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ON A FABRIC OF KISSING CIRCLES

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Abstract

Applying circle inversion on a square grid filled with circles, we obtain a configuration that we call a fabric of kissing circles. We focus on the curvature inside the individual components of the fabric, which are two orthogonal frames and two orthogonal families of chains. We show that the curvatures of the frame circles form a doubly infinite arithmetic sequence (bi-sequence), whereas the curvatures in each chain are arranged in a quadratic bi-sequence. We also prove a sufficient condition for the fabric to be integral.

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Keywords and phrases: curvature, fabric of kissing circles, frame, inversion in a circle, PL chain.

1. Introduction

Classical geometric configurations of circles, such as the arbelos or the Pappus chain, and their numerous fascinating properties, have been known for about two thousand years. At the beginning of the 19th century, the discovery of a transformation called *inversion in a circle* (briefly *inversion*) significantly simplified the solution of classical geometric problems. The last few decades have seen an expansion in the study of geometric configurations, such as circle-packing problems, including the *Apollonian packing*. The latter is a fractal that begins with three externally tangent circles inscribed in one common circle, and is constructed by repeatedly inscribing circles into the triangular interstices in the configuration. Integral Apollonian packings, that is, those in which the curvature of each circle is an integer, are of particular interest. The reader will find an engaging overview of the Apollonian packing in [7].

In this paper we present a novel configuration that we call a *fabric (of kissing circles)*, where the circles are organised in frames or in chains. Like the arbelos, the Pappus chain or the Apollonian packing, the fabric is closely related to the problem of Apollonius: given three generalised mutually touching circles with three points of tangency, there exist two other generalised circles touching the three.

The paper is organised as follows. In Section 2 the fabric is obtained by inverting a square grid filled with circles. Section 3 brings our main results concerning the curvatures in the fabric, and a sufficient condition for the fabric to be integral.

To illustrate the power of the results from Section 3, we use them to solve two old sangaku problems in the last section (Section 4).

We first introduced the fabric in [1], but our approach in Section 2 is different, and the relations between curvatures that we present in Section 3 have not been published.

2. Structure of the fabric

Consider an infinite grid with evenly spaced horizontal and vertical lines, and a circle inscribed in each square cell. Let A be an arbitrary point in the plane of the grid and let C be a circle centred at A. The inversion with respect to C transforms every element of this infinite grid filled with circles into a generalised circle. Throughout this paper, we call each of them a *circle*, even though it may be a straight line.

2.1. Frames. With respect to C, the grid lines are inverted to two infinite families of circles passing through A: vertical lines into circles centred on a horizontal line p, and horizontal lines into circles centred on a vertical line p', both lines passing through A. We denote the two families $\mathcal{V}$ and $\mathcal{H}$, respectively. Inversion preserves angles; thus, $v \perp h$ for each pair of circles $(v, h) \in (\mathcal{V}, \mathcal{H})$. We will call $\mathcal{V}$ a *vertical frame* and $\mathcal{H}$ a *horizontal frame*. Because the frames $\mathcal{V}$ and $\mathcal{H}$ are subsets of two orthogonal parabolic pencils with the common carrier A, we will say that the vertical and horizontal frames are *orthogonal*. Figure 1 represents a pair of orthogonal frames, together with the reference circle and the carrier, which are red.[1] The vertical line is a generalised circle belonging to the vertical frame; similarly, the horizontal line belongs to the horizontal frame.

2.2. Chains. Because a circle is inscribed in each cell of the grid, a chain of tangent congruent circles is inscribed in each strip between two adjacent lines of the grid. The inversion with respect to C turns this chain into a chain of circles, each of which is tangent to its two neighbours in the chain and to two bounding circles. If the central line p passes through the centre of a circle in the chain, a part of the inverted chain is the Pappus chain. This drives us to call the chain of circles obtained as above a *Pappus-like*

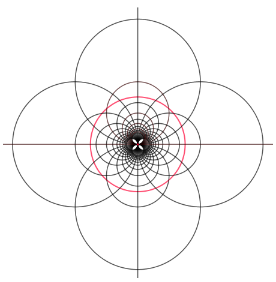

FIGURE 1. Orthogonal frames.

