NATURAL HEDGES WITH IMMUNIZATION STRATEGIES OF MORTALITY AND INTEREST RATES

BY

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ABSTRACT

In this paper, we first derive closed-form formulas for mortality-interest durations and convexities of the prices of life insurance and annuity products with respect to an instantaneously proportional change and an instantaneously parallel movement, respectively, in μ^* (the force of mortality-interest), the addition of μ (the force of mortality) and δ (the force of interest). We then build several mortality-interest duration and convexity matching strategies to determine the weights of whole life insurance and deferred whole life annuity products in a portfolio and evaluate the value at risk and the hedge effectiveness of the weighted portfolio surplus at time zero. Numerical illustrations show that using the mortality-interest duration and convexity matching strategies with respect to an instantaneously proportional change in μ^* can more effectively hedge the longevity risk and interest rate risk embedded in the deferred whole life annuity products than using the mortality-only duration and convexity matching strategies with respect to an instantaneously proportional shift or an instantaneously constant movement in μ only.

KEYWORDS

Interest rate risk, longevity risk, duration, convexity, natural hedge.

1. INTRODUCTION

Interest rate immunization whereby the value of a portfolio will be little affected in response to a change in interest rates has been studied and applied widely in the assets and liabilities management of life insurance policies for life insurers, see, for example, Redington (1952), Fisher and Weil (1971), Shiu (1987, 1990), and Courtois and Denuit (2007). Over the past decades, mortality rates have displayed dramatic improvement, which is reflected in increased

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life expectancies. Because life annuity contracts and pension plans often span decades, the trend in most developed countries toward a gradual increase in life expectancy implies that annuity providers, retirement programs, and long-term care systems face a significant threat from this longevity. On the other hand, life insurers face the risk of financial loss due to a sudden jump in mortality caused by catastrophes (e.g., the 1918 Spanish flu pandemic or the 2004 Indian Ocean earthquake and tsunami). As a result, insurers issuing life insurance and annuity policies bear not only interest rate risk but also mortality and longevity risks.

Following interest rate immunization, mortality rate immunization has also aroused some attention. Mortality rate immunization creates natural hedge opportunities for life insurers and annuity providers through a proper allocation of life insurance and annuity products in a portfolio. Wang *et al.* (2010) adopt the effective mortality duration to determine the weights of two life insurance and annuity products in a portfolio. Li and Hardy (2011) and Li and Luo (2012) define the measure so called "key q-duration," a variation of the effective duration, and then build a longevity hedge with q-forward contracts under the proposed measure. Tsai and Chung (2013) derive the closed-form formulas for mortality durations and convexities with respect to an instantaneously proportional shift and an instantaneously parallel movement, respectively, in μ (the force of mortality). Lin and Tsai (2013, 2014) further define and derive mortality durations and convexities with respect to an instantaneously proportional change and an instantaneously constant movement, respectively, in q (the one-year death probability), p (the one-year survival probability), $\ln(\mu)$, (q/p), and $\ln(q/p)$; they also propose mortality duration and convexity matching strategies to determine the weights of two or three life insurance and annuity products and achieve quite satisfactory hedge effectiveness (HE). Wong *et al.* (2017) quantify the benefits of natural hedges for a range of different types of life insurance product designs and risk measures based on the probability of insurer's solvency. Luciano *et al.* (2017) provide natural hedging strategies with delta and gamma hedges for life insurance and annuity businesses on a single generation or on different generations in the presence of both longevity and interest rate risks. Levantesi and Menzietti (2018) investigate the application of natural hedging strategies for long-term care insurers by diversifying both longevity and disability risks affecting long-term care annuities.

Following the two strands of literature in interest rate immunization and mortality rate immunization, we further develop mortality-interest duration and convexity matching strategies for hedging mortality, longevity, and interest rate risks at the same time. We make the contribution to the literature in two manners. First, motivated by the linear relationship between two sequences of μ^* (the force of mortality-interest), the addition of μ (the force of mortality) and δ (the force of interest), we assume that μ^* moves approximately linearly and then derive closed-form formulas for mortality-interest durations and convexities with respect to an instantaneously proportional change and an instantaneously parallel movement, respectively, in μ^*. The linear assumption contributes to the closed-form formulas for mortality-interest durations and convexities, whereas most papers, for example, Wang *et al.* (2010), Li and Hardy (2011), Li and Luo (2012), Luciano *et al.* (2017), and Levantesi and Menzietti (2018), cannot achieve closed-form formulas for their measures.

Second, adopting the proposed mortality-interest duration or convexity, we can determine the unique solution to a single unknown weight of life insurance for a portfolio to achieve a better natural hedge when considering mortality, longevity, and interest rate risks together. More importantly, one can apply the proposed strategies to a portfolio where a single policy (multiple policies) from each of life insurance and annuity products is (are) issued to policyholders of the same or different ages (ranges of ages). Moreover, the unique solution to a single unknown weight of life insurance under most conditions is feasible (i.e., the weight is between zero and one). Nevertheless, even if one obtains an infeasible weight, we propose to charge risk premiums on insurance products under the allocation of a feasible weight to achieve the same level of unexpected extreme loss with the allocation of the infeasible weight.

In the numerical illustrations, using U.S. mortality and interest rate data, we first compute mortality-interest durations and convexities of the surpluses at time zero for whole life insurance (WL) and deferred whole life annuity (DWA) products with the forecasted mortality rates under the Lee–Carter model and the predicted interest rates under the Cox–Ingersoll–Ross (CIR) model. Next, we illustrate how to allocate a portfolio of two-policy or multipolicy WL and annuity products. Finally, we evaluate hedging performances of the proposed mortality-interest duration and convexity matching strategies based on some risk measures and compared with those of mortality-only duration and convexity matching strategies. We also assess hedging performances to calibrate the robustness of the results with a range of assumptions on alternative stochastic mortality and interest rate models, population basis risks (country basis, gender basis, and age basis risks), and the payment period.

The remainder of this paper proceeds as follows. In Section 2, we define and derive mortality-interest durations and convexities of the prices of life insurance and annuity products. Section 3 introduces mortality-interest duration and convexity matching strategies to determine the weight of a portfolio of life insurance and annuity exposures. In Section 4, we provide numerical illustrations. We compare the value at risk (VaR) and the HE of the simulated surpluses at time zero for a portfolio of two-policy or multipolicy life insurance and annuity products based on the weights resulting from different matching strategies. Section 5 concludes this paper.

2. Mortality-Interest Durations and Convexities

In finance, interest rate duration (convexity) measures the sensitivity (curvature) of the price of a bond in response to a parallel change in interest rate.

Mortality durations and convexities have been defined and derived with applications to natural hedge by mortality immunization in Lin and Tsai (2013, 2014). In this section, combining mortality duration (convexity) with interest rate duration (convexity), we further define and derive mortality-interest durations (convexities) of the prices of life insurance and annuity products.

Consider the price of a more general annuity product, ${}_{h|}\ddot{a}_{x,\,t:\bar{j}|}$, the net single premium (NSP) of one-unit h-year deferred and j-year temporary life annuity-due issued to an insured aged x in year t at time 0,

$$ {}_{h|}\ddot{a}_{x,\,t:\bar{j}|} = \sum_{k=h}^{h+j-1} {}_kp_{x,\,t} \cdot e^{-\int_0^k \delta_s\, ds} = \sum_{k=h}^{h+j-1} e^{-\int_0^k \mu_{x,\,t}(s)\, ds} \cdot e^{-\int_0^k \delta_s\, ds} = \sum_{k=h}^{h+j-1} e^{-\int_0^k [\mu_{x,\,t}(s)+\delta_s]\, ds}, \tag{2.1} $$

where ${}_kp_{x,\,t} = \prod_{i=0}^{k-1} p_{x+i,\,t+i}$ for $k \geq 1$ is the probability that the insured aged x in year t survives k years to age $x+k$ in year $t+k$, $p_{x+i,\,t+i} = e^{-\int_i^{i+1} \mu_{x,\,t}(s)\, ds}$ for $i = 0, 1, \cdots k-1$, $\mu_{x,\,t}(s) > 0$ for $s \geq 0$ is the force of mortality for age $x+s$ in year $t+s$, and δ_s is the force of interest at time s.

We assume that both $\mu_{x,\,t}(s)$ and δ_s are piecewise constant for $s \in [i,\ i+1)$, $i = 0, 1, \cdots$, that is, $\mu_{x,\,t}(s) = \mu_{x,\,t}(i) = -\ln(p_{x+i,\,t+i})$ and $\delta_s = \delta_i$ for $s \in [i,\ i+1)$. Then, (2.1) can be rewritten as

$$ \begin{aligned} {}_{h|}\ddot{a}_{x,\,t:\bar{j}|} &= \sum_{k=h}^{h+j-1} e^{-\int_0^k [\mu_{x,\,t}(s)+\delta_s]\, ds} = \sum_{k=h}^{h+j-1} e^{-\sum_{i=0}^{k-1} [\mu_{x,\,t}(i)+\delta_i]} \\ &= \sum_{k=h}^{h+j-1} e^{-\sum_{i=0}^{k-1} \mu^*_{x,\,t}(i)} = \sum_{k=h}^{h+j-1} {}_kp^*_{x,\,t}, \end{aligned} $$

where ${}_kp^*_{x,\,t} = \prod_{i=0}^{k-1} p^*_{x+i,\,t+i} = e^{-\sum_{i=0}^{k-1} \mu^*_{x,\,t}(i)}$ for $k = 0, 1, \cdots$ with ${}_0p^*_{x,\,t} = e^{-\sum_{i=0}^{-1} \mu^*_{x,\,t}(i)} = 1$, $p^*_{x+i,\,t+i} = e^{-\mu^*_{x,\,t}(i)}$, and $\mu^*_{x,\,t}(i) = \mu_{x,\,t}(i) + \delta_i$ for $i = 0, 1, \cdots$. Note that we use piecewise level forces of mortality and interest to calculate ${}_{h|}\ddot{a}_{x,\,t:\bar{j}|}$. Since ${}_{h|}\ddot{a}_{x,\,t:\bar{j}|}$ is a function of $\mu^*_{x,\,t}$, we append $(\mu^*_{x,\,t})$ to ${}_{h|}\ddot{a}_{x,\,t:\bar{j}|}$ and denote it as ${}_{h|}\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})$ hereafter.

Motivated by a linear relationship between two sequences of mortality rates (see Tsai and Yang (2015)), which has been applied to modeling mortality rates (see Tsai and Yang (2015)) and natural hedges with mortality immunization (see Tsai and Chung (2013), and Lin and Tsai (2013, 2014)), we firstly conjecture that the force of mortality along with the force of interest could move approximately linearly, that is, $\mu^*_{x,\,t}(i)$ could move approximately to $(1+\alpha)\,\mu^*_{x,\,t}(i) + \beta$ for $i = 0, 1, \cdots$. Figure 1 gives the plots of one simulated path (regarded as the realized path) $\tilde{\mu}^*_{x,\,t}(i)$ against the forecasted one $\hat{\mu}^*_{x,\,t}(i)$ for the next $(100-x)$ years, starting 2016 for ages $x = 40$, 50, and 60, respectively, that is, $\tilde{\mu}^*_{x,\,2016}(i)$ against $\hat{\mu}^*_{x,\,2016}(i)$, $i = 0, \cdots, (99-x)$, for $x = 40, 50, 60$ (please refer to the fourth section for the details of simulating and forecasting $\mu_{x,\,t}(i)$ and δ_i sequences). We find that there is an obvious linear relationship between the simulated and forecasted paths of $\mu^*_{x,\,2016}(i)$ for ages 40, 50, and

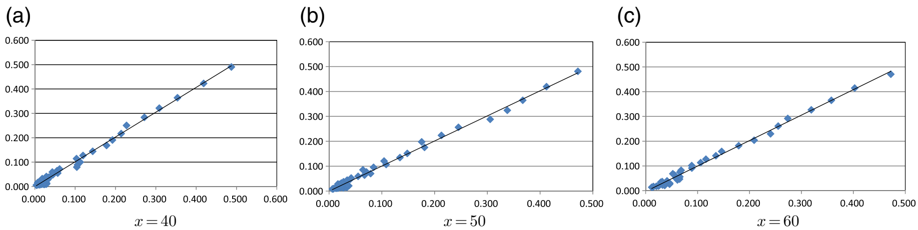

FIGURE 1: Sample paths of $\tilde{\mu}^*_{x,t}(i)$ against $\hat{\mu}^*_{x,t}(i)$, $t = 2016$ and $i = 0, 1, \ldots, (99 - x)$.

