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CONGRUENCE COHERENT DOUBLE MS-ALGEBRAS

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Dedicated to the memory of Dr. Rodney Beazer (1946–1998).

If A is an algebra and ϑ is a congruence on A then A is said to be ϑ-*coherent* provided that, for every subalgebra B of A, if B contains some ϑ-class then B is a union of ϑ-classes. An algebra A is said to be *congruence coherent* if it is ϑ-coherent for every $\vartheta \in \mathrm{Con} A$. This notion was investigated by Beazer [**2**] in the context of de Morgan algebras. Specifically, he showed that a de Morgan algebra is congruence coherent if and only if it is boolean, or simple, or the 4-element de Morgan chain. He also showed that if an algebra in the Berman class $\mathbf{K}_{1,1}$ of Ockham algebras is congruence coherent then it is necessarily a de Morgan algebra; and that a p-algebra is congruence coherent if and only if it is boolean. This notion has also been considered in the context of distributive double p-algebras by Adams, Atallah and Beazer [**1**] who showed that particular examples of congruence coherent double p-algebras are those that are congruence regular (in the sense that if two congruences have a class in common then they coincide). In this paper[†] we extend the results of Beazer to the class of double MS-algebras.

We recall that an *Ockham algebra* $(L; f)$ is a bounded distributed lattice L with a dual endomorphism f. An *MS-algebra* is an Ockham algebra $(L;{}^\circ)$ in which $x \mapsto x^{\circ\circ}$ is a closure. A double *MS-algebra* is an algebra $(L;{}^\circ,{}^+)$ in which $(L;{}^\circ)$ is an MS-algebra, $(L;{}^+)$ is a dual MS-algebra, and the unary operations are linked by the identities $x^{+\circ} = x^{++}$ and $x^{\circ+} = x^{\circ\circ}$. For the basic properties of double MS-algebras we refer the reader to [**3**]. The variety of double MS-algebras is denoted by **DMS**. A fundamental congruence on a double MS-algebra is the relation Φ°_+ defined by

$$(x, y) \in \Phi^\circ_+ \Leftrightarrow x^\circ = y^\circ,\ x^+ = y^+.$$

By [**3**, Theorem 13.4] a double MS-algebra is semisimple if and only if Φ°_+ reduces to equality. Of considerable importance in a double MS-algebra L is the de Morgan subalgebra

$$S(L) = \{x \in L; x = x^{\circ\circ}\} = \left\{x \in L; x = x^{++}\right\} = \left\{x \in L; x^\circ = x^+\right\}.$$

THEOREM 1. *If* $L \in$ **DMS** *then the following statements are equivalent:*
(1) L *is* Φ°_+*-coherent;*
(2) L *is semisimple.*

Proof. (1)⇒(2): Supposing that (1) holds, we shall show that $[y]\Phi^\circ_+ = \left\{y\right\}$ for every $y \in L$ whence Φ°_+ reduces to equality and (2) follows.

First we observe that for every $y \in S(L)$ we have $[y]\Phi^\circ_+ = \left\{y\right\}$. In fact, if $x \in [y]\Phi^\circ_+$ then $x^\circ = y^\circ$ and $x^+ = y^+$ whence $x^{\circ\circ} = y^{\circ\circ} = y = y^{++} = x^{++}$. Since $x^{++} \leq x \leq x^{\circ\circ}$ we deduce that $x = y$. Suppose now that $y \in L \backslash S(L)$ and consider the

