LIMIT THEOREMS FOR THE FRACTIONAL NONHOMOGENEOUS POISSON PROCESS

NIKOLAI LEONENKO,* *Cardiff University*
ENRICO SCALAS** AND
MAILAN TRINH,** *University of Sussex*

Abstract

The fractional nonhomogeneous Poisson process was introduced by a time change of the nonhomogeneous Poisson process with the inverse α-stable subordinator. We propose a similar definition for the (nonhomogeneous) fractional compound Poisson process. We give both finite-dimensional and functional limit theorems for the fractional nonhomogeneous Poisson process and the fractional compound Poisson process. The results are derived by using martingale methods, regular variation properties and Anscombe's theorem. Eventually, some of the limit results are verified in a Monte Carlo simulation.

Keywords: Fractional point processes; limit theorem; Poisson process; additive process; Lévy process; time change; subordination

2010 Mathematics Subject Classification: Primary 60F17
Secondary 60F05; 60G55

1. Introduction

The (one-dimensional) homogeneous Poisson process can be defined as a renewal process by specifying the distribution of the waiting times J_i to be i.i.d. and to follow an exponential distribution with parameter λ. The sequence of associated arrival times

$$T_n = \sum_{i=1}^{n} J_i, \ \ n \in \mathbb{N}, \ T_0 = 0,$$

gives a renewal process and its corresponding counting process

$$N(t) = \sup\{n : T_n \le t\} = \sum_{n=0}^{\infty} n \mathbf{1}_{\{T_n \le t < T_{n+1}\}}$$

is the Poisson process with parameter $\lambda > 0$. Alternatively, $N(t)$ can be defined as a Lévy process with Poisson distributed increments. Among other approaches, both of these representations have been used in order to introduce a fractional homogenous Poisson process (FHPP). As a renewal process, the waiting times are chosen to be i.i.d. Mittag-Leffler distributed instead of exponentially distributed, i.e.

$$\mathbb{P}(J_1 \le t) = 1 - E_\alpha(-(\lambda t)^\alpha), \qquad t \ge 0 \tag{1}$$

where $E_\alpha(z)$ is the one-parameter Mittag-Leffler function defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}, \alpha \in [0, 1).$$

The Mittag-Leffler distribution was first considered in [26] and [32]. A comprehensive treatment of the FHPP as a renewal process can be found in [39] and [44].

Starting from the standard Poisson process $N(t)$ as a point process, the FHPP can also be defined as $N(t)$ time-changed by the inverse α-stable subordinator. Meerschaert et al. [40] showed that both the renewal and the time-change approach yield the same stochastic process (in the sense that both processes have the same finite-dimensional distributions). Laskin [35] and Beghin and Orsingher [4], [5] derived the governing equations associated with the one-dimensional distribution of the FHPP.

In [37], we introduced the fractional nonhomogeneous Poisson process (FNPP) as a generalization of the FHPP. The nonhomogeneous Poisson process is an additive process with deterministic, time-dependent intensity function and thus generally does not allow a representation as a classical renewal process. However, following the construction in [23], [24] we can define the FNPP as a general renewal process. This is done in the next Section 2. Following the time-change approach, the FNPP is defined as a nonhomogeneous Poisson process time-changed by the inverse α-stable subordinator.

Among other results, we have discussed in our previous work that the FHPP can be seen as a Cox process. Following up on this observation, in this article, we will show that, more generally, the FNPP can be treated as a Cox process discussing the required choice of filtration. Cox processes or doubly stochastic processes [15], [33] are relevant for various applications such as filtering theory [10], repeat-buy consumer behavior [19], credit risk theory [8] or actuarial risk theory [28] and, in particular, ruin theory [6], [7]. Moreover, the fractional Poisson process has been recently applied to queueing theory in [11]. Subsequently, using the Cox process theory we are able to identify the compensator of the FNPP. A similar generalization of the original Watanabe characterization [50] of the Poisson process can be found in case of the FHPP in [1].

Limit theorems for Cox processes have been studied by [27] and [48], [49]. Specifically for the FHPP, long-range dependence has been discussed in [38], scaling limits have been derived in [41] and discussed in the context of parameter estimation in [12].

The rest of the article is structured as follows: In Section 2 we give a short overview of definitions and notation concerning the fractional Poisson process. Section 3 is devoted to the application of the Cox process theory to the fractional Poisson process which allows us to identify its compensator and thus derive limit theorems via martingale methods. A different approach to deriving asymptotics is followed in Section 4 and requires a regular variation condition imposed on the rate function of the Poisson process before time change. The fractional compound Poisson process is discussed in Section 5, where we derive both a one-dimensional limit theorem using Anscombe's theorem and a functional limit. Finally, we give a brief discussion of simulation methods for the FHPP and corroborate some of our theoretical results using a Monte Carlo experiment.

2. The fractional Poisson process

This section serves as a brief revision of the fractional Poisson process, both in the homogeneous and the nonhomogeneous case as well as a setup of notation.

Let $(N_1(t))_{t \geq 0}$ be a standard Poisson process with parameter 1. Define the function

$$\Lambda(s, t) := \int_s^t \lambda(\tau) \, d\tau,$$

where $s, t \geq 0$ and $\lambda : [0, \infty) \longrightarrow (0, \infty)$ is locally integrable. For shorthand $\Lambda(t) := \Lambda(0, t)$ and we assume $\Lambda(t) \to \infty$ for $t \to \infty$. We get a nonhomogeneous Poisson process $(N(t))_{t \geq 0}$, by a time-transformation of the homogeneous Poisson process with Λ:

$$N(t) := N_1(\Lambda(t)).$$

The α-stable subordinator is a Lévy process $(L_\alpha(t))_{t \geq 0}$ defined via the Laplace transform

$$\mathbb{E}[\exp(-uL_\alpha(t))] = \exp(-tu^\alpha), \quad u > 0.$$

The inverse α-stable subordinator $(Y_\alpha(t))_{t \geq 0}$ (see e.g. [9]) is defined by

$$Y_\alpha(t) := \inf\{v \geq 0 : L_\alpha(v) > t\}.$$

We assume $(Y_\alpha(t))_{t \geq 0}$ to be independent of $(N(t))_{t \geq 0}$. For $\alpha \in (0, 1)$, the fractional nonhomogeneous Poisson process (FNPP) $(N_\alpha(t))_{t \geq 0}$ is defined as

$$N_\alpha(t) := N(Y_\alpha(t)) = N_1(\Lambda(Y_\alpha(t))) \tag{2}$$

(see [37]). Note that the fractional homogeneous Poisson process (FHPP) is a special case of the nonhomogeneous Poisson process with $\Lambda(t) = \lambda t$, where $\lambda(t) \equiv \lambda > 0$ a constant. Recall that the density $h_\alpha(t, \cdot)$ of $Y_\alpha(t)$ can be expressed as (see e.g. [36], [43])

$$h_\alpha(t, x) = \frac{t}{\alpha x^{1+1/\alpha}} g_\alpha\left(\frac{t}{x^{1/\alpha}}\right), \quad x \geq 0, t \geq 0, \tag{3}$$

where $g_\alpha(z)$ is the density of $L_\alpha(1)$ given by

$$g_\alpha(z) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma(\alpha k + 1)}{k!} \frac{1}{z^{\alpha k + 1}} \sin(\pi k \alpha)$$

The Laplace transform of h_α can be given in terms of the Mittag-Leffler function

$$\tilde{h}_\alpha(t, y) = \int_0^\infty e^{-xy} h_\alpha(t, x) \, dx = E_\alpha(-yt^\alpha), \quad y > 0, \tag{4}$$

and for the FNPP the one-dimensional marginal distribution is given by

$$\mathbb{P}(N_\alpha(t) = k) = \int_0^\infty e^{-\Lambda(u)} \frac{\Lambda(u)^k}{k!} h_\alpha(t, u) \, du, \ k = 0, 1, 2, \ldots.$$