60. Since the force of mortality $\mu_{x,2016}(i)$ at older ages becomes much larger than the force of interest δ_i (i.e., $\mu_{x,2016}(i)$ dominates δ_i at lager values of $x + i$) and there is a linear relationship between the simulated and forecasted paths of $\mu_{x,2016}(i)$ with evidence support from empirical data, $\mu^*_{x,t}(i)$ (an addition of δ_i to $\mu_{x,t}(i)$) still moves linearly over time. Therefore, we think that matching mortality-interest durations or convexities with respect to an instantaneously proportional change or an instantaneously constant movement in $\mu^*_{x,t}(i)$ is appropriate for natural hedge of WL and whole life annuity products.

When $\mu^*_{x,t}(i)$ has a proportional change or a constant movement of size γ to $(1+\gamma)\,\mu^*_{x,t}(i)$ or $\mu^*_{x,t}(i) + \gamma$, the change in the ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ and the corresponding mortality-interest duration and convexity of ${}_h|\ddot{a}_{x,t:\overline{j}|}$ with respect to an instantaneously proportional change or an instantaneously parallel movement in $\mu^*_{x,t}$ are given in Proposition 1 (see Appendix A for the proof).

Proposition 1. *The change in* ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$*, in response to a proportional change* $(\lambda = p)$ *or a constant movement* $(\lambda = c)$ *of size* γ *in* $\mu^*_{x,t}$*, is given by*

$$\Delta^{\lambda}\, {}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}) = D^{\lambda}[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] \cdot \gamma + C^{\lambda}[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] \cdot \frac{\gamma^2}{2} + \cdots .$$

(a) *The mortality-interest duration* (D^p) *and convexity* (C^p) *of* ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ *with respect to an instantaneously proportional change in* $\mu^*_{x,t}$ *are given by*

$$D^p[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] = \sum_{k=h}^{h+j-1} [\ln({}_kp^*_{x,t})] \cdot {}_kp^*_{x,t}$$

and

$$C^p[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] = \sum_{k=h}^{h+j-1} [\ln({}_kp^*_{x,t})]^2 \cdot {}_kp^*_{x,t}.$$

(b) *The mortality-interest duration* (D^c) *and convexity* (C^c) *of* ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ *with respect to an instantaneously constant movement in* $\mu^*_{x,t}$ *are given by*

$$D^c[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] = \sum_{k=h}^{h+j-1} (-k) \cdot {}_kp^*_{x,t} \text{ and } C^c[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] = \sum_{k=h}^{h+j-1} k^2 \cdot {}_kp^*_{x,t}.$$

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Similarly, the corresponding mortality duration and convexity of ${}_{h|}\ddot{a}_{x,\,t:\overline{j|}}$ with respect to an instantaneously proportional shift or an instantaneously parallel movement in $\mu_{x,\,t}$ only (excluding δ) is given in Corollary 1 (the proof is similar to that of Proposition 1).

Corollary 1. *The change in ${}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})$, in response to a proportional change ($\lambda = p$) or a constant movement ($\lambda = c$) of size γ in $\mu_{x,\,t}$, is given by*

$$\Delta^{\lambda}\,{}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t}) = D^{\lambda}[{}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})]\cdot\gamma + C^{\lambda}[{}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})]\cdot\frac{\gamma^2}{2} + \cdots .$$

(a) *The mortality duration (D^p) and convexity (C^p) of ${}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})$ with respect to an instantaneously proportional change in $\mu_{x,\,t}$ are given as*

$$D^{p}[{}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})] = \sum_{k=h}^{h+j-1}[\ln({}_{k}p_{x,\,t})]\cdot {}_{k}p_{x,\,t}\cdot e^{-\sum_{i=0}^{k-1}\delta_i},$$

and

$$C^{p}[{}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})] = \sum_{k=h}^{h+j-1}[\ln({}_{k}p_{x,\,t})]^2\cdot {}_{k}p_{x,\,t}\cdot e^{-\sum_{i=0}^{k-1}\delta_i}.$$

(b) *The mortality duration (D^c) and convexity (C^c) of ${}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})$ with respect to an instantaneously constant movement in $\mu_{x,\,t}$ are given by*

$$D^{c}[{}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})] = \sum_{k=h}^{h+j-1}(-k)\cdot {}_{k}p_{x,\,t}\cdot e^{-\sum_{i=0}^{k-1}\delta_i},$$

and

$$C^{c}[{}_{h|}\ddot{a}_{x,\,t:\overline{j|}}(\mu_{x,\,t})] = \sum_{k=h}^{h+j-1}k^2\cdot {}_{k}p_{x,\,t}\cdot e^{-\sum_{i=0}^{k-1}\delta_i}.$$

Note that ${}_{h|}\ddot{a}_{x,\,t:\overline{j|}}$ turns out to ${}_{n}E_{x,\,t}$ (the NSP of one-unit n-year pure endowment with $(h, j) = (n, 1)$), $\ddot{a}_{x,t:\overline{n|}}$ (the NSP of one-unit n-year temporary life annuity-due with $(h, j) = (0, n)$), $\ddot{a}_{x,\,t}$ (the NSP of one-unit whole life annuity-due with $(h, j) = (0, \infty)$), or ${}_{n|}\ddot{a}_{x,\,t}$ (the NSP of one-unit n-year DWA-due with $(h, j) = (n, \infty)$). Therefore, the mortality(-interest) duration and convexity for typical annuity products can be obtained by Proposition 1 and Corollary 1.

Proposition 2 provides the mortality-interest duration and convexity for life insurance products, with the proof given in Appendix B.

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Proposition 2. *The mortality-interest duration and convexity of* $A_{x,t}(\mu'_{x,t},\ \mu^*_{x,t})$ *with respect to an instantaneously proportional change or an instantaneously constant movement in both* $\mu'_{x,t}$ *and* $\mu^*_{x,t}$ *are given by*

$$B^{\lambda}[A_{x,t}(\mu'_{x,t},\ \mu^*_{x,t})] = e^{-\delta_0} \cdot B^{\lambda}[\ddot{a}_{x,t}(\mu'_{x,t})] - B^{\lambda}[\ddot{a}_{x,t}(\mu^*_{x,t})], \tag{2.2}$$

where $B = D$ *(duration),* C *(convexity);* $\lambda = p$ *(proportional),* c *(constant);* $B^{\lambda}[\ddot{a}_{x,t}(\mu^*_{x,t})]$ *are given in Proposition 1 (a) and (b) with* $(h,\ j) = (0,\ \infty)$*;*

(a) $D^p[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} [\ln({}_kp'_{x,t})] \cdot {}_kp'_{x,t}$;
(b) $C^p[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} [\ln({}_kp'_{x,t})]^2 \cdot {}_kp'_{x,t}$;
(c) $D^c[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} (-k) \cdot {}_kp'_{x,t}$; *and*
(d) $C^c[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} k^2 \cdot {}_kp'_{x,t}$;

${}_kp'_{x,t} = \prod_{i=0}^{k-1} p'_{x+i,\,t+1} = e^{-\sum_{i=0}^{k-1} \mu'_{x,t}(i)}$ *for* $k = 0, 1, \cdots$ *with* ${}_0p'_{x,t} = e^{-\sum_{i=0}^{-1} \mu'_{x,t}(i)} = 1$; $\mu'_{x,t}(i) = \mu_{x,t}(i) + \delta_{i+1}$; *and* $p'_{x+i,\,t+i} = e^{-\mu'_{x,t}(i)}$ *for* $i = 0,\ 1, \cdots, k-1$.

Similar to (2.2), we can derive $B^{\lambda}[A^1_{x,t:\overline{n}|}(\mu'_{x,t},\ \mu^*_{x,t})] = e^{-\delta_0} \cdot B^{\lambda}[\ddot{a}_{x,t:\overline{n}|}(\mu'_{x,t})] - B^{\lambda}[\ddot{a}_{x,t:\overline{n+1}|}(\mu^*_{x,t})]$.

Corollary 2 gives the mortality duration and convexity of $A_{x,t}(\mu_{x,t})$ with respect to an instantaneously proportional shift or an instantaneously parallel movement in $\mu_{x,t}$ only (excluding δ), with the proof given in Appendix C.

Corollary 2.

(a) $D^p[A_{x,t}(\mu_{x,t})] = \sum_{k=0}^{\infty} [\ln({}_kp_{x,t})] \cdot {}_kp_{x,t} \cdot e^{-\sum_{i=0}^{k} \delta_i} - D^p[\ddot{a}_{x,t}(\mu_{x,t})]$;
(b) $C^p[A_{x,t}(\mu_{x,t})] = \sum_{k=0}^{\infty} [\ln({}_kp_{x,t})]^2 \cdot {}_kp_{x,t} \cdot e^{-\sum_{i=0}^{k} \delta_i} - C^p[\ddot{a}_{x,t}(\mu_{x,t})]$;
(c) $D^c[A_{x,t}(\mu_{x,t})] = \sum_{k=0}^{\infty} (-k) \cdot {}_kp_{x,t} \cdot e^{-\sum_{i=0}^{k} \delta_i} - D^c[\ddot{a}_{x,t}(\mu_{x,t})]$; *and*
(d) $C^c[A_{x,t}(\mu_{x,t})] = \sum_{k=0}^{\infty} k^2 \cdot {}_kp_{x,t} \cdot e^{-\sum_{i=0}^{k} \delta_i} - C^c[\ddot{a}_{x,t}(\mu_{x,t})]$.

The following corollary provides the relationships between the mortality duration (convexity) and mortality-interest duration (convexity) with respect to an instantaneously parallel movement, with the proof given in Appendix D.

Corollary 3.

(a) $B^c[{}_h|\ddot{a}_{x,t:\bar{j}|}(\mu_{x,t})] = B^c[{}_h|\ddot{a}_{x,t:\bar{j}|}(\mu^*_{x,t})]$,
(b) $B^c[A_{x,t}(\mu_{x,t})] = B^c[A_{x,t}(\mu'_{x,t},\ \mu^*_{x,t})]$, *where* $B = D, C$.

To sum up, the first contribution of this paper to the literature is that the linear assumption of μ^* leads to the closed-form formulas for mortality-interest durations and convexities of the NSPs of life insurance and annuity products

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with respect to an instantaneously proportional change and an instantaneously parallel movement, respectively, in μ^* and μ'; see Propositions 1 and 2.

3. Natural Hedges With Duration And Convexity Matching Strategies

In this section, we study mortality-interest immunization strategies for two portfolios of life insurance and annuity products. The first portfolio P_1^{WA} consists of a one-unit discrete m-payment WL policy and a one-unit n-payment and n-year DWA-due policy issued to policyholders aged x_l and x_a in year t with the net level premiums, $P^{WL}_{x_l,t:m}$ $(= A_{x_l,t}/\ddot{a}_{x_l,t:\overline{m}|})$ and $P^{DA}_{x_a,t:n}$ $(= {}_{n}|\ddot{a}_{x_a,t}/\ddot{a}_{x_a,t:\overline{n}|})$, and the surplus (negative reserve) at time zero,

$$ {}_0S^{WL}_{x_l,t:m}(\mu'_{x_l,t},\ \mu^*_{x_l,t}) = P^{WL}_{x_l,t:m}\cdot \ddot{a}_{x_l,t:\overline{m}|}(\mu^*_{x_l,t}) - A_{x_l,t}(\mu'_{x_l,t},\mu^*_{x_l,t}) = 0 $$

and

$$ {}_0S^{DA}_{x_a,t:n}(\mu^*_{x_a,t}) = P^{DA}_{x_a,t:n}\cdot \ddot{a}_{x_a,t:\overline{n}|}(\mu^*_{x_a,t}) - {}_{n}|\ddot{a}_{x_a,t}(\mu^*_{x_a,t}) = 0, $$

respectively. Note that $(\mu^*_{x,t})$ (and $(\mu'_{x,t},\ \mu^*_{x,t})$) is appended to $\ddot{a}_{x,t:\overline{m}|}$, $A_{x,t}$, $\ddot{a}_{x,t:\overline{n}|}$, ${}_{n}|\ddot{a}_{x,t}$, ${}_0S^{WL}_{x,t:m}$ and ${}_0S^{DA}_{x,t:n}$ to indicate that these quantities will fluctuate if there is a change in $\mu^*_{x,t}$ (and $\mu'_{x,t}$), whereas $P^{WL}_{x,t:m}$ and $P^{DA}_{x,t:n}$ do not involve $(\mu^*_{x,t})$ or $(\mu'_{x,t},\mu^*_{x,t})$ since the premiums are predetermined and cannot be changed once the policies are issued. Moreover, we use the projected/expected $\hat{\mu}^*_{x,t}$ (and $\hat{\mu}'_{x,t}$) to calculate the premiums and use the simulated/realized $\tilde{\mu}^*_{x,t}$ (and $\tilde{\mu}'_{x,t}$) to calculate the surpluses.

