HOLOMORPHIC CONVEXITY FOR GENERAL FUNCTION ALGEBRAS

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Introduction. In previous papers (**7**; **8**), we have investigated certain properties of general function algebras which may be regarded as generalizations or analogues of familiar results in the theory of analytic functions of several complex variables. This investigation is continued and expanded in the present paper. The main results concern a notion of holomorphic convexity for the general situation. We also extend somewhat several of the results obtained in the earlier papers.

The setting for our investigations is a "system" $[\Sigma, \mathfrak{A}]$ consisting of a Hausdorff space Σ and an algebra $\mathfrak{A}$ of complex-valued continuous functions on Σ. It is always assumed that $\mathfrak{A}$ contains the constant functions and also determines the topology in Σ. In addition, for the most important results, it is assumed that every continuous homomorphism of $\mathfrak{A}$ onto the complex numbers is given by evaluation at a point of Σ. Then $[\Sigma, \mathfrak{A}]$ is called a "natural system." The prototype of a natural system is $[C^n, \mathfrak{P}]$, where C^n is ordinary n-dimensional complex space and $\mathfrak{P}$ is the algebra of all polynomials in n complex variables. Important examples are provided by a Stein space with its algebra of holomorphic functions (**3**, p. 222) and the Gelfand representation of a commutative Banach algebra on its space of maximal ideals (**6**, 3.1.20). Various properties of natural systems are discussed in §1. Included, for instance, is a notion of Šilov boundary for the non-compact case. In §2 a class of "$\mathfrak{A}$-holomorphic" functions is defined and some of its properties obtained. These functions are derived from the algebra $\mathfrak{A}$ by a succession of local approximations. The important fact concerning $\mathfrak{A}$-holomorphic functions is that they satisfy a local maximum modulus principle. In §3, "$\mathfrak{A}$-holomorphic convexity" for subsets of Σ is defined in terms of the $\mathfrak{A}$-holomorphic functions, and conditions for certain sets in Σ to be $\mathfrak{A}$-holomorphically convex are obtained. For example, let G be an open set in Σ and denote by $\mathcal{O}_G$ the algebra of all $\mathfrak{A}$-holomorphic functions defined on G. If $[\Sigma, \mathfrak{A}]$ is natural, then G is $\mathfrak{A}$-holomorphically convex if and only if the system $[G, \mathcal{O}_G]$ is natural. This contains as a special case the theorem proved in (**8**). It also implies a known result for Stein spaces (**3**, VII, A7). The proof depends on convexity properties of $\mathfrak{A}$-analytic varieties which were obtained in (**7**, Theorem 3.2). Finally, we show that under suitable countability assumptions an open $\mathfrak{A}$-holomorphically convex set is

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actually a region of holomorphy; i.e. it is the domain of definition of an $\mathfrak{A}$-holomorphic function which cannot be holomorphically extended to a larger region.

All proofs of the above results are "function algebraic" in character, the only contact with the theory of several complex variables being through Hugo Rossi's local maximum modulus principle for function algebras **(9)**.

1. Natural systems, $[\Sigma, \mathfrak{A}]$**.** Let Σ be a Hausdorff space and $\mathfrak{A}$ a subalgebra of the algebra $C(\Sigma)$ of complex-valued continuous functions on Σ which contains the constant functions. The pair $[\Sigma, \mathfrak{A}]$ will be called a *system* if $\mathfrak{A}$ determines the topology of Σ; i.e. the topology of Σ is the weakest for which all functions in $\mathfrak{A}$ are continuous. The topology assumed for $\mathfrak{A}$ is the compact-open topology given by uniform convergence on compact subsets of Σ. We shall always assume that $[\Sigma, \mathfrak{A}]$ denotes a system in the above sense. Observe that no compactness conditions are imposed on Σ. We did assume in **(7)** that the spaces involved were locally compact; however, the condition was not actually needed.

If X is an arbitrary subset of Σ, we shall denote by $[X, \mathfrak{A}]$ the system obtained by taking the topology in X to be that induced by Σ and the algebra on X to be that obtained by restriction of functions in $\mathfrak{A}$ to the set X. We also call $[X, \mathfrak{A}]$ a *subsystem* of $[\Sigma, \mathfrak{A}]$. Two systems are called *isomorphic* if there exists a homeomorphism between their topological spaces which induces an isomorphism of the two algebras.

Let $\phi: a \rightarrow \hat{a}(\phi)$ denote a homomorphism of $\mathfrak{A}$ onto the complex numbers C. Then it is readily verified that ϕ is continuous with respect to the compact-open topology in $\mathfrak{A}$ if and only if there exists a compact set $K_\phi \subseteq \Sigma$ such that, for each $a \in \mathfrak{A}$, $|\hat{a}(\phi)| \leqslant |a|_{K_\phi}$, where, for arbitrary $X \subset \Sigma$, $|a|_X$ denotes the supremum of numbers $|a(\xi)|$ for $\xi \in X$. The compact set K_ϕ is said to *dominate* ϕ, and we denote by $\mathscr{K}_\phi$ the collection of all such compact sets. A standard application of Zorn's lemma shows that each element of $\mathscr{K}_\phi$ contains a minimal element of $\mathscr{K}_\phi$. A minimal element of $\mathscr{K}_\phi$ is called a *support* for ϕ. A given homomorphism may, of course, have many different supports.

Each point $\sigma \in \Sigma$ determines a homomorphism ϕ_σ defined by evaluation at σ; i.e. $\hat{a}(\phi_\sigma) = a(\sigma)$, $a \in \mathfrak{A}$. It is obvious that ϕ_σ is continuous, with the set consisting of the single point σ as a support. For these special homomorphisms, we write $\mathscr{K}_\sigma$ in place of $\mathscr{K}_{\phi_\sigma}$ and say that elements of $\mathscr{K}_\sigma$ *dominate the point* σ. Similarly, a minimal element of $\mathscr{K}_\sigma$ is called a *support for* σ.

We now define the class of systems with which we are primarily concerned in the following discussion.

1.1. *Definition.* The system $[\Sigma, \mathfrak{A}]$ is said to be *natural* if each continuous homomorphism of $\mathfrak{A}$ onto C is given by evaluation at a point of Σ. If every homomorphism (continuous or not) is so given, then $[\Sigma, \mathfrak{A}]$ is said to be *strictly natural.*

In **(7)** we used the term "natural" to refer to systems which we now call "strictly natural." Quigley **(5)**, in a similar context, has called such function algebras "generic."

If $\mathfrak{B}$ is a commutative Banach algebra with an identity element and $\Phi_{\mathfrak{B}}$ is its space of maximal ideals, then $[\Phi_{\mathfrak{B}}, \hat{\mathfrak{B}}]$, where $\hat{\mathfrak{B}}$ is the Gelfand representation of $\mathfrak{B}$ as an algebra of continuous functions on $\Phi_{\mathfrak{B}}$, is a strictly natural system **(6)**. If C^n is n-dimensional complex space and $\mathfrak{P}$ is the algebra of all polynomials in n complex variables, then $[C^n, \mathfrak{P}]$ is also a strictly natural system **(7)**. More generally, let Λ be an arbitrary index set and associate with each $\lambda \in \Lambda$ a complex plane C_λ. Denote by C^Λ the ordinary cartesian product of the planes C_λ. Also denote by $\mathfrak{P}$ the algebra of all polynomials in variables $\{\zeta_\lambda: \lambda \in \Lambda\}$. Thus each $P \in \mathfrak{P}$ is an ordinary polynomial in a finite number of the variables ζ_λ. We regard $\mathfrak{P}$ as an algebra of continuous functions on C^Λ and obtain a system $[C^\Lambda, \mathfrak{P}]$ which is strictly natural **(8)**.