Alternatively, we can construct a nonhomogeneous Poisson process as follows (see [23]). Let $\xi_1, \xi_2, \ldots$ be a sequence of independent non-negative random variables with identical continuous distribution function

$$F(t) = \mathbb{P}(\xi_1 \leq t) = 1 - \exp(-\Lambda(t)), \ t \geq 0.$$

Define

$$\zeta'_n := \max\{\xi_1, \ldots, \xi_n\}, \quad n = 1, 2, \ldots$$

and

$$\varkappa_n = \inf\{k \in \mathbb{N} : \zeta'_k > \zeta'_{\varkappa_{n-1}}\}, \quad n = 2, 3, \ldots$$

with $\varkappa_1 = 1$. Then, let $\zeta_n := \zeta'_{\varkappa_n}$. The resulting sequence $\zeta_1, \zeta_2, \ldots$ is strictly increasing, since it is obtained from the non-decreasing sequence $\zeta'_1, \zeta'_2, \ldots$ by omitting all repeating elements. Now, we define

$$N(t) := \sup\{k \in \mathbb{N} : \zeta_k \le t\} = \sum_{n=0}^{\infty} n \mathbf{1}_{\{\zeta_n \le t < \zeta_{n+1}\}}, \quad t \ge 0$$

where $\zeta_0 = 0$. By Theorem 1 in [23], we have that $(N(t))_{t \ge 0}$ is a nonhomogeneous Poisson process with independent increments and

$$\mathbb{P}(N(t) = k) = \exp\left(-\Lambda(t)\right) \frac{\Lambda(t)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

It follows via the time-change approach that the FNPP can be written as

$$N_\alpha(t) = \sum_{n=0}^{\infty} n \mathbf{1}_{\{\zeta_n \le Y_\alpha(t) < \zeta_{n+1}\}} \overset{\text{a.s.}}{=} \sum_{n=0}^{\infty} n \mathbf{1}_{\{L_\alpha(\zeta_n) \le t < L_\alpha(\zeta_{n+1})\}},$$

where we have used that $L_\alpha(Y_\alpha(t)) = t$ if and only if t is not a jump time of L_α (see [20]).

3. Martingale methods for the FNPP

Cox processes go back to [15] who proposed to replace the deterministic intensity of a Poisson process by a random one. In this section, we discuss the connection between FNPP and Cox processes. Cox processes are also known as conditional Poisson processes.

Definition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(N(t))_{t \ge 0}$ be a point process adapted to a filtration $(\mathcal{F}_t^N)_{t \ge 0}$. $(N(t))_{t \ge 0}$ is a Cox process if there exist a right-continuous, increasing process $(A(t))_{t \ge 0}$ such that for any $0 < s < t$*

$$\mathbb{P}(N(t) - N(s) = k | \mathcal{F}_t) = \mathrm{e}^{-(A(t) - A(s))} \frac{(A(t) - A(s))^k}{k!}, \quad k = 0, 1, 2, \ldots,$$

where

$$\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{F}_t^N, \quad \mathcal{F}_0 = \sigma(A(t), t \ge 0). \tag{5}$$

Then the Cox process N is said to be directed by A.

In particular we have by definition $\mathbb{E}[N(t) | \mathcal{F}_t] = A(t)$.

Since FHPP is also a renewal process, it can be shown that it is also a Cox process by using the Laplace transform of the waiting time distributions (see Section 2 in [37]). However, in the nonhomogeneous case, we cannot apply the theorems which characterize Cox renewal processes as the FNPP cannot be represented as a classical renewal process. We will follow the

construction of doubly stochastic processes given in Section 6.6 in [8] and verify Definition 1. Let $(\mathcal{F}_t^{N_\alpha})_{t\geq 0}$ be the natural filtration of the FNPP $(N_\alpha(t))_{t\geq 0}$

$$\mathcal{F}_t^{N_\alpha} := \sigma(\{N_\alpha(s){:}s \leq t\})$$

and define

$$\mathcal{F}_0 := \sigma(\{Y_\alpha(t), t \geq 0\}). \tag{6}$$

We refer to this choice of initial σ-algebra $\mathcal{F}_0$ as *non-trivial initial history* as opposed to the case of *trivial initial history*, which is $\mathcal{F}_0 = \{\varnothing, \Omega\}$. The overall filtration $(\mathcal{F}_t)_{t\geq 0}$ is then given by

$$\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{F}_t^{N_\alpha}, \tag{7}$$

which is sometimes referred to as *intrinsic history*. If we choose a trivial initial history, the intrinsic history will coincide with the natural filtration of the FNPP.

Proposition 1. *Let the FNPP be adapted to the filtration $(\mathcal{F}_t)$ as in (7) with non-trivial initial history $\mathcal{F}_0 := \sigma(\{Y_\alpha(t), t \geq 0\})$. Then the FNPP is an $(\mathcal{F}_t)$-Cox process directed by $(\Lambda(Y_\alpha(t)))_{t\geq 0}$.*

Proof. This follows from Proposition 6.6.7. on p. 195 in [8]. We give a similar proof for completeness: As $(Y_\alpha(t))_{t\geq 0}$ is $\mathcal{F}_0$-measurable we have

$$\begin{aligned}
&\mathbb{E}[\exp\{iu(N_\alpha(t) - N_\alpha(s))\}|\mathcal{F}_s] \\
&\quad = \mathbb{E}[\exp\{iu(N_\alpha(t) - N_\alpha(s))\}|\mathcal{F}_0 \vee \mathcal{F}_s^{N_\alpha}] \\
&\quad = \mathbb{E}[\exp\{iu(N_1(\Lambda(Y_\alpha(t))) - N_1(\Lambda(Y_\alpha(s))))\}|\mathcal{F}_0 \vee \mathcal{F}_{\Lambda(Y_\alpha(s))}^{N_1}] \tag{8} \\
&\quad = \mathbb{E}[\exp\{iu(N_1(\Lambda(Y_\alpha(t))) - N_1(\Lambda(Y_\alpha(s))))\}|\mathcal{F}_0] \tag{9} \\
&\quad = \exp\left[\Lambda(Y_\alpha(s), Y_\alpha(t))(e^{iu} - 1)\right],
\end{aligned}$$

where in (8) we used the time-change theorem (see for example Thm. 7.4.I. p. 258 in [17]) and in (9) the fact that the standard Poisson process has independent increments. This means, conditional on $(\mathcal{F}_t)_{t\geq 0}$, $(N_\alpha(t))$ has independent increments and

$$(N_\alpha(t) - N_\alpha(s))|\mathcal{F}_s \sim \mathrm{Poi}(\Lambda(Y_\alpha(s), Y_\alpha(t))) \stackrel{d}{=} \mathrm{Poi}(\Lambda(Y_\alpha(t)) - \Lambda(Y_\alpha(s))).$$

Thus, $(N(Y_\alpha(t)))$ is a Cox process directed by $\Lambda(Y_\alpha(t))$ by definition. □

The identification of the FNPP as a Cox process in the previous section allows us to determine its compensator. In fact, the compensator of a Cox process coincides with its directing process. From Lemma 6.6.3. p.194 in [8] we have the result

Proposition 2. *Let the FNPP be adapted to the filtration $(\mathcal{F}_t)$ as in (7) with non-trivial initial history $\mathcal{F}_0 := \sigma(\{Y_\alpha(t), t \geq 0\})$. Assume $\mathbb{E}[\Lambda(Y_\alpha(t))] < \infty$ for $t \geq 0$. Then the FNPP has $\mathcal{F}_t$-compensator $(A(t))_{t\geq 0}$, where $A(t) := \Lambda(Y_\alpha(t))$, i.e. the stochastic process $(M(t))_{t\geq 0}$ defined by $M(t) := N(Y_\alpha(t)) - \Lambda(Y_\alpha(t))$ is an $\mathcal{F}_t$-martingale.*

3.1. A central limit theorem

Using the compensator of the FNPP, we can apply martingale methods in order to derive limit theorems for the FNPP. For the sake of completeness, we restate the definition of $\mathcal{F}_0$-stable convergence along with a lemma which will be used later.

Definition 2. *If $(X_n)_{n\in\mathbb{N}}$ and X are $\mathbb{R}$-valued random variables on a probability space $(\Omega, \mathcal{E}, \mathbb{P})$ and $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{E}$, then $X_n \to X$ ($\mathcal{F}$-stably) in distribution if for all $B \in \mathcal{F}$ and all $A \in \mathcal{B}(\mathbb{R})$ with $\mathbb{P}(X \in \partial A) = 0$,*

$$\mathbb{P}(\{X_n \in A\} \cap B) \xrightarrow[n\to\infty]{} \mathbb{P}(\{X \in A\} \cap B)$$

(see Definition A.3.2.III. in [17]*).*

Note that $\mathcal{F}$-stable convergence implies weak convergence/convergence in distribution. We can derive a central limit theorem for the FNPP using Corollary 14.5.III. in [17] which we state here as a lemma for convenience.

