mu)}+\nu)$ and derive a contradiction. Take $z_1 \in \Omega(\chi_{\Omega(\mu)}+\nu) \setminus \Omega(\mu+\nu)$. Then, by the argument in the previous paragraph, $u(z_1) > 0$, i.e.

$$\int_{\Omega(\mu)} \log|z_1-\zeta|\,\mathrm{d}m(\zeta) + \int \log|z_1-\zeta|\,\mathrm{d}\nu(\zeta) < \int_{\Omega(\chi_{\Omega(\mu)}+\nu)} \log|z_1-\zeta|\,\mathrm{d}m(\zeta). \tag{5.3}$$

On the other hand,

$$\int \log|z_1-\zeta|\,\mathrm{d}\mu(\zeta) + \int \log|z_1-\zeta|\,\mathrm{d}\nu(\zeta) = \int_{\Omega(\mu+\nu)} \log|z_1-\zeta|\,\mathrm{d}m(\zeta) \tag{5.4}$$

since $\log|z_1-\zeta|$ is harmonic in $\Omega(\mu+\nu)$. Therefore, by (5.3), (5.4) and the fact that $\chi_{\Omega(\chi_{\Omega(\mu)}+\nu)} = \chi_{\Omega(\mu+\nu)}$, we obtain

$$\int_{\Omega(\mu)} \log|z_1-\zeta|\,\mathrm{d}m(\zeta) < \int \log|z_1-\zeta|\,\mathrm{d}\mu(\zeta).$$

This contradicts the definition of $\Omega(\mu)$. □

With the above lemmas and theorem 4.4 we give the following estimate for the distances from the barycentre w_l defined by (1.7) to the boundary points of quadrature domains of a linear combination of the Dirac measures.

THEOREM 5.3. *Let $\alpha_1, \dots, \alpha_l$ be positive numbers and let $c_1, \dots, c_l \in \mathbb{C}$ with $l \geqslant 2$, and define $w_1, \dots, w_l$ by (1.7). Then there exists a non-negative function $\varepsilon_l(t)$ such that, for any quadrature domain $\Omega_\Delta(t)$ of the measure $t\sum_{j=1}^{l}\alpha_j\delta_{c_j}$ for subharmonic functions, the inequality*

$$\sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} - \varepsilon_l(t) \leqslant |z-w_l| \leqslant \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \varepsilon_l(t) \tag{5.5}$$

holds for all $z \in \partial\Omega_\Delta(t)$, $t > 0$, and it has the following asymptotic behaviour:

$$\varepsilon_l(t) = \sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\left(\sum_{k=2}^{l}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{(\sum_{j=1}^{k}\alpha_j)^2}|w_{k-1}-c_k|^2\right)t^{-1/2} + O(t^{-1}) \quad \text{as } t \to \infty. \tag{5.6}$$

Proof. As in the proof of proposition 2.1, it is sufficient to prove estimate (5.5) for the minimum quadrature domain

$$\Omega_\Delta(t) = \Omega\biggl(t\sum_{j=1}^{l}\alpha_j\delta_{c_j}\biggr).$$

We prove the theorem by induction on l.

STEP 1. In the case where $l = 2$, first we assume that $\kappa := |c_1 - c_2|/2$, $c_1 = \kappa\mathrm{i}$, $c_2 = -\kappa\mathrm{i}$. We note here that

$$\alpha_1 t\delta_{c_1} + \alpha_2 t\delta_{c_2} = \kappa^2\pi\biggl(\frac{\alpha_1 t}{\pi\kappa^2}\biggr)\delta_{\kappa\mathrm{i}} + \kappa^2\pi\biggl(\frac{\alpha_2 t}{\pi\kappa^2}\biggr)\delta_{-\kappa\mathrm{i}}.$$