The system $[C^\Lambda, \mathfrak{P}]$ may be regarded as a product of the one-dimensional systems $[C_\lambda, \mathfrak{P}_\lambda]$ where $\mathfrak{P}_\lambda$ is the algebra of polynomials in one variable ζ_λ. In an analogous way, one may define the product of an arbitrary family, $\{[\Sigma_\lambda, \mathfrak{A}_\lambda]: \lambda \in \Lambda\}$, of systems. First take Σ^Λ to be the ordinary cartesian product of the spaces Σ_λ and let $\check{\sigma} = \{\sigma_\lambda\}$ denote an arbitrary element of Σ^Λ. Then take $\mathfrak{A}^\Lambda$ to be the Kronecker (or tensor) product of the algebras $\mathfrak{A}_\lambda$. The algebra $\mathfrak{A}^\Lambda$ may be identified with the algebra of functions on Σ^Λ generated by all functions of the form

$$A_\lambda(\check{\sigma}) = a_\lambda(\sigma_\lambda), \qquad \check{\sigma} \in \Sigma^\Lambda,$$

for arbitrary $\lambda \in \Lambda$ and $a_\lambda \in \mathfrak{A}_\lambda$. Observe that the mapping $a_\lambda \rightarrow A_\lambda$ defines an isomorphism of $\mathfrak{A}_\lambda$ into $\mathfrak{A}^\Lambda$. It follows that every continuous homomorphism of $\mathfrak{A}^\Lambda$ onto C induces a continuous homomorphism of $\mathfrak{A}_\lambda$ onto C. This implies that, if each of the systems $[\Sigma_\lambda, \mathfrak{A}_\lambda]$ is natural, then the product system $[\Sigma^\Lambda, \mathfrak{A}^\Lambda]$ is also natural.

Of special importance for our purposes is the product of a general system $[\Sigma, \mathfrak{A}]$ with one of the form $[C^\Lambda, \mathfrak{P}]$. In this case, the product $\mathfrak{A} \times \mathfrak{P}$ of the two algebras may be regarded as the algebra of all polynomials in the variables $\{\zeta_\lambda: \lambda \in \Lambda\}$ with coefficients in the algebra $\mathfrak{A}$. Since $[C^\Lambda, \mathfrak{P}]$ is strictly natural, the system $[\Sigma \times C^\Lambda, \mathfrak{A} \times \mathfrak{P}]$ will be natural if and only if $[\Sigma, \mathfrak{A}]$ is natural.

We observe next that an arbitrary system $[\Sigma, \mathfrak{A}]$ may be represented as a subsystem of one of the form $[C^\Lambda, \mathfrak{P}]$. Assume that $\{z_\lambda: \lambda \in \Lambda\} = \mathfrak{A}$ and to each point $\sigma \in \Sigma$ associate the point $\sigma^* = \{z_\lambda(\sigma)\}$ in C^Λ. Then the mapping $\sigma \rightarrow \sigma^*$ is a homeomorphism on Σ to a subset Σ^* of C^Λ. Let P be an arbitrary element of $\mathfrak{P}$, with

$$P(\check{\zeta}) = \sum \alpha_{k_1 \ldots k_n} \zeta_{\lambda_1}^{k_1} \ldots \zeta_{\lambda_n}^{k_n},$$

set $\check{z} = \{z_\lambda\}$, and define

$$P(\check{z}) = \sum \alpha_{k_1 \ldots k_n} z_{\lambda_1}^{k_1} \ldots z_{\lambda_n}^{k_n}.$$

Then the mapping $P \to P(\check{z})$ is a homomorphism of $\mathfrak{P}$ onto $\mathfrak{A}$. The kernel of the homomorphism is the ideal $\mathfrak{K}$ in $\mathfrak{P}$ consisting of all polynomial relations among the elements z_λ. Also, since $P(\sigma^*) = P(\check{z})(\sigma)$ for each $\sigma \in \Sigma$, we have

$$\mathfrak{K} = \{P: P \in \mathfrak{P}, P|\Sigma^* = 0\}.$$

Thus $\mathfrak{P}$ modulo $\mathfrak{K}$ is isomorphic with $\mathfrak{P}|\Sigma^*$, and the mapping $P|\Sigma^* \to P(\check{z})$ is an isomorphism between the algebra $\mathfrak{P}|\Sigma^*$ and $\mathfrak{A}$ which is induced by the homeomorphism $\sigma \to \sigma^*$. Therefore the systems $[\Sigma, \mathfrak{A}]$ and $[\Sigma^*, \mathfrak{P}]$ are isomorphic under this homeomorphism. Now let K be a compact set in Σ and define

$$\Delta_K = \{\check{\zeta}: \check{\zeta} \in C^\Lambda, |\zeta_\lambda| \leqslant |z_\lambda|_K, \lambda \in \Lambda\}.$$

Then Δ_K is a compact polydisk in C^Λ. Denote by Δ the union of all of the polydisks Δ_K. Also let

$$\mathscr{V}(\mathfrak{K}) = \{\check{\zeta}: \check{\zeta} \in C^\Lambda, P(\check{\zeta}) = 0, P \in \mathfrak{K}\}.$$

Then $\mathscr{V}(\mathfrak{K})$ is an example of a "subvariety" of C^Λ (see §2), also called the "hull" of the ideal $\mathfrak{K}$. It is not difficult to prove that, if $[\Sigma, \mathfrak{A}]$ is natural, then $\Sigma^* = \mathscr{V}(\mathfrak{K}) \cap \Delta$ and, if $[\Sigma, \mathfrak{A}]$ is strictly natural, then $\Sigma^* = \mathscr{V}(\mathfrak{K})$.

Next let us recall the definition of $\mathfrak{A}$-convexity for a system $[\Sigma, \mathfrak{A}]$. This notion is a generalization of polynomial convexity for $[C^n, \mathfrak{P}]$. If K is a compact subset of Σ, then the set

$$\hat{K} = \{\sigma: \sigma \in \Sigma, |a(\sigma)| \leqslant |a|_K, a \in \mathfrak{A}\}$$

is called the $\mathfrak{A}$-*convex hull* of K. Observe that $\hat{K}$ is always a closed subset of Σ and $|a|_{\hat{K}} = |a|_K$ for each $a \in \mathfrak{A}$. Hence, if K' is any compact subset of $\hat{K}$, then $\hat{K}' \subseteq \hat{K}$. Thus $\hat{K}$ is "convex" in the sense of the following definition:

1.2. *Definition.* A subset Ω of Σ is said to be $\mathfrak{A}$-*convex* if for each compact set $K \subseteq \Omega$ it is true that $\hat{K} \subseteq \Omega$. If X is an arbitrary subset of Σ, then the smallest $\mathfrak{A}$-convex set that contains X is called the $\mathfrak{A}$-*convex hull* of X and denoted by $\hat{X}$.

The entire space Σ and the empty set $\emptyset$ are obviously $\mathfrak{A}$-convex. Thus every subset of Σ is contained in at least one $\mathfrak{A}$-convex set. Observe also that the intersection of an arbitrary family of $\mathfrak{A}$-convex sets is $\mathfrak{A}$-convex. Therefore the $\mathfrak{A}$-convex hull $\hat{X}$ always exists and may be defined as the intersection of all $\mathfrak{A}$-convex sets that contain X. Note that a set X is $\mathfrak{A}$-convex if and only if $\hat{X} = X$.

1.3. PROPOSITION. *If* $[\Sigma, \mathfrak{A}]$ *is natural and* $\Omega \subseteq \Sigma$, *then the subsystem* $[\Omega, \mathfrak{A}]$ *will be natural if and only if* Ω *is* $\mathfrak{A}$-*convex. If* $[\Sigma, \mathfrak{A}]$ *is strictly natural, then* $[\Omega, \mathfrak{A}]$ *will be strictly natural if and only if* $\Omega = \mathscr{V}(\mathfrak{K})$, *where* $\mathfrak{K}$ *is the kernel of the homomorphism* $a \to a|\Omega$, $a \in \mathfrak{A}$.