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subalgebra $\langle y\rangle$ that is generated by $\{y\}$. Since, for example, $\langle y\rangle \supset \{0\} = [0]\Phi_+^\circ$ it follows by (1) that $[y]\Phi_+^\circ \subseteq \langle y\rangle$. Suppose that $x \in [y]\Phi_+^\circ$, so that we have $x \in \langle y\rangle$ with $x^\circ = y^\circ$ and $x^+ = y^+$. If $x \in S(L)$ then, from the above, $[y]\Phi_+^\circ = [x]\Phi_+^\circ = \{x\}$ whence $y = x$ and we have the contradiction $y \in S(L)$. Hence $x \notin S(L)$. Nevertheless, since $x \in \langle y\rangle$, it must be of the form $(y \wedge a) \vee b$ where $a, b \in S(L)$. Then $y^+ = x^+ = (y^+ \vee a^+) \wedge b^+$ so $y^+ \leq b^+$ whence $y \geq y^{++} \geq b^{++} = b$. Then $x = (y \vee b) \wedge (a \vee b) = y \wedge (a \vee b)$ and so $x \leq y$. Likewise, $y \leq x$ and therefore $x = y$. Hence we conclude that in all cases $[y]\Phi_+^\circ = \{y\}$, as required.

(2)⇒(1): Since Φ_+° is equality when L is semisimple, this is trivial. □

The variety **DMS** of double MS-algebras intersects the variety of distributive double p-algebras in the variety **DS** of double Stone algebras. For such algebras we have the following summary.

Theorem 2. *For $L \in$ **DS** the following statements are equivalent:*

(1) *L is congruence coherent;*

(2) *L is congruence regular;*

(3) *L is a trivalent Lukasiewicz algebra;*

(4) *L is a subdirect product of copies of the algebra SID_2 which consists of the 3-element chain $0 < d < 1$ with $d^\circ = 0$ and $d^+ = 1$.*

Proof. (1)⇒(2): This follows by [**1**, Theorem 3.4] since L is of finite range.

(2)⇒(1): This follows by [**1**, Theorem 3.3].

(2)⇔(3): This follows by [**4**, Theorem 1] and the fact that, as observed in [**3**, page 206], the trivalent Lukasiewicz algebras are precisely the semisimple double Stone algebras.

(3)⇔(4): As observed in [**3**], the class of trivalent Lukasiewicz algebras is generated by the subdirectly irreducible algebra SID_2. □

Our objective now is to determine necessary and sufficient conditions for $L \in$ **DMS****DS** to be congruence coherent. For this purpose, we shall make use of the following general result.

Theorem 3. *Let $L \in$ DMS be congruence coherent.*

(1) *If $\varphi \in \mathrm{Con}L$ with $\mathrm{Ker}\,\varphi \neq \{0\}$ then $(\forall a \in L)\, a^\circ \wedge a^{++} \in \mathrm{Ker}\,\varphi$.*

(2) *If $x, y \in L$ with $x \neq 0$ then $x^\circ \wedge y^\circ \wedge y^{++} \leq x^{\circ\circ}$.*

Proof. (1) Suppose that $\varphi \in \mathrm{Con}L$ is such that there exists $a \in L$ with $a^\circ \wedge a^{++} \notin \mathrm{Ker}\,\varphi$. Let A be the sublattice of $S(L)$ that is generated by $\{a^\circ, a^+, a^{\circ\circ}, a^{++}\}$. Observe that if $x \in A$ then $x^\circ, x^+ \in A$ and that the smallest element of A is $a^\circ \wedge a^{++} \neq 0$. Consider the set $K = \{0, 1\} \cup \bigcup_{x \in A}[x]\varphi$. It is readily seen that K is a subalgebra of L and so, since L is congruence coherent, we have $\mathrm{Ker}\,\varphi \subseteq K$. It now follows from the definition of K that $\mathrm{Ker}\,\varphi = \{0\}$.