By lemma 5.1 we see that $\kappa^{-1}\Omega_\Delta(t) := \{\kappa^{-1}z \mid z \in \Omega_\Delta(t)\}$ is the minimum quadrature domain of the measure $\pi\{(\alpha_1 t/(\pi\kappa^2))\delta_{\mathrm{i}} + (\alpha_2 t/(\pi\kappa^2))\delta_{-\mathrm{i}}\}$ for subharmonic functions. Hence, by applying theorem 4.4 with $\alpha = \alpha_1 t/(\pi\kappa^2)$ and $\beta = \alpha_2 t/(\pi\kappa^2)$, we obtain

$$\min_{z\in\partial(\kappa^{-1}\Omega_\Delta(t))}\biggl|z - \frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\mathrm{i}\biggr| = \sqrt{(\alpha_1+\alpha_2)\frac{t}{\pi\kappa^2}} - \frac{4\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^{5/2}}\sqrt{\frac{\pi\kappa^2}{t}} + O(t^{-1}), \tag{5.7}$$

$$\max_{z\in\partial(\kappa^{-1}\Omega_\Delta(t))}\biggl|z - \frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\mathrm{i}\biggr| = \sqrt{(\alpha_1+\alpha_2)\frac{t}{\pi\kappa^2}} + \frac{4\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^{5/2}}\sqrt{\frac{\pi\kappa^2}{t}} + O(t^{-1}) \tag{5.8}$$

as $t \to \infty$. Noting that $\kappa\mathrm{i}(\alpha_1-\alpha_2)/(\alpha_1+\alpha_2) = w_2$, we multiply both sides of the equalities (5.7), (5.8) by κ, and obtain

$$\min_{z\in\partial\Omega_\Delta(t)}|z - w_2| = \sqrt{\frac{t}{\pi}(\alpha_1+\alpha_2)} - \sqrt{\frac{\pi}{\alpha_1+\alpha_2}}\biggl(\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}(2\kappa)^2\biggr)\frac{1}{\sqrt{t}} + O(t^{-1}),$$

$$\max_{z\in\partial\Omega_\Delta(t)}|z - w_2| = \sqrt{\frac{t}{\pi}(\alpha_1+\alpha_2)} + \sqrt{\frac{\pi}{\alpha_1+\alpha_2}}\biggl(\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}(2\kappa)^2\biggr)\frac{1}{\sqrt{t}} + O(t^{-1})$$

as $t \to \infty$. We define a function $\varepsilon_2(t)$ by

$$\varepsilon_2(t) := \max\biggl\{\sqrt{\frac{t}{\pi}(\alpha_1+\alpha_2)} - \min_{z\in\partial\Omega_\Delta(t)}|z-w_2|,\ \max_{z\in\partial\Omega_\Delta(t)}|z-w_2| - \sqrt{\frac{t}{\pi}(\alpha_1+\alpha_2)}\biggr\}. \tag{5.9}$$

Note that $m(\Omega_\Delta(t)) = t(\alpha_1+\alpha_2)$ by (2.2), so that $\varepsilon_2(t)$ has to be non-negative. It also holds that

$$\sqrt{\frac{t}{\pi}(\alpha_1+\alpha_2)} - \varepsilon_2(t) \leqslant |z - w_2| \leqslant \sqrt{\frac{t}{\pi}(\alpha_1+\alpha_2)} + \varepsilon_2(t)$$

for all $z \in \partial\Omega_\Delta(t)$, $t > 0$, and

$$\varepsilon_2(t) = \sqrt{\frac{\pi}{\alpha_1+\alpha_2}}\biggl(\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}|c_1-c_2|^2\biggr)\frac{1}{\sqrt{t}} + O(t^{-1}) \quad \text{as } t \to \infty.$$

For general $c_1, c_2 \in \mathbb{C}$ and $\alpha_1, \alpha_2 > 0$, by using a suitable rotation or translation, we can reduce the case to the previous one. Hence, ε_2 defined by (5.9) satisfies (5.5) and (5.6) in the case where $l = 2$.

Note that, in view of theorem 4.4, the above argument is applicable even if α_1 or α_2 depends on $t > 0$ in the case where $\alpha_j(t) \to \alpha_{j,0} > 0$ as $t \to \infty$.

