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ON SEQUENCES OF EXPECTED MAXIMA AND EXPECTED RANGES

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Abstract

We investigate conditions in order to decide whether a given sequence of real numbers represents expected maxima or expected ranges. The main result provides a novel necessary and sufficient condition, relating an expected maxima sequence to a translation of a Bernstein function through its Lévy–Khintchine representation.

Keywords: Expected maxima; expected range; Bernstein function; Lévy–Khintchine representation; order statistics

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1. Introduction

Let X be an integrable random variable (RV) and suppose that $X_{1:k} \leq \cdots \leq X_{k:k}$ are the order statistics arising from k independent copies of X. Based solely on the expected values of order statistics,

$$\mu_{i:k} = \mathbb{E}X_{i:k}, \qquad i = 1, 2, \ldots, k,\ k = 1, 2, \ldots,$$

Hoeffding (1953) constructed a sequence of RVs X_k that converge weakly to X, and thus, characterized the distribution function (DF) F of X through the triangular array $\mu_{i:k}$. Since each $\mu_{i:k}$ is a linear function of $\mu_{i:i}$, $1 \leq i \leq k$ (see Arnold *et al.* (1992, p. 112) or David and Nagaraja (2003, p. 45)), it immediately follows that the sequence $\{\mu_k\}_{k=1}^{\infty}$ of the expected maxima $\mu_k = \mu_{k:k}$ uniquely determines the DF. Hill and Spruill (1994), using a theorem of Müntz (1914), improved this result by showing that F is characterized by any subsequence $\{\mu_{k(j)}\}_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty} 1/k(j) = \infty$.

Moreover, Hill and Spruill (1994) proved the following continuity result.

Theorem 1.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of integrable RVs, $\{\mu_k\}_{k=1}^{\infty}$ a sequence of real numbers, and write $\mu_k(X_n)$ for the expected maxima of k independent and identically distributed (i.i.d.) copies of X_n. If $\mu_k(X_n) \to \mu_k$ as $n \to \infty$ for all $k \geq 1$ then the following are equivalent:*

(i) *there exists an integrable RV X such that $X_n \xrightarrow{\text{w}} X$ as $n \to \infty$ ('$\xrightarrow{\text{w}}$' denotes weak convergence) and $\mu_k(X) = \mu_k$ for all $k \geq 1$;*

(ii) *$\mu_k = o(k)$ and $\sum_{j=1}^{k} (-1)^j \binom{k}{j} \mu_j = o(k)$ as $k \to \infty$.*

To conclude weak convergence based on this result, it is helpful to recognize whether a given sequence $\{\mu_k\}_{k=1}^{\infty}$ represents the expected maxima of some RV. This question received its own interest, going back to Kadane (1971), (1974), Mallows (1973), Huang (1998), and Kolodynski

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(2000). In the sequel, a sequence that represents expected maxima of some RV will be called the *expected maxima sequence* (EMS). Kadane (1974) proved that a necessary and sufficient condition for the EMS is that the sequence $\{\mu_{k+2} - \mu_{k+1}\}_{k=0}^{\infty}$ is the moment sequence of a finite measure in the *open* interval (0, 1); i.e. there exists a finite measure τ in [0, 1] such that

$$\tau(\{0\}) = \tau(\{1\}) = 0 \quad \text{and} \quad \mu_{n+2} - \mu_{n+1} = \int_{[0,1]} u^n \, \mathrm{d}\tau(u), \qquad n = 0, 1, \ldots. \tag{1.1}$$

According to the well known characterization by Hausdorff (1921), this is equivalent to

$$(-1)^s \Delta^s (\mu_{k+2} - \mu_{k+1}) \geq 0, \qquad s \geq 0,\ k \geq 0,$$

(see Huang (1998)), where 'Δ' is the forward difference operator ($\Delta^0 \alpha_k = \alpha_k$, $\Delta^1 \alpha_k = \Delta\alpha_k = \alpha_{k+1} - \alpha_k$, $\Delta^{s+1} = \Delta\Delta^s$), plus conditions on the sequence μ_k that guarantee $\tau(\{0\}) = \tau(\{1\}) = 0$. Kolodynski (2000) completed Huang's result, proving that the boundary conditions on the measure τ are equivalent to $\mu_k = o(k)$ and $\sum_{j=1}^{k} (-1)^j \binom{k}{j} \mu_j = o(k)$ as $k \to \infty$. Hence, another complete characterization of EMSs is as follows (see Kolodynski (2000)).

Theorem 1.2. *A sequence $\{\mu_k\}_{k=1}^{\infty}$ represents the expected maxima of a nondegenerate integrable RV if and only if the following three conditions are satisfied:*

(i) $(-1)^{s+1} \Delta^s \mu_k > 0$ *for all* $s \geq 1$ *and* $k \geq 1$*;*

(ii) $\mu_k = o(k)$ *as* $k \to \infty$*;*

(iii) $\sum_{j=1}^{k} (-1)^j \binom{k}{j} \mu_j = o(k)$ *as* $k \to \infty$*.*

The aim of this paper is to shed some light on these necessary and sufficient conditions, noting that it is rather difficult to check either Kadane's condition (1.1) or Theorem 1.2(i)–(iii) in practical situations; we thus provide a much easier sufficient condition (of a different nature) in Corollary 3.3. In Section 2 we present an alternative proof of Theorem 1.2; the interest in this proof lies in its constructive part (see Remark 2.1(ii)).

Section 3 contains the main results, Theorems 3.1 and 3.2, with illustrative examples indicating their usefulness. The main result of Theorem 3.1 characterizes the EMSs using a novel method that relates any such sequence to a translation of a suitable Bernstein function through its Lévy–Khintchine representation. Finally, in Section 4 we provide similar results concerning sequences of expected ranges. Several examples are given.

2. A probabilistic proof of Theorem 1.2

For completeness of the presentation, we state a probabilistic proof that only uses the result from Hill and Spruill (see Theorem 1.1, above) plus the Hoeffding construction; thus, we do not invoke results from the moment problem.

Proof of Theorem 1.2. Assume first that $\mu_k = \mathbb{E}X_{k:k} = \mu_k(X)$ for some integrable and nondegenerate RV X with DF F. Then we have

$$\mu_k = \int_{-\infty}^{\infty} [\mathbf{1}(x > 0) - F^k(x)] \, \mathrm{d}x$$

($\mathbf{1}$ denotes an indicator function), and, thus, $(-1)^{s+1} \Delta^s \mu_k = \int_{-\infty}^{\infty} F^k(x)(1 - F(x))^s \, \mathrm{d}x > 0$. Also,

$$\frac{\mu_k}{k} = \int_0^1 u^{k-1} F^{-1}(u) \, \mathrm{d}u,$$

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where $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $0 < u < 1$, is the left-continuous inverse of F. Thus, by dominated convergence, we conclude that $\lim_{k\to\infty} \mu_k/k = 0$. Similarly,

$$\lim_{k\to\infty} \int_0^1 (1-u)^{k-1} F^{-1}(u)\,\mathrm{d}u = 0,$$

and it is easily seen that $\int_0^1 (1-u)^{k-1} F^{-1}(u)\,\mathrm{d}u = -(1/k)\sum_{j=1}^{k}(-1)^j\binom{k}{j}\mu_j$.

Conversely, assume that (i)–(iii) are satisfied and define the numbers

$$\beta_{i,n} = \frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{i+j}\mu_{i+j}, \qquad 1 \leq i \leq n,\ n \geq 1. \tag{2.1}$$

It is easily checked that, for every $n \geq 2$ and $1 \leq i \leq n-1$,

$$\beta_{i+1,n} - \beta_{i,n} = \binom{n}{i}(-1)^{n-i+1}\Delta^{n-i}\mu_i > 0.$$

Therefore, we can define the sequence of discrete uniform RVs X_n by

$$\mathbb{P}(X_n = \beta_{i,n}) = \frac{1}{n}, \qquad 1 \leq i \leq n,$$

noting that the support of X_n is the set $\{\beta_{1,n}, \ldots, \beta_{n,n}\}$ with $\beta_{1,n} < \beta_{2,n} < \cdots < \beta_{n,n}$. Fix now $k \geq 1$ and set $Z_{n,k} = \max\{X_{n,1}, \ldots, X_{n,k}\}$, where $X_{n,1}, \ldots, X_{n,k}$ are i.i.d. copies of X_n. It is clear that $\mathbb{P}(Z_{n,k} = \beta_{i,n}) = (i/n)^k - ((i-1)/n)^k$. Thus,

$$\begin{aligned}
\mu_k(X_n) &= \mathbb{E}Z_{n,k} \\
&= \sum_{i=1}^{n}\beta_{i,n}\bigg[\bigg(\frac{i}{n}\bigg)^k - \bigg(\frac{i-1}{n}\bigg)^k\bigg] \\
&= \sum_{i=1}^{n}\bigg[\bigg(\frac{i}{n}\bigg)^k - \bigg(\frac{i-1}{n}\bigg)^k\bigg]\frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{i+j}\mu_{i+j}.
\end{aligned}$$

Substituting $s = i+j$ so that $s \in \{1, \ldots, n\}$ and $j = s-i$, we obtain

$$\begin{aligned}
\mu_k(X_n) &= \sum_{s=1}^{n}\binom{n}{s}\frac{\mu_s}{n^k}\sum_{i=1}^{s}(-1)^{s-i}\binom{s-1}{i-1}[i^k - (i-1)^k] \\
&= \sum_{s=1}^{n}\binom{n}{s}\frac{\mu_s}{n^k}\sum_{i=0}^{s-1}(-1)^{s-1-i}\binom{s-1}{i}[(i+1)^k - i^k] \\
&= \sum_{s=1}^{n}\binom{n}{s}\frac{\mu_s}{n^k}\sum_{m=0}^{k-1}\binom{k}{m}\bigg\{\sum_{i=0}^{s-1}(-1)^{s-1-i}\binom{s-1}{i}i^m\bigg\},
\end{aligned}$$

where the term i^m should be treated as 1 if $i = m = 0$.