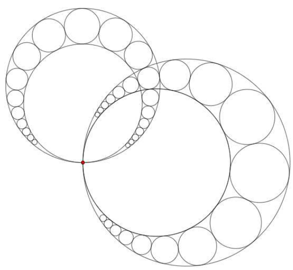

FIGURE 2. Orthogonal chains.

[1]Here and in later figures, the colours in the figures can be seen in the online version of the paper.

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chain, a *PL-chain* or simply a *chain*. Inverting a column of circles gives a *vertical chain*; inverting a row of circles gives a *horizontal chain*.

Each pair of PL-chains, one of which is vertical and the other horizontal, shares a circle. It is so because each row of circles inscribed in the grid shares a circle with each column and *vice versa*. In addition, the bounding circles of the two families of chains are orthogonal. We say that the chains are *orthogonal* (Figure 2).

2.3. The fabric. We call a *fabric of kissing circles* (*fabric*) the union of two orthogonal frames and two orthogonal families of inscribed chains, each of them obtained by the same circle inversion from a square grid filled with circles. The centre A of the reference circle is a *carrier* of the fabric.

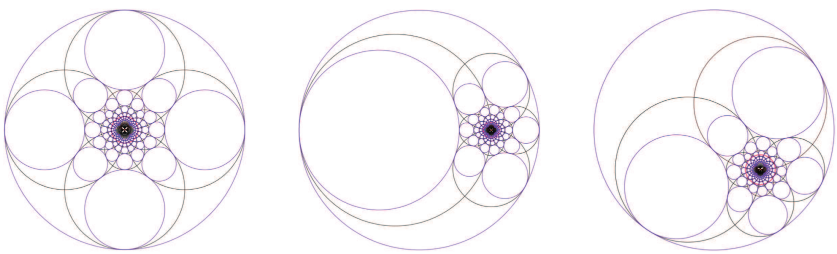
Several properties of the fabric copy those of the grid filled with circles, which is a direct consequence of the construction by inversion. Among others:

(i) each circle inscribed in a frame is shared by two orthogonal chains;
(ii) each region bounded by two frame circles contains a chain;
(iii) the touch points of the circles within a chain lie on a circle passing through A;
(iv) every fabric is invariant under reflections in $4, 2, 1$ or 0 axes that pass through the carrier A, and under $4, 2, 1$ or 1 rotations about A, respectively. This number is determined only by the position in the grid of A, which is also the centre of the reference circle.

The set of reflections and rotations corresponding to the fabric, endowed with the group operation that is composition of transformations, is the symmetry group of the fabric. The group is:

- D_4, if A is a vertex or the centre of a grid cell (Figure 3(a));
- D_2, if A is the mid point of a side of a grid cell (will be shown in Figure 6);
- D_1, if A lies on a side or a diagonal of a grid cell, except the above (Figure 3(b));
- C_1, in all other cases (Figure 3(c)).

Figure 3 shows three fabrics with different symmetry groups. For the convenience of the reader, the frames are black, the chains are purple and the reference circle is red. Notice that the frame circles pass through the carrier, whereas the chain circles do not.



(a) (b) (c)

FIGURE 3. From the left: fabrics with symmetry groups $D_4,\ D_1,\ C_1$.

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3. Curvatures in the fabric

Unlike the grid filled with circles, the fabric exhibits visually endless variability of shapes, many of which are very pleasing to the eye. This visual variability of the fabric is numerically manifested by the curvature of its components. Theorems 3.1 and 3.4, which are the main results in this paper, reveal the basic relations between the curvatures in the frame and in the chain, respectively. These relations apply to every fabric, regardless of its appearance or symmetry.

3.1. Curvatures in the frame.

THEOREM 3.1. *In a plane, let C be a circle of radius r centred at A. Let s,t be two adjacent parallel lines in a square grid and let s',t' be the inverses of s and t with respect to C. Then the curvatures of s',t' differ, in absolute value, by a constant independent of the choice of s and t.*

PROOF. Trivially, s',t' are generalised circles tangent at A. The line passing through A and perpendicular to s and t meets s,t,s',t' at S,T,S',T', respectively, all of which are not necessarily different. Denote by $\overline{XY}$ the distance between points X, Y. By definition of circle inversion,

$$\overline{AS}\cdot\overline{AS'}=\overline{AT}\cdot\overline{AT'}=r^2.$$

(1) Assume that the situation is as in Figure 4. If $\kappa_{s'},\kappa_{t'}$ are the curvatures of s',t', respectively, then

$$|\kappa_{t'}-\kappa_{s'}|=\left|\frac{2}{\overline{AT'}}-\frac{2}{\overline{AS'}}\right|=\left|\frac{2\overline{AT}-2\overline{AS}}{r^2}\right|=\frac{2d}{r^2},$$

which is determined only by the spacing $d=\overline{ST}$ between grid lines and the radius r of the reference circle C.