Lemma 1. *Let N be a simple point process on $\mathbb{R}_+$, $(\mathcal{F}_t)_{t\geq0}$-adapted and with continuous $(\mathcal{F}_t)_{t\geq0}$-compensator A. Suppose for each $T > 0$ an $(\mathcal{F}_t)_{t\geq0}$-predictable process $f_T(t)$ is given such that*

$$B_T^2 = \int_0^T [f_T(u)]^2 \, \mathrm{d}A(u) > 0.$$

and define

$$X_T := \int_0^T f_T(u)[\mathrm{d}N(u) - \mathrm{d}A(u)].$$

Then the randomly normed integrals X_T/B_T converge $\mathcal{F}_0$-stably to a standard normal variable $W \sim N(0, 1)$ for $T \to \infty$.

The above lemma allows us to show the following result for the FNPP.

Proposition 3. *Let $(N(Y_\alpha(t)))_{t\geq0}$ be the FNPP adapted to the filtration $(\mathcal{F}_t)_{t\geq0}$ as defined in Section 3. Then,*

$$\frac{N(Y_\alpha(T)) - \Lambda(Y_\alpha(T))}{\sqrt{\Lambda(Y_\alpha(T))}} \xrightarrow[T\to\infty]{} W \sim N(0, 1) \qquad \mathcal{F}_0\text{-stably.} \tag{10}$$

Proof. First note that the compensator $A(t) := \Lambda(Y_\alpha(t))$ is continuous in t. Let $f_T(u) \equiv 1$, then

$$B_T^2 = \int_0^T [f_T(u)]^2 \, \mathrm{d}A(u) = \Lambda(Y_\alpha(T)) > 0, \ \forall T > 0$$

and

$$X_T := \int_0^T f_T(u)[\mathrm{d}N(Y_\alpha(u)) - \mathrm{d}A(u)] = [N(Y_\alpha(T)) - \Lambda(Y_\alpha(T))].$$

It follows from Lemma 1 above that

$$\frac{X_T}{B_T} = \frac{N(Y_\alpha(T)) - \Lambda(Y_\alpha(T))}{\sqrt{\Lambda(Y_\alpha(T))}} \xrightarrow[T\to\infty]{} W \sim N(0, 1) \qquad \mathcal{F}_0\text{-stably.} \qquad \square$$

3.2. Limit $\alpha \to 1$

In Section 3.2(ii) in [37], in the context of the governing equations for the FNPP, we have argued that for $\alpha = 1$ the FNPP simplifies to the non-fractional nonhomogeneous Poisson process. In the following, we can show that under certain conditions we have convergence of $N_\alpha \to N$ for $\alpha \to 1$. Concerning the type of convergence, we consider the Skorokhod space $\mathcal{D}([0, \infty))$ endowed with a suitable topology (we will focus on the J_1 and M_1 topologies). For more details see [42].

Proposition 4. *Let $(N_\alpha(t))_{t \geq 0}$ be the FNPP as defined in* (2). *Let the FNPP be adapted to the filtration $(\mathcal{F}_t)$ as in* (7) *with non-trivial initial history $\mathcal{F}_0 := \sigma(\{Y_\alpha(t), t \geq 0\})$. Then, we have the limit*

$$N_\alpha \xrightarrow[\alpha \to 1]{J_1} N \quad in \quad D([0, \infty)).$$

Proof. By Proposition 2 we see that $(\Lambda(Y_\alpha(t)))_{t \geq 0}$ is the compensator of $(N_\alpha(t))_{t \geq 0}$. According to Theorem VIII.3.36 on p. 479 in [31] if suffices to show the following convergence in probability

$$\Lambda(Y_\alpha(t)) \xrightarrow[\alpha \to 1]{\mathcal{P}} \Lambda(t) \qquad \text{for all } t \in \mathbb{R}_+.$$

We can check that the Laplace transform of the density of the inverse α-stable subordinator converges to the Laplace transform of the delta distribution:

$$\mathcal{L}\{h_\alpha(\cdot, y)\}(s, y) = E_\alpha(-ys^\alpha) \xrightarrow{\alpha \to 1} \mathrm{e}^{-ys} = \mathcal{L}\{\delta_0(\cdot - y)\}(s, y). \tag{11}$$

We may take the limit as the power series representation of the (entire) Mittag-Leffler function is absolutely convergent. Thus (11) implies the following convergence in distribution

$$Y_\alpha(t) \xrightarrow[\alpha \to 1]{d} t \qquad \text{for all } t \in \mathbb{R}_+.$$

As convergence in distribution to a constant automatically improves to convergence in probability, we have

$$Y_\alpha(t) \xrightarrow[\alpha \to 1]{\mathcal{P}} t \qquad \text{for all } t \in \mathbb{R}_+.$$

By the continuous mapping theorem, it follows that

$$\Lambda(Y_\alpha(t)) \xrightarrow[\alpha \to 1]{\mathcal{P}} \Lambda(t) \qquad \text{for all } t \in \mathbb{R}_+,$$

which concludes the proof. $\square$

4. Regular variation and scaling limits

In this section, we will work with the trivial initial filtration setting $(\mathcal{F}_0 = \{\varnothing, \Omega\})$, i.e. $\mathcal{F}_t$ is assumed to be the natural filtration of the FNPP. We follow the approach of results given in [27], [48], [49], which require conditions on the function Λ. Recall that a function Λ is *regularly varying with index $\beta \in \mathbb{R}$* if

$$\frac{\Lambda(xt)}{\Lambda(t)} \xrightarrow[t \to \infty]{} x^\beta \qquad \text{for all } x > 0. \tag{12}$$

Example 1. *We check whether typical rate functions (taken from Remark 2 in [37]) fulfill the regular variation condition.*

(i) *Weibull's rate function*

$$\Lambda(t) = \left(\frac{t}{b}\right)^c, \quad \lambda(t) = \frac{c}{b}\left(\frac{t}{b}\right)^{c-1}, \quad c \geq 0, \ b > 0$$

is regulary varying with index c. This can be seen as follows

$$\frac{\Lambda(xt)}{\Lambda(t)} = \frac{(xt)^c}{t^c} = x^c, \quad \forall x > 0.$$

(ii) *Makeham's rate function*

$$\Lambda(t) = \frac{c}{b}\mathrm{e}^{bt} - \frac{c}{b} + \mu t, \quad \lambda(t) = c\mathrm{e}^{bt} + \mu, \quad c > 0, \ b > 0, \ \mu \geq 0$$

is not regulary varying, since

$$\frac{\Lambda(xt)}{\Lambda(t)} = \frac{(c/b)\mathrm{e}^{bxt} - (c/b) + \mu xt}{(c/b)\mathrm{e}^{bt} - (c/b) + \mu t} = \frac{(c/b)\mathrm{e}^{bt(x-1)} - (c/b)\mathrm{e}^{-bt} + \mu xt\mathrm{e}^{-bt}}{(c/b) - (c/b)\mathrm{e}^{-bt} + \mu t\mathrm{e}^{-bt}}$$

$$\xrightarrow{t\to\infty} \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ +\infty & \text{if } x > 1 \end{cases}$$

does not fulfill (12).

In the following, the condition that Λ is regularly varying is useful for proving limit results. We will first discuss a one-dimensional limit theorem before moving on to its functional analogue.

4.1. A one-dimensional limit theorem

Proposition 5. *Let the FNPP $(N_\alpha(t))_{t\geq 0}$ be defined as in Equation* (2). *Suppose the function $t \mapsto \Lambda(t)$ is regularly varying with index $\beta \in \mathbb{R}$. Then the following limit holds for the FNPP:*

$$\frac{N_\alpha(t)}{\Lambda(t^\alpha)} \xrightarrow[t\to\infty]{d} (Y_\alpha(1))^\beta. \tag{13}$$

Idea of proof. The result can be directly shown by invoking Lévy's continuity theorem, i.e. one only needs to prove that the characteristic function of the random variables on the left hand side of (13) converges to the characteristic function of $(Y_\alpha(1))^\beta$. Alternatively, the result follows from Theorem 3.4 in [48] or Theorem 1 on pp. 69–70 in [27]. □

Remark 1. As a special case of the theorem we get for $\Lambda(t) = \lambda t$, for constant $\lambda > 0$

$$\frac{\Lambda(xt)}{\Lambda(t)} = x^1$$

which means Λ is regularly varying with index $\beta = 1$. It follows that

$$\frac{N_1(\lambda Y_\alpha(t))}{\lambda t^\alpha} \xrightarrow[t\to\infty]{d} Y_\alpha(1).$$

This is in agreement with the scaling limit given in [12] who showed that

$$\frac{N_1(\lambda Y_\alpha(t))}{\mathbb{E}[N_1(\lambda Y_\alpha(t))]} = \frac{N_1(\lambda Y_\alpha(t))}{\lambda t^\alpha / \Gamma(1+\alpha)} \xrightarrow[t\to\infty]{d} \Gamma(1+\alpha)Y_\alpha(1).$$

4.2. A functional limit theorem

The one-dimensional result in Proposition 5 can be extended to a functional limit theorem.