The second portfolio P_2^{WA} is a mixture of a life portfolio and an annuity portfolio. The life portfolio consists of one-unit discrete m-payment WL policies issued to policyholders for a set of ages $\underline{x}_l$ in year t, and the annuity portfolio is composed of one-unit n-payment and n-year DWA-due policies issued to policyholders for a set of ages $\underline{x}_a$ in year t. Specifically, the surpluses of the life and annuity portfolios at time zero are

$$ {}_0S^{WL}_{\underline{x}_l,t:m}(\mu'_{\underline{x}_l,t},\ \mu^*_{\underline{x}_l,t}) = \sum_{x\in\underline{x}_l} p^{WL}_x \cdot {}_0S^{WL}_{x,t:m}(\mu'_{x,t},\ \mu^*_{x,t}) = 0, $$

and

$$ {}_0S^{DA}_{\underline{x}_a,t:n}(\mu^*_{\underline{x}_a,t}) = \sum_{x\in\underline{x}_a} p^{DA}_x \cdot {}_0S^{DA}_{x,t:n}(\mu^*_{x,t}) = 0, $$

where p^{WL}_x and p^{DA}_x are the percentages or numbers of life and annuity policyholders aged x, respectively.

The corresponding weights of life insurance and annuity policies of each portfolio are denoted by w_{WL} and w_{DA} $(= 1 - w_{WL})$, respectively. The following theorem, with the proof given in Appendix E, gives the weight of the

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life insurance of the portfolio P_i^{WA}, which is determined by $B^\lambda(\mu', \mu^*)$ and called the mortality-interest duration ($B = D$) or convexity ($B = C$) matching strategy with respect to an instantaneously proportional change ($\lambda = p$) or an instantaneously constant movement ($\lambda = c$) in each of $\mu'_{x,t}$ and $\mu^*_{x,t}$.

Theorem 1. *Adopting the mortality-interest duration* ($B = D$) *or convexity* ($B = C$) *matching strategy, the weight of life insurance for the portfolio* P_i^{WA} *is given by*

(a) *for* $i = 1$,

$$w_{WL}\left[B^\lambda(\mu', \mu^*)\right] = \frac{B^\lambda\left[{}_0S^{DA}_{x_a,t:n}(\mu^*_{x_a,t})\right]}{B^\lambda\left[{}_0S^{DA}_{x_a,t:n}(\mu^*_{x_a,t})\right] - B^\lambda\left[{}_0S^{WL}_{x_l,t:m}(\mu'_{x_l,t}, \mu^*_{x_l,t})\right]}, \quad (3.1)$$

where

$$B^\lambda\left[{}_0S^{WL}_{x_l,t:m}(\mu'_{x_l,t}, \mu^*_{x_l,t})\right] = P^{WL}_{x_l,t:m} \cdot B^\lambda\left[\ddot{a}_{x_l,t:\overline{m}|}(\mu^*_{x_l,t})\right] - B^\lambda\left[A_{x_l,t}(\mu'_{x_l,t}, \mu^*_{x_l,t})\right],$$

and

$$B^\lambda\left[{}_0S^{DA}_{x_a,t:n}(\mu^*_{x_a,t})\right] = P^{DA}_{x_a,t:n} \cdot B^\lambda\left[\ddot{a}_{x_a,t:\overline{n}|}(\mu^*_{x_a,t})\right] - B^\lambda\left[{}_{n|}\ddot{a}_{x_a,t}(\mu^*_{x_a,t})\right];$$

(b) *for* $i = 2$,

$$w_{WL}\left[B^\lambda(\mu', \mu^*)\right] = \frac{B^\lambda\left[{}_0S^{DA}_{\underline{x}_a,t:n}(\mu^*_{\underline{x}_a,t})\right]}{B^\lambda\left[{}_0S^{DA}_{\underline{x}_a,t:n}(\mu^*_{\underline{x}_a,t})\right] - B^\lambda\left[{}_0S^{WL}_{\underline{x}_l,t:m}(\mu'_{\underline{x}_l,t}, \mu^*_{\underline{x}_l,t})\right]}, \quad (3.2)$$

where

$$B^\lambda\left[{}_0S^{WL}_{\underline{x}_l,t:m}(\mu'_{\underline{x}_l,t}, \mu^*_{\underline{x}_l,t})\right] = \sum_{x \in \underline{x}_l} p^{WL}_x \cdot B^\lambda\left[{}_0S^{WL}_{x,t:m}(\mu'_{x,t}, \mu^*_{x,t})\right],$$

and

$$B^\lambda\left[{}_0S^{DA}_{\underline{x}_a,t:n}(\mu^*_{\underline{x}_a,t})\right] = \sum_{x \in \underline{x}_a} p^{DA}_x \cdot B^\lambda\left[{}_0S^{DA}_{x,t:n}(\mu^*_{x,t})\right].$$

Similarly, Corollary 4 provides the weight of the life insurance of the portfolio P_i^{WA}, which is determined by $B^\lambda(\mu)$ and called the mortality duration ($B = D$) or convexity ($B = C$) matching strategy with respect to an instantaneously proportional change ($\lambda = p$) or an instantaneously constant movement ($\lambda = c$) in $\mu_{x,t}$ only. The proof of Corollary 4 is similar to that in Theorem 1.

Corollary 4. *Adopting the mortality duration* ($B = D$) *or convexity* ($B = C$) *matching strategy, the weight of life insurance for the portfolio* P_i^{WA} *is given by*

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(a) *for* $i = 1$,

$$w_{WL}\left[B^{\lambda}(\mu)\right] = \frac{B^{\lambda}\left[{}_0S^{DA}_{x_a,\, t:\, n}(\mu_{x_a,\, t})\right]}{B^{\lambda}\left[{}_0S^{DA}_{x_a,\, t:\, n}(\mu_{x_a,\, t})\right] - B^{\lambda}\left[{}_0S^{WL}_{x_l,\, t:\, m}(\mu_{x_l,\, t})\right]}, \tag{3.3}$$

where

$$B^{\lambda}\left[{}_0S^{WL}_{x_l,\, t:\, m}(\mu_{x_l,\, t})\right] = P^{WL}_{x_l,\, t:\, m} \cdot B^{\lambda}\left[\ddot{a}_{x_l,\, t:\, \overline{m}|}(\mu_{x_l,\, t})\right] - B^{\lambda}\left[A_{x_l,\, t}(\mu_{x_l,\, t})\right],$$

and

$$B^{\lambda}\left[{}_0S^{DA}_{x_a,\, t:\, n}(\mu_{x_a,\, t})\right] = P^{DA}_{x_a,\, t:\, n} \cdot B^{\lambda}\left[\ddot{a}_{x_a,\, t:\, \overline{n}|}(\mu_{x_a,\, t})\right] - B^{\lambda}\left[{}_{n|}\ddot{a}_{x_a,\, t}(\mu_{x_a,\, t})\right];$$

(b) *for* $i = 2$,

$$w_{WL}\left[B^{\lambda}(\mu)\right] = \frac{B^{\lambda}\left[{}_0S^{DA}_{\underline{x}_a,\, t:\, n}(\mu_{\underline{x}_a,\, t})\right]}{B^{\lambda}\left[{}_0S^{DA}_{\underline{x}_a,\, t:\, n}(\mu_{\underline{x}_a,\, t})\right] - B^{\lambda}\left[{}_0S^{WL}_{\underline{x}_l,\, t:\, m}(\mu_{\underline{x}_l,\, t})\right]}, \tag{3.4}$$

where

$$B^{\lambda}\left[{}_0S^{WL}_{\underline{x}_l,\, t:\, m}(\mu_{\underline{x}_l,\, t})\right] = \sum_{x \in \underline{x}_l} p^{WL}_x \cdot B^{\lambda}\left[{}_0S^{WL}_{x,\, t:\, m}(\mu_{x,\, t})\right],$$

and

$$B^{\lambda}\left[{}_0S^{DA}_{\underline{x}_a,\, t:\, n}(\mu_{\underline{x}_a,\, t})\right] = \sum_{x \in \underline{x}_a} p^{DA}_x \cdot B^{\lambda}\left[{}_0S^{DA}_{x,\, t:\, n}(\mu_{x,\, t})\right].$$

The second contribution of this paper is that adopting the proposed mortality-interest duration or convexity matching strategy, we can determine the unique solution to a single unknown weight of life insurance for each of the two portfolios above to achieve a better natural hedge when considering mortality, longevity, and interest rate risks together, and the unique solution to a single unknown weight under most cases is feasible. In Luciano *et al.* (2017), there are at least two unknown quantities of life insurance products or zero coupon bonds solved by at least two equations when considering longevity and interest rate risks together; moreover, one could obtain multiple solutions to these unknown quantities. Most of the solutions could be negative, which is not easy to implement in product allocations for the life insurance industry since short selling is impractical in life insurance products. One can apply Theorem 1(a) to a portfolio where there are a single life insurance policy and a single annuity policy for two issue ages x_l and x_a. One can also apply Theorem 1(b) to a mixed portfolio where there are two identical multi-policy life insurance and annuity products for two sets of issue ages, $\underline{x}_l$ and $\underline{x}_a$. When the life insurance and annuity products are replaced with other identical or mixed kinds of life insurance and annuity products, Theorem 1 still applies.

Similarly, considering longevity and mortality risks but ignoring interest rate risk, one can apply Corollary 4(a) to a portfolio of a single life insurance policy and a single annuity policy for two issue ages x_l and x_a and apply Corollary 4(b) to a mixed portfolio of multiple identical life insurance policies and multiple identical annuity products for two issue age sets $\underline{x}_l$ and $\underline{x}_a$. Finally, the weights based on the mortality and mortality-interest duration/convexity matching strategies with respect to an instantaneously constant movement are equal. That is, for Theorem 1 and Corollary 4,

$$w_{WL}\left[B^c(\mu)\right] = w_{WL}\left[B^c(\mu', \mu^*)\right] \tag{3.5}$$

for $B = D, C$. For the case of portfolio P_1^{WA}, we have, by Corollary 3,

$$\begin{aligned} B^c\left[{}_0S^{WL}_{x_l,t:m}(\mu_{x_l,t})\right] &= P^{WL}_{x_l,t:m}\cdot B^c\left[\ddot{a}_{x_l,t:\overline{m}|}(\mu^*_{x_l,t})\right] - B^c\left[A_{x_l,t}(\mu'_{x_l,t}, \mu^*_{x_l,t})\right] \\ &= B^c\left[{}_0S^{WL}_{x_l,t:m}(\mu'_{x_l,t}, \mu^*_{x_l,t})\right] \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} B^c\left[{}_0S^{DA}_{x_a,t:n}(\mu_{x_a,t})\right] &= P^{DA}_{x_a,t:n}\cdot B^c\left[\ddot{a}_{x_a,t:\overline{n}|}(\mu^*_{x_a,t})\right] - B^c\left[{}_n|\ddot{a}_{x_a,t}(\mu^*_{x_a,t})\right] \\ &= B^c\left[{}_0S^{DA}_{x_a,t:n}(\mu^*_{x_a,t})\right] \end{aligned} \tag{3.7}$$

which imply that (3.5) holds; similarly, (3.5) applies to the case of portfolio P_2^{WA}.

4. Numerical Illustration and Analysis

In this section, we first compute mortality(-interest) durations and convexities for the WL and the DWA products, respectively, with the forecasted mortality rates under the Lee and Carter (1992) mortality model and the forecasted interest rates under the CIR model. Then, we obtain the weights for a portfolio of two single-policy or multipolicy life and annuity products and evaluate their performances of hedging mortality, longevity, and interest rate risks with simulations.