(2) If A lies on the open segment ST, it also lies on $S'T'$. The circles s' and t' belong to opposite sides of the frame with respect to the carrier A. The signs of the curvatures

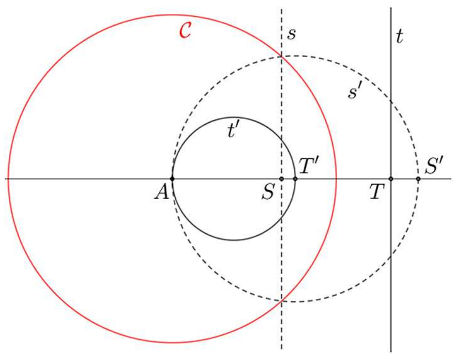

FIGURE 4. Reference circle (red), adjacent grid lines and their inverses.

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$\kappa_{s'}$ and $\kappa_{t'}$ are opposite (see Remark 3.2). Again, $|\kappa_{t'} - \kappa_{s'}| = 2d/r^2$, as

$$|\kappa_{t'} - \kappa_{s'}| = \left|\frac{2\overline{AT} + 2\overline{AS}}{r^2}\right| = \frac{2d}{r^2}.$$

(3) If one of the lines s, t passes through A, then still $|\kappa_{t'} - \kappa_{s'}| = 2d/r^2$ because either $\overline{AS} = 0, s' = s$ and $\kappa_{s'} = 0$, or $\overline{AT} = 0, t' = t$ and $\kappa_{t'} = 0$. □

According to Theorem 3.1, the curvatures of frame circles are arranged in an arithmetic bi-sequence. In addition, Theorem 3.1 implicitly states that the common difference is the same in both frames, and is independent of the position of A in the grid. We will label the common difference

$$\Delta = |\kappa_{t'} - \kappa_{s'}| = \frac{2d}{r^2}.$$

REMARK 3.2. The curvatures of circles located on opposite sides of the parabolic pencil with respect to the carrier have opposite signs. This complies with the phenomenon known in optics: the sign of the curvature depends on the position of the vertex (in our case, the point A) relative to the centre (which is the centre of the relevant circle in the pencil). Without loss of generality, we take these signs to be the same as in the coordinate system, in which A is the origin and central lines p and p' of the frames are coordinate axes.

3.2. Curvatures in the chain. To prove Theorem 3.4 we will use relation (3.1) that connects radii of four circles touching each other externally as in Figure 5(a), and which is known from a 1643 letter written by René Descartes [2].

THEOREM 3.3 (Descartes circle theorem). *If four circles with curvatures k_1, k_2, k_3, k_4 are tangent to each other at six distinct points, then*

$$\left(\sum_{i=1}^{4} k_i\right)^2 = 2\sum_{i=1}^{4} k_i^2. \tag{3.1}$$

According to Apollonius, if the curvatures k_1, k_2, k_3 of three mutually tangent generalised circles with three points of tangency are known, then the quadratic equation (3.1) has two real roots, say k, k' (double roots are counted twice), which

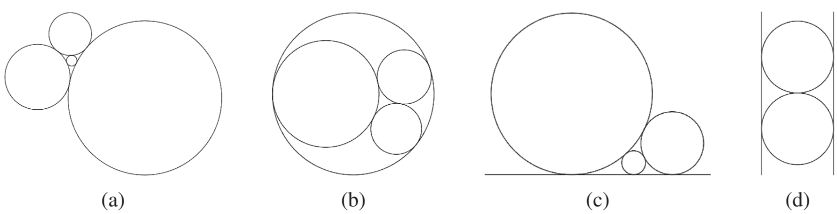

FIGURE 5. Descartes configurations.