Proof. If ϕ is any homorphism of $\mathfrak{A}|\Omega$ onto C, then the mappings

$$a \to a|\Omega \to \widehat{a|\Omega}(\phi), \qquad a \in \mathfrak{A},$$

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define a homomorphism of $\mathfrak{A}$ onto C. Moreover, if ϕ is continuous, then the homomorphism of $\mathfrak{A}$ is also continuous. Therefore, in the present situation, the only homomorphisms of $\mathfrak{A}|\Omega$ that we have to do with are of the form $a|\Omega \rightarrow a(\sigma_0)$ for some $\sigma_0 \in \Sigma$. Furthermore, a mapping of this kind determines a homomorphism of $\mathfrak{A}|\Omega$ if and only if $\sigma_0 \in \mathscr{V}(\mathfrak{K})$, i.e., if and only if $a|\Omega = 0$ implies $a(\sigma_0) = 0$. This already shows that, if $[\Sigma, \mathfrak{A}]$ is strictly natural, then $[\Omega, \mathfrak{A}]$ will be strictly natural if and only if $\Omega = \mathscr{V}(\mathfrak{K})$. The mapping $a|\Omega \rightarrow a(\sigma_0)$ will be a continuous homomorphism if and only if $\sigma_0 \in \hat{K}$ for some compact $K \subseteq \Omega$. Therefore, if $[\Sigma, \mathfrak{A}]$ is natural, then $[\Omega, \mathfrak{A}]$ will be natural if and only if Ω is $\mathfrak{A}$-convex.

For an arbitrary system $[\Sigma, \mathfrak{A}]$, the $\mathfrak{A}$-convex hull of a compact set need not be compact. However, if $[\Sigma, \mathfrak{A}]$ is natural, then $\hat{K}$ is always compact for compact K. This result, which is fundamental in the study of natural systems, is proved by showing that $\hat{K}$ is the space of maximal ideals of a commutative Banach algebra with identity element (**7**, Lemma 1.1). In fact, it is in this way that the theory of Banach algebras becomes available to us. More precisely, if $[\Sigma, \mathfrak{A}]$ is natural and Ω is a compact $\mathfrak{A}$-convex set in Σ, then Ω is the space of maximal ideals of the Banach algebra $\mathfrak{B}$ obtained by closing $\mathfrak{A}|\Omega$ in $C(\Omega)$. In particular, since a homomorphism of a Banach algebra onto C is automatically continuous, it follows that $[\Omega, \mathfrak{B}]$ is strictly natural.

The following proposition contains an extension of a familiar result for the compact case.

1.4. PROPOSITION. *Assume* $[\Sigma, \mathfrak{A}]$ *to be natural and let* Ω *be an* $\mathfrak{A}$*-convex subset of* Σ *which decomposes in the form* $\Omega = \Omega_1 \cup \Omega_2$ *where* $\bar{\Omega}_1 \cap \Omega_2 = \Omega_1 \cap \bar{\Omega}_2 = \emptyset$. *Then* Ω_1 *and* Ω_2 *are also* $\mathfrak{A}$*-convex.*

Proof. Let K be compact in Ω_1. Then $\hat{K}$ is compact, contained in Ω, and is the space of maximal ideals of the Banach algebra $\mathfrak{B}$ obtained by closing $\mathfrak{A}|\hat{K}$ in $C(\hat{K})$. Also $\hat{K} = (\hat{K} \cap \Omega_1) \cup (\hat{K} \cap \Omega_2)$ is a decomposition of $\hat{K}$ into disjoint closed sets. It follows from the Šilov decomposition theorem (**6**, 3.6.3) applied to $[\hat{K}, \mathfrak{B}]$ that $\hat{K} \cap \Omega_1$ and $\hat{K} \cap \Omega_2$ are $\mathfrak{B}$-convex in $\hat{K}$. Since $\mathfrak{A}|\hat{K}$ is dense in $\mathfrak{B}$ and $\hat{K}$ is $\mathfrak{A}$-convex, we conclude that these sets are $\mathfrak{A}$-convex in Σ. Since $K \subseteq \hat{K} \cap \Omega_1$, it follows that $\hat{K} \subseteq \Omega_1$. Therefore Ω_1, and similarly Ω_2, is $\mathfrak{A}$-convex.

Next we introduce a definition of Šilov boundary for a general system $[\Sigma, \mathfrak{A}]$. Note first that if δ is any point of Σ, then a compact set K will dominate δ (i.e. $K \in \mathscr{K}_\delta$) if and only if $\delta \in \hat{K}$. When Σ is compact, a point δ is called a strong boundary point of Σ with respect to $\mathfrak{A}$ if there exists for each neighbourhood U of δ an element $u \in \bar{\mathfrak{A}}$ such that $|u|_{\Sigma-U} < |u|_\Sigma = |u(\delta)|$. Thus every compact set K that belongs to $\mathscr{K}_\delta$ must contain δ. In particular, the only support for the point δ is the set consisting of δ itself. Recall also that, in the compact case, strong boundary points are dense in the Šilov boundary (**6**, 3.3.15). These observations suggest the following definition which applies to an arbitrary system $[\Sigma, \mathfrak{A}]$.

1.5. *Definition.* A point $\delta \in \Sigma$ is called an *independent point* of $[\Sigma, \mathfrak{A}]$ if it is supported only by itself. The closure of the set I of independent points is called the *Šilov boundary* of $[\Sigma, \mathfrak{A}]$ and denoted by $\partial[\Sigma, \mathfrak{A}]$.

If Σ is compact, then $\partial[\Sigma, \mathfrak{A}]$ clearly reduces to the ordinary Šilov boundary. On the other hand, if Σ is not compact, then $\partial[\Sigma, \mathfrak{A}]$ may be empty. This is the case, for example, with $[C^n, \mathfrak{P}]$. However, we have the following generalization of a familiar property of the Šilov boundary in the compact case.

1.6. THEOREM. *Let X be an arbitrary subset of Σ. Then every independent point of $[\hat{X}, \mathfrak{A}]$ is contained in X. If X is closed, then $\partial[\hat{X}, \mathfrak{A}] \subseteq X$.*

Proof. Let δ be an independent point of $[\hat{X}, \mathfrak{A}]$ and consider the set $Y = \hat{X} - \{\delta\}$. For any compact set $K \subseteq Y$, we have $\hat{K} \subseteq \hat{X}$. Furthermore, since $\delta \notin K$ and δ is an independent point of $[\hat{X}, \mathfrak{A}]$, it follows that also $\delta \notin \hat{K}$. Hence $\hat{K} \subseteq Y$. But this means that Y is a proper $\mathfrak{A}$-convex subset of $\hat{X}$. Therefore Y cannot contain X. In other words, $\delta \in X$ and the proof is complete.

A fundamental result in the theory of function algebras is a local maximum modulus principle which was proved by Hugo Rossi (**9**; **3**, p. 62). A version of the principle which is appropriate for our purposes may be stated as follows: Let $[\Sigma, \mathfrak{A}]$ be a natural system with compact Σ and let U be an open subset of Σ which does not intersect $\partial[\Sigma, \mathfrak{A}]$. Then, for every $a \in \mathfrak{A}$, we have $|a|_U = |a|_{\text{bdry } U}$. This says, in particular, that if $\Gamma = \text{bdry } U$, then $U \subseteq \hat{\Gamma}$. Observe that if U is any open set in Σ for which there exists $u \in \mathfrak{A}$ with $|u|_U > |u|_{\text{bdry } U}$, then U must contain points of $\partial[\bar{U}, \mathfrak{A}]$. Hence U must contain a strong boundary point of $\bar{U}$. Such points are obviously independent points of $[U, \mathfrak{A}]$. This suggests the following definition for an arbitrary system $[\Sigma, \mathfrak{A}]$.

1.8. *Definition.* A point $\delta \in \Sigma$ is called a *locally independent* point of $[\Sigma, \mathfrak{A}]$ if there exists a neighbourhood V of δ such that δ is an independent point of $[V, \mathfrak{A}]$.

Every independent point of $[\Sigma, \mathfrak{A}]$ is obviously locally independent. The example $[C^n, \mathfrak{P}]$ shows, as in the case of independent points, that there may be no locally independent points. In any case, we have the following generalization of the local maximum modulus principle to general systems.