The mortality data for both genders of the U.S., 56 yearly observations of the age-specific death numbers and the corresponding population sizes exposed to the risk of death from 1960 to 2015, are taken from the Human Mortality Database (www.mortality.org). We estimate the parameters of the Lee–Carter model with the data for the age–year window [25, 99] × [1960, 2015] and forecast the mortality rates for [25, 99] × [2016, 2090]. We also set $q_{100,t} = 1$. To consider mortality improvement and deterioration, we take cohort mortality sequences and forecast ${}_kp_{x,2016}$ (the k-year survival probability for age x in year 2016) for the calculations of durations and convexities by ${}_k\hat{p}_{x,2016} = \prod_{i=0}^{k-1}\hat{p}_{x+i,2016+i}$, where $\hat{p}$ denotes the predicted one-year survival probability.

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The well-known CIR (1985) model assumes that the interest rate i_t at time t follows the stochastic process as

$$d\, i_t = a \cdot (b - i_t)\, d\, t + c \cdot \sqrt{i_t}\, dz, \tag{4.1}$$

where a, b, and c are constants and dz follows a standard Brownian motion. Then, we calibrate the parameters by using the yearly interest rates of one-year U.S. Treasury securities over the period 1976–2015 available from the website (http://federalreserve.gov/default.htm) of Board of Governors of the Federal Reserve System. The parameters are estimated as $\hat{a} = 0.0332$, $\hat{b} = 0.0117$, and $\hat{c} = 0.0633$. Denote $\hat{i}_t$ the forecasted one-year interest rate for the interval $[t,\ t+1]$ with (4.1), and $\hat{B}_0(T) = \prod_{t=0}^{T-1} [1/(1+\hat{i}_t)]$ the forecasted present value of \$1 payable at time T, which is used to discount the cash flows at time T.

4.1. Product matching

In this subsection, firstly, we compute the mortality(-interest) durations and convexities of the surpluses at time zero for the one-unit discrete 20-payment WL issued to age x and for the one-unit (65–x)-payment (65–x)-year DWA-due also issued to the same age x, with the forecasted survival probabilities $({}_k\hat{p}_{x,2016})$ and the predicted interest rates $(\hat{i}_t)$. People who purchase annuities may tend to live longer than average, a typical "adverse selection" problem (see, e.g., Warshawsky, 1988; Mitchell *et al.*, 1999; Finkelstein and Poterba, 2002); in practice, an annuitant population actually displays a longer life expectancy than a general population. Since we do not have two different and recent mortality tables for life insurance and life annuity, we assume a male (female) mortality table for life insurance (life annuity) for numerical illustrations.

Table 1 displays the durations and convexities of the surpluses at time zero for the one-unit discrete 20-payment WL and the one-unit (65–x)-payment (65–x)-year DWA-due based on the $B^{\lambda}(\mu)$ and $B^{\lambda}(\mu',\ \mu^*)$ matching strategies and the corresponding weights of WL for age $x = 40$, 50, and 60. Observations are summarized as follows.

1. The absolute values of the durations and convexities with respect to an instantaneously constant movement are much larger than those with respect to an instantaneously proportional shift, that is, $|B^c[{}_0S^{WL}_{x,t:m}]| > |B^p[{}_0S^{WL}_{x,t:m}]|$ and $|B^c[{}_0S^{DA}_{x,t:n}]| > |B^p[{}_0S^{DA}_{x,t:n}]|$, implying that the surpluses at time zero for WL and DWA with respect to an instantaneously constant movement in mortality curve or mortality-interest curve are much more sensitive than those with respect to an instantaneously proportional shift.
2. The weight of WL, w_{WL}, determined by (3.1) or (3.3), implies that $w_{WL} \in (0,1)$ if and only if the signs of $B^{\lambda}[{}_0S^{WL}_{x,t:m}]$ and $B^{\lambda}[{}_0S^{DA}_{x,t:n}]$ are opposite. Observing from Table 1, only one pair, $C^p[{}_0S^{WL}_{x,t:m}(\mu'_{x,t},\ \mu^*_{x,t})]$ and

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TABLE 1

DURATIONS AND CONVEXITIES OF THE SURPLUSES AND WEIGHTS OF WHOLE LIFE INSURANCE.

Age x	$D^p(\mu)$	$D^c(\mu)$ $D^c(\mu', \mu^*)$	$C^p(\mu)$	$C^c(\mu)$ $C^c(\mu', \mu^*)$	$D^p(\mu', \mu^*)$	$C^p(\mu', \mu^*)$
			Panel A: $B^\lambda[{}_0S^{WL}_{x,t:m}]$			
40	−0.0902	−13.3298	0.0570	302.3481	−0.0302	−0.0229
50	−0.1170	−11.1584	0.0647	195.9155	−0.0828	0.0034
60	−0.1766	−9.5080	0.0862	134.1131	−0.1675	0.0501
			Panel B: $B^\lambda[{}_0S^{DA}_{x,t:n}]$			
40	5.7498	359.0786	−5.5011	−17336.7479	8.5186	−10.3180
50	6.2719	298.0299	−6.2552	−10484.6346	8.3137	−9.7443
60	6.7201	235.4050	−6.7709	−5217.0401	8.0775	−9.0010
			Panel C: w_{WL}			
40	0.9841	0.9696	0.9877	0.9859	0.9966	1.0024
50	0.9812	0.9698	0.9879	0.9853	0.9905	0.9996
60	0.9738	0.9680	0.9854	0.9806	0.9801	0.9939

$C^p[{}_0S^{DA}_{x,t:n}(\mu^*_{x,t})]$ for age $x = 40$, has the same sign, which produces a weight w_{WL} larger than 1 as expected.

3. From (3.5), both $B^\lambda(\mu)$ and $B^\lambda(\mu', \mu^*)$ matching strategies yield the equal weight w_{WL} for $B = D, C$ and $\lambda = c$. For $\lambda = p$, Panel C of Table 1 shows that $B^p(\mu', \mu^*)$ leads to a heavier weight w_{WL} than $B^p(\mu)$, that is, when a matching strategy B^p is adopted to hedge mortality and longevity risks of a portfolio, a larger weight should be placed on the WL with the intention of hedging the additional interest rate risk. Moreover, among all of the mortality and mortality-interest duration and convexity matching strategies, $C^p(\mu', \mu^*)$, the mortality-interest convexity matching strategy with respect to an instantaneously proportional change produces the largest weight w_{WL}.

Next, we simulate 10,000 forecasted cohort mortality paths and interest rate paths, respectively. Each cohort mortality path is used to calculate the simulated k-year survival probability ${}_k\tilde{p}_{x,2016} = \prod_{i=0}^{k-1} \tilde{p}_{x+i,2016+i}$, where $\tilde{p}_{x+i,2016+i} = \exp(-\tilde{m}_{x+i,2016+i})$ and $\tilde{m}$ denotes the simulated central death rate under the stochastic Lee–Carter model. Denote $\tilde{i}_t$ the simulated one-year interest rate at time t with (4.1) and $\tilde{B}_0(T) = \prod_{t=0}^{T-1} [1/(1+\tilde{i}_t)]$ the simulated present value of \$1 payable at time T. In finance, α-VaR (value at risk) of the surplus is the negative α-percentile (the lower the α-VaR value, the smaller the unexpected loss amount). Then, we evaluate the hedging performance of these matching strategies by comparing the 5%-VaR values of the surpluses at time zero for a matched portfolio, shown in Table 2. We find that a heavier weight of WL results in a lower VaR value for each fixed age x. We infer that selling whole life annuity policies exposes life insurers to higher longevity risk and interest

TABLE 2

5% VaR values of the weighted surpluses for a matched portfolio.

Age x	$D^p(\mu)$	$D^c(\mu)$ $D^c(\mu', \mu^*)$	$C^p(\mu)$	$C^c(\mu)$ $C^c(\mu', \mu^*)$	$D^p(\mu', \mu^*)$	$C^p(\mu', \mu^*)$
40	0.1559	0.1853	0.1486	0.1524	0.1306	0.1190
50	0.1177	0.1343	0.1079	0.1119	0.1040	0.0905
60	0.0822	0.0878	0.0708	0.0754	0.0758	0.0627

rate risk, so selling more WL policies can naturally hedge longevity and interest rate risks more efficiently. Consequently, we can conclude that matching the mortality-interest convexity (duration) with respect to an instantaneously proportional change, which produces the heaviest (the second heaviest) weight of the WL, can lead to an optimal allocation of WL and annuity products and then reduce more efficiently the unexpected extreme losses from mortality, longevity, and interest rate risks.

To further analyze the hedge efficiency, we define HE in terms of variance reduction ratio (see Li and Hardy, 2011; Li and Luo, 2012; Lin and Tsai (2014)) as follows:

$$HE(\sigma^2) = \frac{\sigma^2(S) - \sigma^2(S^*)}{\sigma^2(S)} = 1 - \frac{\sigma^2(S^*)}{\sigma^2(S)},$$

where S and S^* represent the simulated surpluses for a single product (either WL or DWA in this paper) and the matched insurance portfolio, respectively. Since the natural hedge is a hedging vehicle against both mortality risk and longevity risk at the same time, the HE can measure the effectiveness of hedging mortality (longevity) risk along with interest rate risk when the single product is the WL (the DWA) product. Table 3 exhibits summary statistics as well as the HE values of the matched portfolio surpluses at time zero. Observations are summarized as follows.

1. All of the matching strategies result in positive means (gains) of the portfolio surpluses at time zero. Matching the mortality-interest convexity $C^p(\mu', \mu^*)$ with respect to an instantaneously proportional change contributes to the smallest standard deviation and the lowest positive mean.
2. All of the matching strategies achieve at least 55.38% for the probability of gain (positive surplus), but which strategy produces the highest probability of gain cannot be concluded.
3. All of the HE values for longevity risk are very close to one (at least 99.36%); thus, all of the matching strategies are very effective in hedging longevity risk of the DWA product. The mortality-interest convexity $C^p(\mu', \mu^*)$ matching strategy with respect to an instantaneously proportional change produces the highest HE value for longevity risk.

TABLE 3

SUMMARY STATISTICS OF THE WEIGHTED SURPLUSES FOR THE MATCHED PORTFOLIOS.

Age x	$D^p(\mu)$	$D^c(\mu)$ $D^c(\mu', \mu^*)$	$C^p(\mu)$	$C^c(\mu)$ $C^c(\mu', \mu^*)$	$D^p(\mu', \mu^*)$	$C^p(\mu', \mu^*)$
Panel A: Mean						
40	0.0938	0.1133	0.0890	0.0915	0.0771	0.0694
50	0.0806	0.0939	0.0729	0.0759	0.0699	0.0592
60	0.0626	0.0674	0.0529	0.0569	0.0573	0.0458
Panel B: Standard deviation						
40	0.2134	0.2573	0.2027	0.2081	0.1759	0.1587
50	0.1825	0.2114	0.1658	0.1723	0.1593	0.1366
60	0.1402	0.1503	0.1201	0.1283	0.1291	0.1055
Panel C: Probability of gain (%)						
40	57.57	57.50	57.60	57.60	57.86	57.88
50	56.41	56.46	56.46	56.49	56.35	56.33
60	56.38	56.65	55.95	56.19	56.23	55.38
Panel D: HE for mortality risk (%)						
40	−65.75	−141.02	−49.50	−57.65	−12.57	8.37
50	−80.97	−142.73	−49.46	−61.27	−37.89	−1.37
60	−115.95	−148.40	−58.49	−80.91	−83.26	−22.21
Panel E: HE for longevity risk (%)						
40	99.56	99.36	99.60	99.58	99.70	99.76
50	99.55	99.40	99.63	99.60	99.66	99.75
60	99.47	99.40	99.61	99.56	99.55	99.70

4. The HE values for mortality risk are all negative except for one strategy $C^p(\mu', \mu^*)$ and age 40 which has $w_{WL} > 1$, and the highest HEs for mortality risk come from $C^p(\mu', \mu^*)$. The reason of negative HE values for mortality risk is that the simulated distribution of the WL with a death benefit of \$1 is much narrower than that of the DWA with annual survival benefits of \$1 so that the weighted portfolio surplus has a wider distribution than the WL.