(2) If $x, y \in L$ with $x > 0$ then $\mathrm{Ker}\,\vartheta(0, x) \neq \{0\}$ and so, by (1), we have $y^\circ \wedge y^{++} \in \mathrm{Ker}\,\vartheta(0, x)$. It follows by [**3**, Theorem 14.1(8)] that $(y^\circ \wedge y^{++} \wedge x^\circ) \vee x^{\circ\circ} = x^{\circ\circ}$ and therefore $x^\circ \wedge y^\circ \wedge y^{++} \leq x^{\circ\circ}$. □

In what follows we shall make use of the subset

$$C(L) = \{x \in L; x \wedge x^\circ = 0\}.$$

In this connection, we note the following property:

$$L \in \mathbf{DS} \Leftrightarrow C(S(L)) = S(L).$$

If fact, by [**3**], an equational basis for **DS** is the identity $x \wedge x^\circ = 0$. Consequently, if $S(L) = C(S(L))$ then for every $x \in L$ we have $x^{\circ\circ} \wedge x^\circ = 0$ whence $x \wedge x^\circ = 0$ and therefore $L \in \mathbf{DS}$. The converse is trivial since if $L \in \mathbf{DS}$ then $S(L)$ is boolean.

We shall also make use of the relation ϑ_a defined for each $a \in L$ by

$$(x, y) \in \vartheta_a \Leftrightarrow x \wedge a^{\circ\circ} = y \wedge a^{\circ\circ}, \quad x \vee a^\circ = y \vee a^\circ.$$

Clearly, $\vartheta_a \in \mathrm{Con}L$.

THEOREM 4. *If* $L \in \mathbf{DMS}$ *is congruence coherent and* $a \in S(L)\backslash C(S(L))$ *then* $\mathrm{Ker}\,\vartheta_a = \{0\}$.

Proof. Since $a \wedge a^\circ \neq 0$ we have $a \wedge a^\circ \notin \mathrm{Ker}\,\vartheta_a$. But since $a \in S(L)$ we have $a \wedge a^\circ = a^{++} \wedge a^\circ$. It follows by Theorem 3(1) that $\mathrm{Ker}\,\vartheta_a = \{0\}$. □

The following three technical results will lead us to our goal.

THEOREM 5. *If* $L \in \mathbf{DMS}\backslash\mathbf{DS}$ *is congruence coherent then*

(1) $C(S(L)) = \{0, 1\}$;

(2) *L has at most two (complementary) fixed points.*

Proof. (1) Suppose, by way of obtaining a contradiction, that $C(S(L)) \neq \{0, 1\}$. Let $x \in C(S(L))\backslash\{0, 1\}$, so that we have $x = x^{\circ\circ} = x^{++}$ with $x \wedge x^\circ = 0$ and $x \vee x^\circ = 1$. Since by hypothesis $L \notin \mathbf{DS}$, we may choose $a \in S(L)\backslash C(S(L))$. By Theorem 3(2), we have $x^\circ \wedge a^\circ \wedge a \leq x^{\circ\circ} = x$ whence, since $x \wedge x^\circ = 0$, we obtain $x^\circ \wedge a^\circ \wedge a = 0$. It follows that $x^\circ \wedge a^\circ \in \mathrm{Ker}\,\vartheta_a = \{0\}$ by Theorem 4, and so $x^\circ \wedge a^\circ = 0$. Hence $a^\circ = a^\circ \wedge 1 = a^\circ \wedge (x \vee x^\circ) = a^\circ \wedge x$ and so $a^\circ \leq x$. Since $x^\circ \in C(S(L))\backslash\{0, 1\}$, a similar argument produces $a^\circ \leq x^\circ$. Hence $a^\circ \leq x \wedge x^\circ = 0$ and we have the contradiction $a = a^{\circ\circ} = 1$.

(2) Let $\alpha, \beta \in \mathrm{Fix}\,L$ be such that $\alpha \neq \beta$. If $\alpha \wedge \beta \neq 0$ then by Theorem 4 we have $\mathrm{Ker}\,\vartheta_{\alpha\wedge\beta} = \{0\}$. Consider the subalgebra $\{0, \alpha \wedge \beta, \alpha \vee \beta, 1\}$. Since L is congruence coherent, we have $[\alpha \wedge \beta]\vartheta_{\alpha\wedge\beta} \subseteq \{0, \alpha \wedge \beta, \alpha \vee \beta, 1\}$. Since $\alpha, \beta \in [\alpha \wedge \beta, \alpha \vee \beta] = [\alpha \wedge \beta]\vartheta_{\alpha\wedge\beta}$, we must have $\alpha = \alpha \wedge \beta$ or $\alpha = \alpha \vee \beta$ whence the contradiction $\alpha \not\parallel \beta$. Hence we must have $\alpha \wedge \beta = 0$, whence $\alpha \vee \beta = 1$ and the result follows. □