The expression in the curly brackets is a multiple of a Stirling number of the second kind; see Charalambides (2002, Theorem 8.4 and p. 164). Despite this, we can assign a simple probabilistic meaning to the sum, showing that it vanishes whenever $1 \le m < s-1$. Indeed, define

$$S(s,m) := \sum_{i=0}^{s-1} (-1)^{s-1-i} \binom{s-1}{i} i^m,$$

and consider m distinct balls and $s-1$ distinct cells ($s \ge 2$, $m \ge 1$). If we put the balls into the cells at random, then the probability that every cell is occupied by at least one ball is given by the inclusion-exclusion principle:

$$p(s,m) := \mathbb{P}(\text{every cell contains at least one ball}) = \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \left(\frac{s-1-i}{s-1}\right)^m.$$

Hence,

$$p(s,m) = \frac{1}{(s-1)^m} \sum_{i=0}^{s-1} (-1)^{s-1-i} \binom{s-1}{i} i^m = \frac{1}{(s-1)^m} S(s,m).$$

Since the probability $p(s,m)$ is obviously zero whenever $1 \le m < s-1$, we conclude that $S(s,m) = 0$ for $s \ge 3$ and $m = 1, \dots, s-2$. In other words, and since $S(s,0) = 0$ for $s \ge 2$, we can write $S(s,m) = S(s,m)\mathbf{1}(s \le m+1)$, $m \ge 0$, $s \ge 2$. Moreover, since $p(s,s-1) = (s-1)!/(s-1)^{s-1}$ (for $s \ge 2$), and $S(1,0) = 1$ by convention, we also have

$$S(s,s-1) = (s-1)! \quad \text{and} \quad S(s,0) = \mathbf{1}(s=1), \qquad s \ge 1.$$

Therefore,

$$S(s,m) = S(s,m)\mathbf{1}(s \le m+1), \qquad m \ge 0,\ s \ge 1.$$

Using this observation, we see that, for $n \ge k$,

$$\begin{aligned} \mu_k(X_n) &= \sum_{s=1}^{n} \left(\sum_{m=0}^{k-1} \binom{k}{m} S(s,m)\mathbf{1}(s \le m+1) \right) \binom{n}{s} \frac{\mu_s}{n^k} \\ &= \sum_{s=1}^{k} \left(\sum_{m=s-1}^{k-1} \binom{k}{m} S(s,m) \right) \binom{n}{s} \frac{\mu_s}{n^k}, \end{aligned}$$

since, for $s > k$, we have $\mathbf{1}(s \le m+1) = 0$ for all $m = 0, \dots, k-1$. Hence,

$$\lim_{n\to\infty} \mu_k(X_n) = \sum_{s=1}^{k} \left(\sum_{m=s-1}^{k-1} \binom{k}{m} S(s,m) \right) \lim_{n\to\infty} \binom{n}{s} \frac{\mu_s}{n^k}.$$

Clearly, $\lim_{n\to\infty} \binom{n}{s} \mu_s / n^k = 0$ for $s < k$. Thus, only the last term ($s = k$) survives, yielding

$$\lim_{n\to\infty} \mu_k(X_n) = \binom{k}{k-1} S(k,k-1) \lim_{n\to\infty} \binom{n}{k} \frac{\mu_k}{n^k} = k S(k,k-1) \frac{\mu_k}{k!} = \mu_k.$$

Since $\mu_k(X_n) \to \mu_k$ as $n \to \infty$ for all $k \ge 1$ and, by assumption, (ii) and (iii) are satisfied, it follows from Theorem 1.1 that there exists an integrable X such that $X_n \xrightarrow{\text{w}} X$ and $\mu_k(X) = \mu_k$ for all k, completing the proof. □

Remark 2.1. (i) The construction used in the proof follows the line of Hoeffding (1953); the difference here is that the numbers $\beta_{i,n}$ in (2.1) are not assumed to be expectations of (some) order statistics.

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(ii) The proof shows that, under (i), we can always construct a sequence X_n such that $\mu_k(X_n) \to \mu_k$ for all $k \geq 1$. However, without (ii) and (iii) it is possible that $X_n \xrightarrow{\text{w}} Y$ with $\mu_k(Y) \neq \mu_k$; see the examples given in Kolodynski (2000) and in Hill and Spruill (1994).

Example 2.1. Let $\mu_k = k - 1/(k+1)$. Then the values $m_k = \mu_{k+2} - \mu_{k+1} = 1 + 1/((k+2)(k+3))$ correspond to the moments of a finite measure in the interval [0, 1]. More specifically, one can verify that $m_k = \frac{7}{6}\mathbb{E}Y^k$, where $F_Y = \frac{6}{7}F_1 + \frac{1}{7}F_2$ with F_1 being the degenerate DF at 1 (the Dirac measure) and F_2 is the DF of a beta(2, 2) RV with density $f_2(y) = 6y(1-y)$, $0 < y < 1$. Also, a direct calculation using Newton's formula shows that, for $k \geq 0$ and $s \geq 1$,

$$\begin{aligned}\sum_{j=0}^{s}(-1)^{j+1}\binom{s}{j}\Big(k+j-\frac{1}{k+j+1}\Big) &= 0 + s\sum_{j=0}^{s-1}(-1)^j\binom{s-1}{j} + \sum_{j=0}^{s}(-1)^j\binom{s}{j}\int_0^1 x^{k+j}\,\mathrm{d}x \\ &= s\mathbf{1}(s=1) + \int_0^1 x^k(1-x)^s\,\mathrm{d}x.\end{aligned}$$

Using the above calculation, it is seen that

$$(-1)^{s+1}\Delta^s\mu_k = \sum_{j=0}^{s}(-1)^{j+1}\binom{s}{j}\mu_{k+j} = \mathbf{1}(s=1) + \frac{k!\,s!}{(k+s+1)!} > 0, \qquad k \geq 1,\ s \geq 1,$$

and $\sum_{j=1}^{k}(-1)^j\binom{k}{j}\mu_j = k/(k+1) - \mathbf{1}(k=1)$. Thus, μ_k satisfies (i) and (iii), but it is not an EMS since it fails to satisfy (ii). After some algebra it can be seen that the numbers $\beta_{i,n}$ in (2.1) are given by $\beta_{i,n} = i/(n+1) - 1 + n\mathbf{1}(i=n)$ and the sequence of discrete uniform RVs X_n, constructed in the proof, converges weakly to a uniform$(-1,0)$ RV X with $\mu_k(X) = -1/(k+1)$; thus, as $n \to \infty$, $\mu_k(X_n) \to \mu_k$ for all $k \geq 1$ (because (i) is satisfied; see Remark 2.1(ii)), $X_n \xrightarrow{\text{w}} X$ and $\mu_k(X) \neq \mu_k$ for all k. A similar calculation reveals that the sequence $\widetilde{\mu_k} = k/(k+1) - \mathbf{1}(k=1) = 1 - 1/(k+1) - \mathbf{1}(k=1)$ satisfies

$$(-1)^{s+1}\Delta^s\widetilde{\mu_k} = \mathbf{1}(k=1) + \frac{k!\,s!}{(k+s+1)!},$$
$$\sum_{j=1}^{k}(-1)^j\binom{k}{j}\widetilde{\mu_j} = k - \frac{1}{k+1}, \qquad k \geq 1,\ s \geq 1.$$

Therefore, (i) and (ii) hold but (iii) fails for $\widetilde{\mu_k}$. Now, the corresponding RVs X_n are uniformly distributed over $\{i/(n+1) - n\mathbf{1}(i=1)\}_{i=1}^{n}$ and, as $n \to \infty$, $\mu_k(X_n) \to \widetilde{\mu_k}$ for all $k \geq 1$, $X_n \xrightarrow{\text{w}} X$ which is uniform(0, 1) and, of course, $\mu_k(X) = k/(k+1) \neq \widetilde{\mu_k}$ only for $k = 1$; see the example in Hill and Spruill (1994. Erratum). Note that $\widetilde{\mu_k}$ and μ_k are dual sequences in the sense that if μ_k were the EMS for some RV X then $\widetilde{\mu_k}$ would be the EMS for $-X$ and vice versa; see Kolodynski (2000, p. 297).

3. Necessary and sufficient conditions via integral forms

Although the problem of characterizing sequences that represent expected maxima is completely solved by Theorem 1.2 (or (1.1)), it is usually a difficult task to check conditions (i)–(iii)

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(equivalently, to verify the existence of τ in (1.1)) for a given sequence, e.g. $\mu_k = \sqrt{k}$ or $\mu_k = \log k$. In this section we seek a different kind of necessary and sufficient condition, involving the notion of *integral forms*, according to the following definition (see also Definition 3.2, below).

Definition 3.1. We say that a function $g\colon [1,\infty) \to \mathbb{R}$ admits a generalized integral form if there exists a finite (positive) measure μ in $(0,\infty)$, and measurable functions h and s, with $h \geq 0$, such that

$$\begin{aligned} \int_{(0,\infty)} h(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y})\,\mathrm{d}\mu(y) &< \infty, \\ g(x) = \int_{(0,\infty)} h(y)(s(y)-\mathrm{e}^{-xy})\,\mathrm{d}\mu(y), &\qquad x \geq 1. \end{aligned} \tag{3.1}$$

We shall denote by $\mathcal{G}$ the class of all such functions and by $\mathcal{G}^*$ the subset of $\mathcal{G}$ that contains all nonconstant functions $g \in \mathcal{G}$; (3.1) will be denoted by $g = G_s(h;\ \mu)$. In the particular case where $h(y) = h_0(y)$, with

$$h_0(y) = \frac{\mathrm{e}^y}{1-\mathrm{e}^{-y}}, \qquad 0 < y < \infty, \tag{3.2}$$

we say that g is written in canonical form, and we denote (3.1) by $g = G_s(\mu) \equiv G_s(h_0;\ \mu)$.

Before proceeding to the main result we present some auxiliary results.

Lemma 3.1. *Every $g \in \mathcal{G}$ can be written in canonical form.*

Proof. For $g = G_s(h;\ \mu) \in \mathcal{G}$, we can define the measure ν by

$$\nu(A) = \int_A \mathrm{e}^{-y}(1-\mathrm{e}^{-y})h(y)\,\mathrm{d}\mu(y), \qquad A \text{ Borel},\ A \subseteq (0,\infty).$$

By (3.1), ν is finite, since $\int_{(0,\infty)} \mathrm{d}\nu(y) = \int_{(0,\infty)} \mathrm{e}^{-y}(1-\mathrm{e}^{-y})h(y)\,\mathrm{d}\mu(y) < \infty$. Thus,

$$\begin{aligned} g(x) &= \int_{(0,\infty)} h_0(y)(s(y)-\mathrm{e}^{-xy})(\mathrm{e}^{-y}(1-\mathrm{e}^{-y})h(y))\,\mathrm{d}\mu(y) \\ &= \int_{(0,\infty)} h_0(y)(s(y)-\mathrm{e}^{-xy})\,\mathrm{d}\nu(y) \quad \text{for all } x \geq 1, \end{aligned}$$

yielding $g = G_s(\nu)$. □

Lemma 3.2. *Let $g_1 = G_{s_1}(\mu_1)$ and $g_2 = G_{s_2}(\mu_2)$ be two functions in $\mathcal{G}$. Then, the following are equivalent:*

(i) *$g_1(k) - g_2(k) = c$ (constant), $k = 1, 2, \ldots$;*

(ii) *$g_1(x) - g_2(x) = c$ (constant) for all $x \geq 1$;*

(iii) *$\mu_1 = \mu_2$.*

Proof. Since the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are trivial, we show (i) $\Longrightarrow$ (iii). Clearly, (i) implies that $g_2(k) - g_2(1) = g_1(k) - g_1(1)$, i.e.

$$\int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y}-\mathrm{e}^{-ky})\,\mathrm{d}\mu_1(y) = \int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y}-\mathrm{e}^{-ky})\,\mathrm{d}\mu_2(y), \qquad k = 2, 3, \ldots. \tag{3.3}$$

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Consider the measures ν_i $(i = 1, 2)$ defined by $\nu_i((0, u]) = \mu_i([-\log u, \infty))$, $0 < u < 1$. Changing variables $y = -\log u$ in (3.3), and since $h_0(-\log u) = 1/(u(1-u))$, we obtain

$$\int_{(0,1)} (1 + u + \cdots + u^n)\, \mathrm{d}\nu_1(u) = \int_{(0,1)} (1 + u + \cdots + u^n)\, \mathrm{d}\nu_2(u), \qquad n = 0, 1, \ldots .$$

By induction on n, it follows that the finite measures ν_1 and ν_2 have all their moments equal, and since they have bounded supports, they are identical; see, e.g. Billingsley (1995, p. 388, Theorem 30.1). Therefore, for every $y \in (0, \infty)$, $\mu_1((0, y]) = \nu_1([\mathrm{e}^{-y}, 1)) = \nu_2([\mathrm{e}^{-y}, 1)) = \mu_2((0, y])$, yielding $\mu_1 = \mu_2$. □

Corollary 3.1. *The measure μ in the canonical form of $g \in \mathcal{G}$ is unique. In particular, $g(x) = G_s(\mu)(x) = 0$ if and only if $\mu = 0$; any nonvanishing constant function $g \notin \mathcal{G}$.*

In the following proposition we show that every function $g \in \mathcal{G}^*$ is a translation of a *Bernstein function*. Recall that a nonnegative function $\beta\colon [0, \infty) \to [0, \infty)$ is called Bernstein if it is continuous on $[0, \infty)$, infinitely differentiable in $(0, \infty)$, and its nth order derivative $\beta^{(n)}$ satisfies $(-1)^{n+1}\beta^{(n)}(x) \ge 0$ $(n = 1, 2, \ldots, x > 0)$; see Schilling *et al.* (2012, p. 21, Definition 3.1) (in the sequel, the value $\beta(0)$ will be defined by continuity as $\beta(0+)$).