One may ask how to resolve the impossible phenomenon in the insurance industry that the weight w_{WL} equals 1.0024 (or the weight w_{DA} equals −0.0024) for age 40 with a 5%-VaR value of 0.1190 from the $C^p(\mu', \mu^*)$ matching strategy, which would imply a short selling of the DWA policy by 0.24% in the capital market. To make the weight equal to a feasible value such as $w_{WL} = 0.9966$ from the $D^p(\mu', \mu^*)$ matching strategy and keep the same 5%-VaR value of 0.1190 from the $C^p(\mu', \mu^*)$ matching strategy, we add a risk margin or loading rate to the net level premiums of the DWA and the DW.

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TABLE 4

ESTIMATES OF LOADING RATES AND MONEY'S WORTH VALUES OF ANNUITY.

$\hat{\theta}_l$	$\hat{\theta}_a$	Money's worth value of annuity $(1/(1+\hat{\theta}_a))$
0.00%	20.40%	0.8305
1.00%	7.40%	0.9311

More specifically, we add loading rates θ_l to the DW and θ_a to the DWA in the weighted surplus at time 0 as follows:

$$\Delta\,_0S^{WA}(\tilde{\mu}',\tilde{\mu}^*) = 0.9966\cdot\,[(1+\theta_l)P^{WL}_{x,t:m}\cdot\ddot{a}_{x,t:\overline{m}|}(\tilde{\mu}^*_{x,t}) - A_{x,t}(\tilde{\mu}'_{x,t},\tilde{\mu}^*_{x,t})] + 0.0034\cdot\,[(1+\theta_a)P^{DA}_{x,t:n}\cdot\ddot{a}_{x,t:\overline{n}|}(\tilde{\mu}^*_{x,t}) - \,_n|\ddot{a}_{x,t}(\tilde{\mu}^*_{x,t})].$$

Then, we obtain estimates of θ_l and θ_a, shown in Table 4, by setting 0.1190 for the 5%-VaR value of the simulated $\Delta\,_0S^{WA}(\tilde{\mu}',\tilde{\mu}^*)$. When $\hat{\theta}_l = 0$, then $\hat{\theta}_a =$ 20.40%; when $\hat{\theta}_l = 1\%$, then $\hat{\theta}_a = 7.40\%$. The opposite direction of the changes in $\hat{\theta}_l$ and $\hat{\theta}_a$ is reasonable since an insurer gets more risk premiums from the position of WL and then less risk premiums from the position of DWA to reach the same VaR value.

However, a question arises about whether a high loading rate of 20.40% for annuity policy is reasonable. In practice, we actually see that an insurer usually charges policyholders of annuity products a high loading rate since some studies (see, e.g., Warshawsky (1988); Mitchell *et al.* (1999); Finkelstein and Poterba, 2002; Einav *et al.*, 2010) have shown that buying annuities may cost as much as 20% of an annuitant's wealth by examining money's worth values – how much a person gets when he or she pays \$1 for the premium of annuity. In this case, the policyholder pays the premium $[(1+\hat{\theta}_a)\cdot P^{DA}_{x,t:n}\cdot\ddot{a}_{x,t:\overline{n}|}]$ in exchange for his/her annuity benefit of $_n|\ddot{a}_{x,t}$, which leads to the money's worth value of annuity $= \,_n|\ddot{a}_{x,t}/[(1+\hat{\theta}_a)\cdot P^{DA}_{x,t:n}\cdot\ddot{a}_{x,t:\overline{n}|}] = 1/(1+\hat{\theta}_a)$, reported in Table 4. We can see that the costs of purchasing annuity policies are 16.95% (= 1 − 0.8305) and 6.89% (= 1 − 0.9311) under $\hat{\theta}_a = 20.40\%$ and 7.40%, respectively, both of which fall in a reasonable range.

We plot the distributions of the simulated weighted surpluses at time zero for age 40 based on the mortality-interest duration and convexity matching strategies, respectively, with respect to an instantaneously proportional change along with the one with loading rates of $\hat{\theta}_l = 0$ and $\hat{\theta}_a = 20.40\%$ in Figure 2. We observe that the loading rate can reshape the left tail of the distribution for the $D^p(\mu',\mu^*)$ matching strategy or shift the distribution for $D^p(\mu',\mu^*)$ to the right to achieve a similar heaviness of the left tail for $C^p(\mu',\mu^*)$. We also plot in Figure 3(a) the distributions of the simulated surpluses at time zero based on $D^c(\mu)$ (or $D^c(\mu',\mu^*)$, the worst) and $D^p(\mu',\mu^*)$ with loading rates of $\hat{\theta}_l = 0$ and $\hat{\theta}_a = 20.40\%$ (the best) and those for two single products (WL and DWA), all for age 40. The distributions are almost skewed to the right.

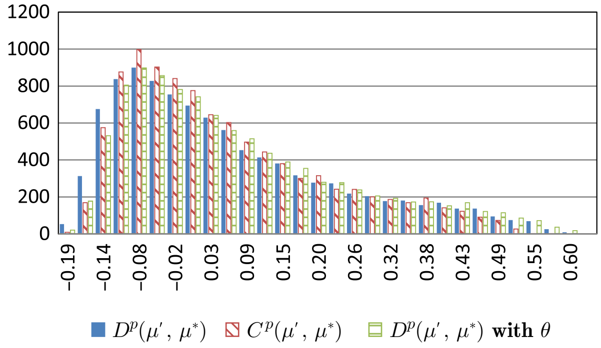

FIGURE 2: Distributions of the surpluses with and without the loading rates.

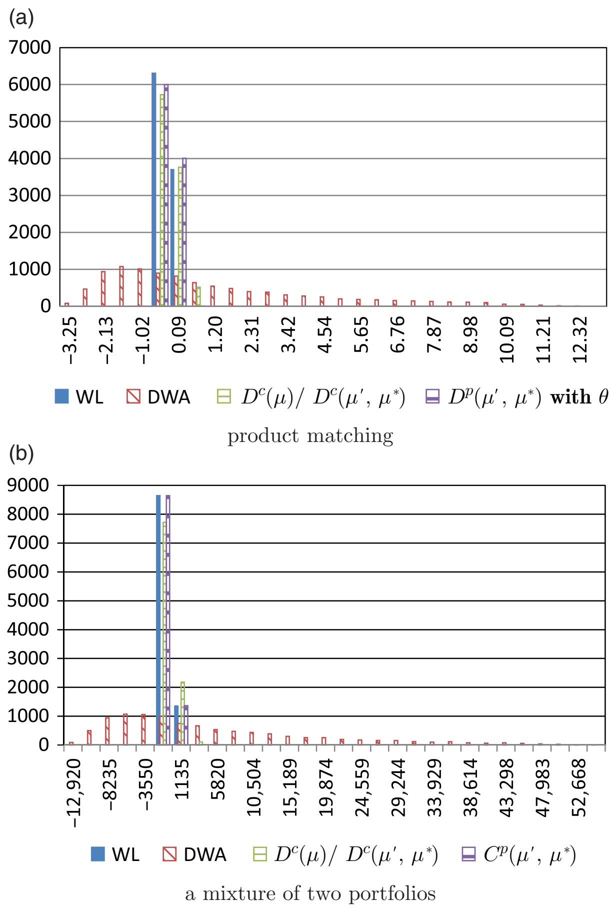

FIGURE 3: Distributions of the simulated surpluses for two single products and their weighted portfolios.

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TABLE 5

$p_{x,gender}$, PERCENTAGES (%) OF AGE, AND GENDER FOR TWO PORTFOLIOS.

Age x	35	40	45	50	55	60
WL (*Male*)	8.39	7.76	8.27	8.44	8.64	7.76
DWA (*Female*)	8.37	7.85	8.40	8.68	9.06	8.37

The distributions for the WL and annuity products spread to both positive and negative territories; the distribution for the latter is much wider than that for the former. The matching strategies can reduce the risk (variance) of the surplus for the single whole life annuity product and, more importantly, also remove the downside risk of loss (the probability of large loss) due to longevity risk from the single whole life annuity product. The distribution for $D^p(\mu', \mu^*)$ with loading is more centered around [−0.47, 0.65] than that for $D^c(\mu)$ (or $D^c(\mu', \mu^*)$), which implies more efficient performances of hedging longevity, mortality, and interest rate risks.

4.2. A mixture of two portfolios

In the preceding subsection, six mortality and mortality-interest duration and convexity matching strategies are applied to a portfolio of DW and DWA for a single and equal age. However, the one-age by one-age product matching is not practical in the real world. In this subsection, we show that the matching strategies based on the proposed mortality-interest duration or convexity can be applied to determine the weight for a mixed portfolio where there are two identical multipolicy life insurance and annuity products for a same set of issue ages.

We build two hypothetical portfolios and want to mix the two portfolios to achieve the best natural hedge: one is a life insurance portfolio consisting of the one-unit discrete 20-payment WL policies for six representative ages $x = 35, 40, 45, 50, 55$, and 60 (i.e., $\underline{x}_l = \{35, 40, 45, 50, 55, 60\}$) for simplicity; the other is an annuity portfolio including the one-unit discrete $(65-x)$-payment $(65-x)$-year DWA-due policies for the same six ages ($\underline{x}_a = \underline{x}_l$). We mimic the demographic structure of the U.S. to build the two portfolios. That is, we distribute totally 10,000 policyholders to the two portfolios with the percentages of policyholders for each age and each gender shown in Table 5. The life insurance portfolio and the annuity portfolio use the male and female percentages, respectively. The percentages for the six ages and each gender are computed by the ratios of the six population sizes for ages 33–37, 38–42, 43–47, 48–52, 53–57, and 58–62 of each gender to the total population size for ages 33–62 of both genders using the U.S. data in 2015 from the Human Mortality Database.

Then, six mortality and mortality-interest duration and convexity matching strategies can be adopted to calculate the weight of the whole life portfolio in a mixture of WL portfolio and annuity portfolio by applying (3.2) and (3.4). Table 6 presents the weights w_{WL} and summary statistics for two single-product portfolios and six matched portfolios. The observations that are similar to the case of one-age by one-age matching can be summarized again as follows.

1. The descending order of value in matching strategy, $C^p(\mu', \mu^*)$, $D^p(\mu', \mu^*)$, $C^p(\mu)$, $C^c(\mu)$ (or $C^c(\mu', \mu^*)$), $D^p(\mu)$, and $D^c(\mu)$ (or $D^c(\mu', \mu^*)$), applies to the w_{WL}, HE_M (HE for mortality risk), and HE_L (HE for longevity risk), and its reversed order applies to the mean, standard deviation, and 5%-VaR. The larger the weight w_{WL}, the bigger the HE_M and HE_L values, and the smaller the mean, standard deviation, and 5%-VaR. Among the six matching strategies, $C^p(\mu', \mu^*)$ ($D^p(\mu', \mu^*)$) produces the heaviest (the second heaviest) $\hat{w}_{WL}$ of 0.9999 (0.9914), the largest (the second largest) HE_M of −0.42% (−33.87%), and HE_L of 99.74% (99.65%) and also contributes to the lowest (the second lowest) mean of 301 (351), a standard deviation of 678 (783), and 5%-VaR value of 473 (540).
2. The means of the surpluses at time zero are all positive for two single-product portfolios and six matched portfolios. The larger the mean gain (reward), the higher the standard deviation (risk) − consistent with the rule of typical investment strategies. The largest mean gain (6209) from the single-product annuity portfolio also accompanies with the highest standard deviation (13,305).
3. The HE values for mortality risk are all negative. The reason of negative HE values for mortality risk is the same with that in the preceding subsection. The HE values for longevity risk are all very close to one (at least 99.38%). The mortality-interest duration and convexity matching strategies have better performances of hedging mortality risk and longevity risk (larger HE_M and HE_L values) than the mortality duration and convexity matching strategies.

Figure 3(b) displays the distributions of the simulated portfolio surpluses at time zero based on $C^p(\mu', \mu^*)$ (the best) and $D^c(\mu)$ (or $D^c(\mu', \mu^*)$, the worst) matching strategies along with those for 100% of the WL portfolio and 100% of the whole life annuity portfolio. The shapes of the distributions are similar to those in Figure 3(a) for one-age by one-age product matching. As consistent with Table 6, our matching strategies reduce the risk (variance) and the downside risk of loss (probability of large loss) due to longevity risk from the whole life annuity portfolio. Especially, the distribution for the $C^p(\mu', \mu^*)$ matching strategy is very centered around $[-1208, 3477]$; it has a much smaller variance than that for 100% of the whole life annuity portfolio and a similar sized variance to that for 100% of the WL portfolio. It implies that this matching strategy largely reduces longevity and interest rate risks of the whole life annuity portfolio but does not increase much mortality risk of the WL portfolio. As a result, we conclude that the proposed mortality-interest convexity matching

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TABLE 6

SUMMARY STATISTICS FOR SINGLE-PRODUCT PORTFOLIOS AND SIX MATCHED PORTFOLIOS.