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are the curvatures of two generalised circles touching the three. From (3.1),

$$k + k' = 2(k_1 + k_2 + k_3). \tag{3.2}$$

Relation (3.1) also applies to all configurations of four two-by-two tangent generalised circles (called *Descartes configurations* in [5]), in which no three share a point of tangency (see Figure 5). Curvatures meet the following rules.

(i) If three circles are internally tangent to the fourth as in Figure 5(b), the curvature of the outermost circle enters (3.1) with a sign opposite to the three, which is poetically explained in [6].
(ii) The curvature of a straight line is zero.

Let us now look at the curvatures in a chain.

THEOREM 3.4. *Let γ_0, γ_1 be two touching circles inscribed in a region bounded by two adjacent frame circles s', t' in a fabric. Let $\{\gamma_n\}_{n=-\infty}^{\infty}$ be a chain generated by the four circles. Let us label with $\kappa_0, \kappa_1, \kappa_{s'}, \kappa_{t'}$ the respective curvatures of $\gamma_0, \gamma_1, s', t'$. Then the curvature κ_n of γ_n satisfies*

$$\kappa_n = \kappa_0 + D \cdot n + \Delta \cdot n(n-1), \quad n \in \mathbb{Z}, \tag{3.3}$$

where $D = \kappa_1 - \kappa_0$ and $\Delta = |\kappa_{t'} - \kappa_{s'}|$.

PROOF. Each circle γ_n is tangent to s' and t', which are internally tangent at A; therefore, the Descartes Theorem 3.3 can be applied. Due to (3.2), where one of $\kappa_{s'}, \kappa_{t'}$ is zero or enters with a negative sign, the curvatures $\kappa_{n+1}, \kappa_{n-1}$ of two chain circles adjacent to γ_n satisfy

$$\kappa_{n+1} + \kappa_{n-1} = 2(\kappa_n + \Delta)$$

or equivalently

$$\kappa_{n+1} - \kappa_n = \kappa_n - \kappa_{n-1} + 2\Delta.$$

Therefore, $\{\kappa_n\}_{n=-\infty}^{\infty}$ is a quadratic bi-sequence with a recurrence relation

$$\kappa_{n+1} = \kappa_n + D + 2n\Delta, \quad n \in \mathbb{Z},$$

and the closed-form expression (3.3) of the general term. □

REMARK 3.5. The curvature of each circle γ_n, as obtained by (3.3), is independent of the choice of γ_0, γ_1 because the quadratic bi-sequence of curvatures is doubly infinite.

3.3. Integral fabric. We will say that the fabric is *integral* if the curvature of each single circle, either in the frame or in the fabric chain, is an integer. In Theorem 3.6 we state a sufficient condition for the fabric to be integral.

THEOREM 3.6. *In a fabric $\mathcal{F}$, let γ_0, γ_1 be a pair of touching circles inscribed in the region bounded by frame circles s', t'. Label with Ψ the chain generated by $s', t', \gamma_0, \gamma_1$. Let α and β be arbitrary neighbours of γ_0 and γ_1, respectively, in the chains orthogonal* *to* Ψ. *Denote the two chains by* Ψ'_0, Ψ'_1. *If the curvatures of* $s', t', \gamma_0, \gamma_1, \alpha$ *and* β *are integers, then the fabric* $\mathcal{F}$ *is integral.*

PROOF. As the curvatures of two adjacent frame circles are integers, by Theorem 3.1 it is the same for each circle in the vertical or horizontal frame.

According to Theorem 3.4 all the circles belonging to Ψ'_0 or Ψ'_1 have integral curvatures. Consequently, each circle in an arbitrary chain orthogonal to Ψ'_0 or Ψ'_1 has an integral curvature. Because every chain circle belongs to a chain orthogonal to the given chain, the proof is complete. □

Theorem 3.6 states that six circles with integer curvature suitably distributed in the fabric are sufficient for the fabric to be integral. Recall that, unlike the fabric, the Apollonian packing is integral if its generating quad, the configuration of four circles as in Figure 5(b), is integral.

EXAMPLE 3.7. The fabric with symmetry group D_2 pictured in Figure 6, where chain circles are purple and frame circles black, is integral. We will derive all the curvatures in the fabric; several circles are already labelled with their curvature. Figure 7 shows the corresponding grid filled with circles, in which each line and each circle is labelled with the curvature of its inverse shape in the fabric. The reference circle and its centre are red in both figures.