1.8. THEOREM. *If $[\Sigma, \mathfrak{A}]$ is a natural system, then every locally independent point of $[\Sigma, \mathfrak{A}]$ is independent.*

Proof. Let δ be a locally independent point of $[\Sigma, \mathfrak{A}]$ with V a neighbourhood of δ such that δ is independent in $[V, \mathfrak{A}]$. Suppose that δ is not independent in

$[\Sigma, \mathfrak{A}]$. Then there exists a compact set K in Σ such that $\delta \in \hat{K} - K$. Since K is compact, there exists a neighbourhood U of δ such that

$$\bar{U} \cap \hat{K} \subseteq V \cap (\hat{K} - K).$$

Note that, since $\partial[\hat{K}, \mathfrak{A}] \subseteq K$, the neighbourhood U is disjoint from $\partial[\hat{K}, \mathfrak{A}]$. Let $\Gamma = \text{bdry}_{\hat{K}}\ U \cap \hat{K}$. Since $[\hat{K}, \mathfrak{A}]$ is natural, it follows from the Rossi local maximum principle that $U \cap \hat{K} \subseteq \hat{\Gamma}$. In particular, $\delta \in \hat{\Gamma} - \Gamma$. But Γ is a compact set in V, so we have a contradiction of the assumption that δ is an independent point of $[V, \mathfrak{A}]$. Therefore δ must be independent in $[\Sigma, \mathfrak{A}]$ and the theorem is proved.

It is not difficult to obtain the following corollary to the above theorem.

1.9. Corollary. *If $\delta \in \Sigma - \partial[\Sigma, \mathfrak{A}]$ and U is any neighbourhood of δ disjoint from $\partial[\Sigma, \mathfrak{A}]$, then there exists a neighbourhood V of δ contained in U such that δ is in the $\mathfrak{A}$-convex hull of* bdry V.

We close this section with a result which shows that a natural system may be "normalized" so as to have an empty Šilov boundary.

1.11. Theorem. *Let $[\Sigma, \mathfrak{A}]$ be natural and denote by Σ_0 the $\mathfrak{A}$-convex hull of the set $\Sigma - \partial[\Sigma, \mathfrak{A}]$ in Σ. Then $[\Sigma_0, \mathfrak{A}]$ is natural and $\partial[\Sigma_0, \mathfrak{A}]$ is empty.*

Proof. Since Σ_0 is $\mathfrak{A}$-convex, $[\Sigma_0, \mathfrak{A}]$ is natural by Proposition 1.4. Let I_0 denote the set of independent points of $[\Sigma_0, \mathfrak{A}]$ and define $\Sigma'_0 = \Sigma_0 - I_0$. If K is a compact set in Σ'_0, then $\hat{K} \subseteq \Sigma_0$. Furthermore, since $K \cap I_0 = \emptyset$, also $\hat{K} \cap I_0 = \emptyset$. Hence $\hat{K} \subseteq \Sigma'_0$, so Σ'_0 is $\mathfrak{A}$-convex. Similarly, we conclude that $\Sigma - I$ is $\mathfrak{A}$-convex. Since $\Sigma - \partial[\Sigma, \mathfrak{A}] \subseteq \Sigma - I$, it follows that $\Sigma_0 \subseteq \Sigma - I$. Now suppose there exists a point $\delta \in I_0 - \partial[\Sigma, \mathfrak{A}]$. Then, in particular, $\delta \notin I$, so there exists a compact set $K \subseteq \Sigma$ such that $\delta \in \hat{K} - K$. Choose an open set U such that $\delta \in U$ and

$$\bar{U} \cap K = \bar{U} \cap (\hat{K} \cap \partial[\Sigma, \mathfrak{A}]) = \emptyset.$$

Then $\bar{U} \cap \hat{K} \subseteq \Sigma - \partial[\Sigma, \mathfrak{A}] \subseteq \Sigma_0$, and hence, if

$$\Gamma = \text{bdry}_{\hat{K}}\ U \cap \hat{K},$$

then $\Gamma \subseteq \Sigma_0$ and Γ is compact. Now since $\delta \notin \Gamma$ and δ is independent in $[\Sigma_0, \mathfrak{A}]$, we conclude that $\delta \notin \hat{\Gamma}$. On the other hand, $U \cap \hat{K}$ is open in $\hat{K}$ and

$$U \cap \hat{K} \subseteq \hat{K} - K \subseteq \hat{K} - \partial[\hat{K}, \mathfrak{A}].$$

Hence $\delta \notin \hat{\Gamma}$ contradicts the local maximum principle for $[\hat{K}, \mathfrak{A}]$. Therefore $I_0 \subseteq \partial[\Sigma, \mathfrak{A}]$ and we have

$$\Sigma - \partial[\Sigma, \mathfrak{A}] \subseteq \Sigma_0 - I_0 = \Sigma'_0.$$

Since Σ'_0 is $\mathfrak{A}$-convex, it follows that $\Sigma_0 \subseteq \Sigma'_0$ and hence $\Sigma_0 = \Sigma'_0$. In other words, $I_0 = \emptyset$ and therefore $\partial[\Sigma_0, \mathfrak{A}]$ is empty.

2. $\mathfrak{A}$-holomorphic functions. Let Σ be any Hausdorff space and consider an arbitrary family $\mathscr{F}$ of functions defined on subsets of Σ. The domain of definition $\mathscr{D}_f$ of a function f in $\mathscr{F}$ may be an arbitrary subset of Σ, and elements of $\mathscr{F}$ need not have a common domain. We call $\mathscr{F}$ a *partial algebra* if it contains, along with elements f and g, the functions $f + g$ and fg (defined on $\mathscr{D}_f \cap \mathscr{D}_g$) whenever the latter exist. A function g is said to be *locally approximable* by elements of $\mathscr{F}$ if there exists for each point of $\mathscr{D}_g$ a neighbourhood U such that g is uniformly approximable on $U \cap \mathscr{D}_g$ by elements of $\mathscr{F}$. The family of all functions that are locally approximable by elements of $\mathscr{F}$ is called the *local extension* of $\mathscr{F}$ and denoted by $\operatorname{loc}\mathscr{F}$. It is obvious that $\operatorname{loc}\mathscr{F}$ contains $\mathscr{F}$. If $\operatorname{loc}\mathscr{F} = \mathscr{F}$, then $\mathscr{F}$ is said to be *locally closed*. The family of all functions defined on subsets of Σ is clearly locally closed. Moreover, the intersection of an arbitrary collection of locally closed families is also locally closed. Therefore, although the local extension of $\mathscr{F}$ need not be locally closed, there always exists a smallest locally closed family $\mathscr{F}_{\text{loc}}$ that contains $\mathscr{F}$; viz., the intersection of *all* locally closed families that contain $\mathscr{F}$. We call $\mathscr{F}_{\text{loc}}$ the *local closure* of $\mathscr{F}$.

If $\mathscr{E} \subseteq \mathscr{F}$, then it is immediate from the definition that $\operatorname{loc}\mathscr{E} \subseteq \operatorname{loc}\mathscr{F}$ and $\mathscr{E}_{\text{loc}} \subseteq \mathscr{F}_{\text{loc}}$. Since $\mathscr{F} \subseteq \mathscr{F}_{\text{loc}}$, it follows that $\operatorname{loc}\mathscr{F} \subseteq \mathscr{F}_{\text{loc}}$. Note that the local closure of any locally closed family $\mathscr{F}$ is equal to $\mathscr{F}$; i.e., $\operatorname{loc}\mathscr{F} = \mathscr{F}$ implies $\mathscr{F}_{\text{loc}} = \mathscr{F}$. In particular, for arbitrary $\mathscr{F}$, we have $(\mathscr{F}_{\text{loc}})_{\text{loc}} = \mathscr{F}_{\text{loc}}$. Next we observe that the local closure of $\mathscr{F}$ may be described in terms of local extensions by a process of transfinite induction.