	Single product		Six matched portfolios					
	WL	DWA	$D^c(\mu)$ $D^c(\mu', \mu^*)$	$D^p(\mu)$	$C^c(\mu)$ $C^c(\mu', \mu^*)$	$C^p(\mu)$	$D^p(\mu', \mu^*)$	$C^p(\mu', \mu^*)$
w_{WL}	1.0000	0.0000	0.9700 $<$	0.9812 $<$	0.9854 $<$	0.9876 $<$	0.9914 $<$	0.9999
HE_M(%)	–	–	−140.69 $<$	−80.82 $<$	−60.85 $<$	−50.56 $<$	−33.87 $<$	−0.42
HE_L(%)	–	–	99.38 $<$	99.53 $<$	99.58 $<$	99.61 $<$	99.65 $<$	99.74
Mean	301	6209	478 $>$	412 $>$	387 $>$	374 $>$	351 $>$	301
std	677	13, 305	1050 $>$	910 $>$	858 $>$	830 $>$	783 $>$	678
5%-VaR	472	8518	710 $>$	622 $>$	588 $>$	570 $>$	540 $>$	473
Pr(gain) (%)	57.40	58.10	57.79	57.70	57.74	57.76	57.57	57.41

strategy $C^p(\mu', \mu^*)$ has the most efficient performance of hedging longevity, mortality, and interest rate risks.

Note that $w_{WL}[B^\lambda(\mu', \mu^*)]$ in (3.2) and $w_{WL}[B^\lambda(\mu)]$ in (3.4) are based on \$1 sum assumed (SA) for life insurance and \$1 annual payment (AP) for all life annuities. If the sum assumed is SA for life insurance and the annual payment is AP for all life annuities, then the formulas for weights $w_{WL}[B^\lambda(\mu', \mu^*)]$ and $w_{WL}[B^\lambda(\mu)]$ are adjusted to

$$w_{WL}[B^\lambda(\mu', \mu^*)] = \frac{AP \cdot B^\lambda[{}_0S^{DA}_{\underline{x}_a,\, t:n}(\mu^*_{\underline{x}_a,\, t})]}{AP \cdot B^\lambda[{}_0S^{DA}_{\underline{x}_a,\, t:n}(\mu^*_{\underline{x}_a,\, t})] - SA \cdot B^\lambda[{}_0S^{WL}_{\underline{x}_l,\, t:m}(\mu'_{\underline{x}_l,\, t},\ \mu^*_{\underline{x}_l,\, t})]},$$

and

$$w_{WL}[B^\lambda(\mu)] = \frac{AP \cdot B^\lambda[{}_0S^{DA}_{\underline{x}_a,\, t:n}(\mu_{\underline{x}_a,\, t})]}{AP \cdot B^\lambda[{}_0S^{DA}_{\underline{x}_a,\, t:n}(\mu_{\underline{x}_a,\, t})] - SA \cdot B^\lambda[{}_0S^{WL}_{\underline{x}_l,\, t:m}(\mu_{\underline{x}_l,\, t})]}.$$

If $SA = 300,000$ and $AP = 30,000$, then the sequence of increasing weights 0.9700, 0.9812, 0.9854, 0.9876, 0.9914, and 0.9999 in Table 6 are modified to 0.7641, 0.8394, 0.8708, 0.8886, 0.9204, and 0.9989. The small difference 0.0299 between 0.9700 and 0.9999 is enlarged to a big gap 0.2348 between 0.7641 and 0.9989.

4.3. Robustness checking: A summary

To calibrate the robustness of the results above, in the subsection, we report the weights of WL for the same portfolio consisting of the one-unit discrete 20-payment WL policies and the one-unit discrete $(65-x)$-payment $(65-x)$-year DWA-due policies, with one-age by one-age matching as in Section 4.1 for a wide range of ages, 25–64, shown in Figure 4(a) based on the Lee–Carter model using the U.S. female data for annuity and the U.S. male data for life insurance, which is regarded as the base scenario when compared with the recalculated weights under six different scenarios displayed in Figure 4(b)–(g).

First, we consider the population basis risk which is the risk for the mismatch between the population of exposures and that of hedges. Coughlan *et al.* (2011) classify four types of population basis risks – gender basis risk, age basis risk, subpopulation basis risk, and country basis risk. As a result, we recalculate the weights using the durations and convexities of the surpluses for annuity products with the U.K. female data and life insurance ones with the U.S. male data in Figure 4(b); we also recalculate the weights of a two-product portfolio shown in Figure 4(c), where the WL products for ages 25–44 match, one-age by one-age, with the annuity ones for ages 45–64. Second, we adopt another mortality model, the Cairns–Blake–Dowd (2006) model, and redo the same process to obtain the weights in Figure 4(d). Third, we consider two different payment periods of WL by setting $m = 1$ and $m = 30$ in Figure 4(e) and (f), respectively.

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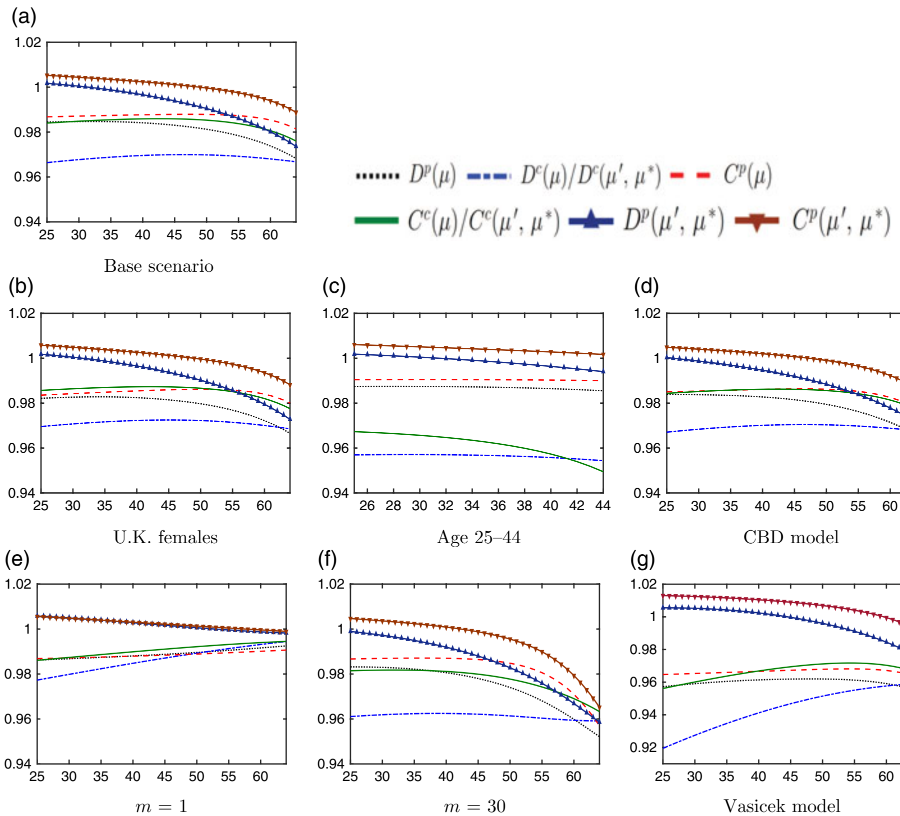

FIGURE 4: Weights w_{WL} under the base scenario and six different scenarios.

Fourth, we replace the CIR model with the Vasicek model to redo the same process to obtain the weights in Figure 4(g). Figure 5 plots the corresponding 5%-VaR values under the base scenario and six different scenarios. Note that the method of charging loading rates in Section 4.1 and a mixture of two portfolios in Section 4.2 can still be applied to these scenarios to resolve an infeasible weight and meet the practical need in the life insurance industry.

We confirm again that the mortality-interest duration $D^p(\mu', \mu^*)$ and convexity $C^p(\mu', \mu^*)$ matching strategies with respect to an instantaneously proportional change in $\mu'_{x,t}$ and $\mu^*_{x,t}$ lead to heavier weights of life insurance products and then lower unexpected extreme losses and can hedge longevity and interest rate risks of long-term annuity products more efficiently. Especially, the mortality-interest convexity matching strategy $C^p(\mu', \mu^*)$ contributes to the heaviest weight of life insurance product and the lowest VaR value, so it can provide the most efficient performance in natural hedges for a portfolio of WL and annuity products.

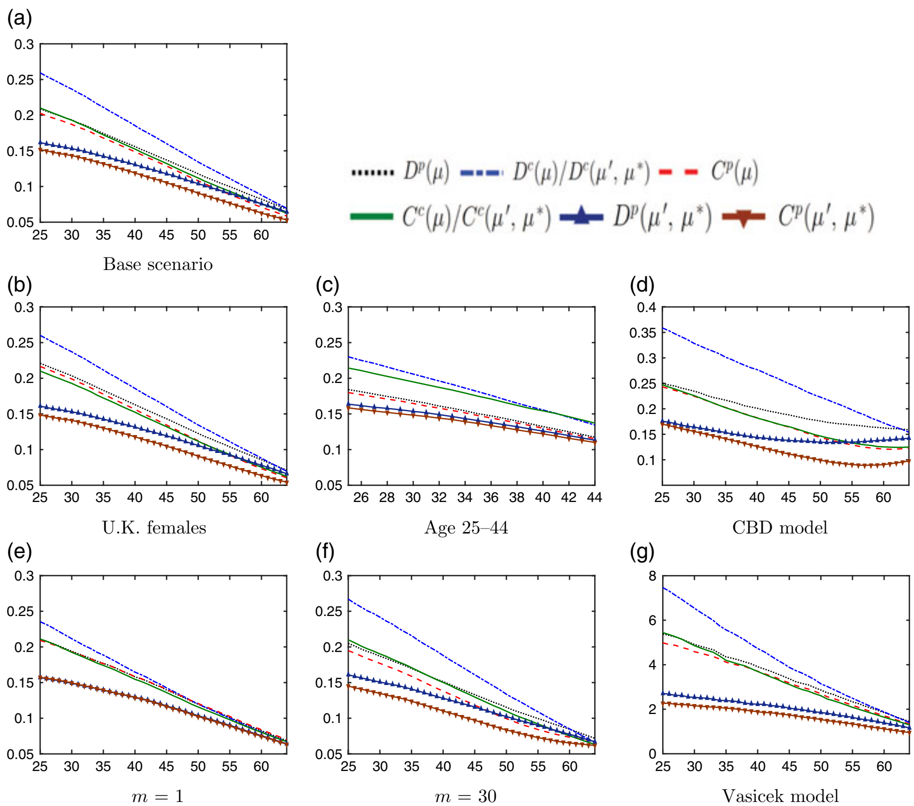

FIGURE 5: 5%-VaRs of the weighted surpluses under the base and six different scenarios.

5. CONCLUSION

Interest rate risk has been always a very important factor in risk management for life insurers. A dramatic improvement in mortality rates that has occurred over the past decades has led to an urgent demand for methodologies which can effectively hedge longevity risk for annuity providers, retirement programs, and social security systems. Interest rate immunization is more commonly adopted than mortality immunization by insurance companies. Combining the interest rate immunization and mortality rate immunization, we propose the mortality-interest immunization and quantify the extent to which life insurers and annuity providers can benefit from hedging not only longevity and mortality risks but also interest rate risk by adopting the proposed mortality-interest duration and convexity matching strategies.