THEOREM 6. *Let* $L \in \mathbf{DMS}\backslash\mathbf{DS}$ *be congruence coherent.*

(1) *If* $a^\circ, a^+ \in S(L)\backslash\{0, 1\}$ *then* $a \in S(L)\backslash\{0, 1\}$.

(2) *If* $a \in S(L)\backslash\{0, 1\}$ *then* $a \preceq a^\circ$ or $a^\circ \preceq a$.

Proof. (1) By the hypothesis, Theorem 5(1) and Theorem 4, we have $\mathrm{Ker}\,\vartheta_{a^\circ} = \{0\} = \mathrm{Ker}\,\vartheta_{a^{++}}$. Since L is congruence coherent it follows that for every $x \in S(L)$ we have $[x]\vartheta_{a^\circ} \subseteq S(L)$ and $[x]\vartheta_{a^{++}} \subseteq S(L)$. Consequently, $a \vee a^\circ \in [a_\circ]\vartheta_{a_\circ} \subseteq S(L)$ and $a \wedge a \in [a^+]\vartheta_{a^{++}} \subseteq S(L)$. It follows that $a^{++} \vee a^\circ = a^{\circ\circ} \vee a^\circ$ and $a^{++} \wedge a^+ = a^{\circ\circ} \wedge a^+$, the latter giving $a^{++} \wedge a^\circ = a^{\circ\circ} \wedge a^\circ$. Since L is distributive, we deduce that $a^{++} = a^{\circ\circ}$ whence $a \in S(L)$.

(2) By Theorem 4 we have $\mathrm{Ker}\,\vartheta_a = \{0\}$. Consider the subalgebra $\{0, a \wedge a^\circ, a \vee a^\circ, 1\}$. Since L is congruence coherent we have

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$$[a]\vartheta_a, [a^\circ]\vartheta_{a^\circ} \subseteq \{0, a \wedge a^\circ, a \vee a^\circ, 1\}.$$

Now since $[a]\vartheta_a = [a, a \vee a^\circ]$ and $[a^\circ]\vartheta_{a^\circ} = [a^\circ, a \vee a^\circ]$ we must have $a = a \wedge a^\circ$ or $a^\circ = a \wedge a^\circ$ whence $a \not\parallel a^\circ$. If $a < a^\circ$ then $[a, a^\circ] = [a]\vartheta_a \subseteq \{a, a^\circ\}$ whence $a \prec a^\circ$, and similarly if $a > a^\circ$ then $a \succ a^\circ$. □

THEOREM 7. *Let* $L \in \mathbf{DMS}\backslash\mathbf{DS}$ *be congruence coherent.*
(1) *If* $a^\circ = 0$ *and* $a^+ \notin \{0, 1\}$ *then* $a^+ = a^{++} \prec a \prec 1$;
(2) *If* $a^+ = 1$ *and* $a^\circ \notin \{0, 1\}$ *then* $0 \prec a \prec a^{\circ\circ} = a^\circ$.

Proof. We establish (1), the proof of (2) being dual.

Suppose then that $a^\circ = 0$ and $a^+ \notin \{0, 1\}$, Then by Theorem 6(2) we have either $a^+ \preceq a^{++}$ or $a^{++} \preceq a^+$. Now by the hypotheses we have $(a \wedge a^+)^+, (a \wedge a^+)^\circ \notin \{0, 1\}$ and so, by Theorem 6(1), we have $a \wedge a^+ \in S(L)$ whence $(a \wedge a^+)^\circ = (a \wedge a^+)^+$. Thus $a^{++} = a^+ \vee a^{++}$ and so we deduce that $a^+ \preceq a^{++}$.