2.1. LEMMA. *Let $\mathscr{F}$ be an arbitrary family of functions defined on subsets of Σ. Then there exists an ordinal μ and, for each $\nu \leqslant \mu$, a family $\mathscr{F}_\nu$ of functions with the following properties:*

(i) $\mathscr{F}_0 = \mathscr{F}$ *and* $\mathscr{F}_\mu = \mathscr{F}_{\text{loc}}$.

(ii) *If* $\alpha < \beta \leqslant \mu$, *then* $\mathscr{F}_\alpha \subsetneqq \mathscr{F}_\beta$.

(iii) *For each* $\nu \leqslant \mu$,

$$\mathscr{F}_\nu = \operatorname{loc}\,(\cup_{\alpha<\nu}\mathscr{F}_\alpha).$$

Proof. Define $\mathscr{F}_0 = \mathscr{F}$ and then, by transfinite induction, define

$$\mathscr{F}_\nu = \operatorname{loc}\,(\cup_{\alpha<\nu}\mathscr{F}_\alpha)$$

for each ordinal ν. Then $\alpha < \beta$ implies that $\mathscr{F}_\alpha \subseteq \operatorname{loc}\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$. Furthermore, a simple cardinality argument shows that not all of the classes $\mathscr{F}_\nu$ can be distinct. Hence there exists a first ordinal μ such that $\mathscr{F}_\mu = \mathscr{F}_\rho$ for some $\rho > \mu$. Since $\mu < \mu + 1 \leqslant \rho$, we have

$$\mathscr{F}_\mu \subseteq \mathscr{F}_{\mu+1} \subseteq \mathscr{F}_\rho = \mathscr{F}_\mu.$$

Therefore $\mathscr{F}_{\mu+1} = \mathscr{F}_\mu$ and hence

$$\mathscr{F}_\mu \subseteq \operatorname{loc}\mathscr{F}_\mu \subseteq \mathscr{F}_{\mu+1} = \mathscr{F}_\mu,$$

so $\operatorname{loc}\mathscr{F}_\mu = \mathscr{F}_\mu$ and $\mathscr{F}_\mu$ is locally closed. Moreover, μ is the first ordinal for which $\mathscr{F}_\mu$ is locally closed. It remains to show that $\mathscr{F}_\mu = \mathscr{F}_{\text{loc}}$. Since $\mathscr{F} \subseteq \mathscr{F}_\mu$, it follows by definition that $\mathscr{F}_{\text{loc}} \subseteq \mathscr{F}_\mu$. Also, $\mathscr{F}_0 = \mathscr{F} \subseteq \mathscr{F}_{\text{loc}}$ and, if ν is an ordinal such that $\mathscr{F}_\alpha \subseteq \mathscr{F}_{\text{loc}}$ for $\alpha < \nu$, then

$$\mathscr{F}_\nu = \operatorname{loc}\left(\bigcup_{\alpha<\nu}\mathscr{F}_\alpha\right) \subseteq \operatorname{loc}\mathscr{F}_{\text{loc}} = \mathscr{F}_{\text{loc}}.$$

Therefore, by induction, $\mathscr{F}_\nu \subseteq \mathscr{F}_{\text{loc}}$ for all ν. In particular, $\mathscr{F}_\mu \subseteq \mathscr{F}_{\text{loc}}$ and the desired result, $\mathscr{F}_\mu = \mathscr{F}_{\text{loc}}$, follows.

We are now ready to apply these concepts to the algebra $\mathfrak{A}$ involved in a system $[\Sigma, \mathfrak{A}]$.

2.2. *Definition.* Let $[\Sigma, \mathfrak{A}]$ be an arbitrary system. Then elements of the local closure $\mathfrak{A}_{\text{loc}}$ of $\mathfrak{A}$ are called *$\mathfrak{A}$-holomorphic* functions. Elements of $\mathfrak{A}_\nu$ are said to be *$\mathfrak{A}$-holomorphic of class ν.*

The family of all $\mathfrak{A}$-holomorphic functions with a common domain of definition G is denoted by $\mathscr{O}_G$. A function f such that $G \subseteq \mathscr{D}_f$ and $f|G \in \mathscr{O}_G$ is said to be *$\mathfrak{A}$-holomorphic on G.* If f is $\mathfrak{A}$-holomorphic, then it is automatically $\mathfrak{A}$-holomorphic on every subset of its domain.

In **(7)**, the term "$\mathfrak{A}$-holomorphic" was used to refer to functions that are $\mathfrak{A}$-holomorphic of class 1 according to Definition 2.2. The principal motivation for the introduction of the more extensive class of $\mathfrak{A}$-holomorphic functions is the fact that the class $\mathfrak{A}_1$ need not be closed under uniform convergence, let alone under local extension **(8**, §3**)**. A beautiful example due to Eva Kallin **(4)** shows that, even when $[\Sigma, \mathfrak{A}]$ is natural with compact Σ and $\mathfrak{A}$ is closed in $C(\Sigma)$, there may exist functions that belong locally to $\mathfrak{A}$ but do not belong to $\mathfrak{A}$. This already shows that we may have $\mathfrak{A}_1 \neq \mathfrak{A}$. The example in **(8)**, whose construction is based on the Kallin example, shows that we may also have $\mathfrak{A}_2 \neq \mathfrak{A}_1$. Unfortunately this construction does not generalize and we do not have examples that distinguish between classes $\mathfrak{A}_\nu$ for $\nu \geqslant 2$. Nevertheless, it is plausible to conjecture that examples do exist showing that the classes $\mathfrak{A}_\nu$ may be distinct, at least for finite ν. However, the construction is likely to be difficult.

Since $\mathfrak{A}$ is an algebra of continuous functions, it follows easily by an induction argument that each of the classes $\mathfrak{A}_\nu$, and hence the class of all $\mathfrak{A}$-holomorphic functions, is a partial algebra of continuous functions. It follows that for arbitrary $G \subseteq \Sigma$ the family $\mathscr{O}_G$ of $\mathfrak{A}$-holomorphic functions defined in G is an algebra of continuous functions. We also introduce the algebra $\mathscr{O}_G{}^*$ of all functions that are defined and continuous on the closure $\bar{G}$ of G and $\mathfrak{A}$-holomorphic on G. The algebra of all functions continuous on Σ and $\mathfrak{A}$-holomorphic on $\Sigma - \partial[\Sigma, \mathfrak{A}]$ will be denoted by $\mathscr{O}^*$.

2.3. THEOREM. *Let $h_1, \ldots, h_n \in \mathscr{O}_G{}^*$ and set*

$$\Gamma = \{(h_1(\sigma), \ldots, h_n(\sigma)) \colon \sigma \in \bar{G}\}.$$

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Also let F be an ordinary holomorphic function of n complex variables defined in an open neighbourhood of Γ *in* C^n. *Set*

$$h(\sigma) = F(h_1(\sigma), \ldots, h_n(\sigma)), \qquad \sigma \in \bar{G}.$$

Then h also belongs to $\mathcal{O}_G^*$.

Proof. It is obvious that h is continuous on $\bar{G}$. By the preceding remark, we note that if P is any polynomial in n variables, then $P(h_1, \ldots, h_n) \in \mathcal{O}_G^*$. Now let $\delta \in G$ and choose a closed polydisk Δ in C^n with centre $(h_1(\delta), \ldots, h_n(\delta))$ and contained in the domain on which F is holomorphic. Next choose a neighbourhood V of δ such that $(h_1(\sigma), \ldots, h_n(\sigma)) \in \Delta$ for $\sigma \in V \cap G$. Since F is holomorphic in a neighbourhood of Δ, there exist polynomials $\{P_k\}$ which converge uniformly on Δ to F. Hence

$$h(\sigma) = \lim_{k\to\infty} P_k(h_1(\sigma), \ldots, h_n(\sigma))$$

uniformly in $V \cap G$. But since $P_k(h_1, \ldots, h_n) \in \mathcal{O}_G^*$, this implies that h is locally approximable on G by $\mathfrak{A}$-holomorphic functions and so belongs to $\mathcal{O}_G^*$.