Proposition 3.1. *Let $g = G_s(h; \mu) \in \mathcal{G}^*$. Then g is continuous on $[1, \infty)$, infinitely differentiable in $(1, \infty)$, and its nth order derivative is given by*

$$(-1)^{n+1} g^{(n)}(x) = \int_{(0,\infty)} y^n h(y) \mathrm{e}^{-xy}\, \mathrm{d}\mu(y) > 0, \qquad n = 1, 2, \ldots, \ x > 1. \tag{3.4}$$

Proof. Note that the right-hand side of (3.4) is strictly positive for all $x > 1$, because it can be written as $\int_{(0,\infty)} y^n h_0(y)\mathrm{e}^{-xy}\, \mathrm{d}\nu(y)$, where $\nu \ne 0$ is the measure in the canonical form of g; see Lemma 3.1 and Corollary 3.1. Also, the function g is continuous at $x = 1$ since for $y > 0$ and $\varepsilon \in (0, 1)$, $1 - \mathrm{e}^{-\varepsilon y} \le 1 - \mathrm{e}^{-y}$. Hence, by (3.1) and dominated convergence, $g(1+\varepsilon) - g(1) = \int_{(0,\infty)} h(y)\mathrm{e}^{-y}(1 - \mathrm{e}^{-\varepsilon y})\, \mathrm{d}\mu(y) \to 0$ as $\varepsilon \searrow 0$.

Regarding (3.4), we see that

$$\frac{\partial^n}{\partial x^n}(h(y)(s(y) - \mathrm{e}^{-xy})) = (-1)^{n+1} y^n h(y)\mathrm{e}^{-xy} \quad (n = 1, 2, \ldots)$$

is continuous in $x > 1$ for every fixed $y > 0$. Fix $\delta > 1$. Then, with $\theta = \delta - 1 > 0$,

$$\begin{aligned} y^n h(y)\mathrm{e}^{-xy} &\le h(y)\mathrm{e}^{-y}(1 - \mathrm{e}^{-y})\frac{y^n \mathrm{e}^{-\theta y}}{1 - \mathrm{e}^{-y}} \\ &\le h(y)\mathrm{e}^{-y}(1 - \mathrm{e}^{-y}) \sup_{y>0} \frac{y^n \mathrm{e}^{-\theta y}}{1 - \mathrm{e}^{-y}}, \qquad x > \delta,\ y > 0. \end{aligned}$$

The (positive) function $t(y) = y^n \mathrm{e}^{-\theta y}/(1 - \mathrm{e}^{-y})$ is bounded:

$$t(y) \le \begin{cases} \dfrac{y}{1 - \mathrm{e}^{-y}} \le \dfrac{1}{1 - \mathrm{e}^{-1}}, & 0 < y \le 1, \\[2ex] \dfrac{y^n \mathrm{e}^{-\theta y}}{1 - \mathrm{e}^{-1}} \le \dfrac{\max\{\mathrm{e}^{-\theta}, (n/\theta)^n \mathrm{e}^{-n}\}}{1 - \mathrm{e}^{-1}}, & y > 1. \end{cases}$$

Thus, choosing, e.g. $C = \max\{1, (n/\theta)^n \mathrm{e}^{-n}\}/(1-\mathrm{e}^{-1})$, we see that

$$\left|\frac{\partial^n}{\partial x^n}(h(y)(s(y)-\mathrm{e}^{-xy}))\right| = y^n h(y)\mathrm{e}^{-xy} \leq Ch(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y}), \qquad y>0,\ x>\delta.$$

Since the dominant function $K(y) = Ch(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y})$ is integrable with respect to μ, it is permitted to differentiate (3.1) under the integral sign (see, e.g. Ferguson (1996, p. 124)), yielding (3.4) for $x > \delta > 1$; and since $\delta > 1$ is arbitrary, we conclude (3.4). □

From Proposition 3.1, we see that if $g \in \mathcal{G}^*$ then the function $B(x) := g(x+1) - g(1)$, $x \geq 0$, is Bernstein (of a particular form). It is known that every Bernstein function β can be expressed by its *Lévy–Khintchine representation* (LKR)

$$\beta(x) = a_0 + a_1 x + \int_{(0,\infty)} (1-\mathrm{e}^{-xy})\,\mathrm{d}\nu(y), \qquad x \geq 0; \tag{3.5}$$

see Schilling *et al.* (2012, p. 21, Theorem 3.2). Of course, it is much simpler to verify the converse, i.e. every function that is expressed as in (3.5) is Bernstein (see the proof of Proposition 3.1). The triplet $(a_0, a_1;\ \nu)$ in the LKR is uniquely determined by β, the measure ν satisfies $\int_{(0,\infty)} \min\{1, y\}\,\mathrm{d}\nu(y) < \infty$, and the constants a_0, a_1 are nonnegative. Comparing the LKR of B with the canonical form of $g = G_s(\mu) \in \mathcal{G}^*$, we see that (see (3.1))

$$\begin{aligned} a_0 + a_1 x + \int_{(0,\infty)} (1-\mathrm{e}^{-xy})\,\mathrm{d}\nu(y) &= g(x+1) - g(1) \\ &= \int_{(0,\infty)} \mathrm{e}^{-y} h_0(y)(1-\mathrm{e}^{-xy})\,\mathrm{d}\mu(y), \qquad x \geq 0. \end{aligned}$$

That is, $a_0 = a_1 = 0$ and $\mathrm{d}\nu(y) = \mathrm{e}^{-y} h_0(y)\,\mathrm{d}\mu(y)$ is the LKR of $B(x) = g(x+1) - g(1)$. Conversely, if $\mathcal{B}^*$ denotes the class of Bernstein functions with LKR triplet $(0, 0;\ \nu)$, $\nu \neq 0$, it is not difficult to show that $g(x+1) - g(1) \in \mathcal{B}^*$ implies that $g \in \mathcal{G}^*$. Hence, $g \in \mathcal{G}^*$ if and only if $B \in \mathcal{B}^*$, and we conclude with the following proposition.

Proposition 3.2. *A function $g\colon [1,\infty) \to \mathbb{R}$ belongs to $\mathcal{G}^*$ if and only if $B(x) := g(x+1) - g(1)$, $x \geq 0$, is a Bernstein function that admits an LKR of the form (3.5) with $a_0 = a_1 = 0$, $\nu \neq 0$.*

We are now in a position to state and prove the main result.

Theorem 3.1. *For a real sequence $\{\mu_k\}_{k=1}^{\infty}$ the following are equivalent:*

(i) *there exists a nondegenerate integrable RV X such that $\mu_k(X) = \mu_k$ for $k = 1, 2, \ldots$;*

(ii) *the sequence $\{\mu_k\}_{k=1}^{\infty}$ is the restriction to the natural numbers of a function $g \in \mathcal{G}^*$ (for $\mathcal{G}^*$, see Definition 3.1), i.e. $\mu_k = g(k)$, $k = 1, 2, \ldots$;*

(iii) *there exists a Bernstein function B with LKR triplet $(0, 0;\ \nu)$, $\nu \neq 0$ (see (3.5)), such that $\mu_k = \mu_1 + B(k-1)$, $k = 1, 2, \ldots$.*

If one of (i), (ii), or (iii) is fulfilled by $\{\mu_k\}_{k=1}^{\infty}$ then the function $g \in \mathcal{G}^*$ in (ii) is unique, and admits the representation

$$g(x) = \int_{(0,\infty)} \frac{\lambda \mathrm{e}^{y}}{1-\mathrm{e}^{-y}} \left(\frac{\mu_1}{\lambda}\mathrm{e}^{-y}(1-\mathrm{e}^{-y}) + \mathrm{e}^{-y} - \mathrm{e}^{-xy}\right) \mathrm{d}F_Y(y), \qquad x \geq 1, \tag{3.6}$$

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where $\lambda = \mu_2 - \mu_1$, F_Y *is the DF of the RV* $Y = -\log F(V)$, F *is the DF of* X, *and the RV* V *has density*

$$f_V(x) = \frac{1}{\lambda} F(x)(1 - F(x)), \qquad -\infty < x < \infty;$$

the Bernstein function B *in (iii), which is also unique, is related to* g *by* $B(x) = g(x+1) - \mu_1$, $x \geq 0$.

Proof. (ii) $\Longrightarrow$ (i). Suppose that $\mu_k = g(k)$, $k = 1, 2, \ldots$, for some $g = G_s(h;\ \mu) \in \mathcal{G}^*$. It suffices to verify conditions (i)–(iii) of Theorem 1.2 for μ_k. From (3.4), $g'(x) > 0$ for $x > 1$. Hence, by monotone convergence and by continuity of g at $1+$,

$$\int_1^x g'(t)\,\mathrm{d}t = \lim_{\varepsilon \searrow 0} \int_{1+\varepsilon}^x g'(t)\,\mathrm{d}t = \lim_{\varepsilon \searrow 0} [g(x) - g(1+\varepsilon)] = g(x) - g(1), \qquad x > 1. \quad (3.7)$$

It should be noted that the differentiability of g in $(1, \infty)$ plus continuity at 1 are not sufficient for concluding (3.7), as the example $g(x) = (x-1)\sin(1/(x-1))$ shows. Now, by induction on s (and by using (3.7) when $k = 1$), it is easily seen that

$$\sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} g(k+j) = \int_k^{k+1} \int_{t_1}^{t_1+1} \cdots \int_{t_{s-1}}^{t_{s-1}+1} g^{(s)}(t_s)\,\mathrm{d}t_s \cdots \mathrm{d}t_2\,\mathrm{d}t_1, \qquad s \geq 1,\ k \geq 1.$$

Therefore, since $\mu_{k+j} = g(k+j)$,

$$(-1)^{s+1} \Delta^s \mu_k = \sum_{j=0}^{s} (-1)^{j+1} \binom{s}{j} g(k+j) = \int_k^{k+1} \int_{t_1}^{t_1+1} \cdots \int_{t_{s-1}}^{t_{s-1}+1} (-1)^{s+1} g^{(s)}(t_s)\,\mathrm{d}t_s \cdots \mathrm{d}t_2\,\mathrm{d}t_1;$$

the last expression verifies condition (i) of Theorem 1.2, because the integrand is strictly positive (see (3.4)). Condition (ii) of Theorem 1.2 is simply deduced from dominated convergence since $(1 - \mathrm{e}^{-ky})/k \leq 1 - \mathrm{e}^{-y}$ and, obviously, $(1 - \mathrm{e}^{-ky})/k \to 0$ as $k \to \infty$. Hence,

$$\lim_{k\to\infty} \frac{\mu_k}{k} = \lim_{k\to\infty} \frac{\mu_{k+1} - \mu_1}{k} = \lim_{k\to\infty} \int_{(0,\infty)} \mathrm{e}^{-y} h(y) \left(\frac{1 - \mathrm{e}^{-ky}}{k} \right) \mathrm{d}\mu(y) = 0.$$