Theorem 1. *Let the FNPP* $(N_\alpha(t))_{t \geq 0}$ *be defined as in Equation* (2). *Suppose the function* $t \mapsto \Lambda(t)$ *is regularly varying with index* $\beta \in \mathbb{R}$. *Then the following limit holds for the FNPP:*

$$\left(\frac{N_\alpha(t\tau)}{\Lambda(t^\alpha)} \right)_{\tau \geq 0} \xrightarrow[t \to \infty]{J_1} ([Y_\alpha(\tau)]^\beta)_{\tau \geq 0}. \tag{14}$$

Remark 2. As the limit process has continuous paths the mode of convergence improves to local uniform convergence. Also in this theorem, we will denote the homogeneous Poisson process with intensity parameter $\lambda = 1$ with N_1.

In order to prove the above theorem, we need Theorem 2 on p. 81 in [27], which we will state here for convenience as a lemma.

Lemma 2. *Let* $\bar{\Lambda}$ *be a stochastic process in* $\mathcal{D}([0, \infty))$ *with* $\bar{\Lambda}(0) = 0$ *and let* $N = N_1(\bar{\Lambda})$ *be the corresponding doubly stochastic process. Let* $a \in \mathcal{D}([0, \infty))$ *with* $a(0) = 0$ *and* $t \mapsto b_t$ *a positive regularly varying function with index* $\rho > 0$ *such that*

$$\frac{a(t)}{b_t} \xrightarrow[t \to \infty]{} \kappa \in [0, \infty) \text{ and}$$

$$\left(\frac{\bar{\Lambda}(t\tau) - a(t\tau)}{b_t} \right)_{\tau \geq 0} \xrightarrow[t \to \infty]{J_1} (S(\tau))_{\tau \geq 0},$$

where S *is a stochastic process in* $\mathcal{D}([0, \infty))$. *Then*

$$\left(\frac{N(t\tau) - a(t\tau)}{b_t} \right)_{\tau \geq 0} \xrightarrow[t \to \infty]{J_1} (S(\tau) + h(B(\tau)))_{\tau \geq 0},$$

where $h(\tau) = \kappa \tau^{2\rho}$ *and* $(S(t))_{t \geq 0}$ *and* $(B(t))_{t \geq 0}$ *are independent.* $(B(t))_{t \geq 0}$ *is the standard Brownian motion in* $\mathcal{D}([0, \infty))$.

Proof of Theorem 1. We apply Lemma 2 and choose $a \equiv 0$ and $b_t = \Lambda(t^\alpha)$. Then it follows that $\kappa = 0$ and it can be checked that b_t is regularly varying with index $\alpha\beta$:

$$\frac{b_{xt}}{b_t} = \frac{\Lambda(x^\alpha t^\alpha)}{\Lambda(t^\alpha)} \xrightarrow[t \to \infty]{} x^{\alpha\beta}$$

by the regular variation property in (12).
We are left to show that

$$\tilde{\Lambda}_t(\tau) := \left(\frac{\Lambda(Y_\alpha(t\tau))}{\Lambda(t^\alpha)} \right)_{\tau \geq 0} \xrightarrow[t \to \infty]{J_1} ([Y_\alpha(\tau)]^\beta)_{\tau \geq 0}. \tag{15}$$

This can be done by following the usual technique of first proving convergence of the finite-dimensional marginals and then tightness of the sequence in the Skorokhod space $\mathcal{D}([0, \infty))$. Concerning the convergence of the finite-dimensional marginals we show convergence of their respective characteristic functions. Let $t > 0$ be fixed at first, $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}_+^n$ and $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^n$. Then, we can write the characteristic function of the joint distribution of the vector

$$\frac{\Lambda(t^\alpha Y_\alpha(\tau))}{\Lambda(t^\alpha)} = \left(\frac{\Lambda(t^\alpha Y_\alpha(\tau_1))}{\Lambda(t^\alpha)}, \frac{\Lambda(t^\alpha Y_\alpha(\tau_2))}{\Lambda(t^\alpha)}, \ldots, \frac{\Lambda(t^\alpha Y_\alpha(\tau_n))}{\Lambda(t^\alpha)} \right) \in \mathbb{R}_+^n$$

as

$$\varphi_t(u) := \mathbb{E}\left[\exp\left(i\left\langle u, \frac{\Lambda(Y_\alpha(t\tau))}{\Lambda(t^\alpha)}\right\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\left\langle u, \frac{\Lambda(t^\alpha Y_\alpha(\tau))}{\Lambda(t^\alpha)}\right\rangle\right)\right] \tag{16}$$

$$= \int_{\mathbb{R}_+^n} \exp\left(i\left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)}\right\rangle\right) h_\alpha(\tau, x)\, dx$$

$$= \int_{\mathbb{R}_+^n} \left[\prod_{k=1}^n \exp\left(iu_k \frac{\Lambda(t^\alpha x_k)}{\Lambda(t^\alpha)}\right)\right] h_\alpha(\tau_1, \ldots, \tau_n; x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

where $u \in \mathbb{R}^n$ and $h_\alpha(\tau, x) = h_\alpha(\tau_1, \tau_2, \ldots, \tau_n; x_1, x_2 \ldots, x_n)$ is the density of the joint distribution of $(Y_\alpha(\tau_1), Y_\alpha(\tau_2), \ldots, Y_\alpha(\tau_n))$. In (16), we use self-similarity. We can find a dominating function by the following estimate:

$$\left| \exp\left(i\left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)}\right\rangle\right) h_\alpha(\tau, x)\right| \le h_\alpha(\tau, x).$$

The upper bound is an integrable function which is independent of t. By dominated convergence we may interchange limit and integration:

$$\lim_{t \to \infty} \varphi_n(u) = \lim_{t \to \infty} \int_{\mathbb{R}_+^n} \exp\left(i\left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)}\right\rangle\right) h_\alpha(\tau, x)\, dx$$

$$= \int_{\mathbb{R}_+^n} \lim_{t \to \infty} \exp\left(i\left\langle u, \frac{\Lambda(t^\alpha x)}{\Lambda(t^\alpha)}\right\rangle\right) h_\alpha(\tau, x)\, dx$$

$$= \int_{\mathbb{R}_+^n} \exp\left(i\langle u, x^\beta\rangle\right) h_\alpha(\tau, x)\, dx$$

$$= \mathbb{E}[\exp\left(i\langle u, (Y_\alpha(\tau))^\beta\rangle\right)],$$

where in the last step we used the continuity of the exponential function and the scalar product to calculate the limit. By Lévy's continuity theorem we may conclude that for $n \in \mathbb{N}$

$$\left(\frac{\Lambda(Y_\alpha(t\tau_k))}{\Lambda(t^\alpha)}\right)_{k=1,\ldots,n} \xrightarrow[t \to \infty]{d} ([Y_\alpha(\tau_k)]^\beta)_{k=1,\ldots,n}.$$

In order to show tightness, first observe that for fixed t both the stochastic process $\tilde{\Lambda}_t$ on the left hand side and the limit candidate $([Y_\alpha(\tau)]^\beta)_{\tau \ge 0}$ have increasing paths. Moreover, the limit candidate has continuous paths. Therefore we are able to invoke Thm. VI.3.37(a) in [31] to ensure tightness of the sequence $(\tilde{\Lambda}_t)_{t \ge 0}$ and thus the thesis follows. $\qquad \square$

By applying the transformation theorem for probability densities to (3), we can write for the density $h_\alpha^\beta(t, \cdot)$ of the one-dimensional marginal of the limit process $([Y_\alpha(t)]^\beta)_{t \ge 0}$ as

$$h_\alpha^\beta(t, x) = \frac{1}{\beta} x^{1/\beta - 1} h_\alpha(t, x^{1/\beta})$$

$$= \frac{1}{\beta} x^{1/\beta - 1} \frac{t}{\alpha x^{1/\beta(1+1/\alpha)}} g_\alpha\left(\frac{t}{y^{1/(\alpha\beta)}}\right)$$

$$= \frac{t}{\alpha\beta x^{1+1/(\alpha\beta)}} g_\alpha\left(\frac{t}{y^{1/(\alpha\beta)}}\right), \quad x > 0. \tag{17}$$

Note that this is *not* the density of $Y_{\alpha\beta}(t)$.