2.4. Corollary. $\mathcal{O}_G^*$ *is an algebra of continuous functions that contains inverses (i.e., if* $h \in \mathcal{O}_G^*$ *and* $h(\sigma) \neq 0$ *for* $\sigma \in \bar{G}$, *then* $h^{-1} \in \mathcal{O}_G^*$).

Next we establish a local maximum principle for $\mathfrak{A}$-holomorphic functions. As in the case of the algebra $\mathfrak{A}$ itself, the system involved must be natural. We first prove a lemma that is essentially an extension to $\mathfrak{A}$-holomorphic functions of the version of the Rossi theorem used in the discussion of $\mathfrak{A}$.

2.5. Lemma. *Assume* $[\Sigma, \mathfrak{A}]$ *natural with compact* Σ, *and let* $\bar{U}$ *be an open set in* $\Sigma - \partial[\Sigma, \mathfrak{A}]$. *If h is any function continuous on* $\bar{U}$ *and* $\mathfrak{A}$*-holomorphic in U, then* $|h|_{\bar{U}} = |h|_{\text{bdry } U}$.

Proof. Observe that h is not assumed to be defined outside of $\bar{U}$. Let us refer to this lemma as Lemma "ν" if h is restricted to be $\mathfrak{A}$-holomorphic of class ν in U. The proof of the lemma will consist in proving by an induction argument that Lemma "ν" is valid for all ν. Observe that Lemma "0" is valid by the Rossi theorem. Therefore assume that Lemma "α" has been established for all $\alpha < \nu$ and suppose that Lemma "ν" were false. Then there exists an open set U in Σ, with $U \cap \partial[\Sigma, \mathfrak{A}] = \emptyset$, and a function h continuous on $\bar{U}$ and $\mathfrak{A}$-holomorphic of class ν in U such that $|h|_{\bar{U}} > |h|_{\text{bdry } U}$. Denote by $\mathfrak{B}$ the subalgebra of $C(\bar{U})$ generated by h plus all functions that are continuous on $\bar{U}$ and $\mathfrak{A}$-holomorphic of class $\alpha < \nu$ in U. From the assumption on h, it follows that U must contain an independent point δ of the system $[\bar{U}, \mathfrak{B}]$. Now choose a neighbourhood V of δ such that $\bar{V} \subset U$ and h is a uniform limit on $\bar{V}$ of functions from $\bigcup_{\alpha<\nu} \mathfrak{A}_\alpha$. Then every function in $\mathfrak{B}$ clearly has this same property on $\bar{V}$. Since δ is an independent point of $[\bar{U}, \mathfrak{B}]$, there exists $f \in \mathfrak{B}$ such that

$$|f(\delta)| > |f|_{\bar{U}-V} \geqslant |f|_{\text{bdry } V}.$$

Hence there exists for some $\alpha < \nu$ a g in $\mathfrak{A}_\alpha$ defined on $\bar{V}$ such that

$$|g(\delta)| > |g|_{\text{bdry } V}.$$

But this contradicts Lemma "α" which was assumed true for all $\alpha < \nu$. Therefore Lemma "ν" must be true for all ν and the lemma is proved.

Glicksberg **(2)** proved essentially the above lemma for functions h which are $\mathfrak{A}$-holomorphic of class 1 on all of Σ. The proof of Lemma "1" was given in **(7)**. As in the case of $\mathfrak{A}$, there is a local maximum principle for $\mathfrak{A}$-holomorphic functions that holds without compactness restrictions on Σ. First a definition is needed.

2.6. *Definition.* A point δ is called an *$\mathfrak{A}$-holomorphically independent point* of $[\Sigma, \mathfrak{A}]$ if there exists an open neighbourhood U of δ such that δ is an independent point of the system $[U, \mathscr{O}_U]$.

Since $\mathfrak{A}|U \subseteq \mathscr{O}_U$, it is obvious that every locally independent point of $[\Sigma, \mathfrak{A}]$ (Definition 1.8) is also $\mathfrak{A}$-holomorphically independent. When $[\Sigma, \mathfrak{A}]$ is natural, the following converse is true.

2.7. Theorem. *If* $[\Sigma, \mathfrak{A}]$ *is natural, then each $\mathfrak{A}$-holomorphically independent point of* $[\Sigma, \mathfrak{A}]$ *is an independent point of* $[\Sigma, \mathfrak{A}]$.

Proof. Let δ be an $\mathfrak{A}$-holomorphically independent point of $[\Sigma, \mathfrak{A}]$ with U an open neighbourhood of δ such that δ is an independent point of $[U, \mathscr{O}_U]$. Suppose that δ is not independent in $[\Sigma, \mathfrak{A}]$. Then there exists a compact set K such that $\delta \in \hat{K} - K$. Now choose a neighbourhood V of δ such that $\bar{V} \cap \hat{K} \subset U$ and $V \cap K = \emptyset$. Set $\Gamma = \text{bdry}_{\hat{K}} (V \cap \hat{K})$. Then Γ is a compact subset of U and $\delta \notin \Gamma$. Hence there exists $h \in \mathscr{O}_U$ such that $|h(\delta)| > |h|_\Gamma$. Thus we have $|h|_{\bar{V}\cap\hat{K}} > |h|_\Gamma$. Since h is $\mathfrak{A}$-holomorphic on $\bar{V}$, it is continuous on $\bar{V} \cap \hat{K}$ and $\mathfrak{A}$-holomorphic in $V \cap \hat{K}$. Furthermore, since $\partial[\hat{K}, \mathfrak{A}] \subseteq K$, also $V \cap \partial[\hat{K}, \mathfrak{A}] = \emptyset$. We thus have a contradiction of Lemma 2.5 (applied to $[\hat{K}, \mathfrak{A}]$), so the theorem follows.

2.8. Corollary. *If G is an open set disjoint from* $\partial[\Sigma, \mathfrak{A}]$, *then* $\partial[G, \mathscr{O}_G]$ *is empty and* $\partial[\bar{G}, \mathscr{O}_G^*] \subseteq \text{bdry}\, G$. *Also* $\partial[\Sigma, \mathscr{O}^*] = \partial[\Sigma, \mathfrak{A}]$.

The notion of $\mathfrak{A}$-holomorphic function is used to extend the concept of an analytic variety to a general system **(7)**. For this purpose, we may in fact use a wider class of functions. Let us call a continuous function f *almost $\mathfrak{A}$-holomorphic* if it is $\mathfrak{A}$-holomorphic on that portion of its domain of definition where it is non-zero. In the case of $[C^n, \mathfrak{P}]$, and in certain more general situations (see Glicksberg **(2)**), it follows from the Radó theorem that an almost holomorphic function (defined, say, on a polycylinder) is actually holomorphic. The crucial fact concerning almost $\mathfrak{A}$-holomorphic functions for $[\Sigma, \mathfrak{A}]$ is that they satisfy the local maximum modulus principle of Lemma 2.5. This follows easily from Lemma 2.5 and the definition.