Assume that the distance between two extreme horizontal lines in Figure 6 is 2. We can easily determine the smallest curvatures.

The black central line passing through the carrier (curvature 0) is the largest generalised circle in the horizontal frame. The two frame circles adjacent to the line are congruent (radius 1/2) and tangent from outside. Therefore, their curvatures ± 2 have opposite signs. We find that $\Delta = |2 - 0| = 2$. The arithmetic bi-sequence of curvatures in this frame is $\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$. These are the labels of horizontal lines in Figure 7.

Now look at the vertical frame. The two largest black circles are congruent (radius 1) and touch each other from the outside, so their curvatures ± 1 are opposite. As they are adjacent frame circles, the value $\Delta = 2$ is confirmed. We notice that the curvatures in the horizontal frame are all even integers, whereas in the vertical frame they are all odd integers. The latter are the labels of the vertical lines in Figure 7.

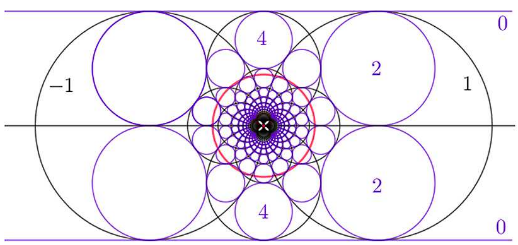

FIGURE 6. A fabric with symmetry group D_2.

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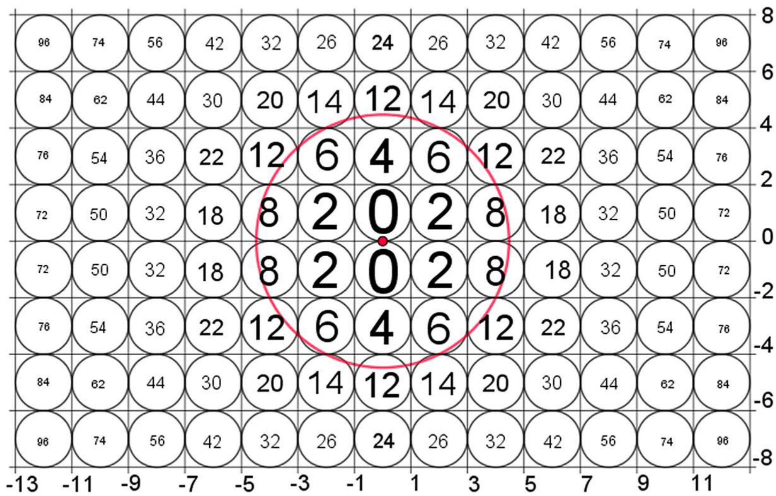

FIGURE 7. A grid labelled with curvatures relative to the red reference circle.

Most chains in the fabric are easy to identify. Consider the vertical chain inscribed in the rightmost large circle. Two congruent touching circles (curvature 2) belonging to the chain give $D = |2 - 2| = 0$. With the use of (3.3) we find the bi-sequence of curvatures $\{\dots, 26, 14, 6, 2, 2, 6, 14, 26, \dots\}$ (see the labels in a column in Figure 7).

The strangest chain in Figure 6 is coming out of the carrier upwards. It includes the upper horizontal line followed by the lower one, and the chain continues in smaller and smaller circles upwards, back to the carrier. This is a vertical chain inscribed in the region 'between' two large congruent circles, members of the vertical frame. Because the two lines are neighbours in the chain, $D = |0 - 0| = 0$. The bi-sequence of curvatures $\{\dots, 12, 4, 0, 0, 4, 12, 24, \dots\}$ appears in the middle column in Figure 7.

4. Two sangaku problems

Solving sangaku problems is growing in popularity. In Japanese tradition, mathematical problems drawn on wooded tablets (sangaku) originally hung in shrines or temples. For a rich and recognised source of information on Japanese *sacred mathematics* with many problems and solutions, see [3, 4]. Below we use Theorem 3.4 to solve two sangaku problems with chains of circles. Problem 4.1 dated 1814 from the Gumma prefecture is available in [3] as Problem 1.8.6.