A further limit result can be obtained for the FHPP via a continuous mapping argument.

Proposition 6. *Let $(N_1(t))_{t \geq 0}$ be a homogeneous Poisson process and $(Y_\alpha(t))_{t \geq 0}$ be the inverse α-stable subordinator. Then*

$$\left(\frac{N_1(Y_\alpha(t)) - \lambda Y_\alpha(t)}{\sqrt{\lambda}} \right)_{t \geq 0} \xrightarrow[\lambda \to \infty]{J_1} (B(Y_\alpha(t)))_{t \geq 0},$$

where $(B(t))_{t \geq 0}$ is a standard Brownian motion.

Proof. The classical result

$$\left(\frac{N_1(t) - \lambda t}{\sqrt{\lambda}} \right)_{t \geq 0} \xrightarrow[\lambda \to \infty]{J_1} (B(t))_{t \geq 0}$$

can be shown by using that $(N_1(t) - \lambda t)_{t \geq 0}$ is a martingale. As $(B(t))_{t \geq 0}$ has continuous paths and $(Y_\alpha(t))_{t \geq 0}$ has increasing paths we can use Theorem 13.2.2 in [51] to obtain the result. $\quad \square$

The above proposition can be compared with Lemma 3 in the next section and a similar continuous mapping argument is applied in the proof of Theorem 4.

5. The fractional compound Poisson process

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables. The fractional compound Poisson process is defined analogously to the standard compound Poisson process where the Poisson process is replaced by a FNPP:

$$Z_\alpha(t) := \sum_{k=1}^{N_\alpha(t)} X_k, \tag{18}$$

where $\sum_{k=1}^{0} X_k := 0$. The process N_α is not necessarily independent of the X_i's unless stated otherwise.

In the following, we need to discuss stable laws as we are dealing with limit theorems. Stable laws can be defined via the form of their characteristic function.

Definition 3. *A random variable S is said to have a stable distribution if there are parameters $0 < \tilde{\alpha} \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that its characteristic function has the following form:*

$$\mathbb{E}[\exp(i\theta S)] = \begin{cases} \exp\left(-\sigma^{\tilde{\alpha}} |\theta|^{\tilde{\alpha}} \left[1 - i\beta \, sign(\theta) \tan\left(\frac{\pi\tilde{\alpha}}{2} \right) \right] + i\mu\theta \right) & \text{if } \tilde{\alpha} \neq 1, \\ \exp\left(-\sigma |\theta| \left[1 + i\beta \frac{2}{\pi} \, sign(\theta) \ln(|\theta|) \right] + i\mu\theta \right) & \text{if } \tilde{\alpha} = 1 \end{cases}$$

(see Definition 1.1.6 in [47]). We will assume a limit result for the sequence of partial sums without time change

$$S_n := \sum_{k=1}^{n} X_k. \tag{19}$$

There exist sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ and and a random variable following a stable distribution S such that

$$\bar{S}_n := a_n S_n - b_n \xrightarrow[n \to \infty]{d} S.$$

(for details see Chapter XVII in [21] for example). In other words the distribution of the X_k is in the domain of attraction of a stable law.

In the following, we will derive limit theorems for the fractional compound Poisson process. In Section 5.2, we assume N_α to be independent of the X_k's and use a continuous mapping theorem argument to show functional convergence w.r.t. a suitable Skorokhod topology. A corresponding one-dimensional limit theorem would follow directly from the functional one. However, in the special case of N_α being a FHPP, using Anscombe type theorems in Section 6.1 allows us to drop the independence assumption between N_α and the X_k's and thus strengthen the result for the one-dimensional limit.

5.1. A one-dimensional limit result

The following theorem is due to [2] and can be found slightly reformulated in [46].

Theorem 2. *We assume that the following conditions are fulfilled:*

(i) *The sequence of random variables R_n such that*

$$R_n \xrightarrow[n \to \infty]{d} R,$$

for some random variable R.

(ii) *Let the family of integer-valued random variables $(\tilde{N}(t))_{t \geq 0}$ be relatively stable, i.e. for a real-valued function ψ with $\psi(t) \xrightarrow[t \to \infty]{} +\infty$ it holds that*

$$\frac{\tilde{N}(t)}{\psi(t)} \xrightarrow[t \to \infty]{P} 1.$$

(iii) *(Uniform continuity in probability) For every $\varepsilon > 0$ and $\eta > 0$ there exists a $c = c(\varepsilon, \eta)$ and a $t_0 = t_0(\varepsilon, \eta)$ such that for all $t \geq t_0$*

$$\mathbb{P}\Big(\max_{m:|m-t|<ct} |R_m - R_t| > \varepsilon \Big) < \eta.$$

Then,

$$R_{\tilde{N}(t)} \xrightarrow[t \to \infty]{d} R.$$

Concerning the condition (ii), note that the required convergence in probability is stronger than the convergence in distribution we have derived in the previous sections for the FNPP. Nevertheless, in the special case of the FHPP, we can prove the following lemma.

Lemma 3. *Let N_α be a FHPP, i.e. $\Lambda(t) = \lambda t$. Then with $C := \frac{\lambda}{\Gamma(1+\alpha)}$ it holds that*

$$\frac{N_\alpha(t)}{Ct^\alpha} \xrightarrow[t \to \infty]{P} 1.$$

Proof. According to Proposition 4.1 from [18] we have the result that for fixed $t > 0$ the convergence

$$\frac{N_1(\lambda Y_\alpha(t))}{\mathbb{E}[N_1(\lambda Y_\alpha(t))]} = \frac{N_1(\lambda Y_\alpha(t))}{\lambda t^\alpha / \Gamma(1 + \alpha)} \xrightarrow[\lambda \to \infty]{L^1} 1 \tag{20}$$

holds and therefore also in probability.

It can be shown by using the fact that the moments and the waiting time distribution of the FHPP can be expressed in terms of the Mittag-Leffler function.

Let $\varepsilon > 0$. We have

$$\lim_{t \to \infty} \mathbb{P}\Big(\Big|\frac{N_1(\lambda Y_\alpha(t))}{Ct^\alpha} - 1\Big| > \varepsilon\Big) = \lim_{t \to \infty} \mathbb{P}\Big(\Big|\frac{N_1(\lambda t^\alpha Y(1))}{\lambda t^\alpha / \Gamma(1 + \alpha)} - 1\Big| > \varepsilon\Big) \tag{21}$$

$$= \lim_{\tau \to \infty} \mathbb{P}\Big(\Big|\frac{N_1(\tau Y(1))}{\tau \cdot 1^\alpha / \Gamma(1 + \alpha)} - 1\Big| > \varepsilon\Big) = 0, \tag{22}$$

where in (21) we used the self-similarity property of Y_α and in (22) we applied (20) with $t = 1$. $\qquad\square$

As a direct application of Theorem 2 we can prove the following lemma.

Lemma 4. *Let N_α be a FHPP and $X_1, X_2, \ldots$ be a sequence of i.i.d. variables in the DOA of a stable law μ. Then, for the partial sums S_n defined in (19) there exist sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ such that*

$$a_{N_\alpha(t)} S_{N_\alpha(t)} - b_{N_\alpha(t)} \xrightarrow[t \to \infty]{d} S,$$

where $S \sim \mu$.

Proof. We would like to use the above theorem for $R_n = \bar{S}_n$ and $\tilde{N} = N_\alpha$. Indeed, condition (i) follows from the assumption that the law of X_1 lies in the domain of attraction of a stable law and condition (ii) follows from Lemma 3. It is readily proven in Theorem 3 in [2] that $(\bar{S}_n)$ satisfies the condition (iii), if condition (i) and (ii) are fulfilled. Therefore, it follows from Theorem 2 that

$$\bar{S}_{N_\alpha(t)} = a_{N_\alpha(t)} \sum_{k=1}^{N_\alpha(t)} X_k - b_{N_\alpha(t)} \xrightarrow[t \to \infty]{d} S. \qquad\square$$

Finally, we would like to replace $N_\alpha(t)$ with $\lfloor Ct^\alpha \rfloor$ in the index of a and b. This requires additional conditions. The following theorem is a slight modification of Theorem 3.6 in Chapter 9 of [30].