Let f be a function defined on all of Σ. Then we say that $\lim_{\sigma\to\infty} f(\sigma) = \beta$ if for arbitrary $\epsilon > 0$ there exists a compact set K such that

$$|f(\sigma) - \beta| < \epsilon, \qquad \sigma \in \Sigma - K.$$

2.9. THEOREM. *Let* $[\Sigma, \mathfrak{A}]$ *be natural with empty Šilov boundary and let* $h \in \mathcal{O}_\Sigma$. *Then* $\lim_{\sigma\to\infty} h(\sigma)$ *exists if and only if* h *is constant.*

Proof. If h is constant, then it is obvious that $\lim_{\sigma\to\infty} h(\sigma)$ exists. Therefore assume that h is not constant and that $\lim_{\sigma\to\infty} h(\sigma)$ exists. Without loss of generality we may assume that $\lim_{\sigma\to\infty} h(\sigma) = 0$. Since h is continuous and small outside compact sets, it is bounded and the maximum set

$$M = \{\sigma : \sigma \in \Sigma, |h(\sigma)| = |h|_\Sigma\}$$

is compact and non-empty. Let $g = |h|_\Sigma^{-1} h$. Then $|g(\sigma)| = 1$ for $\sigma \in M$ and $|g(\sigma)| < 1$ for $\sigma \notin M$. Since M is compact, the Šilov boundary $\partial[M, \mathfrak{A}]$ exists so there exists an independent point δ for $[M, \mathfrak{A}]$. Now let K be an arbitrary compact set in Σ with $\delta \notin K$ and let $K_0 = K \cap M$. Then there exists $u \in \mathfrak{A}$ such that $|u(\delta)| > 1 > |u|_{K_0}$ (where $|u|_{K_0} = 0$ if $K_0 = \emptyset$). Let $K_1 = \{\sigma : \sigma \in K, |u(\sigma)| \geqslant 1\}$. Then $K_1 \cap M = \emptyset$ so $|g|_{K_1} < 1$. Choose k such that $|g|_{K_1}{}^k < (|u|_K + 1)^{-1}$ and define $f = ug^k$. Note that $|f(\delta)| = 1$. Also, if $\sigma \in K - K_1$, then $|u(\sigma)| < 1$ so

$$|f(\sigma)| = |u(\sigma)|\,|g(\sigma)|^k < 1.$$

If $\sigma \in K_1$, then

$$|f(\sigma)| \leqslant |u|_{K_1} |g|_{K_1}{}^k < |u|_K (|u|_K + 1)^{-1} < 1.$$

Therefore

$$|f|_K < |f(\delta)| = 1.$$

Since $f \in \mathcal{O}_\Sigma$, it follows that δ is an independent point of $[\Sigma, \mathcal{O}_\Sigma]$ and hence is also an independent point of $[\Sigma, \mathfrak{A}]$. This contradicts the assumption that $\partial[\Sigma, \mathfrak{A}] = \emptyset$, so we conclude that h is constant.

2.10. *Definition.* Let Θ and Ω be subsets of Σ with $\Theta \subseteq \Omega$. Then Θ is called an $\mathfrak{A}$-*analytic subvariety* of Ω if each $\omega \in \Omega$ has a neighbourhood U such that $U \cap \Theta$ consists of the common zeros of a family of functions which are almost $\mathfrak{A}$-holomorphic in $U \cap \Omega$.

Observe that in this definition the intersection $U \cap \Theta$ may be empty (so the family of functions could reduce to a single non-zero constant). Also, the families of functions involved in the definition may be infinite in number. Since almost $\mathfrak{A}$-holomorphic functions are continuous, it follows that a subvariety of Ω must be relatively closed in Ω. Moreover, if a subset of Ω is known to be relatively closed in Ω, then in order to show that it is a subvariety of Ω one need only verify the condition of the definition at points of the subset. For $[\mathbf{C}^n, \mathfrak{P}]$ the above definition gives the usual notion of subvariety (**3**, p. 86).

The following lemma records a few of the elementary properties of $\mathfrak{A}$-analytic varieties.

2.11. Lemma.

(i) *The empty set* $\emptyset$ *and the set* Ω *are subvarieties of* Ω.

(ii) *If* Θ_1 *and* Θ_2 *are subvarieties of* Ω, *then* $\Theta_1 \cap \Theta_2$ *is also a subvariety of* Ω.

(iii) *If* Θ *is a subvariety of* Ω *and* $\Omega' \subseteq \Omega$, *then* $\Theta \cap \Omega'$ *is a subvariety of* Ω'.

(iv) *If* G *is relatively open in* Ω *and* Θ *is a subvariety of* G *which is relatively closed in* Ω, *then* Θ *is also a subvariety of* Ω.

(v) *If* Θ *is a subvariety of* Σ *which decomposes in the form* $\Theta = \Theta_1 \cup \Theta_2$, *where* $\bar{\Theta}_1 \cap \Theta_2 = \Theta_1 \cap \bar{\Theta}_2 = \emptyset$, *then* Θ_1 *and* Θ_2 *are also subvarieties of* Σ.

Proof. Properties (i), (ii), and (iii) follow immediately from the definition. For the proof of (iv), since Θ is relatively closed in Ω, we need only consider points of Θ. Since G is open in Ω, there exists an open set W in Σ such that $G = W \cap \Omega$. Also, since Θ is a subvariety of G, there exists a neighbourhood V of $\theta \in \Theta$ such that $V \cap \Theta$ consists of the common zeros of functions that are almost $\mathfrak{A}$-holomorphic in $V \cap G$. Set $U = V \cap W$. Then U is an open neighbourhood of θ with $U \cap \Omega = V \cap G$ and $U \cap \Theta = V \cap \Theta$. Thus $U \cap \Theta$ consists of the common zeros of functions that are almost $\mathfrak{A}$-holomorphic in $U \cap \Omega$. Hence Θ is a subvariety of Ω. Property (v) follows easily from (iv). In fact, since Θ is a closed set in Σ, the component sets Θ_1 and Θ_2 are also closed. Therefore, if $G = \Sigma - \Theta_2$, then G is open in Σ and $\Theta_1 = G \cap \Theta$. Hence Θ_1 is a subvariety of G by (iii). Finally, since Θ_1 is closed in Σ, it must be a subvariety of Σ by (iv). Similarly, Θ_2 is a subvariety of Σ and the proof is complete.

In **(7**, Theorem 3.2**)** we proved that if $[\Sigma, \mathfrak{A}]$ is natural, then an $\mathfrak{A}$-analytic subvariety (determined by $\mathfrak{A}$-holomorphic functions of class 1) of a compact $\mathfrak{A}$-convex set is also $\mathfrak{A}$-convex. Although only $\mathfrak{A}$-holomorphic functions of class 1 were considered in **(7)**, the proof of this result depends primarily on the fact that these functions satisfy the local maximum modulus principle. Therefore an identical proof yields the same result for the more general varieties considered here. We now remove the compactness restriction.

2.12. Theorem. *Assume* $[\Sigma, \mathfrak{A}]$ *to be natural and let* Ω *be an arbitrary* $\mathfrak{A}$*-convex set in* Σ. *Then every* $\mathfrak{A}$*-analytic subvariety* Θ *of* Ω *is* $\mathfrak{A}$*-convex*.

Proof. Let K be a compact set in Θ. Then, since $\Theta \subseteq \Omega$ and Ω is $\mathfrak{A}$-convex, $\hat{K} \subseteq \Omega$. Also $\hat{K}$ is a compact $\mathfrak{A}$-convex set. By Lemma 2.11(iii), $\Theta \cap \hat{K}$ is an $\mathfrak{A}$-analytic subvariety of $\hat{K}$. Therefore it follows from the theorem for the compact case that $\Theta \cap \hat{K}$ is $\mathfrak{A}$-convex. Since $K \subseteq \Theta \cap \hat{K}$, this implies that $\hat{K} \subseteq \Theta \cap \hat{K}$. In other words, if K is any compact set in Θ, then $\hat{K} \subseteq \Theta$ so Θ is $\mathfrak{A}$-convex by definition.

As a consequence of the above theorem plus Lemma 2.11(iv), we have the following corollary.

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2.13. COROLLARY. *Let G be an open set in Σ, and Θ an $\mathfrak{A}$-analytic subvariety of G. If Θ is closed in Σ, then it is $\mathfrak{A}$-convex.*

3. $\mathfrak{A}$-holomorphic convexity. We shall assume throughout this section that the system $[\Sigma, \mathfrak{A}]$ is natural. Let G be a subset of Σ and recall that $\mathcal{O}_G$ denotes the algebra of all $\mathfrak{A}$-holomorphic functions defined on G. Let $\{h_\lambda: \lambda \in \Lambda\}$ be an arbitrary subset of $\mathcal{O}_G$ and, for K a compact set in G, let

$$\hat{K}(\{h_\lambda\}) = \{\sigma: \sigma \in G, |h_\lambda(\sigma)| \leqslant |h_\lambda|_K, \lambda \in \Lambda\}.$$

When there is no chance of confusion, we write simply $\hat{K}$ in place of $\hat{K}(\{h_\lambda\})$. Since $\mathfrak{A}$-holomorphic functions are continuous, the set $\hat{K}$ is always relatively closed in G. If also $\mathfrak{A}|G \subseteq \{h_\lambda\}$, then $\hat{K} \subseteq \hat{K}$, so the closure of $\hat{K}$ in Σ is a compact set. In general, however, $\hat{K}$ itself will not be compact even when $\{h_\lambda\}$ contains $\mathfrak{A}|G$.

