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A link between the shape of the austenite–martensite interface and the behaviour of the surface energy

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Let $\Omega \subset \mathbb{R}^2$ denote a bounded Lipschitz domain and consider some portion Γ_0 of $\partial\Omega$ representing the austenite–twinned-martensite interface which is not assumed to be a straight segment. We prove that

$$\inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y = 0 \qquad (*)$$

for an elastic energy density $\varphi : \mathbb{R}^2 \to [0,\infty)$ such that $\varphi(0,\pm 1) = 0$. Here, $\mathcal{W}(\Omega)$ consists of all functions u from the Sobolev class $W^{1,\infty}(\Omega)$ such that $|u_y| = 1$ almost everywhere on Ω together with $u = 0$ on Γ_0. We will first show that, for Γ_0 having a vertical tangent, one cannot always expect a finite surface energy, i.e. in the above problem, the condition

$$u_{yy} \text{ is a Radon measure such that } \int_\Omega |u_{yy}(x,y)|\,\mathrm{d}x\mathrm{d}y < +\infty$$

in general cannot be included. This generalizes a result of [12] where Γ_0 is a vertical straight line. Property $(*)$ is established by constructing some minimizing sequences vanishing on the whole boundary $\partial\Omega$, that is, one can even take $\Gamma_0 = \partial\Omega$. We also show that the existence or non-existence of minimizers depends on the shape of the austenite–twinned-martensite interface Γ_0.

1. Introduction

In solid–solid phase transformations, one often observes certain characteristic microstructural features involving fine mixtures of the phases. If we consider martensitic phase transformations, then one usually has a plane interface that separates one homogeneous phase, called austenite, from a very fine mixture of twins of the other phase, termed martensite. We now consider a two-dimensional section and assume that, for some physical reasons, the interface that separates the two phases is not a segment but a curve, not necessarily being smooth (see figure 1).

For instance, it is known that some applied small loads easily change the austenite–martensite interface. For further details concerning the physical background of martensitic phase transformation and also the mathematical modelling, we refer the reader to the papers [2] and [3] and the references quoted therein. To give a more precise formulation of the problem, let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ representing the martensitic configuration and let Γ_0 denote a part of

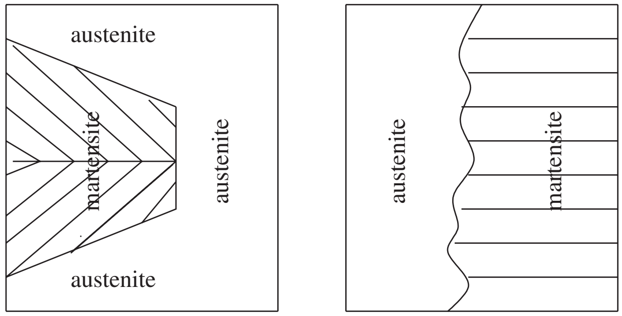

Figure 1. The austenite–twinned-martensite interface.

$\partial\Omega$ with positive measure having the meaning of the austenite–twinned-martensite interface. Let $\varphi : \mathbb{R}^2 \to [0, \infty)$ denote a Borel function such that

$$\varphi(0, 1) = \varphi(0, -1) = 0. \tag{1.1}$$

For example, φ could be the elastic energy density of the martensite with wells in $(0, \pm 1)$ corresponding to the stress-free states of two possible variants of the martensite. We would then like to consider the problem

$$I^\infty := \inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x, y))\,\mathrm{d}x\mathrm{d}y \tag{1.2}$$

in the class of admissible comparison functions

$$\mathcal{W} := \mathcal{W}(\Omega) := \{u \in W^{1,\infty}(\Omega) : |u_y| = 1 \text{ a.e. in } \Omega \text{ and } u = 0 \text{ on } \Gamma_0\}.$$

Here, $W^{1,\infty}(\Omega)$ is the Sobolev space of all weakly differentiable functions $u : \Omega \to \mathbb{R}$ such that $u, |\nabla u| \in L^\infty(\Omega)$. Since Ω is a bounded Lipschitz domain, Sobolev's embedding theorem implies that $W^{1,\infty}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, and the requirement $u = 0$ on Γ_0 has to be understood in the pointwise sense. If $u = 0$ on the whole of $\partial\Omega$, we just say that u is of class $W_0^{1,\infty}(\Omega)$. For a further discussion of Sobolev spaces, we refer the reader to [1].

We remark that the boundary condition occurring in $\mathcal{W}$ refers to elastic compatibility with the austenitic phase in the extreme case of complete rigidity of the austenite (see [2,3,7]). Problems of the type (1.2) have been investigated by Chipot and Collins (cf. [4, 5]), but without the constraint $|u_y| = 1$. This constraint was introduced by Kohn and Müller (see [8,9]): they considered a functional consisting of an elastic energy plus a surface energy term for the case that the martensitic configuration is a rectangle like $(0, L) \times (0, 1)$ and the austenite–martensite interface is the segment $\{0\} \times (0, 1)$.