Theorem 3. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and set*

$$S_n := \sum_{k=1}^n X_k, \quad n \geq 1.$$

Suppose that $(a_n)_{n \geq 0}$ is a sequence of positive norming constants such that

$$\frac{S_n}{a_n} \xrightarrow[n \to \infty]{d} S,$$

where S follows a stable law with index $\alpha \in (1, 2]$. Let $(N(t))_{t \geq 0}$ be a sequence of integer-valued random variables such that (ii) in Theorem 2 is fulfilled. Then,

$$a_{\lfloor Ct^\alpha \rfloor} \sum_{k=1}^{N_\alpha(t)} X_k = a_{\lfloor Ct^\alpha \rfloor} Z_\alpha(t) \xrightarrow[t \to \infty]{d} S.$$

Idea of proof. By Lemma 4 we have

$$a_{N_\alpha(t)} \sum_{k=1}^{N_\alpha(t)} X_k \xrightarrow[t \to \infty]{d} S,$$

as $b_n = 0$ by assumption. In order to replace $N_\alpha(t)$ with $\lfloor Ct^\alpha \rfloor$ in the index of a one has to show that

$$\frac{N_\alpha(t)}{Ct^\alpha} \xrightarrow[t \to \infty]{P} 1$$

implies

$$\frac{a_{N_\alpha(t)}}{a_{\lfloor Ct^\alpha \rfloor}} \xrightarrow[t \to \infty]{P} 1.$$

The derivation of suitable estimates relies on the fact that $n \mapsto a_n$ is regularly varying (for details see Lemma 2.9 (a) in [29]). $\qquad \square$

Remark 3. (i) The conditions restrict to the centered, symmetric case (i.e. $\mathbb{E}[X_1] = 0$, $b_n = 0$) and $\alpha \in (1, 2]$ as the mean exists. While it can be shown that $a_n \in \mathcal{R}_{-1/\alpha}$, in the non-symmetric case, we generally do not have a regular variation property for b_n.

(ii) Note that this convergence result does not require N_α to be independent of the X_k's. The above derivation also works for mixing sequences $X_1, X_2, \ldots$ instead of i.i.d. (see [16] for a generalization of Anscombe's theorem for mixing sequences).

5.2. A functional limit theorem

Theorem 4. *Let the FNPP $(N_\alpha(t))_{t \geq 0}$ be defined as in Equation ([2]) and suppose the function $t \mapsto \Lambda(t)$ is regularly varying with index $\beta \in \mathbb{R}$. Moreover let $X_1, X_2, \ldots$ be i.i.d. random variables independent of N_α. Assume that the law of X_1 is in the domain of attraction of a stable law, i.e. there exist sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ and a stable Lévy process $(S(t))_{t \geq 0}$ such that the partial sums S_n defined in ([19]) satisfy*

$$(a_n S_{\lfloor nt \rfloor} - b_n)_{t \geq 0} \xrightarrow[n \to \infty]{J_1} (S(t))_{t \geq 0}. \tag{23}$$

Then the fractional compound Poisson process Z_α defined in ([18]) satisfies the following limit:

$$(c_n Z_\alpha(nt) - d_n)_{t \geq 0} \xrightarrow[n \to \infty]{M_1} (S([Y_\alpha(t)]^\beta))_{t \geq 0},$$

where $c_n := a_{\lfloor \Lambda(n) \rfloor}$ and $d_n := b_{\lfloor \Lambda(n) \rfloor}$.

Proof. The proof follows the technique proposed by [41]: By Theorem [1] we have

$$\left(\frac{N_\alpha(t\tau)}{\Lambda(t^\alpha)} \right)_{\tau \geq 0} \xrightarrow[t \to \infty]{J_1} ([Y_\alpha(\tau)]^\beta)_{\tau \geq 0}.$$

By the independence assumptions we can combine this with ([23]) to get

$$(a_{\lfloor \Lambda(n^\alpha) \rfloor} S_{\lfloor \Lambda(n^\alpha)t \rfloor} - b_{\lfloor \Lambda(n^\alpha) \rfloor}, [\Lambda(n^\alpha)]^{-1} N_\alpha(nt))_{t \geq 0} \xrightarrow[n \to \infty]{J_1} (S(t), [Y_\alpha(t)]^\beta)_{t \geq 0}$$

in the space $\mathcal{D}([0, \infty), \mathbb{R} \times [0, \infty))$. Note that $([Y_\alpha(t)]^\beta)_{t \geq 0}$ is non-decreasing. Moreover, due to independence the Lévy processes $(S(t))_{t \geq 0}$ and $(D_\alpha(t))_{t \geq 0}$ do not have simultaneous jumps (for details see [3] and more generally [14]). This allows us to apply Theorem 13.2.4 in [51] to get the thesis by means of a continuous mapping argument since the composition mapping is continuous in this setting. $\qquad \square$

6. Numerical experiments

6.1. Simulation methods

In the special case of the FHPP, the process is simulated by sampling the waiting times J_i of the overall process $N(Y_\alpha(t))$, which are Mittag-Leffler distributed (see Equation (1)). Direct sampling of the waiting times of the FHPP can be done via a transformation formula due to [34]

$$J_1 = -\frac{1}{\lambda} \log(U) \left[\frac{\sin(\alpha\pi)}{\tan(\alpha\pi V)} - \cos(\alpha\pi) \right]^{1/\alpha},$$

where U and V are two independent random variables uniformly distributed on $[0, 1]$. For futher discussion and details on the implementation see [22] and [25].

As the above method is not applicable for the FNPP, we draw samples of $Y_\alpha(t)$ first, before sampling N. The Laplace transform w.r.t. the time variable of $Y_\alpha(t)$ is given by

$$\int_0^\infty e^{-st} h_\alpha(t, x) \, dt = s^{\alpha-1} \exp(-xs^\alpha).$$

We evaluate the density h_α by inverting the Laplace transform numerically using the Post-Widder formula ([45] and [52]):

Theorem 5. *If the integral*

$$\bar{f}(s) = \int_0^\infty e^{-su} f(u) \, du$$

converges for every $s > \gamma$, then

$$f(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \bar{f}^{(n)}\left(\frac{n}{t}\right),$$

for every point $t > 0$ of continuity of $f(t)$ (cf. p. 37 in [13]).

This evaluation of the density function allows us to sample $Y_\alpha(t)$ using discrete inversion.

6.2. Numerical results

Figure 1 shows the shape and time-evolution of the densities for different values of α. As Y_α is a non decreasing process, the densities spread to the right as time passes. We performed a small Monte Carlo simulation in order to illustrate the one-dimensional convergence results of Lemma 1 and Proposition 5. In Figures 2, 3 and 4, we can see that the simulated values for the probability density $x \mapsto \varphi_\alpha(t, x)$ of $[N(Y_\alpha(t)) - \Lambda(Y_\alpha(t))]/\sqrt{\Lambda(Y_\alpha(t))}$ approximate the density of a standard normal distribution for increasing time t. In a similar manner, Figure 5 depicts how the probability density function $x \mapsto \phi_\alpha(t, x)$ of $N_\alpha(t)/\Lambda(t^\alpha)$ approximates the density of $(Y_\alpha(t))^\beta$ given in (17), where Λ has regular variation index $\beta = 0.7$.