Now $a \neq 1$ (since otherwise $a^+ = 0$); and if $a \leq x < 1$ then $x^\circ = 0 = a^\circ$, $x^+ \notin \{0, 1\}$, and $x^+ \leq a^+ \preceq a^{++} \leq x^{++}$. Similar to the above, we have $x^+ \preceq x^{++}$ whence it follows that $x^+ = a^+$. Since $\Phi_+^\circ = \omega$ we obtain $x = a$ and consequently $a \prec 1$.

If now $a^{++} \leq y < a$ then $y^+ = a^+ \notin \{0, 1\}$. Also, $y^\circ \notin \{0, 1\}$ (since $y^\circ = 0$ together with $\Phi_+^\circ = \omega$ gives $y = a$, and $y^\circ = 1$ gives $a^+ = 1$). It follows by Theorem 6(1) that $y \in S(L)$. Hence $y = y^{++} = a^{++} \preceq a$.

From the above we see that

$$(\star) \qquad a^+ \preceq a^{++} \prec a \prec 1.$$

Our objective now is to show that $a^+ = a^{++}$. For this purpose suppose, by way of obtaining a contradiction, that $a^+ \prec a^{++}$. Consider the congruence $\vartheta(a, 1)$. We show first that

$$\text{Coker}\,\vartheta(a, 1) = \{a^{++}, a, 1\},$$

noting that by [**3**, Theorem 14.1] we have

$$x \in \text{Coker}\,\vartheta(a, 1) \Leftrightarrow (x \vee a^+) \wedge a^{++} = a^{++}.$$

Observe that, under the hypothesis, $a^+ \notin \text{Coker}\,\vartheta(a, 1)$ and therefore $\vartheta(a, 1) \neq \iota$.

Suppose that $x \in \text{Coker}\,\vartheta(a, 1)$ with $x \neq 1$. Then $x^+ \neq 0$ (since otherwise $x^{++} = 1$ whence $x = 1$) and $x^\circ \wedge a^{++} \leq x^+ \wedge a^{++} \leq a^+$ whence $x^\circ \neq 1$ and $x^+ \neq 1$. Moreover, x^+ is not a fixed point (since otherwise $\vartheta(a, 1) = \iota$). There are two cases to consider:

(1) $x^\circ = 0$.

In this case we can use an argument similar to the above to obtain

$$x \prec x^{++} \prec x \prec 1.$$

Since $a \prec 1$ we have $a \vee x = 1$ or $x \leq a$.

Now if $a \vee x = 1$ then, writing $z = a^{++} \wedge x$, we have $z^\circ = a^+ \notin \{0, 1\}$ and $z^+ = a^+ \vee x^+$. Clearly, $z^+ \neq 0$; and $z^+ \neq 1$ since otherwise we would have

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$x^+ \vee a^+ = 1 = x^{++} \vee a^+$ whence, by $(\star)$, $x^+ \vee a^{++} = 1 = x^{++} \vee a^{++}$ and therefore $x^{++} \wedge a^+ = 0 = x^+ \wedge a^+$ and consequently, by distributivity, $x^+ = x^{++}$ which contradicts the fact that x^+ is not a fixed point. It follows by Theorem 6(1) that $z \in S(L)$. Consequently,

$$a^{++} = a^{++} \wedge 1 = a^{++} \wedge x^{\circ\circ} = (a^{++} \wedge x)^{\circ\circ} = z^{\circ\circ} = z = a^{++} \wedge x$$

and therefore $a^{++} \leq x$ which gives the contradiction

$$x = x \vee a^{++} \geq x^{++} \vee a^{++} = (x \vee a)^{++} = 1^{++} = 1.$$

Hence we must have $x \leq a$. Since $x \prec 1$ it follows that $x = a$.

(2) $x^\circ \neq 0$.