Problem (1.2) was studied in [6] for the case when no loads are applied, i.e. the austenite–martensite interface is given by a segment Γ_0. We proved that the value of I^∞ is zero by constructing suitable minimizing sequences from the class $\mathcal{W}(\Omega)$, which represent, according to the Ball–James theory, the microstructure. The minimizing sequences discussed in [6] differ for the case when the segment Γ_0 is vertical and for the case when Γ_0 is oblique. In particular, for non-vertical segments, we could even replace the set $\mathcal{W}(\Omega)$ by a smaller class by adding the additional constraint

$$u_{yy} \text{ is a Radon measure of finite mass,} \tag{1.3}$$

which is not possible in the vertical case (see [12]). In what follows, the term

$$\int_{\Omega} |u_{yy}(x,y)|\,\mathrm{d}x\mathrm{d}y$$

is addressed as the surface energy and it should be understood as the mass $|u_{yy}|(\Omega)$ of the Radon measure u_{yy}. Here, our terminology follows the paper [9], i.e. we use the same formula for the surface energy as Kohn and Müller did for the case when Ω is a rectangle.

In the present paper, we want to extend the results of [6, 12] to the general case of curved boundary portions; precisely, we have the following theorem.

THEOREM 1.1. *There exist domains Ω and boundary portions Γ_0 that do not contain a vertical straight line for which condition (1.3) cannot be included into problem (1.2), i.e. there is no function u in $\mathcal{W}(\Omega)$ satisfying (1.3).*

Indeed, we will exhibit in § 2 some 'bad' curved boundaries for which one cannot incorporate condition (1.3); namely, they only have a vertical tangent forcing the surface energy to tend to infinity. This generalizes a result of [12] where the domain Ω is chosen to be a rectangle and Γ_0 is a vertical straight line. Now, if the condition of finite surface energy is dropped, one has, for any curved boundary, the following result.

THEOREM 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and consider a non-empty portion Γ_0 of $\partial\Omega$ having positive measure. If φ satisfies (1.1), then we have*

$$I^{\infty} := \inf_{u\in\mathcal{W}(\Omega)} \int_{\Omega} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y = 0.$$

Moreover, we can find a minimizing sequence $(u_n)_n \subset \mathcal{W}(\Omega)$ such that $u_n = 0$ on the whole boundary $\partial\Omega$.

For the proof of this result, we will first discuss the case when the Lipschitz domain Ω is replaced by some elementary domain, e.g. the domain enclosed by a triangle or a square. Then we consider the general situation by covering every bounded open set with a countable number of such elementary domains (Vitali's covering lemma).

2. Some curved boundaries leading to infinite surface energies: proof of theorem 1.1

In this section we prove that if the boundary part Γ_0 of Ω has a vertical tangent, one cannot, in general, add the constraint (1.3) to the definition of the class $\mathcal{W}(\Omega)$. Without loss of generality, we assume that the origin lies on Γ_0 and the tangent at this point is vertical. To be more precise, we assume that there exists a continuous function $f : I := [0,T] \to \mathbb{R}$ of class C^1 on $(0,T)$ such that

$$f(0) = 0, \qquad \lim_{t\to 0^+} f'(t) = +\infty, \tag{2.1}$$

$$\{(t, f(t)) \mid t \in I\} \subset \Gamma_0, \tag{2.2}$$

$$\{(t,y) \in [0,T] \times \mathbb{R} \mid 0 \leqslant y \leqslant f(t)\} \subset \Omega. \tag{2.3}$$

Notice that, by (2.1) and by eventually reducing the interval I, we can assume that f is strictly increasing in I. The function f is a bijective mapping $[0,T] \to [0, f(T)]$ and its inverse f^{-1} is also strictly increasing, continuous on $[0, f(T)]$ and of class $C^1(0, f(T))$. Moreover, one has that

$$(f^{-1})'(0) \text{ exists and is equal to } 0.$$

In order to prove that we cannot, in general, expect the surface energy to be finite, we first bound it from below. We have the following estimate.

LEMMA 2.1. *For every $u \in \mathcal{W}(\Omega)$ with (1.3), one has*

$$\int_0^T \int_0^{f(t)} |u_{yy}(t,y)| \,\mathrm{d}y\mathrm{d}t \geqslant \int_0^{f(T)} \left\{ \frac{s^{3/2}}{\sqrt{3K}} \frac{(f^{-1})'(s)}{(\int_0^s (f^{-1}(s) - f^{-1}(y))^2 \,\mathrm{d}y)^{1/2}} - 2 \right\} \mathrm{d}s, \tag{2.4}$$

where $\sqrt{K}$ is the Lipschitz constant of u.

Proof. Let $u \in \mathcal{W}(\Omega)$ satisfy (1.3), $t \in I$ and $y \in (0, f(t))$. Since $(f^{-1}(y), y) \in \Gamma_0$ and $u = 0$ on Γ_0, one has

$$u(t,y) = u(t,y) - u(f^{-1}(y), y) = \int_{f^{-1}(y)}^t u_x(s,y)\,\mathrm{d}s.$$

Hence

$$|u(t,y)| \leqslant \int_{f^{-1}(y)}^t |u_x(s,y)|\,\mathrm{d}s \leqslant \sqrt{K}(t - f^{-1}(y)),$$

and we get

$$\int_0^{f(t)} u(t,y)^2 \,\mathrm{d}y \leqslant K \int_0^{f(t)} (t - f^{-1}(y))^2 \,\mathrm{d}y. \tag{2.5}$$

□

Then we use the following lemma.