STEP 2. Assuming that the assertion of the theorem holds for $l-1$, we prove it for case l. Then we have a non-negative function $\varepsilon_{l-1}(t)$ which satisfies

$$\sqrt{\frac{t}{\pi}\sum_{j=1}^{l-1}\alpha_j} - \varepsilon_{l-1}(t) \leqslant |z - w_{l-1}| \leqslant \sqrt{\frac{t}{\pi}\sum_{j=1}^{l-1}\alpha_j} + \varepsilon_{l-1}(t) \tag{5.10}$$

for all

$$z \in \partial\Omega\biggl(t\sum_{j=1}^{l-1}\alpha_j\delta_{c_j}\biggr), \quad t > 0,$$

and

$$\varepsilon_{l-1}(t) = \sqrt{\frac{\pi}{\sum_{j=1}^{l-1}\alpha_j}}\biggl(\sum_{k=2}^{l-1}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{(\sum_{j=1}^{k}\alpha_j)^2}|w_{k-1} - c_k|^2\biggr)\frac{1}{\sqrt{t}} + O(t^{-1}) \quad \text{as } t \to \infty.$$

Let us prove the estimate (5.5) for $|z - w_l|$ from above. By lemma 5.2 and the estimate (5.10), we see that

$$\begin{aligned}\Omega_\Delta(t) &= \Omega(\chi_{\Omega(t\sum_{j=1}^{l-1}\alpha_j\delta_{c_j})} + t\alpha_l\delta_{c_l}) \\ &\subset \Omega\Bigl(\chi_{D(w_{l-1},\sqrt{t\pi^{-1}\sum_{j=1}^{l-1}\alpha_j}+\varepsilon_{l-1}(t))} + t\alpha_l\delta_{c_l}\Bigr) \\ &= \Omega(t\hat{\alpha}(t)\delta_{w_{l-1}} + t\alpha_l\delta_{c_l}),\end{aligned} \tag{5.11}$$

where

$$\begin{aligned}\hat{\alpha}(t) &:= \frac{1}{t}m\biggl(D\biggl(w_{l-1}, \sqrt{\frac{t}{\pi}\sum_{j=1}^{l-1}\alpha_j} + \varepsilon_{l-1}(t)\biggr)\biggr) \\ &= \sum_{j=1}^{l-1}\alpha_j + \frac{2\pi}{t}\sqrt{\frac{t}{\pi}\sum_{j=1}^{l-1}\alpha_j} \\ &\quad\times \sqrt{\frac{\pi}{\sum_{j=1}^{l-1}\alpha_j}}\biggl(\sum_{k=2}^{l-1}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{(\sum_{j=1}^{k}\alpha_j)^2}|w_{k-1} - c_k|^2\biggr)\frac{1}{\sqrt{t}} + O(t^{-3/2}) \\ &= \sum_{j=1}^{l-1}\alpha_j + 2\pi\biggl(\sum_{k=2}^{l-1}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{(\sum_{j=1}^{k}\alpha_j)^2}|w_{k-1} - c_k|^2\biggr)\frac{1}{t} + O(t^{-3/2}) \quad \text{as } t \to \infty.\end{aligned}$$

In the case where $w_{l-1} = c_l$, it is easy to see that $w_l = w_{l-1}$, and hence

$$\begin{aligned}\Omega(t\hat{\alpha}(t)\delta_{w_{l-1}} + t\alpha_l\delta_{c_l}) &= \Omega(t(\hat{\alpha}(t)+\alpha_l)\delta_{w_l}) \\ &= D\biggl(w_l, \sqrt{\frac{t}{\pi}(\hat{\alpha}(t)+\alpha_l)}\biggr), \end{aligned} \tag{5.12}$$

where

$$\sqrt{\frac{t}{\pi}(\hat{\alpha}(t)+\alpha_l)} = \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\biggl(\sum_{k=2}^{l-1}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{(\sum_{j=1}^{k}\alpha_j)^2}|w_{k-1}-c_k|^2\biggr)\frac{1}{\sqrt{t}} \\ + O(t^{-1}) \quad \text{as } t\to\infty. \tag{5.13}$$

Noting that $\hat{\alpha}(t) \geqslant \sum_{j=1}^{l-1}\alpha_j$, we define a non-negative function $\varepsilon_l^+(t)$ by

$$\varepsilon_l^+(t) := \sqrt{\frac{t}{\pi}(\hat{\alpha}(t)+\alpha_l)} - \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j}.$$

Then, by (5.11)–(5.13), we see that $\varepsilon_l^+(t)$ satisfies the second inequality of (5.5) and (5.6) with $\varepsilon_l(t) = \varepsilon_l^+(t)$.