Set now $\nu_k = \sum_{j=1}^{k} (-1)^j \binom{k}{j} \mu_j$, so that $\nu_1 = -\mu_1$. It is not difficult to check that $\nu_{s+1} - \nu_s = (-1)^{s+1} \Delta^s \mu_1 > 0$, where $\Delta^s \mu_1 = \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} \mu_{j+1}$. Defining $y_s := (-1)^{s+1} \Delta^s \mu_1 > 0$, we have

$$\nu_k = -\mu_1 + \sum_{s=1}^{k-1} (-1)^{s+1} \Delta^s \mu_1 = -\mu_1 + \sum_{s=1}^{k-1} y_s.$$

If it can be shown that $\lim_{k\to\infty} y_k = 0$ then it will follow that

$$\lim_{k\to\infty} \frac{\nu_k}{k} = \lim_{k\to\infty} \frac{y_1 + \cdots + y_{k-1}}{k} = 0,$$

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which means that the sequence μ_k satisfies condition (iii) of Theorem 1.2. Due to (3.1),

$$\begin{aligned} y_k &= \sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}g(j+1) \\ &= \sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}\int_{(0,\infty)} h(y)(s(y)-\mathrm{e}^{-(j+1)y})\,\mathrm{d}\mu(y) \\ &= \int_{(0,\infty)} h(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y})^k\,\mathrm{d}\mu(y) \to 0 \quad \text{as } k\to\infty, \end{aligned}$$

by dominated convergence.

(i) $\Longrightarrow$ (ii). Let F be the DF of X, and set $\alpha = \inf\{x\colon F(x) > 0\}$, $\omega = \sup\{x\colon F(x) < 1\}$. By the assumption that X is nondegenerate, it follows that $-\infty \le \alpha < \omega \le +\infty$, and the open interval (α, ω) has strictly positive (or infinite) length. We define the family of DFs $\{F^t,\ t \ge 1\}$, and denote by X_t a generic RV with DF F^t, so that $X_1 = X$. Since X is integrable, the same holds for each X_t. Indeed, $F^t(x) \le F(x)$ and $1 - F^t(x) \le t(1 - F(x))$ for all $x \in \mathbb{R}$ and $t \ge 1$, yielding

$$\begin{aligned} \mathbb{E}X_t^- &= \int_{-\infty}^{0} F^t(x)\,\mathrm{d}x \le \int_{-\infty}^{0} F(x)\,\mathrm{d}x < \infty, \\ \mathbb{E}X_t^+ &= \int_{0}^{\infty} (1-F^t(x))\,\mathrm{d}x \le t\int_{0}^{\infty} (1-F(x))\,\mathrm{d}x < \infty, \end{aligned}$$

where $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$ denote, respectively, the positive and negative part of any RV X. This enables us to define the function $g\colon [1,\infty) \to \mathbb{R}$ by

$$g(t) := \mathbb{E}X_t = \int_{-\infty}^{\infty} [\mathbf{1}(x>0) - F^t(x)]\,\mathrm{d}x, \qquad t \ge 1;$$

by definition, $g(k) = \mu_k$ for $k = 1, 2, \ldots$. For $t \in [1, \infty)$, write

$$\begin{aligned} g(t) - g(1) &= \int_{\alpha}^{\omega} [F(x) - F^t(x)]\,\mathrm{d}x \\ &= \int_{\alpha}^{\omega} F(x)(1-F(x))\frac{F(x)-F^t(x)}{F(x)(1-F(x))}\,\mathrm{d}x \\ &= \int_{\alpha}^{\omega} F(x)(1-F(x))\frac{\mathrm{e}^{-\delta(x)}-\mathrm{e}^{-t\delta(x)}}{\mathrm{e}^{-\delta(x)}(1-\mathrm{e}^{-\delta(x)})}\,\mathrm{d}x, \end{aligned} \tag{3.8}$$

where $\delta(x) = -\log F(x)$; note that $0 < F(x) < 1$ for all $x \in (\alpha, \omega)$, so that $\delta(x) > 0$. Setting $\lambda = \mu_2 - \mu_1 = g(2) - g(1) = \int_\alpha^\omega F(x)(1 - F(x))\,\mathrm{d}x > 0$, we readily see that $f_V(x) := F(x)(1 - F(x))/\lambda$ defines a probability density on $\mathbb{R}$ with support (α, ω). Consider an RV V with density f_V. Then (3.8) can be written as

$$g(t) - g(1) = \lambda\mathbb{E}\biggl\{\frac{\mathrm{e}^{\delta(V)}}{1-\mathrm{e}^{-\delta(V)}}(\mathrm{e}^{-\delta(V)} - \mathrm{e}^{-t\delta(V)})\biggr\}, \qquad t \ge 1,$$

where $\delta(V) = -\log F(V)$ is a strictly positive RV, because $\alpha < V < \omega$ with probability 1. Setting $Y := \delta(V) > 0$, we obtain

$$g(t)-g(1) = \lambda\mathbb{E}\biggl\{\frac{\mathrm{e}^{Y}}{1-\mathrm{e}^{-Y}}(\mathrm{e}^{-Y}-\mathrm{e}^{-tY})\biggr\} = \lambda\int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y}-\mathrm{e}^{-ty})\,\mathrm{d}F_Y(y), \qquad t \ge 1,$$

where $h_0(y) = \mathrm{e}^y/(1-\mathrm{e}^{-y})$ (see (3.2)) and F_Y is the DF of Y. If we introduce the measure μ defined by $\mu(A) = \lambda\mathbb{P}(Y \in A)$ for a Borel set $A \subseteq (0,\infty)$, the above relation takes the form

$$g(t) - g(1) = \int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y} - \mathrm{e}^{-ty})\,\mathrm{d}\mu(y), \qquad t \geq 1.$$

Moreover, since $h_0(y) > 0$,

$$0 < \int_{(0,\infty)} h_0(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y})\,\mathrm{d}\mu(y) = \int_{(0,\infty)} \mathrm{d}\mu(y) = \mu((0,\infty)) = \lambda < \infty.$$

Observing that

$$g(1) = \mu_1 = \frac{\mu_1}{\lambda}\int_{(0,\infty)} \mathrm{d}\mu(y) = \int_{(0,\infty)} h_0(y)\bigg(\frac{\mu_1}{\lambda}\mathrm{e}^{-y}(1-\mathrm{e}^{-y})\bigg)\,\mathrm{d}\mu(y),$$

we obtain

$$\begin{aligned} g(t) &= g(1) + (g(t) - g(1)) \\ &= \int_{(0,\infty)} h_0(y)\bigg(\frac{\mu_1}{\lambda}\mathrm{e}^{-y}(1-\mathrm{e}^{-y}) + \mathrm{e}^{-y} - \mathrm{e}^{-ty}\bigg)\,\mathrm{d}\mu(y), \qquad t \geq 1, \end{aligned}$$

proving both (3.1) and (3.6).

Finally, the equivalence of (ii) and (iii) follows from Proposition 3.2, and uniqueness (of g and μ) is evident from Lemma 3.2. □

The following definition provides a helpful tool in verifying whether a given function g belongs to $\mathcal{G}^*$.

Definition 3.2. Let $g\colon [1,\infty) \to \mathbb{R}$ be an arbitrary function. We say that g admits an integral form if there exist measurable functions $h_1\colon (0,\infty) \to \mathbb{R}$ and $s\colon (0,\infty) \to \mathbb{R}$, with $h_1 \geq 0$, such that

$$0 < \int_0^\infty h_1(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y})\,\mathrm{d}y < \infty \tag{3.9}$$

and

$$g(x) = \int_0^\infty h_1(y)(s(y) - \mathrm{e}^{-xy})\,\mathrm{d}y, \qquad x \geq 1. \tag{3.10}$$

We shall denote by $\mathcal{I}$ the class of all such functions and, provided that h_1 satisfies (3.9), representation (3.10) will be denoted by $g = I_s(h_1)$.

Lemma 3.3. *It holds that* $\mathcal{I} \subseteq \mathcal{G}^*$.

Proof. Assume that $g = I_s(h_1) \in \mathcal{I}$ and define the (positive) measure μ by

$$\mu((0,y]) = \int_0^y h_1(x)\mathrm{e}^{-x}(1-\mathrm{e}^{-x})\,\mathrm{d}x, \qquad y > 0.$$

By definition, μ is absolutely continuous with respect to the Lebesgue measure on $(0,\infty)$, with Radon–Nikodym derivative

$$\frac{\mathrm{d}\mu(y)}{\mathrm{d}y} = h_1(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y}) \quad \text{for almost all } y > 0.$$

Clearly, μ is finite, and (3.10) can be written as

$$g(x) = \int_0^\infty h_0(y)(s(y) - \mathrm{e}^{-xy})(h_1(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y}))\,\mathrm{d}y$$
$$= \int_{(0,\infty)} h_0(y)(s(y) - \mathrm{e}^{-xy})\,\mathrm{d}\mu(y), \qquad x \geq 1,$$

yielding the integral representation in (3.1). Moreover, from (3.9),

$$0 < \int_{(0,\infty)} h_0(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y})\,\mathrm{d}\mu(y) = \int_{(0,\infty)} \mathrm{d}\mu(y) = \int_0^\infty h_1(y)\mathrm{e}^{-y}(1-\mathrm{e}^{-y})\,\mathrm{d}y < \infty.$$

Hence, $g = G_s(\mu)$ with $\mu \neq 0$. □

Corollary 3.2. *The function h_1 in the integral representation (3.10) of any $g = I_s(h_1) \in \mathcal{I}$ is (almost everywhere) unique.*

Proof. If we express $g = I_s(h_1) \in \mathcal{G}$ in its canonical form as $g = G_s(\mu)$ (see Lemma 3.1), then the function $h_1(y)/h_0(y)$ is a Radon–Nikodym derivative of μ with respect to the Lebesgue measure. The result follows from Corollary 3.1 and the fact that the Radon–Nikodym derivative is almost everywhere unique. □

We can now state the following result which provides a sufficient condition that is useful for most practical situations.

Corollary 3.3. *If a function $g\colon [1,\infty) \to \mathbb{R}$ belongs to $\mathcal{I}$ (see Definition 3.2) then the sequence $\mu_k = g(k)$, $k = 1, 2, \ldots$, represents the expected maxima sequence of an integrable nondegenerate random variable.*

Proof. This is evident from Theorem 3.1 and Lemma 3.3. □

If $g = G_s(\mu) \in \mathcal{G}^*$ (see Definition 3.1) and the measure μ has a Radon–Nikodym derivative h_μ with respect to the Lebesgue measure, condition (3.1) is equivalent to (3.9) and (3.10). Indeed, in this case,

$$g(x) = \int_{(0,\infty)} h_0(y)(s(y) - \mathrm{e}^{-xy})\,\mathrm{d}\mu(y) = \int_0^\infty h_\mu(y)h_0(y)(s(y) - \mathrm{e}^{-xy})\,\mathrm{d}y,$$

and it is sufficient to choose $h_1 = h_0 \cdot h_\mu$. Hence, $g = G_s(\mu) \in \mathcal{I}$ if and only if the measure μ in the canonical form of g is (nonzero and) absolutely continuous with respect to the Lebesgue measure. However, given an *arbitrary* sequence μ_k, even if it can be shown that it is an EMS (using, e.g. Theorem 1.2, Theorem 3.1, Corollary 3.3, or (1.1)), we would like to decide if it corresponds to an absolutely continuous RV. We note at this point that the condition $g \in \mathcal{I}$ is neither necessary nor sufficient for concluding that the EMS $\{g(k)\}_{k=1}^\infty$ corresponds to a density (see Remark 3.1, below). An interesting exception where this fact can be deduced automatically is described in the following definition.