7. Summary and outlook

Due to the nonhomogeneous component of the FNPP, it is not surprising that analytical tractability needed to be compromised in order to derive analogous limit theorems. Most
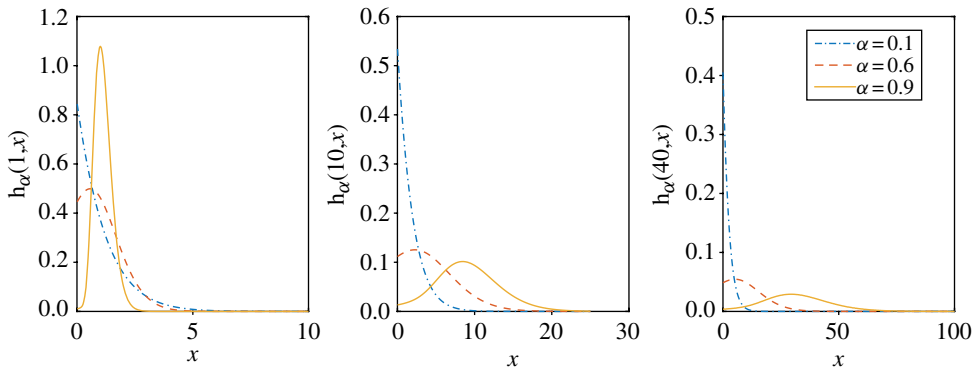
FIGURE 1: Plots of the probability densities $x \mapsto h_\alpha(t, x)$ of the distribution of the inverse α-stable subordinator $Y_\alpha(t)$ for different parameter $\alpha = 0.1, 0.6, 0.9$ indicating the time-evolution: the plot on the left is generated for $t = 1$, the plot in the middle for $t = 10$ and the plot on the right for $t = 40$.
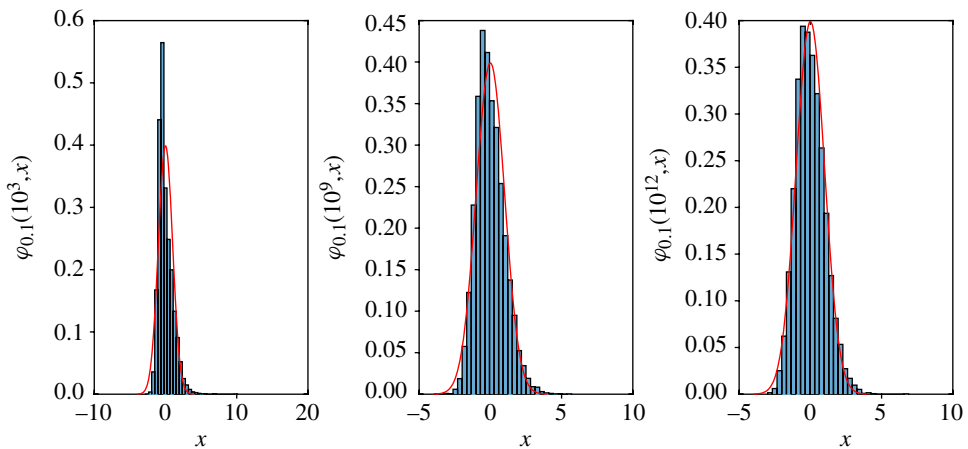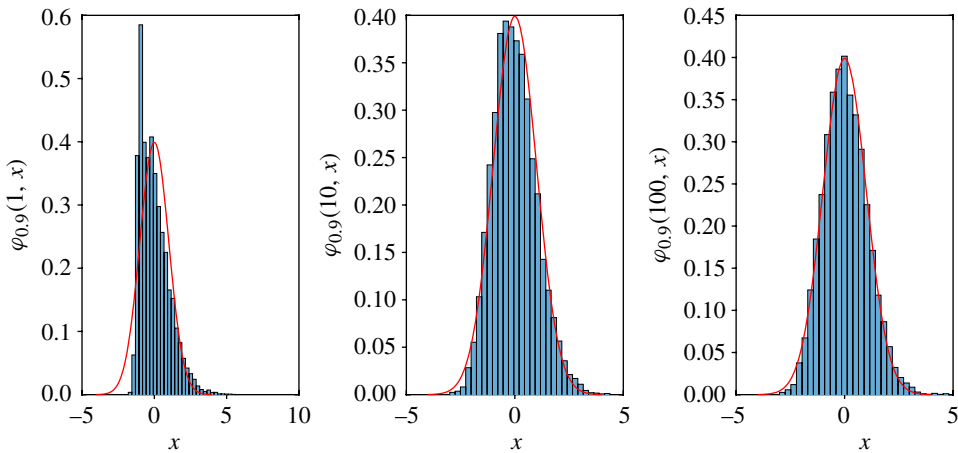

FIGURE 2: The red line shows the probability density function of the standard normal distribution, the limit distribution according to Lemma 1. The blue histograms depict samples of size 10^4 of the right hand side of (10) for different times $t = 10, 10^9, 10^{12}$ to illustrate convergence to the standard normal distribution for $\alpha = 0.1$.

noticeably, the lack of a renewal representation of the FNPP compared to its homogeneous version leads to the requirement of additional conditions on the underlying filtration structure or rate function Λ.

The result in Proposition 4 partly answered an open question that followed after Theorem 1 in [37] concerning the limit $\alpha \to 1$.

Futher research will be directed towards the implications of the limit results for estimation techniques as well as on convergence rates.

FIGURE 3: The red line shows the probability density function of the standard normal distribution, the limit distribution according to Lemma 1. The blue histograms depict samples of size 10^4 of the right hand side of (10) for different times $t = 1, 10, 100$ to illustrate convergence to the standard normal distribution for $\alpha = 0.6$.
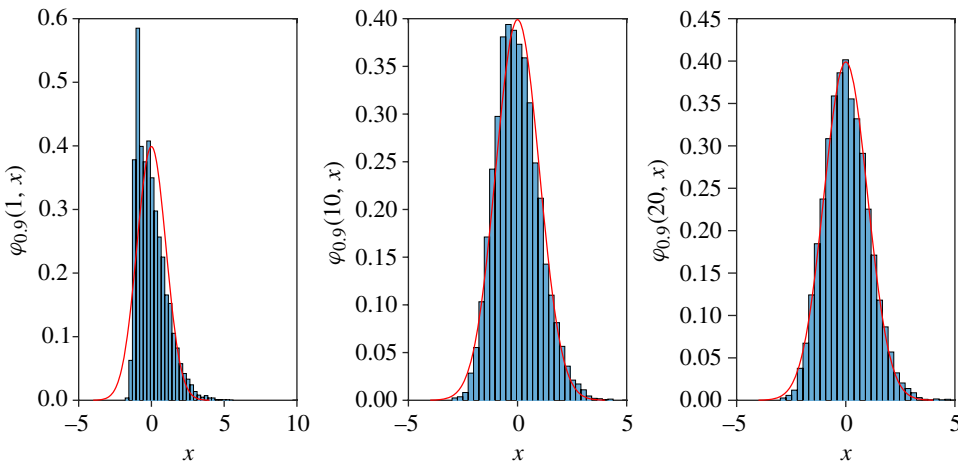

FIGURE 4: The red line shows the probability density function of the standard normal distribution, the limit distribution according to Lemma 1. The blue histograms depict samples of size 10^4 of the right hand side of (10) for different times $t = 1, 10, 20$ to illustrate convergence to the standard normal distribution for $\alpha = 0.9$.
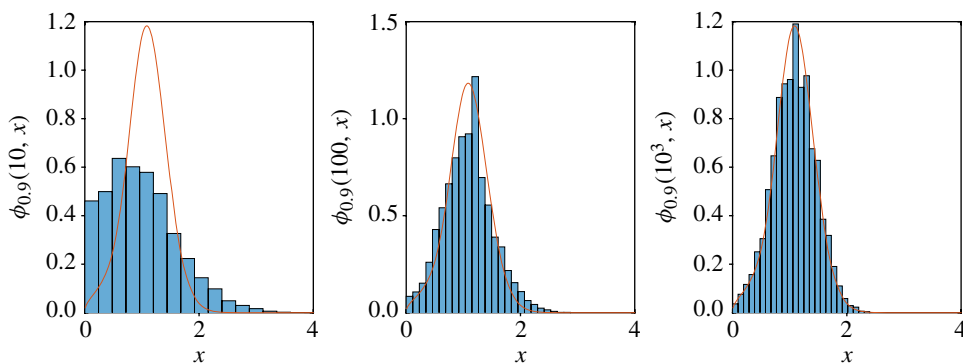
Acknowledgements

FIGURE 5: Red line: probability density function ϕ of the distribution of the random variable $(Y_{0.9}(1))^{0.7}$, the limit distribution according to Proposition 5. The blue histogram is based on 10^4 samples of the random variables on the right hand side of (13) for time points $t = 10, 100, 10^3$ to illustrate the convergence result.

References

[1] ALETTI, G., LEONENKO, N. AND MERZBACH, E. (2018). Fractional Poisson fields and martingales. *J. Statist. Phys.* **170,** 700–730.

[2] ANSCOMBE, F. J. (1952). Large-sample theory of sequential estimation. *Proc. Camb. Phil. Soc.* **48**, 600–607.