PROBLEM 4.1 (Figure 8(a)). *A chain of tangent circles is inscribed in a region bounded by two internally tangent semicircles and a common central line. If the radii of the circles are $r_1 > r_2 > \cdots$, prove that*

$$\frac{7}{r_4} = \frac{2}{r_7} + \frac{5}{r_1}. \tag{4.1}$$

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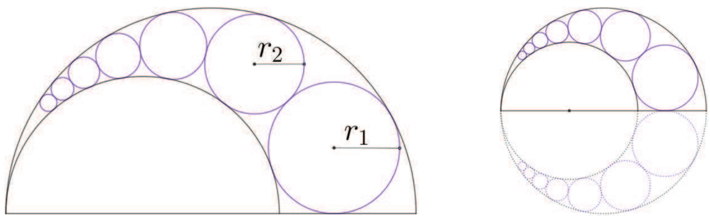

(a) (b)

FIGURE 8. A chain of seven circles.

Solution. The chain is part of a PL-chain with two largest congruent circles symmetric about the horizontal axis, as shown in Figure 8(b). Equal curvatures of congruent neighbours give $D = 0$. We set $\kappa_i = 1/r_i$ and rewrite (4.1) as

$$7\kappa_4 - (2\kappa_7 + 5\kappa_1) = 0.$$

With the use of (3.3),

$$7\kappa_4 - (2\kappa_7 + 5\kappa_1) = 7(\kappa_0 + 4D + 12\Delta) - 2(\kappa_0 + 7D + 42\Delta) - 5(\kappa_0 + D) = 9D = 0,$$

which proves (4.1).

REMARK 4.2. We observe that:

(1) the radii of the semicircles in Figure 8(a) do not affect the result because Δ and κ_0 drop out from $7\kappa_4 - (2\kappa_7 + 5\kappa_1)$;
(2) identity (4.1) applies exclusively to PL-chains that contain a pair of touching congruent circles.

Unlike Problem 4.1, a 3 : 2 ratio of the radii of bounding circles is crucial in Problem 4.3. In [4, page 312], we read: *The problem is from a lost tablet hung by Kanei Teisuke in 1828 in the Menuma temple of Kumagaya city, Saitama prefecture. We know of it from the 1830 manuscript Saishi Shinzan or Collection of Sangaku by Nakamura Tokikazu (?—1880). (Japan Academy)*

PROBLEM 4.3 (Figure 9(a)). *Show that $r_7 = r/7$.*

Solution. The configuration is symmetric about the central line of bounding circles and of three congruent circles. The inscribed chain is also symmetric about this line, as shown in Figure 9(b). Curvatures of the outermost and the inner bounding (frame) circles are $a = 1/(3r)$ and $b = 1/(2r)$; hence, $\Delta = |b - a| = 1/(6r)$. As the largest chain circle with radius r is considered as No. 1 in the chain, we have $\kappa_1 = 1/r$. Due to the symmetry, the neighbours of this circle are congruent: $\kappa_0 = \kappa_2$. The relation (3.2) becomes

$$2\kappa_0 = 2(-a + b + \kappa_1),$$

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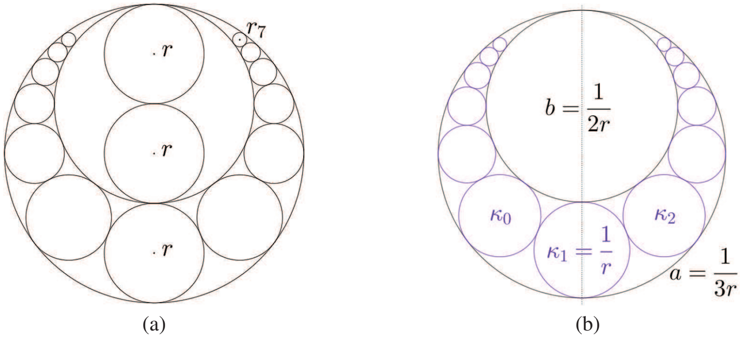

FIGURE 9. A seventh circle problem.

which gives $D = \kappa_1 - \kappa_0 = -\Delta = -1/(6r)$. From this we get $\kappa_0 = \kappa_1 - D = 7/(6r)$. With the use of (3.3),

$$\kappa_7 = \kappa_0 + 7D + 42\Delta = \frac{7}{6r} - \frac{7}{6r} + \frac{42}{6r} = \frac{7}{r}$$

or equivalently $r_7 = r/7$.

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