LEMMA 2.2. *Let $g \in W^{1,\infty}(0,l)$ be such that $|g'| = 1$ a.e. and g' changes sign N times on the open interval $(0,l)$. Then one has*

$$\int_0^l g^2(x)\,\mathrm{d}x \geqslant \tfrac{1}{12} l^3 (N+1)^{-2} = \tfrac{1}{12} l^3 \left(\frac{1}{2} \int_0^l |g''(x)|\,\mathrm{d}x + 1 \right)^{-2}.$$

Proof. The above inequality was proved by Kohn and Müller for $l = 1$ (lemma 2.7 in [9]). One can easily derive the general case by scaling.

Lemma 2.2 then yields

$$\int_0^{f(t)} u(t,y)^2 \,\mathrm{d}y \geqslant \tfrac{1}{12} f(t)^3 \left(\frac{1}{2} \int_0^{f(t)} |u_{yy}(t,y)|\,\mathrm{d}y + 1 \right)^{-2}. \tag{2.6}$$

Combining (2.5) and (2.6), one gets

$$\int_0^{f(t)} |u_{yy}(t,y)|\,\mathrm{d}y \geqslant \frac{1}{\sqrt{3K}}\frac{f(t)^{3/2}}{(\int_0^{f(t)}(t-f^{-1}(y))^2\,\mathrm{d}y)^{1/2}} - 2.$$

Therefore, one obtains

$$\int_0^T\int_0^{f(t)} |u_{yy}(t,y)|\,\mathrm{d}y\mathrm{d}t \geqslant \int_0^T\bigg\{\frac{1}{\sqrt{3K}}\frac{f(t)^{3/2}}{(\int_0^{f(t)}(t-f^{-1}(y))^2\,\mathrm{d}y)^{1/2}} - 2\bigg\}\,\mathrm{d}t,$$

from which the claim of lemma 2.1 follows by the change of variables $s = f(t)$. □

REMARK 2.3. One should observe that, by lemma 2.1, the surface energy that is bounded from below by

$$\int_0^T\int_0^{f(t)} |u_{yy}(t,y)|\,\mathrm{d}y\mathrm{d}t$$

is infinite whenever f is chosen in such a way that

$$\int_0^{f(T)} s^{3/2}\frac{(f^{-1})'(s)}{(\int_0^s(f^{-1}(s)-f^{-1}(y))^2\,\mathrm{d}y)^{1/2}}\,\mathrm{d}s = +\infty.$$

Therefore, theorem 1.1 will be a consequence of the following result.

THEOREM 2.4. *Assume that the function f satisfies, in addition,*

$$s\frac{(f^{-1})'(s)}{f^{-1}(s)} \geqslant \frac{c}{s^{\alpha}} \quad \textit{on } (0, f(T)) \tag{2.7}$$

for a positive constant c and some exponent $\alpha \geqslant 1$. Then one has

$$\int_0^{f(T)} s^{3/2}\frac{(f^{-1})'(s)}{(\int_0^s(f^{-1}(s)-f^{-1}(y))^2\,\mathrm{d}y)^{1/2}}\,\mathrm{d}s = +\infty.$$

Proof. Since

$$0 < f^{-1}(s) - f^{-1}(y) \leqslant f^{-1}(s) \quad \forall y \in (0,s),$$

one has

$$\int_0^{f(T)} s^{3/2}\frac{(f^{-1})'(s)}{(\int_0^s(f^{-1}(s)-f^{-1}(y))^2\,\mathrm{d}y)^{1/2}}\,\mathrm{d}s \geqslant \int_0^{f(T)} s\frac{(f^{-1})'(s)}{f^{-1}(s)}\,\mathrm{d}s.$$

Using (2.7), one deduces that

$$\int_0^{f(T)} s\frac{(f^{-1})'(s)}{f^{-1}(s)}\,\mathrm{d}s = \int_0^{f(T)} \frac{c}{s^{\alpha}}\,\mathrm{d}s = +\infty$$

and the theorem is proved. Notice that, for example,

$$s\frac{(f^{-1})'(s)}{f^{-1}(s)} = \frac{1}{s^{\alpha}}, \quad \alpha \geqslant 1, \quad f^{-1}(s) > 0 \quad \text{on } (0, f(T)), \quad f^{-1}(0) = 0$$

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if and only if

$$f^{-1}(s) = \begin{cases} c\exp\left(-\dfrac{1}{\alpha s^{\alpha}}\right) & \text{if } s \neq 0 \quad (c > 0 \text{ is a constant}), \\ 0 & \text{if } s = 0, \end{cases}$$

or, equivalently,

$$f(t) = \begin{cases} \left(-\dfrac{1}{\alpha \ln(t/c)}\right)^{1/\alpha} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

□

REMARK 2.5. Notice that condition (2.7) is equivalent to

$$f^{-1}(s)\exp\left(\frac{c}{\alpha s^{\alpha}}\right) \quad \text{is increasing on } (0, f(T)).$$

Recall that one has

$$f^{-1}(0) = (f^{-1})'(0) = 0.$$

Now if, in addition, f^{-1} is of class $C^n([0, f(T)])$, one also has

$$(f^{-1})^{(m)}(0) = 0 \quad \forall m \in \{0, 1, 2, \dots, n\}.$$

This result, which is not *a priori* evident, follows from theorem 2.4 and the following lemma.