Next we consider the case where $w_{l-1} \neq c_l$. As mentioned at the end of step 1, we can apply the result for the case where $l = 2$ to the quadrature domain $\Omega(t\hat{\alpha}(t)\delta_{w_{l-1}} + t\alpha_l\delta_{c_l})$ and obtain a non-negative function $\hat{\varepsilon}(t)$ which satisfies

$$\biggl|z - \frac{\hat{\alpha}(t)w_{l-1}+\alpha_l c_l}{\hat{\alpha}(t)+\alpha_l}\biggr| \leqslant \sqrt{\frac{t}{\pi}(\hat{\alpha}(t)+\alpha_l)} + \hat{\varepsilon}(t) \tag{5.14}$$

for all $z \in \partial\Omega(t\hat{\alpha}(t)\delta_{w_{l-1}} + t\alpha_l\delta_{c_l})$, $t > 0$, and

$$\begin{aligned}\hat{\varepsilon}(t) &= \sqrt{\frac{\pi}{\hat{\alpha}(t)+\alpha_l}}\biggl(\frac{\hat{\alpha}(t)\alpha_l}{(\hat{\alpha}(t)+\alpha_l)^2}|w_{l-1}-c_l|^2\biggr)\frac{1}{\sqrt{t}} + O(t^{-1}) \\ &= \biggl(\sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}} + O(t^{-1})\biggr)\biggl(\frac{\alpha_l\sum_{j=1}^{l-1}\alpha_j}{(\sum_{j=1}^{l}\alpha_j)^2} + O(t^{-1})\biggr)|w_{l-1}-c_l|^2\frac{1}{\sqrt{t}} + O(t^{-1}) \\ &= \sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\biggl(\frac{\alpha_l\sum_{j=1}^{l-1}\alpha_j}{(\sum_{j=1}^{l}\alpha_j)^2}|w_{l-1}-c_l|^2\biggr)\frac{1}{\sqrt{t}} + O(t^{-1}) \quad \text{as } t\to\infty. \end{aligned} \tag{5.15}$$

We now define a non-negative function $\varepsilon_l^+(t)$ by

$$\varepsilon_l^+(t) := \biggl|\frac{\hat{\alpha}(t)w_{l-1}+\alpha_l c_l}{\hat{\alpha}(t)+\alpha_l} - w_l\biggr| + \biggl(\sqrt{\frac{t}{\pi}(\hat{\alpha}(t)+\alpha_l)} - \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j}\biggr) + \hat{\varepsilon}(t). \tag{5.16}$$

Then, from (5.14) and (5.16), it follows that

$$|z-w_l| \leqslant \left|z-\frac{\hat{\alpha}(t)w_{l-1}+\alpha_l c_l}{\hat{\alpha}(t)+\alpha_l}\right| + \left|\frac{\hat{\alpha}(t)w_{l-1}+\alpha_l c_l}{\hat{\alpha}(t)+\alpha_l} - w_l\right|$$
$$= \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \varepsilon_l^+(t) \quad \text{for all } z \in \partial\Omega(t\hat{\alpha}(t)\delta_{w_{l-1}} + t\alpha_l\delta_{c_l}),\ t>0.$$

Hence, by (5.11) we see that $\varepsilon_l^+(t)$ satisfies the second inequality of (5.5) with $\varepsilon_l(t) = \varepsilon_l^+(t)$. On the other hand, since

$$\left|\frac{\hat{\alpha}(t)w_{l-1}+\alpha_l c_l}{\hat{\alpha}(t)+\alpha_l} - w_l\right| = O(t^{-1}) \quad \text{as } t \to \infty,$$

by (5.13) and (5.15) we see that (5.6) holds with $\varepsilon_l(t) = \varepsilon_l^+(t)$.

The same argument shows that there exists a non-negative function $\varepsilon_l^-(t)$ which satisfies the first inequality of (5.5) and (5.6) with $\varepsilon_l(t) = \varepsilon_l^-(t)$. Therefore, $\varepsilon_l(t) := \max\{\varepsilon_l^+(t), \varepsilon_l^-(t)\}$ satisfies (5.5) and (5.6). This completes the proof. □

Finally, we prove theorem 1.1 by combining theorem 5.3 with theorem 4.5.