Observing a linear relationship between two sequences of $\mu^*_{x,t}(i)$ $(= \mu_{x,t}(i) + \delta_i)$, we assume μ^* moves approximately linearly. We also introduce $\mu'_{x,t}(i)$ $(= \mu_{x,t}(i) + \delta_{i+1})$ to avoid dealing with the term $e^{-\delta_k}$ which varies in k and thus cannot be taken out from $\sum_{k=0}^{n-1} e^{-\delta_k} \cdot {}_kp^*_{x,t}$ (see the note immediately below

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(B.1)). Then, we derive closed-form formulas for mortality-interest durations and convexities with respect to an instantaneously proportional change and an instantaneously parallel movement, respectively, in μ^* and μ'. Next, we determine the weight of WL in a portfolio for natural hedges of long-term life insurance and annuity exposures by using the mortality(-interest) durations and convexities and compute 5%-VaR and HE (variance reduction ratio) for comparisons of hedging performances. Numerical illustrations show that life insurers can prevent their insurance portfolios from incurring high level of unexpected losses due to longevity risk and interest rate risk from issuing a long-term annuity product by using the mortality-interest duration and convexity matching strategies with respect to an instantaneously proportional change in μ^*, which are more efficient than the mortality ones with respect to an instantaneously proportional or constant change in μ only. It is noticeable that matching the mortality-interest convexity with respect to an instantaneously proportional change in μ^* can provide the most efficient way of hedging longevity risk and interest rate risk embedded in whole life annuity products.

By allocating life insurance and annuity policies in an insurance portfolio with the proposed mortality-interest duration/convexity matching strategies, we can reduce the unexpected loss to a lower level. In practice, it is difficult for a life insurer or annuity provider to reach the optimal weight of its portfolio. In this case, the life insurer or annuity provider may consider external hedging by purchasing some kinds of mortality-linked securities, for example, mortality bonds or longevity bonds. However, the external hedging involves hedge costs and basis risk. Moreover, mortality-linked securities are available only in some past periods and have been rarely traded. The failures of longevity securitization may be due to design or pricing problems (see Zelenko (2014)) or a moral hazard problem (see MacMinn and Brockett (2017)). Cox and Lin (2007) provide empirical evidence supporting that the natural hedge is an important factor contributing to annuity price differences which become more significant for those insurers selling relatively more annuity. As a result, natural hedge is an alternative efficient method of hedging longevity risk for insurers selling annuity.

In summary, our proposed mortality-interest durations and convexities have four features.

1. They are magnitude-free. Unlike the effective mortality duration and convexity used in Wang *et al.* (2010) which have no closed-form formulas and depend on the size of a proportional or constant change in mortality rates, ours provide size-free closed-form formulas.
2. They are based on an instantaneous change in the force of mortality-interest μ^*. Therefore, when we are explicitly hedging mortality or longevity risk with a duration or convexity matching strategy, we are also implicitly hedging the interest rate risk at the same time. Moreover, numerical illustrations show that natural hedging strategies with mortality-interest immunization are more effective in hedging mortality and longevity risks than those with mortality immunization only.

3. They are model- and parameter-free. The duration (key q-duration, a variation of effective duration) defined in Li and Hardy (2011) and Li and Luo (2012) is exclusive to the underlying mortality model and its associated model parameters. However, our mortality-interest durations and convexities are calculated with the given $-\ln(p)$s ($=\mu$s) data, no matter how these ps data are obtained from a mortality table or an insurer's mortality experience, best estimated by an actuary, or projected from a specific mortality model and its associated parameters.
4. They are feasible, applicable, and easy to implement. The natural hedging strategies can be easily applied, without hedge costs, to a portfolio of life insurance and annuity products with any different issue age sets ($\underline{x}_l$ for life and $\underline{x}_a$ for annuity), and sum insured (SA) and annuity payment (AP); and each strategy produces a unique weight $\hat{w}_{WL}$ from its corresponding formula. The natural hedging for both mortality and interest rate risks proposed by Luciano *et al*. (2017) could produce multiple solutions because the number of constraints is less than the number of variables.

Different from the typical immunization strategies in finance where duration matching lies in the first place and convexity matching the second, convexity matching instead plays the most important role in natural hedges using the mortality-interest immunization. Finally, the development of mortality-interest duration and convexity in the paper can be the foundation of further applying the immunization strategies of mortality rates and interest rates to the asset and liability management for life insurance companies. To this end, an optimal allocation of bonds with various maturities, life insurance, and annuity exposures can be determined to achieve the best hedge of longevity, mortality, and interest rate risks.

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APPENDIX A. PROOF OF PROPOSITION 1

When $\mu^*_{x,t}(i)$ has a proportional change of size γ to $(1+\gamma)\,\mu^*_{x,t}(i)$, the ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ becomes

$$ {}_h|\ddot{a}_{x,t:\overline{j}|}((1+\gamma)\,\mu^*_{x,t}) = \sum_{k=h}^{h+j-1} e^{-\sum_{i=0}^{k-1}(1+\gamma)\,\mu^*_{x,t}(i)} = \sum_{k=h}^{h+j-1} ({}_kp^*_{x,t})^{1+\gamma}, $$

and the change in ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ is

$$ \Delta^p\,{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}) = {}_h|\ddot{a}_{x,t:\overline{j}|}((1+\gamma)\,\mu^*_{x,t}) - {}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}) = \sum_{k=h}^{h+j-1} {}_kp^*_{x,t}\cdot[({}_kp^*_{x,t})^{\gamma}-1]. \tag{A.1} $$

We expand $({}_kp^*_{x,t})^{\gamma}$ to $1+\ln({}_kp^*_{x,t})\cdot\gamma+[\ln({}_kp^*_{x,t})]^2\cdot\gamma^2/2+\cdots$, which leads (A.1) to

$$ \Delta^p\,{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}) = \sum_{k=h}^{h+j-1} {}_kp^*_{x,t}\cdot\ln({}_kp^*_{x,t})\cdot\gamma + \sum_{k=h}^{h+j-1} {}_kp^*_{x,t}\cdot[\ln({}_kp^*_{x,t})]^2\cdot\frac{\gamma^2}{2}+\cdots. \tag{A.2} $$

We can obtain the mortality-interest duration and convexity of ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ with respect to an instantaneously proportional change in $\mu^*_{x,t}$ as

$$ D^p[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] = \left.\frac{\partial\,\Delta^p\,{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})}{\partial\gamma}\right|_{\gamma=0} = \sum_{k=h}^{h+j-1}[\ln({}_kp^*_{x,t})]\cdot{}_kp^*_{x,t}, \tag{A.3} $$

and

$$ C^p[{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})] = \left.\frac{\partial^2\,\Delta^p\,{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})}{\partial\gamma^2}\right|_{\gamma=0} = \sum_{k=h}^{h+j-1}[\ln({}_kp^*_{x,t})]^2\cdot{}_kp^*_{x,t}. \tag{A.4} $$

Similarly, when $\mu^*_{x,t}(i)$ has a parallel movement of size γ to $\mu^*_{x,t}(i)+\gamma$, the ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ becomes

$$ {}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}+\gamma) = \sum_{k=h}^{h+j-1} e^{-\sum_{i=0}^{k-1}(\mu^*_{x,t}(i)+\gamma)} = \sum_{k=h}^{h+j-1} {}_kp^*_{x,t}\cdot e^{-\gamma\cdot k}, $$

and the change in ${}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t})$ is

$$ \Delta^c\,{}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}) = {}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}+\gamma) - {}_h|\ddot{a}_{x,t:\overline{j}|}(\mu^*_{x,t}) = \sum_{k=h}^{h+j-1} {}_kp^*_{x,t}\cdot[e^{-\gamma\cdot k}-1]. \tag{A.5} $$

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We expand $e^{-\gamma\cdot k}$ to $=1+(-k)\cdot\gamma+k^2\cdot\gamma^2/2+\cdots$. Then (A.5) can be re-expressed as

$$\Delta^c\ {}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})=\sum_{k=h}^{h+j-1}\ {}_kp^*_{x,\,t}\cdot(-k)\cdot\gamma+\sum_{k=h}^{h+j-1}\ {}_kp^*_{x,\,t}\cdot k^2\cdot\frac{\gamma^2}{2}+\cdots. \tag{A.6}$$

Similarly, we can obtain the mortality-interest duration and convexity of ${}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})$ with respect to an instantaneously parallel movement in $\mu^*_{x,\,t}$ as

$$D^c[{}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})]=\frac{\partial\Delta^c\ {}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})}{\partial\gamma}\bigg|_{\gamma=0}=\sum_{k=h}^{h+j-1}(-k)\cdot\ {}_kp^*_{x,\,t}, \tag{A.7}$$

and

$$C^c[{}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})]=\frac{\partial^2\Delta^c\ {}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})}{\partial\gamma^2}\bigg|_{\gamma=0}=\sum_{k=h}^{h+j-1}k^2\cdot\ {}_kp^*_{x,\,t}. \tag{A.8}$$

Therefore, (A.2) and (A.6) can be re-expressed, using (A.3)–(A.4) and (A.7)–(A.8), as

$$\Delta^\lambda\ {}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})=D^\lambda[{}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})]\cdot\gamma+C^\lambda[{}_h|\ddot{a}_{x,\,t:\bar{j}|}(\mu^*_{x,\,t})]\cdot\frac{\gamma^2}{2}+\cdots, \tag{A.9}$$

where $\lambda=p$ (proportional) or $\lambda=c$ (constant).

APPENDIX B. PROOF OF PROPOSITION 2

For life insurance, the NSP of one-unit discrete n-year term life insurance issued to an insured aged x in year t is $A^1_{x,\,t:\bar{n}|}=\sum_{k=0}^{n-1}[{}_kp_{x,\,t}-{}_{k+1}p_{x,\,t}]\cdot e^{-\sum_{i=0}^{k}\delta_i}$. Since

$${}_kp_{x,\,t}\cdot e^{-\sum_{i=0}^{k}\delta_i}=e^{-\delta_0}\cdot[e^{-\sum_{i=0}^{k-1}[\mu_{x,\,t}(i)+\delta_{i+1}]}]=e^{-\delta_0}\cdot e^{-\sum_{i=0}^{k-1}\mu'_{x,\,t}(i)}=e^{-\delta_0}\cdot\ {}_kp'_{x,\,t} \tag{B.1}$$

(note that in the last two equalities above we do not adopt the expression $e^{-\delta_k}\cdot[e^{-\sum_{i=0}^{k-1}[\mu_{x,\,t}(i)+\delta_i]}]=e^{-\delta_k}\cdot e^{-\sum_{i=0}^{k-1}\mu^*_{x,\,t}(i)}=e^{-\delta_k}\cdot\ {}_kp^*_{x,\,t}$ to avoid dealing with the term $e^{-\delta_k}$ which varies in k and thus cannot be taken out from $\sum_{k=0}^{n-1}e^{-\delta_k}\cdot\ {}_kp^*_{x,\,t}$), and

$${}_{k+1}p_{x,\,t}\cdot e^{-\sum_{i=0}^{k}\delta_i}=[e^{-\sum_{i=0}^{k}[\mu_{x,\,t}(i)+\delta_i]}]=e^{-\sum_{i=0}^{k}\mu^*_{x,\,t}(i)}=\ {}_{k+1}p^*_{x,\,t},$$

we have

$$A^1_{x,\,t:\bar{n}|}=e^{-\delta_0}\cdot\sum_{k=0}^{n-1}\ {}_kp'_{x,\,t}-\sum_{k=0}^{n-1}\ {}_{k+1}p^*_{x,\,t}=e^{-\delta_0}\cdot\ddot{a}_{x,\,t:\bar{n}|}(\mu'_{x,\,t})-[\ddot{a}_{x,\,t:\overline{n+1}|}(\mu^*_{x,\,t})-1],$$

where ${}_kp'_{x,\,t}=\prod_{i=0}^{k-1}p'_{x+i,\,t+i}=e^{-\sum_{i=0}^{k-1}\mu'_{x,\,t}(i)}$ for $k=0,1,\cdots$ with ${}_0p'_{x,\,t}=e^{-\sum_{i=0}^{-1}\mu'_{x,\,t}(i)}=1$, $p'_{x+i,\,t+i}=e^{-\mu'_{x,\,t}(i)}$, $\mu'_{x,\,t}(i)=\mu_{x,\,t}(i)+\delta_{i+1}$, $\mu^*_{x,\,t}(i)=\mu_{x,\,t}(i)+\delta_i$ for $i=0,1,\cdots$, and $\ddot{a}_{x,\,t:\bar{n}|}(\mu'_{x,\,t})=\sum_{k=0}^{n-1}\ {}_kp'_{x,\,t}$. Since $A^1_{x,\,t:\bar{n}|}$ can be expressed in terms of $\mu'_{x,\,t}$ and $\mu^*_{x,\,t}$, we denote it as $A^1_{x,\,t:\bar{n}|}(\mu'_{x,\,t},\ \mu^*_{x,\,t})$ hereafter. Then, the NSP of one-unit discrete WL issued to an insured aged x in year t can be obtained from $A^1_{x,t:\overline{n}|}(\mu'_{x,t}, \mu^*_{x,t})$ by $A_{x,t}(\mu'_{x,t}, \mu^*_{x,t}) = \lim_{n\to\infty} A^1_{x,t:\overline{n}|}(\mu'_{x,t}, \mu^*_{x,t})$ as

$$A_{x,t}(\mu'_{x,t}, \mu^*_{x,t}) = e^{-\delta_0} \cdot \ddot{a}_{x,t}(\mu'_{x,t}) - [\ddot{a}_{x,t}(\mu^*_{x,t}) - 1]. \tag{B.2}$$