In this case it follows by Theorem 6(1) that $x \in S(L)$. Also, by $(\star)$, we have either $a \vee x = 1$ or $x \leq a$.

Now if $a \vee x = 1$ then on the one hand $a^+ \vee x^+ = 0$ which gives $a^{++} \vee x = 1$. Since, by hypothesis, $x \in \text{Coker}\,\vartheta(a, 1)$ we have $(x \vee a^+) \wedge a^{++} = a^{++}$. Combining these observations, we obtain

$$1 = x \vee a^{++} = (x \vee a^+) \wedge (x \vee a^{++}) = x \vee a^+.$$

On the other hand, if we write $p = x \wedge a$ then $p^\circ = x^\circ \notin \{0, 1\}$. Also, $p^+ = x^+ \vee a^+ \neq 1$ since otherwise we would have $1 = x^+ \vee a^+ = x \vee a^+$ which gives $0 = x \wedge a^{++} = x^+ \wedge a^{++}$ and, by $(\star)$, $0 = x \wedge a^+ = x^+ \wedge a^+$. By distributivity, we deduce that $x^{++} = x = x^+$ which contradicts the fact that x^+ is not a fixed point.

Thus we must have $x \leq a$, whence $x = x^{++} \leq a^{++}$. Consequently, $a^+ \leq x \vee a^+ \leq a^{++}$ and so, by $(\star)$, we have either $x \leq a^+$ or $x \vee a^+ = a^{++}$.

Now $x \in \text{Coker}\,\vartheta(a, 1)$ and $x \leq a^+$ would give $a^+ \wedge a^{++} = a^{++}$, contradicting the basic hypothesis that $a^+ \prec a^{++}$.

Hence we must have $x \vee a^+ = a^{++}$ whence $x^+ \wedge a^{++} = a^+$. Now let $q = x \wedge a^+$. Clearly, $q \neq 1$; and $q \neq 0$ since otherwise $0 = x \wedge a^+ = x \wedge x^+ \wedge a^{++} = x \wedge x^+ = x \wedge x^\circ$ and we have the contradiction $x \in C(S(L)) = \{0, 1\}$. It follows from Theorem 4 that $\text{Ker}\vartheta_q = \{0\}$. Consider the subalgebra $K = \{0, a^+, a^{++}, 1\}$. Since L is congruence coherent we have $[a^+]\vartheta_q \subseteq K$. Observe now that $a^+ \wedge q = q$ and $q^\circ \geq a^{++} > a^+$, whence we have that $[q]\vartheta_q = [a^+]\vartheta_q$. It follows from these observations that $q = a^+$. Thus $x \wedge a^+ = a^+$ whence $x \geq a^+$ and consequently $x = x \vee a^+ = a^{++}$.

We conclude from the above that $\text{Coker}\,\vartheta(a, 1) = \{a^{++}, a, 1\}$. Recalling the hypothesis that $a^\circ = 0$, consider the subalgebra $\langle a \rangle = \{0, a^+, a^{++}, a, 1\}$. This contains $\text{Coker}\,\vartheta(a, 1)$ and so, since L is congruence coherent, must contain also $\text{Ker}\vartheta(a, 1)$. Now we have

$$x \in \text{Ker}\,\vartheta(a, 1) \Leftrightarrow (x \vee a^+) \wedge a^{++} = a^+ \wedge a^{++} = a^+$$

from which we see that $a^+ \in \text{Ker}\,\vartheta(a, 1)$. It follows that $\text{Ker}\,\vartheta(a, 1) = \{0, a^+\}$. Consider again the subalgebra $K = \{0, a^+, a^{++}, 1\}$. We have $\text{Ker}\,\vartheta(a, 1) \subseteq K$ but $\text{Coker}\,\vartheta(a, 1) \not\supseteq K$, in contradiction to the hypothesis that L is congruence coherent.