[3] BECKER-KERN, P., MEERSCHAERT, M. M. AND SCHEFFLER, H.-P. (2004). Limit theorems for coupled continuous time random walks. *Ann. Prob.* **32**, 730–756.

[4] BEGHIN, L. AND ORSINGHER, E. (2009). Fractional Poisson processes and related planar random motions. *Electron. J. Prob.* **14,** 1790–1827.

[5] BEGHIN, L. AND ORSINGHER, E. (2010). Poisson-type processes governed by fractional and higher-order recursive differential equations. *Electron. J. Prob.* **15,** 684–709.

[6] BIARD, R. AND SAUSSEREAU, B. (2014). Fractional Poisson process: long-range dependence and applications in ruin theory. *J. Appl. Prob.* **51,** 727–740.

[7] BIARD, R. AND SAUSSEREAU, B. (2016). Correction: "Fractional Poisson process: long-range dependence and applications in ruin theory". *J. Appl. Prob.* **53,** 1271–1272.

[8] BIELECKI, T. R. AND RUTKOWSKI, M. (2002). Credit risk: modelling, valuation and hedging. *Springer Finance*. Springer, Berlin.

[9] BINGHAM, N. H. (1971). Limit theorems for occupation times of Markov processes. *Z. Wahrscheinlichkeitsth.* **14,** 1–22.

[10] BRÉMAUD, P. (1981). *Point Processes and Queues*. Springer, New York.

[11] CAHOY, D. O., POLITO, F. AND PHOHA, V. (2015). Transient behavior of fractional queues and related processes. *Methodol. Comput. Appl. Prob.* **17,** 739–759.

[12] CAHOY, D. O., UCHAIKIN, V. V. AND WOYCZYNSKI, W. A. (2010). Parameter estimation for fractional Poisson processes. *J. Statist. Plann. Infer.* **140,** 3106–3120.

[13] COHEN, A. M. (2007). *Numerical Methods for Laplace Transform Inversion* (Numerical Methods Alg. **5**). Springer, New York.

[14] CONT, R. AND TANKOV, P. (2004). Financial modelling with jump processes. Chapman & Hall/CRC, Boca Raton, FL.

[15] COX, D. R. (1955). Some statistical methods connected with series of events. *J. R. Statist. Soc. Ser. B.* **17**, 129–157; discussion, 157–164.

[16] CSÖRGŐ, M. AND FISCHLER, R. (1973). Some examples and results in the theory of mixing and random-sum central limit theorems. Collection of articles dedicated to the memory of Alfréd 402 Rényi, II. *Period. Math. Hungar.* **3,** 41–57.

[17] DALEY, D. J. AND VERE-JONES, D. (2003). *An Introduction to the Theory of Point Processes*, Vol. I, 2nd edn. Springer, New York.

[18] DI CRESCENZO, A., MARTINUCCI, B. AND MEOLI, A. (2016). A fractional counting process and its connection with the Poisson process. *ALEA Lat. Am. J. Prob. Math. Statist.* **13,** 291–307.

[19] EHRENBERG, A. S. (1988). *Repeat-buying: Facts, Theory and Data*. Oxford University Press, New York.

[20] EMBRECHTS, P. AND HOFERT, M. (2013). A note on generalized inverses. *Math. Methods Oper. Res.* **77,** 423–432.

[21] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd edn. John Wiley, New York.

[22] FULGER, D., SCALAS, E. AND GERMANO, G. (2008). Monte carlo simulation of uncoupled continuous-time random walks yielding a stochastic solution of the space-time fractional diffusion equation. *Physical Review E* **77,** 021122.

[23] GERGELY, T. AND YEZHOW, I. I. (1973). On a construction of ordinary Poisson processes and their modelling. Z. *Wahrscheinlichkeitsth.* **27,** 215–232.

[24] GERGELY, T. AND YEZHOW, I. I. (1975). Asymptotic behaviour of stochastic processes modelling an ordinary Poisson process. *Period. Math. Hungar.* **6,** 203–211.

[25] GERMANO, G., POLITI, M., SCALAS, E. AND SCHILLING, R. L. (2009). Stochastic calculus for uncoupled continuous-time random walks. *Phys. Rev. E* **79,** 066102.

[26] GNEDENKO, B. V. AND KOVALENKO, I. N. (1968). *Introduction to Queueing Theory.* Translated from Russian by R. Kondor. Daniel Davey, Hartford, Conn.

[27] GRANDELL, J. (1976). *Doubly Stochastic Poisson Processes* (Lecture Notes Math. **529**). Springer, Berlin.

[28] GRANDELL, J. (1991). *Aspects of Risk Theory*. Springer, New York.

[29] GUT, A. (1974). On the moments and limit distributions of some first passage times. *Ann. Prob.* **2,** 277–308.

[30] GUT, A. (2013). *Probability: A Graduate Course*, 2nd edn. Springer, New York.

[31] JACOD, J. AND SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes* (Fundamental Principles Math. Sci. **288**), 2nd edn. Springer, Berlin.

[32] KHINTCHINE, A. Y. (1969). *Mathematical Methods in the Theory of Queueing* (Griffin's Statist. Monogr. Courses **7**), 2nd edn. Hafner Publishing, New York.

[33] KINGMAN, J. (1964). On the doubly stochastic Poisson processes. *Proc. Camb. Phil. Soc.* **60,** 923–930.

[34] KOZUBOWSKI, T. J. AND RACHEV, S. T. (1999). Univariate geometric stable laws. *J. Comput. Anal. Appl.* **1,** 177–217.

[35] LASKIN, N. (2003). Fractional Poisson process. *Commun. Nonlinear Sci. Numer. Simul.* **8,** 201–213.

[36] LEONENKO, N. AND MERZBACH, E. (2015). Fractional Poisson fields. *Methodol. Comput. Appl. Prob.* **17,** 155–168.

[37] LEONENKO, N., SCALAS, E. AND TRINH, M. (2017). The fractional non-homogeneous Poisson process. *Statist. Prob. Lett.* **120,** 147–156.

[38] MAHESHWARI, A. AND VELLAISAMY, P. (2016). On the long-range dependence of fractional Poisson and negative binomial processes. *J. Appl. Prob.* **53,** 989–1000.

[39] MAINARDI, F., GORENFLO, R. AND SCALAS, E. (2004). A fractional generalization of the Poisson processes. *Vietnam J. Math.* **32,** 53–64.

[40] MEERSCHAERT, M. M., NANE, E. AND VELLAISAMY, P. (2011). The fractional Poisson process and the inverse stable subordinator. *Electron. J. Prob.* **16,** 1600–1620.

[41] MEERSCHAERT, M. M. AND SCHEFFLER, H.-P. (2004). Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Prob.* **41,** 623–638.

[42] MEERSCHAERT, M. M. AND SIKORSKII, A. (2012). *Stochastic Models for Fractional Calculus* (De Gruyter Stud. Math. **43**). Walter de Gruyter, Berlin.

[43] MEERSCHAERT, M. M. AND STRAKA, P. (2013). Inverse stable subordinators. *Math. Model. Nat. Phenom.* **8,** 1–16.

[44] POLITI, M., KAIZOJI, T. AND SCALAS, E. (2011). Full characterisation of the fractional poisson process. *Europhysics Letters* **96**.

[45] POST, E. L. (1930). Generalized differentiation. *Trans. Amer. Math. Soc.* **32,** 723–781.

[46] RICHTER, W. (1965). Übertragung von Grenzaussagen für Folgen von zufälligen Grössen auf Folgen mit zufälligen Indizes. *Teor. Veroyat. Primen.* **10,** 82–94. English translation: *Theoret. Prob. Appl.* **10,** 74–84.

[47] SAMORODNITSKY, G. AND TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York.

[48] SERFOZO, R. F. (1972a). Conditional Poisson processes. *J. Appl. Prob.* **9,** 288–302.

[49] SERFOZO, R. F. (1972b). Processes with conditional stationary independent increments. *J. Appl. Prob.* **9,** 303–315.

[50] WATANABE, S. (1964). On discontinuous additive functionals and Lévy measures of a Markov process. *Japan. J. Math.* **34,** 53–70.

[51] WHITT, W. (2002). *Stochastic-Process Limits*. Springer, New York.

[52] WIDDER, D. V. (1941). *The Laplace Transform* (Princeton Math. Ser. **6**). Princeton University Press.