LEMMA 2.6. *Let $h > 0$ and consider a function g of class $C^n([0, h])$ satisfying*

$$g(0) = g'(0) = \cdots = g^{(n-1)}(0) = 0, \qquad g^{(n)}(0) \neq 0. \tag{2.8}$$

Then one has

$$\int_0^h s^{3/2} \frac{|g'(s)|}{(\int_0^s (g(s) - g(y))^2 \,\mathrm{d}y)^{1/2}} \,\mathrm{d}s < +\infty.$$

Proof. One has

$$\int_0^s (g(s) - g(y))^2 \,\mathrm{d}y = sg(s)^2 - 2g(s)\int_0^s g(y)\,\mathrm{d}y + \int_0^s g(y)^2 \,\mathrm{d}y. \tag{2.9}$$

From the assumption on $g^{(k)}(0)$, it follows that

$$\tilde{g}^{(k)}(0) = 0 \quad \text{for all } 0 \leqslant k \leqslant n,$$

where we have set

$$\tilde{g}(t) := g(t) - \frac{1}{n!}g^{(n)}(0)t^n.$$

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If we write

$$\begin{aligned}\tilde{g}(t) &= \int_0^t \tilde{g}'(t_1)\,\mathrm{d}t_1\\ &= \int_0^t \int_0^{t_1} \tilde{g}''(t_2)\,\mathrm{d}t_2\mathrm{d}t_1\\ &= \cdots\\ &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \tilde{g}^{(n)}(t_n)\,\mathrm{d}t_n\cdots\mathrm{d}t_1,\end{aligned}$$

then it easily follows that

$$|\tilde{g}(t)| \leqslant t^n \max_{[0,T]} |\tilde{g}^{(n)}|$$

and, in conclusion,

$$g(s) = \frac{1}{n!} g^{(n)}(0)s^n + o(s^n).$$

Using this formula on the right-hand side of (2.9), we deduce that

$$\int_0^s (g(s) - g(y))^2\,\mathrm{d}y = \left(\frac{g^{(n)}(0)}{n!}\right)^2 \left[1 - \frac{2}{n+1} + \frac{1}{2n+1}\right] s^{2n+1} + o(s^{2n+1}).$$

This implies that

$$\lim_{s\to 0^+} s^{3/2} \frac{g'(s)}{(\int_0^s (g(s) - g(y))^2\,\mathrm{d}y)^{1/2}} = n\frac{g^{(n)}(0)}{|g^{(n)}(0)|}\left(1 - \frac{2}{n+1} + \frac{1}{2n+1}\right)^{-1/2},$$

and the lemma is proved. □

3. Proof of theorem 1.2

First we prove theorem 1.2 for some special domains having ‘nice’ boundaries. Let Δ denote the interior of the triangle with vertices in $(-1,0)$, $(1,0)$ and $(0,1)$.

Theorem 3.1. *Assume that φ satisfies equation (1.1). Then there exists a sequence $v_n \in W_0^{1,\infty}(\Delta)$ satisfying $|\partial_y v_n| = 1$ a.e. for each n and such that*

$$\lim_{n\to\infty} \int_\Delta \varphi(\nabla v_n(x,y))\,\mathrm{d}x\mathrm{d}y = 0.$$

Proof. Given $N \in \mathbb{N}$, we define $u \in W_0^{1,\infty}(\Delta)$, $|u_y| = 1$, such that

$$\int_\Delta \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y$$

is of order $1/N$. Let $\delta := 1/N$ and consider the δ-periodic extension to the whole line of

$$h(t) := \begin{cases} t & \text{if } 0 \leqslant t \leqslant \frac{1}{2}\delta,\\ \delta - t & \text{if } \frac{1}{2}\delta \leqslant t \leqslant \delta.\end{cases}$$

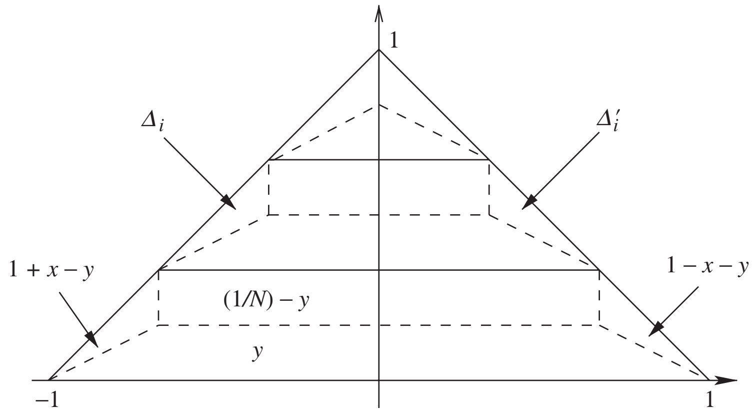

Figure 2. The function u for $N = 3$.

We then let

$$u(x, y) := \begin{cases} (x + 1 - y) \wedge h(y) & \text{if } (x, y) \in \Delta,\ -1 \leqslant x \leqslant 0, \\ (1 - x - y) \wedge h(y) & \text{if } (x, y) \in \Delta,\ 0 \leqslant x \leqslant 1. \end{cases}$$

Here we write $\alpha \wedge \beta$ for the minimum of two numbers $\alpha, \beta \in \mathbb{R}$. Figure 2 shows the situation for $N = 3$.