Proof of theorem 1.1. It is sufficient to prove the estimate (1.10) for the minimum quadrature domain

$$\Omega(t) = \Omega\biggl(\chi_{\Omega(0)} + t\sum_{j=1}^{l}\alpha_j\delta_{c_j}\biggr).$$

Let $\varepsilon_-(t) := \varepsilon_l(t)$, where $\varepsilon_l(t)$ is obtained by theorem 5.3. Then, by the inclusion relation

$$\Omega\biggl(t\sum_{j=1}^{l}\alpha_j\delta_{c_j}\biggr) \subset \Omega(t)$$

we see that

$$\sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} - \varepsilon_-(t) \leqslant |z-w_l| \quad \text{for all } z \in \partial\Omega(t),\ t>0. \tag{5.17}$$

Next we estimate $|z-w_l|$ from above. In definition (1.8) of r_0, we can take the minimum instead of the infimum. To show this, we take sequences $\{c^{(k)}\}$, $\{r^{(k)}\}$ such that $r^{(k)} \to r_0$ and $\Omega(0) \subset D(c^{(k)}, r^{(k)})$. Then $\{c^{(k)}\}$ is bounded since $\{r^{(k)}\}$ is bounded. Hence, there exists a subsequence $\{c^{(k_p)}\}$ of $\{c^{(k)}\}$ that converges to a point $c_0 \in \mathbb{C}$. Therefore,

$$\Omega(0) \subset \bigcap_{p=1}^{\infty} D(c^{(k_p)}, r^{(k_p)}) \subset \bigcap_{p=1}^{\infty} D(c_0, r^{(k_p)} + |c^{(k_p)} - c_0|) \subset \overline{D(c_0, r_0)},$$

so that $\Omega(0) \subset D(c_0, r_0)$. Observe that, by lemma 5.2 and theorem 5.3,

$$\begin{aligned}\Omega(t) &\subset \Omega\biggl(\chi_{D(c_0,r_0)} + t\sum_{j=1}^{l}\alpha_j\delta_{c_j}\biggr) = \Omega(\chi_{D(c_0,r_0)} + \chi_{\Omega(t\sum_{j=1}^{l}\alpha_j\delta_{c_j})}) \\ &\subset \Omega(\chi_{D(c_0,r_0)} + \chi_{D(w_l,R(t))}) = \Omega(\pi {r_0}^2\delta_{c_0} + \pi R(t)^2\delta_{w_l}),\end{aligned} \tag{5.18}$$

where

$$R(t) := \sqrt{t\pi^{-1}\sum_{j=1}^{l}\alpha_j} + \varepsilon_l(t).$$

In the case where $c_0 = w_l$, we see that

$$\begin{aligned}\Omega(\pi {r_0}^2\delta_{c_0+}\pi R(t)^2\delta_{w_l}) &= \Omega(\pi(r_0^2 + R(t)^2)\delta_{w_l}) \\ &= D(w_l, \sqrt{r_0^2 + R(t)^2})\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}&\sqrt{r_0^2 + R(t)^2} \\ &\quad = R(t) + \frac{{r_0}^2}{2R(t)} + O(R(t)^{-2}) \\ &\quad = \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \varepsilon_l(t) + \frac{{r_0}^2}{2}\sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\frac{1}{\sqrt{t}} + O(t^{-1}) \\ &\quad = \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\biggl(\sum_{k=2}^{l}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{(\sum_{j=1}^{k}\alpha_j)^2}|w_{k-1} - c_k|^2 + \frac{{r_0}^2}{2}\biggr)\frac{1}{\sqrt{t}} + O(t^{-1})\end{aligned} \tag{5.20}$$

as $t \to \infty$. Therefore, we define $\varepsilon_+(t)$ by

$$\varepsilon_+(t) := \sqrt{r_0^2 + R(t)^2} - \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j}.$$

Then, by combining (5.18) with (5.19) we see that

$$|z - w_l| \leqslant \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \varepsilon_+(t) \quad \text{for all } z \in \partial\Omega(t),\ t > 0. \tag{5.21}$$

Here $\varepsilon_+(t)$ satisfies

$$\varepsilon_+(t) = \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\biggl(\sum_{k=2}^{l}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{(\sum_{j=1}^{k}\alpha_j)^2}|w_{k-1} - c_k|^2 + \frac{r_0^2}{2}\biggr)\frac{1}{\sqrt{t}} + O(t^{-1}) \quad \text{as } t \to \infty. \tag{5.22}$$