Definition 3.3. Denote by $\mathcal{F}$ the subclass of absolutely continuous RVs X with interval supports $(\alpha, \omega) = (\alpha_X, \omega_X)$, $-\infty \leq \alpha < \omega \leq +\infty$, having a differentiable DF F in (α, ω), and such that their density $f(x) = F'(x)$ is strictly positive and continuous in (α, ω).

Theorem 3.2. *For a given sequence $\{\mu_k\}_{k=1}^\infty$, the following statements are equivalent:*

(i) *the sequence μ_k represents an expected maxima sequence of an integrable RV $X \in \mathcal{F}$;*

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(ii) *there is an extension* $g\colon [1,\infty)\to\mathbb{R}$ *of the sequence* μ_k *(i.e.* $\mu_k = g(k)$, $k=1,2,\dots$*), such that* g *admits an integral representation of the form (3.10), with* h_1 *satisfying (3.9) and, furthermore,* h_1 *is strictly positive and continuous in* $(0,\infty)$.

Moreover, if (i) or (ii) holds then the function g *is unique, and the continuous version of* h_1 *in the integral representation (3.10) is uniquely determined by*

$$h_1(y) = \frac{\mathrm{e}^{-y}}{f(F^{-1}(\mathrm{e}^{-y}))}, \qquad 0 < y < +\infty, \tag{3.11}$$

where f *and* F^{-1} *are, respectively, the density and the inverse DF of the unique RV* $X \in \mathcal{F}$ *with expected maxima* μ_k*; any other version* h_2 *is equal to* h_1 *almost everywhere in* $(0,\infty)$.

Proof. Assume first that (i) holds, let F be the DF of X, and set $(\alpha,\omega) = \{x\colon 0 < F(x) < 1\}$. Since $X \in \mathcal{F}$, $\lambda := \mu_2 - \mu_1 > 0$. Using (3.6) and the fact that V has density

$$f_V(x) = \frac{1}{\lambda}F(x)(1-F(x)), \qquad \alpha < x < \omega,$$

the additional assumption $X \in \mathcal{F}$ implies that $Y = -\log F(V)$ has a continuous, strictly positive, density

$$f_Y(y) = \frac{\mathrm{e}^{-2y}(1-\mathrm{e}^{-y})}{\lambda f(F^{-1}(\mathrm{e}^{-y}))}, \qquad 0 < y < \infty,$$

with f and F^{-1} being, respectively, the derivative and the ordinary inverse of the restriction in (α,ω) of F. Substituting $\mathrm{d}F_Y(y) = f_Y(y)\,\mathrm{d}y$ in (3.6) we obtain (ii) with h_1 as in (3.11).

Assume now that (ii) holds. From (3.10),

$$\mu_k - \mu_1 = g(k) - g(1) = \int_0^\infty h_1(y)(\mathrm{e}^{-y} - \mathrm{e}^{-ky})\,\mathrm{d}y, \qquad k = 1,2,\dots. \tag{3.12}$$

Also, from Corollary 3.3 we see that the sequence $\mu_k = g(k)$ is an EMS of a unique (nondegenerate) RV X. It remains to show that $X \in \mathcal{F}$, i.e. that its DF F belongs to $\mathcal{F}$. To this end, define the function

$$G(u) := \begin{cases} c_1 - \displaystyle\int_u^{1/2} \frac{1}{t} h_1(-\log t)\,\mathrm{d}t, & 0 < u \le \frac{1}{2}, \\ c_1 + \displaystyle\int_{1/2}^{u} \frac{1}{t} h_1(-\log t)\,\mathrm{d}t, & \frac{1}{2} < u < 1, \end{cases} \tag{3.13}$$

where c_1 is a constant to be specified later. By the assumption on h_1, G is strictly increasing and differentiable in the interval $(0,1)$. Moreover, G is integrable, since by (3.9) and Tonelli's theorem,

$$\begin{aligned} \int_0^1 |G(u) - c_1|\,\mathrm{d}u &= \int_0^{1/2}\int_u^{1/2} \frac{1}{t} h_1(-\log t)\,\mathrm{d}t\,\mathrm{d}u + \int_{1/2}^1\int_{1/2}^u \frac{1}{t} h_1(-\log t)\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_0^{1/2} h_1(-\log t)\,\mathrm{d}t + \int_{1/2}^1 \frac{1-t}{t} h_1(-\log t)\,\mathrm{d}t \\ &= \int_{\log 2}^\infty \mathrm{e}^{-y} h_1(y)\,\mathrm{d}y + \int_0^{\log 2} (1-\mathrm{e}^{-y}) h_1(y)\,\mathrm{d}y \\ &< \infty. \end{aligned}$$

Let U be a uniform$(0, 1)$ RV and define the RV $Y := G(U)$ with DF F_Y; i.e. $G = F_Y^{-1}$. Clearly, $\mathbb{E}|Y| = \mathbb{E}|G(U)| < \infty$. We can show that $Y \in \mathcal{F}$. Indeed, setting $\alpha_Y := \lim_{u \searrow 0} G(u)$ and $\omega_Y := \lim_{u \nearrow 1} G(u)$, we see that $G\colon (0, 1) \to (\alpha_Y, \omega_Y)$ is strictly increasing and differentiable, with continuous, strictly positive, derivative $G'(u) = h_1(-\log u)/u$. This means that its inverse, $G^{-1} = F_Y\colon (\alpha_Y, \omega_Y) \to (0, 1)$, has also a continuous, strictly positive, derivative $f_Y(y) = F_Y'(y) = 1/G'(G^{-1}(y))$. Observe that $F_Y(y) = G^{-1}(y)$ tends to 0 as y approaches α_Y from above, so that, by monotone convergence,

$$\int_{\alpha_Y}^{y} f_Y(x)\,\mathrm{d}x = \lim_{a \searrow \alpha_Y} \int_a^y F_Y'(x)\,\mathrm{d}x = \lim_{a \searrow \alpha_Y} [F_Y(y) - F_Y(a)] = F_Y(y), \qquad \alpha_Y < y < \omega_Y.$$

Taking limits as $y \nearrow \omega_Y$ in the above relation, and using again monotone convergence and the fact that $G^{-1}(y)$ tends to 1 as $y \nearrow \omega_Y$, we see that

$$\int_{\alpha_Y}^{\omega_Y} f_Y(x)\,\mathrm{d}x = \lim_{y \nearrow \omega_Y} \int_{\alpha_Y}^{y} f_Y(x)\,\mathrm{d}x = \lim_{y \nearrow \omega_Y} F_Y(y) = \lim_{y \nearrow \omega_Y} G^{-1}(y) = 1;$$

hence, $Y \in \mathcal{F}$. According to the implication (i) $\Longrightarrow$ (ii), the sequence $\widetilde{\mu}_k := \mu_k(Y)$ admits an extension $g_2\colon [1, \infty) \to \mathbb{R}$ of the form

$$g_2(x) = \int_0^\infty h_2(y)(s_2(y) - \mathrm{e}^{-xy})\,\mathrm{d}y, \qquad x \geq 1,$$

such that h_2 satisfies (3.9) (with h_2 in place of h_1) and is continuous and strictly positive in $(0, \infty)$. Therefore, we have

$$\widetilde{\mu}_k - \widetilde{\mu}_1 = g_2(k) - g_2(1) = \int_0^\infty h_2(y)(\mathrm{e}^{-y} - \mathrm{e}^{-ky})\,\mathrm{d}y, \qquad k = 1, 2, \ldots. \tag{3.14}$$

We can calculate the same quantities directly from $G = F_Y^{-1}$ as follows:

$$\begin{aligned}
\widetilde{\mu}_k - \widetilde{\mu}_1 &= \int_0^1 k u^{k-1} G(u)\,\mathrm{d}u - \int_0^1 G(u)\,\mathrm{d}u \\
&= -\int_0^{1/2} k u^{k-1} \int_u^{1/2} \frac{1}{t} h_1(-\log t)\,\mathrm{d}t\,\mathrm{d}u + \int_0^{1/2} \int_u^{1/2} \frac{1}{t} h_1(-\log t)\,\mathrm{d}t\,\mathrm{d}u \\
&\quad + \int_{1/2}^1 k u^{k-1} \int_{1/2}^u \frac{1}{t} h_1(-\log t)\,\mathrm{d}t\,\mathrm{d}u - \int_{1/2}^1 \int_{1/2}^u \frac{1}{t} h_1(-\log t)\,\mathrm{d}t\,\mathrm{d}u.
\end{aligned}$$

Since all integrands in the last four integrals are nonnegative, we can interchange the order of integration. Thus,

$$\begin{aligned}
\widetilde{\mu}_k - \widetilde{\mu}_1 &= -\int_0^{1/2} t^{k-1} h_1(-\log t)\,\mathrm{d}t + \int_0^{1/2} h_1(-\log t)\,\mathrm{d}t \\
&\quad + \int_{1/2}^1 \frac{1 - t^k}{t} h_1(-\log t)\,\mathrm{d}t - \int_{1/2}^1 \frac{1 - t}{t} h_1(-\log t)\,\mathrm{d}t \\
&= \int_{\log 2}^\infty (-\mathrm{e}^{-ky} + \mathrm{e}^{-y}) h_1(y)\,\mathrm{d}y + \int_0^{\log 2} ((1 - \mathrm{e}^{-ky}) - (1 - \mathrm{e}^{-y})) h_1(y)\,\mathrm{d}y \\
&= \int_0^\infty (\mathrm{e}^{-y} - \mathrm{e}^{-ky}) h_1(y)\,\mathrm{d}y, \qquad k = 1, 2, \ldots.
\end{aligned} \tag{3.15}$$

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From (3.12), (3.14), and (3.15), we see that

$$g(k) - g(1) = \mu_k - \mu_1 = \widetilde{\mu}_k - \widetilde{\mu}_1 = g_2(k) - g_2(1), \qquad k = 1, 2, \ldots.$$

Therefore, since g and g_2 belong to $\mathcal{I} \subseteq \mathcal{G}^*$, it follows from Lemma 3.2 that $g(x) - g_2(x) = \mu_1 - \widetilde{\mu}_1$ (constant), $x \ge 1$. Choosing the constant c_1 in (3.13) so that $\widetilde{\mu}_1 = \mu_1$, we obtain $\widetilde{\mu}_k = \mu_k$ for all k, which implies that $g = g_2$ and $F = F_Y \in \mathcal{F}$.

Uniqueness of g and h_1 follows immediately from Theorem 3.1 and Corollary 3.2, respectively. □

Remark 3.1. Assume that $g \in \mathcal{G}^*$. The additional assumption $g \in \mathcal{I}$ is neither necessary nor sufficient for the EMS $\{g(k)\}_{k=1}^{\infty}$ to arise from an absolutely continuous RV.