This contradiction shows that the assumption $a^+ \neq a^{++}$ cannot hold and that therefore $a^+ = a^{++}$ as required. $\square$

The previous technical results come together in establishing the following:

THEOREM 8. *Let* $L \in \mathbf{DMS} \backslash \mathbf{DS}$ *be congruence coherent. If* L *is not a de Morgan algebra then* L is *simple.*

Proof. By hypothesis we have $L \neq S(L)$. We show first that $S(L) = \{0, 1\} \cup \mathrm{Fix} L$. Suppose, by way of obtaining a contradiction, that there exists $a \in S(L)$ with $a \notin \{0, 1\} \cup \mathrm{Fix} L$. By Theorem 6(2) we have either $a \prec a^\circ$ or $a^\circ \prec a$. Without loss of generality, we may assume that $a \prec a^\circ$. We show as follows that $a^\circ \prec 1$.

Suppose that $a^\circ \leq x < 1$. Then we have

$$0 \leq x^\circ \leq x^+ \leq a \prec a^\circ \leq x^{++} \leq x \leq x^{\circ\circ} \leq 1. \qquad (\dagger)$$

Clearly $x^\circ, x^+ \notin \{1\} \cup \mathrm{Fix} L$, and $x^+ \neq 0$ (since otherwise $x^{++} = 1$ whence the contradiction $x = 1$). By Theorem 7(1) we deduce that $x^\circ \neq 0$ and then, by Theorem 6(1), that $x \in S(L)$. By Theorem 6(2) it follows that $x^\circ \prec x$ and then, by ($\dagger$), that $x = a^\circ$.

A similar argument shows that $0 \prec a$, whence we see that

$$0 \prec a \prec a^\circ \prec 1. \qquad (\dagger\dagger)$$

Now by the hypotheses there exists $b \in L \backslash S(L)$, and by Theorem 6(1) we have either $b^\circ = 0$ or $b^+ = 1$.

(1) $b^\circ = 0$.

In this case, by ($\dagger\dagger$), we have $a \wedge b = 0$ or $a \wedge b = a$. The former gives the contradiction $a^\circ = 1$. The latter gives $a \leq b$ whence $a \leq b \wedge a^\circ \leq a^\circ$. Since $a = b \wedge a^\circ$ gives the contradiction $a^\circ = a^{\circ\circ} = a$, we must have $b \wedge a^\circ = a^\circ$ whence $b \geq a^\circ$ and therefore, by the above, $b = a^\circ$ or $b = 1$ whence the contradiction $b \in S(L)$.

(2) $b^+ = 1$.

In this case a similar argument shows that $0 \leq b \leq a$ whence the contradiction $b \in S(L)$.

The above observations therefore show that $S(L) = \{0, 1\} \cup \mathrm{Fix} L$ from which we deduce, using Theorem 5(2), that the de Morgan subalgebra $S(L)$ is simple.

Now let $\varphi \in \mathrm{Con} L$ such that $\varphi \neq \omega$ and let $(a, b) \in \varphi$ with $a < b$. Since, by Theorem 1, $\Phi_+^\circ = \omega$ we have either $a^\circ \neq b^\circ$ or $a^+ \neq b^+$. Since $S(L)$ is simple, we then have either $\vartheta(b^\circ, a^\circ)|_{S(L)} = \iota|_{S(L)}$ or $\vartheta(b^+, a^+)|_{S(L)} = \iota|_{S(L)}$. It follows that from this that $(0, 1) \in \vartheta(b^\circ, a^\circ)$ or $(0, 1) \in \vartheta(b^+, a^+)$ whence $\vartheta(a, b) = \iota$ and consequently L is simple. $\square$

Since every simple double MS-algebra is trivially congruence coherent, we may combine Theorem 2, Theorem 8 and the results of Beazer [**2**] to obtain the following result.

THEOREM 9. $L \in \mathbf{DMS}$ *is congruence coherent if and only if* L *is a trivalent Lukasiewicz algebra, or is simple, or is boolean, or is the 4-element de Morgan chain.* $\square$

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