Let us consider the case where $c_0 \neq w_l$. As in the proof of theorem 5.3, we may assume that $\kappa := |c_0 - w_l|/2$, $c_0 = \kappa\mathrm{i}$ and $w_l = -\kappa\mathrm{i}$. Note that

$$\pi {r_0}^2 \delta_{c_0} + \pi R(t)^2 \delta_{w_l} = \kappa^2 \pi \left(\frac{r_0}{\kappa}\right)^2 \delta_{\kappa\mathrm{i}} + \kappa^2 \pi \left(\frac{R(t)}{\kappa}\right)^2 \delta_{-\kappa\mathrm{i}}.$$

Hence, by lemma 5.1 we see that $\hat{\Omega}(t) := \{\kappa^{-1} z \mid z \in \Omega(\pi {r_0}^2 \delta_{c_0} + \pi R(t)^2 \delta_{w_l})\}$ is the minimum quadrature domain of the measure $\pi((r_0/\kappa)^2 \delta_{\mathrm{i}} + (R(t)/\kappa)^2 \delta_{-\mathrm{i}})$ for subharmonic functions. Since $(R(t)/\kappa)^2 \to \infty$ as $t \to \infty$, applying theorem 4.5 yields

$$\max_{z \in \partial \hat{\Omega}(t)} |z + \mathrm{i}| = \sqrt{\left(\frac{R(t)}{\kappa}\right)^2 + \frac{1}{2}\left(\frac{r_0}{\kappa}\right)^2 \frac{\kappa}{R(t)}} + O(R(t)^{-2}) \quad \text{as } t \to \infty.$$

Consequently,

$$\max_{z \in \partial \Omega(\pi {r_0}^2 \delta_{c_0} + \pi R(t)^2 \delta_{w_l})} |z - w_l| = R(t) + \frac{{r_0}^2}{2R(t)} + O(R(t)^{-2}) \quad \text{as } t \to \infty, \tag{5.23}$$

which can be calculated as in (5.20). Therefore, we define a non-negative function $\varepsilon_+(t)$ by

$$\varepsilon_+(t) := \max_{z \in \partial \Omega(\pi {r_0}^2 \delta_{c_0} + \pi R(t)^2 \delta_{w_l})} |z - w_l| - \sqrt{\frac{t}{\pi} \sum_{j=1}^{l} \alpha_j}$$

Then, by combining (5.18) with (5.23), we obtain (5.21) and (5.22) again.

For any $\sigma \in S_l$, the above argument can clearly be applied to obtain the estimates (5.17) and (5.21) for the case where j is replaced by $\sigma(j)$. Therefore, by taking the minima of $\varepsilon_-(t)$ and $\varepsilon_+(t)$ over $\sigma \in S_l$ and writing them as $\varepsilon_-(t)$ and $\varepsilon_+(t)$ again, we obtain the desired estimate (1.10) with (1.11), since $\Omega(t)$ is irrelevant to the way of numbering the injection points. □

6. Concluding remarks

In the case of spatial dimension 3, the same problem (1.5) appears for the flow of viscous fluid through a porous medium, since, by Darcy's law, its velocity field can be characterized as a potential flow with the potential being its pressure [8]. For other phenomena modelled by (1.5) we refer to the reader to [10]. Therefore, studying the problem in higher-dimensional spaces is also important in applications.

Quadrature domains for subharmonic functions or for harmonic functions in arbitrary dimensions are also defined in the same way. Basic properties such as the existence and the uniqueness of a quadrature domain for subharmonic functions were established [14]. Hence, we can define the weak solution of the problem in a higher-dimensional space by using the same form as (2.1), and the existence and the uniqueness of the weak solution follow. However, the method of this paper depends heavily on two-dimensional structure when we construct the rational mappings from the unit disc onto quadrature domains of the two Dirac measures (lemma 4.1). The explicit representations or the precise estimates of quadrature domains of the two Dirac measures in higher-dimensional spaces, to the best of our knowledge, have not been found. Therefore, we need a new idea in estimating quadrature domains to obtain analogous results in higher-dimensional spaces.

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