We can obtain the mortality-interest duration and convexity of $\ddot{a}_{x,t}(\mu'_{x,t})$ with respect to an instantaneously proportional change or an instantaneously parallel movement in $\mu'_{x,t}$, similar to (A.3)–(A.4) and (A.7)–(A.8) with $(h, j) = (0, \infty)$, as

$$D^p[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} [\ln({}_kp'_{x,t})] \cdot {}_kp'_{x,t},$$

$$C^p[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} [\ln({}_kp'_{x,t})]^2 \cdot {}_kp'_{x,t},$$

$$D^c[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} (-k) \cdot {}_kp'_{x,t}, \tag{B.3}$$

and

$$C^c[\ddot{a}_{x,t}(\mu'_{x,t})] = \sum_{k=0}^{\infty} k^2 \cdot {}_kp'_{x,t}. \tag{B.4}$$

Then, using the linearity preservation property of operators D^p, C^p, D^c, and C^c, the mortality-interest duration and convexity of $A_{x,t}(\mu'_{x,t}, \mu^*_{x,t})$ with respect to an instantaneously proportional change or an instantaneously constant movement in both $\mu'_{x,t}$ and $\mu^*_{x,t}$ in (B.2) are given by

$$B^\lambda[A_{x,t}(\mu'_{x,t}, \mu^*_{x,t})] = e^{-\delta_0} \cdot B^\lambda[\ddot{a}_{x,t}(\mu'_{x,t})] - B^\lambda[\ddot{a}_{x,t}(\mu^*_{x,t})], \tag{B.5}$$

where $B = D$ (duration) or C (convexity), and $\lambda = p$ (proportional) or c (constant).

APPENDIX C. PROOF OF COROLLARY 2

The NSP of one-unit discrete WL issued to an insured aged x in year t is

$$A_{x,t}(\mu_{x,t}) = \sum_{k=0}^{\infty} [{}_kp_{x,t} - {}_{k+1}p_{x,t}] \cdot e^{-\sum_{i=0}^{k}\delta_i} = \sum_{k=0}^{\infty} {}_kp_{x,t} \cdot e^{-\sum_{i=0}^{k}\delta_i} - [\ddot{a}_{x,t}(\mu_{x,t}) - 1]. \tag{C.1}$$

Using the linearity preservation property of operators D^p, C^p, D^c and C^c, we have

$$B^\lambda[A_{x,t}(\mu_{x,t})] = B^\lambda\left[\sum_{k=0}^{\infty} {}_kp_{x,t} \cdot e^{-\sum_{i=0}^{k}\delta_i}\right] - B^\lambda[\ddot{a}_{x,t}(\mu_{x,t})].$$

The expressions for $B^\lambda[\sum_{k=0}^{\infty} {}_kp_{x,t} \cdot e^{-\sum_{i=0}^{k}\delta_i}]$ for $B = D$ (duration) or C (convexity), and $\lambda = p$ (proportional) or c (constant) can be referred to Corollary 1.

APPENDIX D. PROOF OF COROLLARY 3

From Proposition 1(b), Corollary 1(b), and ${}_kp_{x,\,t}\cdot e^{-\sum_{i=0}^{k-1}\delta_i} = {}_kp^*_{x,\,t}$, we can get (a). To prove (b), (C.1) can be re-expressed, using (B.1), as

$$A_{x,\,t}(\mu_{x,\,t}) = e^{-\delta_0}\cdot\sum_{k=0}^{\infty}{}_kp'_{x,\,t} - [\ddot{a}_{x,\,t}(\mu_{x,\,t}) - 1].$$

From (a) and (B.3)–(B.5), we obtain

$$D^c\left[A_{x,\,t}(\mu_{x,\,t})\right] = e^{-\delta_0}\cdot\sum_{k=0}^{\infty}(-k)\cdot{}_kp'_{x,\,t} - D^c\left[\ddot{a}_{x,\,t}(\mu_{x,\,t})\right] = D^c\left[A_{x,\,t}(\mu'_{x,\,t},\ \mu^*_{x,\,t})\right], \quad \text{(D.1)}$$

and

$$C^c\left[A_{x,\,t}(\mu_{x,\,t})\right] = e^{-\delta_0}\cdot\sum_{k=0}^{\infty}k^2\cdot{}_kp'_{x,\,t} - C^c\left[\ddot{a}_{x,\,t}(\mu_{x,\,t})\right] = C^c\left[A_{x,\,t}(\mu'_{x,\,t},\ \mu^*_{x,\,t})\right]. \quad \text{(D.2)}$$

APPENDIX E. PROOF OF THEOREM 1

For the portfolio P_1^{WA}, the weighted surplus (negative reserve) at time 0, expressed in terms of $\mu'_{x,\,t}$ and $\mu^*_{x,\,t}$, is

$$\begin{aligned}
0 = {}_0S^{WA}(\mu',\ \mu^*) &= w_{WL}\cdot{}_0S^{WL}_{x_l,\,t:\,m}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t}) + w_{DA}\cdot{}_0S^{DA}_{x_a,\,t:\,n}(\mu^*_{x_a,\,t})\\
&= w_{WL}\left[P^{WL}_{x_l,\,t:\,m}\cdot\ddot{a}_{x_l,\,t:\,\overline{m}|}(\mu^*_{x_l,\,t}) - A_{x_l,\,t}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t})\right]\\
&\quad + w_{DA}\left[P^{DA}_{x_a,\,t:\,n}\cdot\ddot{a}_{x_a,\,t:\,\overline{n}|}(\mu^*_{x_a,\,t}) - {}_{n|}\ddot{a}_{x_a,\,t}(\mu^*_{x_a,\,t})\right].
\end{aligned}$$

If there is a proportional change ($\lambda = p$) or a constant movement ($\lambda = c$) of size γ in each of $\mu'_{x,\,t}$ and $\mu^*_{x,\,t}$ for any x and t, then

$$\begin{aligned}
&\Delta^\lambda\,{}_0S^{WA}(\mu',\ \mu^*)\\
&= w_{WL}\cdot\Delta^\lambda\,{}_0S^{WL}_{x_l,\,t:\,m}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t}) + w_{DA}\cdot\Delta^\lambda\,{}_0S^{DA}_{x_a,\,t:\,n}(\mu^*_{x_a,\,t})\\
&= \left\{w_{WL}\cdot D^\lambda\left[{}_0S^{WL}_{x_l,\,t:\,m}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t})\right] + (1-w_{WL})\cdot D^\lambda\left[{}_0S^{DA}_{x_a,\,t:\,n}(\mu^*_{x_a,\,t})\right]\right\}\cdot\gamma\\
&\quad + \left\{w_{WL}\cdot C^\lambda\left[{}_0S^{WL}_{x_l,\,t:\,m}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t})\right] + (1-w_{WL})\cdot C^\lambda\left[{}_0S^{DA}_{x_a,\,t:\,n}(\mu^*_{x_a,\,t})\right]\right\}\cdot\frac{\gamma^2}{2} + \cdots.
\end{aligned}$$

We set $w_{WL}\cdot B^\lambda[{}_0S^{WL}_{x_l,\,t:\,m}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t})] + (1-w_{WL})\cdot B^\lambda[{}_0S^{DA}_{x_a,\,t:\,n}(\mu^*_{x_a,\,t})] = 0$ with $B = D$ for the mortality-interest duration matching strategy and $B = C$ for the mortality-interest convexity matching strategy, which leads to

$$w_{WL}\left[B^\lambda(\mu',\ \mu^*)\right] = \frac{B^\lambda\left[{}_0S^{DA}_{x_a,\,t:\,n}(\mu^*_{x_a,\,t})\right]}{B^\lambda\left[{}_0S^{DA}_{x_a,\,t:\,n}(\mu^*_{x_a,\,t})\right] - B^\lambda\left[{}_0S^{WL}_{x_l,\,t:\,m}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t})\right]}, \quad \text{(E.1)}$$

where

$$B^{\lambda}[{}_0S^{DA}_{x_a,\,t:n}(\mu^*_{x_a,\,t})] = P^{DA}_{x_a,\,t:n} \cdot B^{\lambda}[\ddot{a}_{x_a,\,t:\overline{n}|}(\mu^*_{x_a,\,t})] - B^{\lambda}[{}_{n|}\ddot{a}_{x_a,\,t}(\mu^*_{x_a,\,t})]$$

and

$$B^{\lambda}[{}_0S^{WL}_{x_l,\,t:m}(\mu'_{x_l,\,t},\ \mu^*_{x_l,\,t})] = P^{WL}_{x_l,\,t:m} \cdot B^{\lambda}[\ddot{a}_{x_l,\,t:\overline{m}|}(\mu^*_{x_l,\,t})] - B^{\lambda}[A_{x_l,\,t}(\mu'_{x_l,\,t},\,\mu^*_{x_l,\,t})].$$

For the portfolio P_2^{WA}, the weighted surplus (negative reserve) at time 0, expressed in terms of $\mu'_{x,\,t}$ and $\mu^*_{x,\,t}$, is

$$0 = {}_0S^{WA}(\mu',\ \mu^*) = w_{WL} \cdot {}_0S^{WL}_{\underline{x}_l,\,t:m}(\mu'_{\underline{x}_l,\,t},\ \mu^*_{\underline{x}_l,\,t}) + w_{DA} \cdot {}_0S^{DA}_{\underline{x}_a,\,t:n}(\mu^*_{\underline{x}_a,\,t})$$

where

$${}_0S^{DA}_{\underline{x}_a,\,t:n}(\mu^*_{\underline{x}_a,\,t}) = \sum_{x\in\underline{x}_a} p^{DA}_x \cdot {}_0S^{DA}_{x,\,t:n}(\mu^*_{x,\,t})$$

and

$${}_0S^{WL}_{\underline{x}_l,\,t:m}(\mu'_{\underline{x}_l,\,t},\ \mu^*_{\underline{x}_l,\,t}) = \sum_{x\in\underline{x}_l} p^{WL}_x \cdot {}_0S^{WL}_{x,\,t:m}(\mu'_{x,\,t},\ \mu^*_{x,\,t}).$$

Similarly to (E.1), we obtain

$$w_{WL}\left[B^{\lambda}(\mu',\ \mu^*)\right] = \frac{B^{\lambda}\left[{}_0S^{DA}_{\underline{x}_a,\,t:n}(\mu^*_{\underline{x}_a,\,t})\right]}{B^{\lambda}\left[{}_0S^{DA}_{\underline{x}_a,\,t:n}(\mu^*_{\underline{x}_a,\,t})\right] - B^{\lambda}\left[{}_0S^{WL}_{\underline{x}_l,\,t:m}(\mu'_{\underline{x}_l,\,t},\ \mu^*_{\underline{x}_l,\,t})\right]},$$

where using the linearity preservation property of operators D^p, C^p, D^c, and C^c,

$$B^{\lambda}\left[{}_0S^{DA}_{\underline{x}_a,\,t:n}(\mu^*_{\underline{x}_a,\,t})\right] = \sum_{x\in\underline{x}_a} p^{DA}_x \cdot B^{\lambda}\left[{}_0S^{DA}_{x,\,t:n}(\mu^*_{x,\,t})\right]$$

and

$$B^{\lambda}\left[{}_0S^{WL}_{\underline{x}_l,\,t:m}(\mu'_{\underline{x}_l,\,t},\ \mu^*_{\underline{x}_l,\,t})\right] = \sum_{x\in\underline{x}_l} p^{WL}_x \cdot B^{\lambda}\left[{}_0S^{WL}_{x,\,t:m}(\mu'_{x,\,t},\ \mu^*_{x,\,t})\right].$$

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