Clearly, $u \in W_0^{1,\infty}(\Delta)$ and

$$\nabla u(x, y) = (0, \pm 1)$$

for points (x, y) not belonging to the $2N$ triangles

$$\Delta_i = \biggl\{(x, y) \in \biggl[-1 + \frac{i-1}{N}, -1 + \frac{i}{N}\biggr] \times \biggl[\frac{i-1}{N}, \frac{i}{N}\biggr], \\ \frac{1}{2}\biggl(x + 1 + \frac{i-1}{N}\biggr) \leqslant y \leqslant x + 1\biggr\}, \quad i = 1, \dots, N,$$

and their reflexions Δ_i', $i = 1, \dots, N$, with respect to the y-axis. It is easy to check that

$$\nabla u(x, y) = (1, -1) \qquad \text{on } \Delta_i,$$

whereas

$$\nabla u(x, y) = (-1, -1) \quad \text{on } \Delta_i'.$$

Therefore, $|u_y| = 1$ a.e. on Δ and (1.1) implies that

$$\begin{aligned} \int_\Delta \varphi(\nabla u(x, y))\,\mathrm{d}x\mathrm{d}y &= \sum_{i=1}^N \biggl[\int_{\Delta_i} \varphi(\nabla u(x, y))\,\mathrm{d}x\mathrm{d}y + \int_{\Delta_i'} \varphi(\nabla u(x, y))\,\mathrm{d}x\mathrm{d}y\biggr] \\ &= \sum_{i=1}^N [\mathcal{L}^2(\Delta_i)\varphi(1, -1) + \mathcal{L}^2(\Delta_i')\varphi(-1, -1)] \\ &= \tfrac{1}{4} N\delta^2[\varphi(1, -1) + \varphi(-1, -1)]. \end{aligned}$$

Thus

$$0 \leqslant I^{\infty} \leqslant \int_{\Delta} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y = \frac{1}{4N}[\varphi(1,-1)+\varphi(-1,-1)],$$

and theorem 3.1 is established. □

Let S now denote the set of points (x,y) such that $(x,y) \in \bar{\Delta}$ or $(x,-y) \in \bar{\Delta}$, i.e. S is the closed square with vertices in $(\pm 1, 0)$ and $(0, \pm 1)$. Then we have the following result.

Corollary 3.2. *Assume that* φ *satisfies equation (1.1). Then there exists a sequence* $v_n \in W_0^{1,\infty}(\dot{S})$ *satisfying* $|\partial_y v_n| = 1$ *a.e. for each* n *and such that*

$$\lim_{n\to\infty} \int_S \varphi(\nabla v_n(x,y))\,\mathrm{d}x\mathrm{d}y = 0.$$

Proof. Let us define on S the following function,

$$v(x,y) := \begin{cases} u(x,y) & \text{if } (x,y) \in \Delta, \\ u(x,-y) & \text{if } (x,y) \in S \setminus \Delta, \end{cases}$$

where the function $u : \Delta \to \mathbb{R}$ is defined in the proof of theorem 3.1. One can easily check that

$$\int_S \varphi(\nabla v(x,y))\,\mathrm{d}x\mathrm{d}y = \int_{\Delta} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y + \int_{\Delta} \tilde{\varphi}(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y,$$

where

$$\tilde{\varphi}(x,y) = \varphi(x,-y).$$

Thus

$$\begin{aligned} \int_S \varphi(\nabla v(x,y))\,\mathrm{d}x\mathrm{d}y &= \frac{1}{4N}[\varphi(1,-1)+\varphi(-1,-1)+\tilde{\varphi}(1,-1)+\tilde{\varphi}(-1,-1)] \\ &= \frac{1}{4N}[\varphi(1,-1)+\varphi(-1,-1)+\varphi(1,1)+\varphi(-1,1)], \end{aligned}$$

and corollary 3.2 is proved. □

Remark 3.3. Notice that, for the elementary domains we considered above, one can add the constraint

$$|u_{yy}| \text{ is a Radon measure of finite mass.}$$

Recall that, for domains like squares with sides parallel to the x- and y-axis or domains as studied in § 2 (with function f satisfying (2.7)), it is not possible to incorporate the above constraint. For domains with f^{-1} satisfying (2.8) like discs the question is still open, since, by lemma 2.6, the right-hand side of (2.4) is finite; we cannot conclude that the surface energy is $+\infty$ for any $u \in \mathcal{W}(\Omega)$.

In order to prove theorem 1.2 for general domains, we need the following lemmas.

Lemma 3.4 (Vitali's covering lemma). *Let Ω denote a bounded open subset of $\mathbb{R}^2$. Then there exist points $(x_n, y_n) \in \Omega$ and positive numbers r_n such that*

$$S_n := r_n S + (x_n, y_n) \subset \Omega \quad \text{and} \quad \dot{S}_l \cap \dot{S}_k = \emptyset \quad \text{for } l \neq k,$$

where S is the square with vertices in $(\pm 1, 0)$ and $(0, \pm 1)$. Moreover, we have

$$\Omega = \bigcup_{n=0}^{+\infty} S_n.$$

Proof. We refer, for example, to [11] for a proof. □

Applying the construction of lemma 3.4, we find $r_n > 0$, $(x_n, y_n) \in \Omega$ such that the sets $S_n = r_n S + (x_n, y_n) \subset \Omega$ have the stated properties. Given a function $u_0 \in W_0^{1,\infty}(\dot{S})$, we let

$$\left.\begin{aligned} u_n : S_n \to \mathbb{R}, \quad & u_n(x,y) := r_n u_0\left(\frac{1}{r_n}(x - x_n, y - y_n)\right), \\ u : \Omega \to \mathbb{R}, \quad & u(x,y) := \sum_{n=1}^{\infty} (\chi_{\dot{S}_n} u_n)(x,y), \end{aligned}\right\} \tag{3.1}$$

where $\chi_{\dot{S}_n}$ denotes the characteristic function of the set $\dot{S}_n$. We make the following claim.