(i) Consider the RV X with density $f(x) = \frac{1}{2}\mathbf{1}(-2 < x < -1) + \frac{1}{2}\mathbf{1}(1 < x < 2)$ so that $\mu_k = \mu_k(X) = 2(k/(k+1) - 2^{-k})$. Hence, $\mu_1 = 0$, $\lambda = \mu_2 - \mu_1 = \frac{5}{6}$, and, from (3.6), we see that $F_Y = \frac{3}{5}F_1 + \frac{2}{5}F_2$, where F_1 is the degenerate DF at $\log 2$ and the DF F_2 has density $f_2(y) = 6\mathrm{e}^{-2y}(1 - \mathrm{e}^{-y})$, $y > 0$. Since $\mu(\{\log 2\}) = \lambda\mathbb{P}(Y = \log 2) = \frac{1}{2}$, the function $g(x) = 2(x/(x+1) - 2^{-x}) = G_s(\mu)(x) \in \mathcal{G}$ has a non absolutely continuous canonical measure μ. Thus, $g \notin \mathcal{I}$.

(ii) For $h_1(y) = \mathbf{1}(0 < y < 1)$ and $s(y) = 1$, (3.10) yields $I_s(h_1)(x) = g(x) = 1 - (1 - \mathrm{e}^{-x})/x$, $x \ge 1$. It is easily checked that the particular EMS $\{g(k)\}_{k=1}^{\infty}$ corresponds to the DF F with inverse $F^{-1}(u) = (1 + \log u)\mathbf{1}(\mathrm{e}^{-1} < u < 1)$. However, this F does not have a density, since it assigns probability e^{-1} at the point 0.

Example 3.1. Let $\mu_k = k^{\theta}$, $0 < \theta < 1$, and define $g(x) = x^{\theta}$, $x \ge 1$. Representation (3.10) follows from

$$x^{\theta} = \int_0^x \frac{\theta}{t^{1-\theta}}\,\mathrm{d}t = \frac{\theta}{\Gamma(1-\theta)} \int_0^x \int_0^{\infty} y^{-\theta}\mathrm{e}^{-ty}\,\mathrm{d}y\,\mathrm{d}t = \frac{\theta}{\Gamma(1-\theta)} \int_0^{\infty} y^{-\theta} \int_0^x \mathrm{e}^{-ty}\,\mathrm{d}t\,\mathrm{d}y,$$

where the change in the order of integration is justified by Tonelli's theorem. Therefore,

$$x^{\theta} = \frac{\theta}{\Gamma(1-\theta)} \int_0^{\infty} \frac{1 - \mathrm{e}^{-xy}}{y^{1+\theta}}\,\mathrm{d}y, \qquad x \ge 0,\ 0 < \theta < 1. \tag{3.16}$$

Thus, (3.10) is satisfied with $h_1(y) = \beta_{\theta} y^{-1-\theta}$, where $\beta_{\theta} = \theta/\Gamma(1-\theta) > 0$, and $s(y) = 1$. Note that (3.6) suggests using a different function s, namely,

$$\widetilde{s}(y) = \mathrm{e}^{-y} + \frac{\mathrm{e}^{-y} - \mathrm{e}^{-2y}}{2^{\theta} - 1};$$

hence, s in the representations (3.10) or (3.1) need not be unique. Since (3.9) is obviously fulfilled, from Corollary 3.3 we see that the sequence k^{θ} is an EMS. More precisely, from Theorem 3.2 we see that the particular EMS, k^{θ}, corresponds to the RV $X \in \mathcal{F}$ with distribution inverse G given by (3.13) (with $h_1(y) = \beta_{\theta} y^{-1-\theta}$), i.e.

$$F^{-1}(u) = G(u) = \frac{\theta}{\Gamma(1-\theta)} \int \frac{(-\log u)^{-1-\theta}}{u}\,\mathrm{d}u = \frac{(-\log u)^{-\theta}}{\Gamma(1-\theta)} + C.$$

Since $\mu_1 = \int_0^1 F^{-1}(u)\,\mathrm{d}u = 1$, we find that $C = 0$ and the parent DF admits the explicit formula $F(x) = \exp(-\lambda x^{-1/\theta})$, $x > 0$, where $\lambda = \Gamma(1-\theta)^{-1/\theta} > 0$; thus, $1/X$ is Weibull.

Moreover, it is evident from Theorem 3.2 and (3.16) that $\{(k+c)^\theta\}_{k=1}^\infty$ is an EMS for every $c \in [-1, \infty)$ and $\theta \in (0,1)$, and the corresponding functions in the representation (3.10) are $h_1(y) = \beta_\theta \mathrm{e}^{-cy}/y^{1+\theta}$ and $s(y) = \mathrm{e}^{cy}$.

Example 3.2. Let $\mu_k = \log k$ and define $g(x) = \log x$, $x \ge 1$. To see representation (3.10), write (for $x > 0$)

$$\log x = \int_1^x \frac{1}{t}\,\mathrm{d}t = \int_1^x \int_0^\infty \mathrm{e}^{-ty}\,\mathrm{d}y\,\mathrm{d}t = \int_0^\infty \int_1^x \mathrm{e}^{-ty}\,\mathrm{d}t\,\mathrm{d}y = \int_0^\infty \frac{\mathrm{e}^{-y} - \mathrm{e}^{-xy}}{y}\,\mathrm{d}y, \tag{3.17}$$

yielding $h_1(y) = 1/y$ and $s(y) = \mathrm{e}^{-y}$. Again (3.9) is obviously fulfilled and from Corollary 3.3 we see that the sequence $\log k$ is an EMS. More precisely, (3.13) yields $F^{-1}(u) = -\log(-\log u) + C$, $0 < u < 1$. By the substitution $y = -\log u$, we obtain

$$\mu_1 = \int_0^1 F^{-1}(u)\,\mathrm{d}u = C - \int_0^\infty \mathrm{e}^{-y} \log y\,\mathrm{d}y = C + \gamma,$$

where γ is Euler's constant; see, e.g. Lagarias (2013, p. 535). Since $\mu_1 = \log 1 = 0$, it follows that $C = -\gamma$ and $F(x) = \exp(-\mathrm{e}^{-(x+\gamma)})$ is an extreme value (Gumbel) distribution. Furthermore, Theorem 3.2 and (3.17) enable us to verify that $\{\log(k+c)\}_{k=1}^\infty$ is an EMS for every $c \in (-1, \infty)$; the corresponding functions in the representation (3.10) are $h_1(y) = \mathrm{e}^{-cy}/y$ and $s(y) = \mathrm{e}^{(c-1)y}$.

Example 3.3. The harmonic number function was defined by Euler as

$$H(x) = \int_0^1 \frac{1-u^x}{1-u}\,\mathrm{d}u = \int_0^\infty \frac{\mathrm{e}^{-y}}{1-\mathrm{e}^{-y}}(1-\mathrm{e}^{-xy})\,\mathrm{d}y, \qquad x > -1; \tag{3.18}$$

see Lagarias (2013, p. 532). It satisfies

$$H(0) = 0, \qquad H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n = 1, 2, \ldots),$$
$$H(x) = H(x-1) + \frac{1}{x}, \qquad x > 0.$$

From Theorem 3.2, we conclude that, for every $c \in (-2, \infty)$, the sequence $\{H(k+c)\}_{k=1}^\infty$ is an EMS from an absolutely continuous RV; indeed, (3.18) shows that the function $g(x) = H(x+c)$ satisfies (3.9) and (3.10) with $h_1(y) = \mathrm{e}^{-(c+1)y}/(1-\mathrm{e}^{-y})$ and $s(y) = \mathrm{e}^{cy}$. The standard exponential corresponds to $c = 0$ and the standard logistic to $c = -1$; see Example 4.1, below. The function $\psi(x) = (\mathrm{d}/\mathrm{d}x)\log\Gamma(x) = \Gamma'(x)/\Gamma(x)$ admits a similar representation due to Gauss; see Lagarias (2013, p. 557). It follows that $\{\psi(k+c)\}_{k=1}^\infty$ is an EMS for $c > -1$. However, this fact is evident from the corresponding result for H, due to the relationship $\psi(x) + \gamma = H(x-1)$, $x > 0$. Finally, the easily verified identity

$$\mu_k := 1 + \frac{1}{2^\theta} + \cdots + \frac{1}{k^\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty \frac{y^{\theta-1}\mathrm{e}^{-y}}{1-\mathrm{e}^{-y}}(1-\mathrm{e}^{-ky})\,\mathrm{d}y \quad (\theta > 0,\ k = 1, 2, \ldots)$$

yields that this μ_k is an EMS for every $\theta > 0$ (choose $h_1(y) = \Gamma(\theta)^{-1} y^{\theta-1}\mathrm{e}^{-y}/(1-\mathrm{e}^{-y})$ and $s(y) = 1$ in (3.10)).

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Remark 3.2. It is known that the class of Bernstein functions is closed under composition; see Schilling *et al.* (2012, p. 28, Corollary 3.8). Therefore, the connection of EMSs to Bernstein functions (Theorem 3.1) provides an additional tool in verifying that a given sequence is an EMS. For instance, Example 3.1 with $c = -1$ shows that $g_1(x) := (x-1)^\theta$ $(x \ge 1, 0 < \theta < 1)$ belongs to $\mathcal{L}$; thus, from Lemma 3.3 and Proposition 3.2, $B_1(x) := g_1(x+1) - g_1(1) = x^\theta$ $(x \ge 0)$ is Bernstein. By the same reasoning, Example 3.2 (with $c = 0$) shows that $B_2(x) := \log(x+1)$ $(x \ge 0)$ is Bernstein and, hence, $\beta(x) := B_1(B_2(x)) = (\log(x+1))^\theta$ $(x \ge 0)$ is also a Bernstein function with LKR as in (3.5). Observing that $a_0 = \beta(0) = 0$ and $a_1 = \lim_{x\to\infty} \beta(x)/x = 0$, we see that the LKR triplet of β is of the form $(0, 0;\ \nu)$, $\nu \ne 0$. Hence, from Proposition 3.2, we see that, for any $\theta \in (0, 1]$, the function $g(x) := \beta(x-1) = (\log x)^\theta$ $(x \ge 1)$ belongs to $\mathcal{G}^*$, and we conclude, from Theorem 3.1, that $(\log k)^\theta$ is an EMS. Note that, for any $\delta > 0$, $(\log x)^{1+\delta} \notin \mathcal{G}$, since the second derivative changes its sign in the interval $(1, \infty)$; see Proposition 3.1.

4. Sequences of expected ranges

Denote by $R_k(X) = X_{k:k} - X_{1:k} = \max_i X_i - \min_i X_i$ the (sample) range based on k i.i.d. copies $X_1, \dots, X_k$ of an RV X. In this section we consider the similar question concerning expected ranges. That is, we want to decide whether a given sequence $\{\rho_k\}_{k=1}^\infty$ represents an *expected ranges sequence* (ERS), i.e. whether there exists an integrable RV X with

$$\mathbb{E}R_k(X) = \rho_k, \qquad k = 1, 2, \dots.$$

The following result is the range analogue of Theorem 1.2.

Theorem 4.1. *A sequence $\{\rho_k\}_{k=1}^\infty$ is an ERS of a nondegenerate integrable RV if and only if the following three conditions are satisfied:*

(i) $(-1)^{s+1}\Delta^s \rho_k > 0$ *for all* $s \ge 1$ *and* $k \ge 1$;

(ii) $\rho_k = o(k)$ *as* $k \to \infty$;

(iii) $\rho_k = \sum_{j=1}^k (-1)^j \binom{k}{j} \rho_j$ *for all* $k \ge 1$.

Proof. Conditions (i)–(iii) are necessary. Indeed, if $\rho_k = \mathbb{E}R_k(X)$ for some integrable RV X with DF F, then we have

$$\rho_k = \int_{-\infty}^{\infty} [1 - F^k(x) - (1 - F(x))^k]\,\mathrm{d}x, \qquad k \ge 1.$$

Therefore, for all $s \ge 1$, $k \ge 1$,

$$(-1)^{s+1}\Delta^s \rho_k = \int_{-\infty}^{\infty} [F^k(x)(1 - F(x))^s + F^s(x)(1 - F(x))^k]\,\mathrm{d}x > 0,$$

proving (i). With $F^{-1}(u) = \inf\{x\colon F(x) \ge u\}$, $0 < u < 1$, we can write

$$\frac{\rho_k}{k} = \frac{\mathbb{E}R_k(X)}{k} = \int_0^1 [u^{k-1} - (1-u)^{k-1}]F^{-1}(u)\,\mathrm{d}u \to 0 \quad \text{as } k \to \infty,$$

by dominated convergence; this verifies (ii). Finally,

$$\begin{aligned}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\rho_j &= \int_{-\infty}^{\infty}\sum_{j=1}^{k}(-1)^j\binom{k}{j}[1-F^j(x)-(1-F(x))^j]\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty}[1-F^k(x)-(1-F(x))^k]\,\mathrm{d}x\\ &= \rho_k,\end{aligned}$$

which is (iii).

Conversely, assume that (i)–(iii) hold, and consider the sequence $\mu_k = \frac{1}{2}\rho_k$. Obviously, conditions (i)–(iii) of Theorem 1.2 are fulfilled by μ_k. Hence, we can find an integrable RV X such that $\mathbb{E}X_{k:\,k} = \frac{1}{2}\rho_k$ for all $k \geq 1$. Since, however, $\mathbb{E}X_{1:\,k} = -\sum_{j=1}^{k}(-1)^j\binom{k}{j}\mathbb{E}X_{j:j}$ (for any integrable X), condition (iii) yields $\mathbb{E}X_{1:\,k} = -\frac{1}{2}\rho_k$; thus, $\mathbb{E}[X_{k:\,k} - X_{1:\,k}] = \rho_k$, and the proof is complete. □

Remark 4.1. (i) Condition (iii) implies that $\rho_1 = 0$ (trivial) and $\rho_3 = \frac{3}{2}\rho_2$. Condition (i) shows that $0 = \rho_1 < \rho_2 < \cdots$.

(ii) The random variable X, constructed in the sufficiency proof of Theorem 4.1, is symmetric, i.e. $X \overset{\mathrm{D}}{=} -X$ (where '$\overset{\mathrm{D}}{=}$' means equality in distribution). To see this, let $Y_i = -X_i$ with X_i being i.i.d. copies of X used in the proof. Then $\mathbb{E}Y_{k:\,k} = \mathbb{E}\max\{-X_1,\dots,-X_k\} = -\mathbb{E}\min\{X_1,\dots,X_k\} = \frac{1}{2}\rho_k = \mathbb{E}X_{k:\,k}$ for all $k \geq 1$; thus, by the result of Hoeffding, we see that X and Y have the same DF. In fact, this is the *unique symmetric* RV having the given expected ranges. Indeed, if Y is any symmetric RV with $\mathbb{E}R_k(Y) = \rho_k$ then, since $\mathbb{E}Y_{k:\,k} = -\mathbb{E}Y_{1:\,k}$ (by symmetry), we should have $\rho_k = 2\mathbb{E}Y_{k:\,k}$ for all $k \geq 1$.

(iii) For any integrable Y, we can find a *symmetric* integrable X with the same expected ranges. Indeed, if $\rho_k = \mathbb{E}R_k(Y)$ for arbitrary Y (not necessarily symmetric) then the sequence ρ_k satisfies conditions (i)–(iii) of Theorem 4.1. Thus, based on these values ρ_k, we can construct X as in the necessity proof, and this X is symmetric. This fact seems to be quite surprising at first glance. However, we observe that a DF F is symmetric (i.e. it corresponds to a symmetric RV X) if and only if $F^{-1}(u) = -F^{-1}((1-u)+)$, $0 < u < 1$, where $F^{-1}(t+)$ denotes the right-hand limit of F^{-1} at the point $t \in (0,1)$. Using this, it is easy to verify that the left-continuous inverses of the DFs of X and Y are related through

$$F_X^{-1}(u) = \tfrac{1}{2}[F_Y^{-1}(u) - F_Y^{-1}((1-u)+)], \qquad 0 < u < 1. \tag{4.1}$$

We conclude that the RV X, whose distribution inverse is defined by (4.1), is the unique symmetric RV with the same expected ranges as Y.

Example 4.1. It is well known that the order statistics from the exponential distribution have means

$$\mathbb{E}Y_{i:\,k} = \sum_{j=k-i+1}^{k}\frac{1}{j}, \qquad 1 \leq i \leq k,$$

and, therefore,

$$\rho_k = \mathbb{E}R_k(Y) = \mathbb{E}[Y_{k:\,k} - Y_{1:\,k}] = 1 + \frac{1}{2} + \cdots + \frac{1}{k-1} \quad (\rho_1 = 0).$$

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From Theorem 4.1 and Remark 4.1, we know that there exists a unique symmetric RV X with expected ranges ρ_k. Since $F^{-1}(u) = -\log(1-u)$, (4.1) shows that

$$F_X^{-1}(u) = \frac{1}{2}\log\left(\frac{u}{1-u}\right), \qquad 0 < u < 1,$$

which corresponds to a logistic RV with mean 0 and variance $\pi^2/12$. This is in accordance with the recurrence relation $\mu_{k+1} = 1/k + \mu_k$, satisfied by the expected maxima of the standard logistic distribution (with mean 0 and variance $\pi^2/3$), first obtained by Shah (1970); see also Arnold *et al.* (1992, p. 83).

Example 4.2. The expected ranges of a Bernoulli(p) RV are $1 - p^k - (1-p)^k$. The same expected ranges are obtained from a three-valued RV, assigning (equal) probabilities $\min\{p, 1-p\}$ at $\pm\frac{1}{2}$, and the remaining mass $1 - 2\min\{p, 1-p\}$ at 0.

Remark 4.2. If Y is symmetric around its mean μ then, obviously, the symmetric RV with the same expected ranges is $X = Y - \mu$. In particular, if Y is uniform(a, b) then X is uniform$(-\frac{1}{2}(b-a), \frac{1}{2}(b-a))$; if Y is $N(\mu, \sigma^2)$ then X is $N(0, \sigma^2)$. However, it should be noted that there exist nonnormal (nonuniform) RVs with expected ranges like normal (uniform); see Arnold *et al.* (1992, pp. 145–146). To highlight the situation, assume that X is $N(0, 1)$ with density ϕ, and let Φ be its DF with inverse Φ^{-1}. Let $0 < \varepsilon < \sqrt{2\pi}$ and define $h(u) = \Phi^{-1}(u) + u(1-u)\varepsilon$. Then, $h \in L^1(0, 1)$ and $h'(u) = 1/\phi(\Phi^{-1}(u)) + (1-2u)\varepsilon > 0$ for all $u \in (0, 1)$. The fact that $h'(u) > 0$ is obvious for $0 < u \le \frac{1}{2}$ and it remains to verify that

$$\varepsilon < \frac{1}{(2u-1)\phi(\Phi^{-1}(u))}, \qquad \frac{1}{2} < u < 1.$$

This is indeed satisfied because

$$\inf_{1/2<u<1}\left\{\frac{1}{(2u-1)\phi(\Phi^{-1}(u))}\right\} = \frac{1}{\sup_{1/2<u<1}\{(2u-1)\phi(\Phi^{-1}(u))\}} \ge \sqrt{2\pi}$$

since

$$\sup_{1/2<u<1}\{(2u-1)\phi(\Phi^{-1}(u))\} = \sup_{x>0}\{(2\Phi(x)-1)\phi(x)\} \le \sup_{x>0}\phi(x) = \frac{1}{\sqrt{2\pi}}.$$

Defining the RV $Y = h(U)$, where U is uniform$(0, 1)$, we see that $F_Y^{-1} = h$; thus, Y is nonnormal, and

$$\begin{aligned}
\mathbb{E}R_k(Y) &= k\int_0^1 (u^{k-1} - (1-u)^{k-1})\Phi^{-1}(u)\,\mathrm{d}u + k\varepsilon\int_0^1 (u^{k-1} - (1-u)^{k-1})u(1-u)\,\mathrm{d}u \\
&= k\int_0^1 (u^{k-1} - (1-u)^{k-1})\Phi^{-1}(u)\,\mathrm{d}u \\
&= \mathbb{E}R_k(X) \quad \text{for all } k \ge 1.
\end{aligned}$$

Similar examples can be found for most RVs. For example, a uniform$(0, 1)$ RV X has the same expected ranges as a beta$(\frac{1}{2}, 1)$ RV Y with density $f_Y(y) = (2\sqrt{y})^{-1}\mathbf{1}(0 < y < 1)$.

From Remark 4.2, it is clear that, in contrast to the expected maxima sequences, the sequences of expected ranges are far from characterizing the location family of the distribution.

We summarize these facts in the following theorem.

Theorem 4.2. (i) *A sequence $\{\rho_k\}_{k=1}^{\infty}$ represents the expected ranges of an integrable RV if and only if it represents the expected maxima of a symmetric (around 0) integrable RV.*

(ii) *For every integrable Y, there exists a unique symmetric integrable X with the same expected ranges as Y; X and Y are related through (4.1).*

(iii) *The integrable RVs X and Y have the same expected ranges if and only if the (generalized) inverses F_X^{-1} and F_Y^{-1} of their DFs satisfy*

$$F_X^{-1}(u) - F_Y^{-1}(u) = F_X^{-1}((1-u)+) - F_Y^{-1}((1-u)+), \qquad 0 < u < 1, \tag{4.2}$$

i.e. if and only if the function $h(u) = F_X^{-1}(u) - F_Y^{-1}(u)$ is symmetric around $\frac{1}{2}$ for almost all $u \in (0, 1)$.

Proof. (i) and (ii) are discussed in Remark 4.1; note that the symmetric RV X whose expected maxima are the expected ranges of Y is given by (see (4.1))

$$F_X^{-1}(u) = F_Y^{-1}(u) - F_Y^{-1}((1-u)+), \qquad 0 < u < 1.$$

To prove (iii), assume first that $h := F_X^{-1} - F_Y^{-1}$ is almost everywhere symmetric around $\frac{1}{2}$. Then

$$\mathbb{E}R_k(X) - \mathbb{E}R_k(Y) = k \int_0^1 [u^{k-1} - (1-u)^{k-1}] h(u) \, \mathrm{d}u = 0 \quad \text{for all } k \geq 1,$$

because the integrand, $g(u) = [u^{k-1} - (1-u)^{k-1}]h(u)$, is antisymmetric around $\frac{1}{2}$ (i.e. $g(1-u) = -g(u)$ for almost all u).