Lemma 3.5. *The function u defined in (3.1) is in the space $W_0^{1,\infty}(\Omega)$, and we have the following formula:*

$$\nabla u(x,y) = \sum_{n=1}^{\infty} (\chi_{\dot{S}_n} \nabla u_n)(x,y) = \sum_{n=1}^{\infty} \chi_{\dot{S}_n} \nabla u_0\left(\frac{1}{r_n}(x - x_n, y - y_n)\right) \quad \text{a.e. on } \Omega.$$

Remark 3.6. If we know that $|\partial_y u_0| = 1$ a.e. on $\dot{S}$, then we deduce from the disjointness of the family $\{\dot{S}_n\}$ that $|u_y| = 1$ is also true a.e. on Ω.

Proof of lemma 3.5. On account of $(x_n, y_n) \in \Omega$, $S_n \subset \Omega$, the sequence $(r_n)_n$ stays bounded, and thus

$$\|u\|_{L^\infty(\Omega)} \leqslant \sup_{n \in \mathbb{N}} r_n \|u_0\|_{L^\infty(S)} < \infty.$$

In order to prove weak differentiability of the function u, we fix $\psi \in C_0^\infty(\Omega)$ and get, from Lebesgue's theorem on dominated convergence, that

$$\int_\Omega u(x,y) \nabla \psi(x,y) \,\mathrm{d}x\mathrm{d}y = \sum_{n=1}^{\infty} \int_{\dot{S}_n} u_n(x,y) \nabla \psi(x,y) \,\mathrm{d}x\mathrm{d}y.$$

Observing that $u_n = 0$ on ∂S_n, we can write

$$\int_{\dot{S}_n} u_n(x,y) \nabla \psi(x,y) \,\mathrm{d}x\mathrm{d}y = -\int_{\dot{S}_n} \nabla u_n(x,y) \psi(x,y) \,\mathrm{d}x\mathrm{d}y,$$

and, by the same reasoning as above (note that $\|\nabla u_n\|_{L^\infty(S_n)} = \|\nabla u_0\|_{L^\infty(S)}$, and therefore $\|\sum_{n=1}^{M} \chi_{\dot{S}_n} \nabla u_n\|_{L^\infty(\Omega)} = \|\nabla u_0\|_{L^\infty(S)}$ for all $M \geqslant 1$),

$$-\sum_{n=1}^{\infty} \int_{\dot{S}_n} \nabla u_n(x,y)\psi(x,y)\,\mathrm{d}x\mathrm{d}y = -\int_{\Omega} \biggl(\sum_{n=1}^{\infty} \chi_{\dot{S}_n} \nabla u_n(x,y)\biggr)\psi(x,y)\,\mathrm{d}x\mathrm{d}y,$$

which proves that

$$\sum_{n=1}^{\infty} \chi_{\dot{S}_n} \nabla u_n \in L^\infty(\Omega, \mathbb{R}^2)$$

is the weak derivative of u. Again, by dominated convergence, it is obvious that

$$\sum_{n=1}^{M} \chi_{\dot{S}_n} u_n \to u, \qquad \sum_{n=1}^{M} \chi_{\dot{S}_n} \nabla u_n \to \nabla u$$

as M goes to infinity in $L^p(\Omega)$ for any finite p. Since the compact sets S_n are included in Ω, we have

$$\sum_{n=1}^{M} \chi_{\dot{S}_n} u_n \in W_0^{1,p}(\Omega),$$

thus $u \in W_0^{1,p}(\Omega),\ p < \infty$. The Lipschitz boundary of Ω guarantees that

$$W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : B(v) = 0\},$$

where $B : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator. Recalling that, for functions $v \in W^{1,p}(\Omega) \cap C^0(\bar{\Omega})$, $B(v)$ is the pointwise trace, we finally deduce that $u \in W_0^{1,\infty}(\Omega)$.

The proof of theorem 1.2 can now be carried out as follows. Given $N \in \mathbb{N}$, we constructed in the proof of corollary 3.2 a function $u_0 \in W_0^{1,\infty}(\dot{S})$ such that $|\partial_y u_0| = 1$ on S and

$$\int_S \varphi(\nabla u_0(x,y))\,\mathrm{d}x\mathrm{d}y = \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1) + \varphi(1,1) + \varphi(-1,1)].$$

Let us consider the function u defined in (3.1) for this particular choice of u_0. Lemma 3.5 implies that $u \in W_0^{1,\infty}(\Omega)$, and, from the remark after lemma 3.5, we deduce that $|u_y| = 1$ a.e. on Ω, and thus $u \in \mathcal{W}(\Omega)$. Furthermore, we have

$$\begin{aligned}\int_{\Omega} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y &= \sum_{n=1}^{\infty} \int_{\dot{S}_n} \varphi\biggl(\nabla u_0\biggl(\frac{1}{r_n}(x - x_n, y - y_n)\biggr)\biggr)\,\mathrm{d}x\mathrm{d}y \\ &= \sum_{n=1}^{\infty} r_n^2 \int_{\dot{S}} \varphi(\nabla u_0(x,y))\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

so that

$$\int_{\Omega} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y = \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1) + \varphi(1,1) + \varphi(-1,1)] \sum_{n=1}^{\infty} r_n^2.$$

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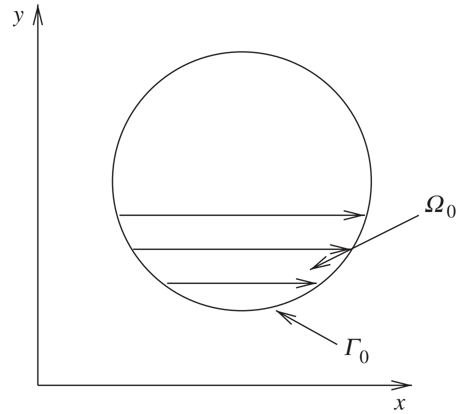

Figure 3. Ω = a disc.

Finally, we observe that

$$\mathcal{L}^2(\Omega) = \sum_{n=1}^{\infty} \mathcal{L}^2(r_n S + (x_n, y_n)) = 2 \sum_{n=1}^{\infty} r_n^2,$$

hence

$$\int_{\Omega} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y = \frac{1}{2N}\mathcal{L}^2(\Omega)[\varphi(1,-1) + \varphi(-1,-1) + \varphi(1,1) + \varphi(-1,1)],$$

and since N was arbitrary, we have shown that $I^{\infty} = 0$. Moreover, it should be obvious how to obtain from the above construction a minimizing sequence in the class $\mathcal{W}(\Omega) \cap W_0^{1,\infty}(\Omega)$. This finishes the proof of theorem 1.2. □

4. Remarks

In addition to (1.1), let us assume that the integrand φ satisfies

$$\varphi(p, \pm 1) = 0 \quad \Rightarrow \quad p = 0. \tag{4.1}$$

Under this condition, we investigate if the infimum $I^{\infty} = 0$ is attained by some function $u \in \mathcal{W}(\Omega)$. This heavily depends on the shape of the boundary portion. For example, if $\Gamma_0 \subset \mathbb{R} \times \{b\}$ for some number $b \in \mathbb{R}$, then clearly $u(x,y) = y - b$ vanishes on Γ_0, $\partial_y u \equiv 1$ and $\nabla u(x,y) = (0,1)$, and hence $\varphi(\nabla u(x,y)) = 0$ by (1.1). In order to exclude such a behaviour, we let Σ denote the union of all rays starting from points $(x_0, y_0) \in \Gamma_0$ into Ω with direction $(1,0)$, and require that

$$\Omega_0 := \Omega \cap \Sigma \text{ is open and non-empty.} \tag{4.2}$$

Of course, condition (4.2) does not hold in case $\Gamma_0 \subset \mathbb{R} \times \{b\}$ (see figure 3).

THEOREM 4.1. *Let (1.1), (4.1) and (4.2) hold. Then we have*

$$\int_{\Omega} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y > 0$$

for any $u \in \mathcal{W}(\Omega)$.

Proof. If we assume that

$$\int_{\Omega} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y = 0$$

for some $u \in \mathcal{W}(\Omega)$, then we get from (4.1) that

$$u_x = 0 \quad \text{on } \Omega.$$

This implies the vanishing of u on any ray of the type defined before, and hence, by (4.2), $u = 0$ on Ω_0, contradicting $u_y = \pm 1$ a.e. □

Next we describe minimizing sequences in terms of Young measures (see [10] for details about the notion of Young measures).

THEOREM 4.2. *Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^2$ and assume that the boundary portion Γ_0 is chosen in such a way that $\Omega_0 = \Omega$ (see (4.2)). Suppose that the integrand $\varphi : \mathbb{R}^2 \to [0,\infty)$ is a continuous function such that*

$$\varphi(p,q) = 0 \quad \textit{if and only if } (p,q) = (0,\pm 1).$$

Let $(u_n)_n$ denote a minimizing sequence of problem (1.2) such that

$$\|u_n\|_{L^\infty(\Omega)}, \|\nabla u_n\|_{L^\infty(\Omega)} \leqslant C \tag{4.3}$$

for a finite constant C independent of n. Then

$$u_n \to 0 \quad \textit{uniformly on } \Omega. \tag{4.4}$$

Moreover, the sequence of gradients $(\nabla u_n)_n$ defines a unique homogeneous Young measure given by

$$\nu_{(x,y)} = \tfrac{1}{2}\delta_{(0,-1)} + \tfrac{1}{2}\delta_{(0,1)} \quad \textit{a.e. in } \Omega, \tag{4.5}$$

where $\delta_{(0,\pm 1)}$ are the Dirac measures at $(0,\pm 1)$.