Conversely, $\mathbb{E}R_k(X) = \mathbb{E}R_k(Y)$ for all k implies that

$$\int_0^1 [u^{k-1} - (1-u)^{k-1}][F_X^{-1}(u) - F_Y^{-1}(u)] \, \mathrm{d}u = \int_0^1 u^{k-1} g(u) \, \mathrm{d}u = 0 \quad \text{for all } k \geq 1,$$

where $g(u) = [F_X^{-1}(u) - F_Y^{-1}(u)] - [F_X^{-1}(1-u) - F_Y^{-1}(1-u)]$. Since $g \in L^1(0, 1)$ and $\int_0^1 u^n g(u) \, \mathrm{d}u = 0$ for $n = 0, 1, \ldots$, it follows that $g = 0$ almost everywhere in $(0, 1)$. This means that, for almost all $u \in (0, 1)$,

$$F_X^{-1}(u) - F_Y^{-1}(u) = F_X^{-1}(1-u) - F_Y^{-1}(1-u),$$

which, taking left limits to both sides, yields (4.2). □

Therefore, every ERS is just a translation of an EMS from a symmetric RV (around its mean), and we can apply Theorem 3.1 to obtain the following characterization.

Theorem 4.3. *Let $\mathcal{X}_s$ be the class of nondegenerate, integrable RVs that are symmetric around their means. A sequence $\{\mu_k\}_{k=1}^{\infty}$ is an EMS from an RV $X \in \mathcal{X}_s$ if and only if it can be extended to a function $g = G_s(\mu) \in \mathcal{G}^*$ and, furthermore, the (unique) measure μ in the canonical form of g satisfies*

$$\mu((0, y]) = \mu([-\log(1 - \mathrm{e}^{-y}), \infty)), \qquad 0 < y < \infty. \tag{4.3}$$

If such an extension g exists, it is unique (and it is given by (3.6)).

Proof. Let $\mu_k = \mu_k(X)$ be the EMS of an RV $X \in \mathcal{X}_s$. By Theorem 3.1, μ_k admits an extension $g = G_s(\mu) \in \mathcal{G}^*$. Also, $X - \mu_1$ is symmetric around 0 and, according to Theorem 4.2(i), $\rho_k = \mu_k - \mu_1$ is an ERS. In particular, $\rho_k = \mu_k - \mu_1$ satisfies condition (iii) of Theorem 4.1, i.e. $(\mu_k - \mu_1) = \sum_{j=1}^{k}(-1)^j\binom{k}{j}(\mu_j - \mu_1)$, $k = 1, 2, \ldots$. Substituting $\mu_j - \mu_1 = g(j) - g(1) = \int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y} - \mathrm{e}^{-jy})\,\mathrm{d}\mu(y)$ $(j = 1, 2, \ldots, k)$, we obtain

$$\begin{aligned}\int_{(0,\infty)} & h_0(y)(\mathrm{e}^{-y} - \mathrm{e}^{-ky})\,\mathrm{d}\mu(y)\\ &= \sum_{j=1}^{k}(-1)^j\binom{k}{j}\int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y} - \mathrm{e}^{-jy})\,\mathrm{d}\mu(y)\\ &= \int_{(0,\infty)} h_0(y)(1 - \mathrm{e}^{-y} - (1 - \mathrm{e}^{-y})^k)\,\mathrm{d}\mu(y), \qquad k = 1, 2, \ldots. \end{aligned} \tag{4.4}$$

Consider the measure ν defined by $\nu((0, y]) = \mu([-\log(1 - \mathrm{e}^{-y}), \infty))$, $0 < y < \infty$. Clearly, $\nu \neq 0$ is finite. Changing variables $y = -\log(1 - \mathrm{e}^{-w})$ in (4.4), and since $h_0(-\log(1 - \mathrm{e}^{-w})) = h_0(w)$, $0 < w < \infty$ (see (3.2)), we obtain

$$\int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y} - \mathrm{e}^{-ky})\,\mathrm{d}\mu(y) = \int_{(0,\infty)} h_0(w)(\mathrm{e}^{-w} - \mathrm{e}^{-kw})\,\mathrm{d}\nu(w), \qquad k = 1, 2, \ldots. \tag{4.5}$$

Setting $s_0(y) = \mathrm{e}^{-y}$, we see that the function $g_2 := G_{s_0}(\nu) \in \mathcal{G}^*$, and (4.5) shows that $g(k) - g_2(k) = \mu_1$ (constant) for $k = 1, 2 \ldots$; hence, $\mu = \nu$ (see Lemma 3.2). Therefore, for all $y \in (0, \infty)$, $\mu((0, y]) = \nu((0, y]) = \mu([-\log(1 - \mathrm{e}^{-y}), \infty))$, $0 < y < \infty$, and (4.3) follows.

Conversely, assume that there exists an extension $g = G_s(\mu) \in \mathcal{G}^*$ of μ_k with μ satisfying (4.3). From Theorem 3.1, we see that μ_k is an EMS and, thus, $\rho_k = \mu_k - \mu_1$ is also an EMS. This means that the sequence ρ_k satisfies conditions (i) and (ii) of Theorem 4.1 (or of Theorem 1.2). Moreover,

$$\begin{aligned}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\rho_j &= \sum_{j=1}^{k}(-1)^j\binom{k}{j}\int_{(0,\infty)} h_0(y)(\mathrm{e}^{-y} - \mathrm{e}^{-jy})\,\mathrm{d}\mu(y)\\ &= \int_{(0,\infty)} h_0(y)(1 - \mathrm{e}^{-y} - (1 - \mathrm{e}^{-y})^k)\,\mathrm{d}\mu(y), \qquad k = 1, 2, \ldots. \end{aligned}$$

Substituting $y = -\log(1 - \mathrm{e}^{-w})$ in the last integral, and in view of (4.3), it is easily seen that this integral is equal to ρ_k, and we conclude that condition (iii) of Theorem 4.1 is also satisfied by ρ_k. Thus, ρ_k is an ERS and, therefore, it is an EMS from a (unique) symmetric (around 0) RV Y (see Theorem 4.2(i)); i.e. $\mu_k = \mu_1 + \rho_k$ is the EMS of $X = \mu_1 + Y$, which is symmetric around its mean μ_1.

Uniqueness follows from Lemma 3.2. □

Corollary 4.1. *A sequence* $\{\rho_k\}_{k=1}^{\infty}$ *is an ERS of a nondegenerate RV if and only if* $\rho_1 = 0$ *and there exists an extension* $g = G_s(\mu) \in \mathcal{G}^*$ *of* ρ_k *such that the measure* μ *satisfies (4.3).*

Corollary 4.2. *Assume that the function g admits an integral representation of the form (3.10) with h_1 satisfying (3.9); i.e. $g = I_s(h_1) \in \mathcal{I}$. Then:*

(i) *the sequence $\mu_k = g(k)$ is an EMS of a symmetric (around its mean) nondegenerate RV if and only if*

$$h_1(-\log(1-\mathrm{e}^{-y})) = (\mathrm{e}^y - 1)h_1(y) \quad \textit{for almost all } y \in (0,\infty); \tag{4.6}$$

(ii) *the sequence $\rho_k = g(k)$ is an ERS of a nondegenerate RV if and only if $\rho_1 = 0$ and (4.6) is satisfied.*

Proof. The assumption on g implies that $g \in \mathcal{I} \subseteq \mathcal{G}^*$ and, thus, $g = G_s(\mu)$ for a unique $\mu \neq 0$ (see Lemmas 3.1 and 3.3 and Corollary 3.1). From (3.9), we see that μ is absolutely continuous with respect to the Lebesgue measure on $(0,\infty)$, with Radon–Nikodym derivative $h_\mu := h_1/h_0$ (where $h_0(y) = \mathrm{e}^y/(1-\mathrm{e}^{-y})$; see (3.2)). Moreover, if ν is the measure defined by $\nu((0,y]) = \mu([-\log(1-\mathrm{e}^{-y}),\infty))$, $0 < y < \infty$, then ν is also absolutely continuous with respect to the Lebesgue measure, since

$$\nu((0,y]) = \mu([-\log(1-\mathrm{e}^{-y}),\infty)) = \int_{-\log(1-\mathrm{e}^{-y})}^{\infty} h_\mu(x)\,\mathrm{d}x, \qquad 0 < y < \infty.$$

From this expression, it follows that a Radon–Nikodym derivative of ν is given by

$$h_\nu(y) := \frac{\mathrm{d}\nu(y)}{\mathrm{d}y} = \frac{\mathrm{e}^{-y}}{1-\mathrm{e}^{-y}} h_\mu(-\log(1-\mathrm{e}^{-y})), \qquad 0 < y < \infty.$$

Since $\mu = \nu$ if and only if $h_\mu = h_\nu$ almost everywhere (a.e.) in $(0,\infty)$, we conclude that (4.3) is equivalent to (4.6). The result follows from Theorems 4.3 and 4.2(i). □

Example 4.3. If H is the harmonic number function then $g(x) := H(x+c) = I_s(h_1)(x)$ $(c > -2)$, where $h_1(y) = \mathrm{e}^{-(c+1)y}/(1-\mathrm{e}^{-y})$ and $s(y) = \mathrm{e}^{cy}$; see (3.18). It is easily seen that (4.6) reduces to $(\mathrm{e}^y - 1)^{c+1} = 1$ a.e., and, thus, it is satisfied if and only if $c = -1$. This shows that the only symmetric RV in this family is the logistic, completing both Examples 3.3 and 4.1.

Example 4.4. For $g(x) := \log(x+c) = I_s(h_1)(x)$ $(c > -1)$, $h_1(y) = \mathrm{e}^{-cy}/y$, and $s(y) = \mathrm{e}^{(c-1)y}$; see (3.17). Hence, (4.6) is written as $(\mathrm{e}^y - 1)^{c-1} = -\log(1-\mathrm{e}^{-y})/y$ a.e. and, obviously, this identity cannot be fulfilled (by any value of $c > -1$). Hence, all EMSs of Example 3.2 correspond to asymmetric RVs.

Example 4.5. For $g(x) := (x+c)^\theta = I_s(h_1)(x)$ $(c \geq -1, \theta \in (0,1))$, $h_1(y) = \beta_\theta \mathrm{e}^{-cy}/y^{1+\theta}$ and $s(y) = \mathrm{e}^{cy}$, where $\beta_\theta > 0$ is a constant; see (3.16). Therefore, (4.6) is now reduced to the identity $(\mathrm{e}^y - 1)^{c-1} = (-\log(1-\mathrm{e}^{-y})/y)^{1+\theta}$ a.e. Obviously, this is impossible (for all values of $c \geq -1$ and $\theta \in (0,1)$). Hence, all EMSs of Example 3.1 correspond to asymmetric RVs.

Example 4.6. For $g(x) := 1 - 1/(x+c) = I_s(h_1)(x)$ $(c > -1)$, $h_1(y) = \mathrm{e}^{-cy}$ and $s(y) = \mathrm{e}^{(c-1)y}$. Therefore, (4.6) is now reduced to the identity $(\mathrm{e}^y - 1)^{c-1} = 1$ a.e. Obviously, this identity is satisfied if and only if $c = 1$ (which corresponds to a standard uniform RV). Hence, $\{g(k)\}_{k=1}^{\infty}$ is an EMS for every $c > -1$ (Theorem 3.2), but the corresponding RV is asymmetric, unless $c = 1$. Using (3.13), it is recognized that $c = 0$ corresponds to the RV $1 - Y$, with Y being a standard exponential.

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