Proof. One proceeds as in [6]. We refer also to [4] for a proof related to multiple-wells problems. □

COROLLARY 4.3. *Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^2$. Suppose that the integrand $\varphi : \mathbb{R}^2 \to [0,\infty)$ is a continuous function such that*

$$\varphi(p,q) = 0 \quad \textit{if and only if } (p,q) = (0,\pm 1).$$

Let $(u_n)_n$ denote a minimizing sequence of problem (1.2) such that

$$\|u_n\|_{L^\infty(\Omega)}, \|\nabla u_n\|_{L^\infty(\Omega)} \leqslant C.$$

Suppose further that (4.2) holds. Then

$$u_n \to 0 \quad \textit{uniformly on } \Omega_0.$$

Moreover, the sequence of gradients $(\nabla u_n)_n$ defines a Young measure given by

$$\nu_{(x,y)} = \alpha(x)\delta_{(0,-1)} + (1-\alpha(x))\delta_{(0,1)} \quad \textit{a.e. in } \Omega,$$

where $\alpha : \Omega \to [0,1]$ is a measurable function such that

$$\alpha(x) = \tfrac{1}{2} \quad \textit{a.e. in } \Omega_0.$$

Proof. The restriction of (u_n) to Ω_0 is a minimizing sequence of

$$I^\infty(\Omega_0) := \inf_{u\in\mathcal{W}(\Omega_0)} \int_{\Omega_0} \varphi(\nabla u(x,y))\,\mathrm{d}x\mathrm{d}y = 0,$$

where $\mathcal{W}(\Omega_0)$ is defined with respect to the boundary portion $\Gamma_0 \cap \partial\Omega_0$. Since $(\Omega_0)_0 = \Omega_0$ with an obvious definition of $(\Omega_0)_0$, one can apply theorem 4.2 to get corollary 4.3. □

REMARK 4.4. Note that $\Omega_0 = \Omega$ holds for the particular case $\Gamma_0 = \partial\Omega$. Now, if $\Omega_0 \neq \Omega$, then the considered minimizing sequences do not necessarily converge to zero uniformly on the whole domain Ω and the related Young measure is, in general, not unique. In order to illustrate this, let us first consider the square $\Omega := (0,1)\times(0,1)$. Then, by § 3, there exists a minimizing sequence (u_n) of (1.2) vanishing on the whole boundary of Ω such that (4.3), (4.4) and (4.5) are satisfied. Now let

$$\Omega := (0,1)\times(0,2), \qquad \Gamma_0 := \{0\}\times[0,1].$$

Let (v_n) be the sequence in $\mathcal{W}(\Omega)$ defined as follows:

$$v_n(x,y) = \begin{cases} u_n(x,y) & \text{if } (x,y)\in(0,1)\times(0,1), \\ y-1 & \text{if } (x,y)\in(0,1)\times(1,2). \end{cases}$$

It is clear that (v_n) is a minimizing sequence of (1.2) that does not converge uniformly to zero. Moreover, the sequence of gradients of (v_n) generates a Young measure $(\mu_X)_{X\in\Omega}$ such that

$$\mu_X = \tfrac{1}{2}\delta_{(0,-1)} + \tfrac{1}{2}\delta_{(0,1)} \quad \text{a.e. in } (0,1)\times(0,1)$$

and

$$\mu_X = \delta_{(0,1)} \qquad\qquad \text{a.e. in } (0,1)\times(1,2).$$

If we let

$$w_n(x,y) = \begin{cases} u_n(x,y) & \text{if } (x,y)\in[0,1]\times[0,1], \\ 1-y & \text{if } (x,y)\in[0,1]\times[1,2], \end{cases}$$

we obtain a new minimizing sequence such that

$$\mu_X = \delta_{(0,-1)} \quad \text{a.e. in } (0,1)\times(1,2).$$

Therefore, there is a lack of uniqueness for the Young measure.

References

1 R. A. Adams. *Sobolev spaces* (Academic, 1975).

2 J. M. Ball and R. D. James. Fine phase mixtures as minimizers of energy. *Arch. Ration. Mech. Analysis* **100** (1987), 13–52.

3 J. M. Ball and R. D. James. Proposed experimental tests of a theory of fine microstructures. *Phil. Trans. R. Soc. Lond.* A **338** (1992), 350–389.

4 M. Chipot. Numerical analysis of oscillations in nonconvex problems. *Numer. Math.* **59** (1991), 747–767.

5 M. Chipot and C. Collins. Numerical approximation in variational problems with potential wells. *SIAM J. Numer. Analysis* **29** (1993), 473–487.

6 A. Elfanni and M. Fuchs. The behaviour of microstructures with small shears of the austenite–martensite interface in martensitic phase transformations. *Z. Angew. Math. Phys.* **54** (2003), 937–953.

7 R. Kohn. The relationship between linear and nonlinear variational models of coherent phase transitions. In *Proc. Seventh Army Conf. on Applied Mathematics and Computing, West Point, June 1989*.

8 R. V. Kohn and S. Müller. Branching of twins near an austenite/twinned martensite interface. *Phil. Mag.* A **66** (1992), 697–715.

9 R. V. Kohn and S. Müller. Surface energy and microstructure in coherent phase transitions. *Commun. Pure Appl. Math.* **47** (1994), 405–435.

10 P. Pedregal. *Parametrized measures and variational principles* (Birkhäuser, 1997).

11 E. Stein. *Singular integrals and differentiability properties of functions* (Princeton, NJ: Princeton University Press, 1970).

12 M. Winter. Lavrentiev phenomenon in microstructure theory. *Electron. J. Diff. Eqns* **6** (1996), 1–12.

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