3.1. *Definition.* A set $\Omega \subseteq G$ is said to be *$\{h_\lambda\}$-convex* if for every compact set $K \subseteq \Omega$ it is true that $\hat{K}$ is compact and contained in Ω. If G itself is $\mathcal{O}_G$-convex, then it is said to be *$\mathfrak{A}$-holomorphically convex.*

Note that if Ω is a compact subset of G, then it will be $\{h_\lambda\}$-convex if and only if $\hat{\Omega} = \Omega$. Since $\mathfrak{A}|G \subseteq \mathcal{O}_G$, it follows that the closure of $\hat{K}(\mathcal{O}_G)$ is compact for every compact set $K \subseteq G$. Therefore, according to the above definition, every closed set is $\mathfrak{A}$-holomorphically convex. Thus, interesting results will hold only for certain special cases, e.g. for open sets. Note, however, that an open set in C^n is holomorphically convex in the usual sense (**3**, I, G4) if and only if it is $\mathfrak{P}$-holomorphically convex in the sense of the above definition. Every $\mathfrak{A}$-convex set is automatically $\mathfrak{A}$-holomorphically convex. If G is $\mathfrak{A}$-holomorphically convex and H is an $\mathcal{O}_G$-convex subset of G, then H is also $\mathfrak{A}$-holomorphically convex. If $J \subseteq H \subseteq G$ and J is $\mathcal{O}_G$-convex, then it is also $\mathcal{O}_H$-convex. These remarks follow easily from the definition.

Let $\{h_\lambda: \lambda \in \Lambda\} \subseteq \mathcal{O}_G$ and denote by $\check{h}$ the function with values in C^Λ defined as follows:

$$\check{h}(\sigma) = \{h_\lambda(\sigma)\}, \qquad \sigma \in G.$$

For each $\sigma \in G$ set $\tilde{\sigma} = (\sigma, \check{h}(\sigma))$. Then the mapping $\sigma \to \tilde{\sigma}$, which is a homeomorphism of G into $\Sigma \times C^\Lambda$, is called a *general Oka mapping*. For any subset Ω of G, the image $\tilde{\Omega}$ in $\Sigma \times C^\Lambda$ under the Oka mapping is the *graph of $\check{h}$ over the set* Ω. Note that the graph $\tilde{\Omega}$ will be compact in $\Sigma \times C^\Lambda$ if and only if Ω is compact in Σ. Recall that $[\Sigma \times C^\Lambda, \mathfrak{A} \times \mathfrak{P}]$ is a natural system, where $\mathfrak{A} \times \mathfrak{P}$ may be regarded as the algebra of all polynomials in the variables $\{\zeta_\lambda: \lambda \in \Lambda\}$ with coefficients in $\mathfrak{A}$.

3.2. THEOREM. *Let G be an open set in Σ and $\{h_\lambda: \lambda \in \Lambda\}$ an arbitrary subset of $\mathcal{O}_G$. If Ω is any $\{h_\lambda\}$-convex subset of G, then the graph $\tilde{\Omega}$ is an $(\mathfrak{A} \times \mathfrak{P})$-convex set in $\Sigma \times C^\Lambda$.*

Proof. Let us assume first that Ω is compact. Define

$$\Delta_\Omega = \{\check{\zeta}: \check{\zeta} \in \mathbf{C}^\Lambda, |\zeta_\lambda| \leqslant |h_\lambda|_\Omega, \lambda \in \Lambda\}.$$

Then Δ_Ω is a compact polydisk in $\mathbf{C}^\Lambda$ and so, in particular, is $\mathfrak{P}$-convex. Next, for each $\lambda \in \Lambda$, define

$$H_\lambda(\sigma, \check{\zeta}) = h_\lambda(\sigma) - \zeta_\lambda, \qquad (\sigma, \check{\zeta}) \in G \times \mathbf{C}^\Lambda.$$

Then $\{H_\lambda: \lambda \in \Lambda\}$ is a family of $(\mathfrak{A} \times \mathfrak{P})$-holomorphic functions defined in $G \times \mathbf{C}^\Lambda$. Therefore

$$\mathscr{V}(\{H_\lambda\}) = \{(\sigma, \check{\zeta}): H_\lambda(\sigma, \check{\zeta}) = 0, \lambda \in \Lambda\}$$

is an $(\mathfrak{A} \times \mathfrak{P})$-analytic subvariety of $G \times \mathbf{C}^\Lambda$. If $(\omega, \check{\zeta}) \in \tilde{\Omega}$, then $\omega \in \Omega$, $h_\lambda(\omega) = \zeta_\lambda$, and hence $|\zeta_\lambda| \leqslant |h_\lambda|_\Omega$ for each $\lambda \in \Lambda$. Therefore

$$\tilde{\Omega} \subseteq \mathscr{V}(\{H_\lambda\}) \cap (G \times \Delta_\Omega).$$

On the other hand, let $(\sigma, \check{\zeta})$ be an arbitrary element of $\mathscr{V}(\{H_\lambda\}) \cap (G \times \Delta_\Omega)$. Then $\sigma \in G$, $h_\lambda(\sigma) = \zeta_\lambda$, and $|\zeta_\lambda| \leqslant |h_\lambda|_\Omega$ for each $\lambda \in \Lambda$. Thus $|h_\lambda(\sigma)| \leqslant |h_\lambda|_\Omega$ for each $\lambda \in \Lambda$ and we have $\sigma \in \hat{\Omega}(\{h_\lambda\}) = \hat{\Omega}$. But Ω is compact and $\{h_\lambda\}$-convex, so $\hat{\Omega} = \Omega$ and hence $\sigma \in \Omega$. This means that $(\sigma, \check{\zeta}) \in \tilde{\Omega}$ and we conclude that

$$\tilde{\Omega} = \mathscr{V}(\{H_\lambda\}) \cap (G \times \Delta_\Omega).$$

In other words, $\tilde{\Omega}$ is a compact $(\mathfrak{A} \times \mathfrak{P})$-analytic subvariety of $G \times \Delta_\Omega$. Since $G \times \Delta_\Omega$ is an open subset of the space $\Sigma \times \Delta_\Omega$ and $\tilde{\Omega}$, being compact, is a closed subset of $\Sigma \times \Delta_\Omega$, it follows by Lemma 2.11(iv) that $\tilde{\Omega}$ is an $(\mathfrak{A} \times \mathfrak{P})$-analytic subvariety of $\Sigma \times \Delta_\Omega$. Since $[\Sigma, \mathfrak{A}]$ is natural and Δ_Ω is $\mathfrak{P}$-convex, the system $[\Sigma \times \Delta_\Omega, \mathfrak{A} \times \mathfrak{P}]$ is also natural. Therefore $\tilde{\Omega}$ is $(\mathfrak{A} \times \mathfrak{P})$-convex in $\Sigma \times \Delta_\Omega$. Since $\Sigma \times \Delta_\Omega$ is $(\mathfrak{A} \times \mathfrak{P})$-convex in $\Sigma \times \mathbf{C}^\Lambda$, we conclude that $\tilde{\Omega}$ is also $(\mathfrak{A} \times \mathfrak{P})$-convex in $\Sigma \times \mathbf{C}^\Lambda$. This completes the proof when Ω is compact.

For the general case, let Γ be any compact subset of $\tilde{\Omega}$. Then Γ is of the form $\tilde{K}$ for some compact set $K \subseteq \Omega$. Since Ω is $\{h_\lambda\}$-convex, the set $\hat{K}$ is compact and contained in Ω. Note that $\hat{K}$ is also $\{h_\lambda\}$-convex. Therefore, by the result for compact Ω, the graph of $\hat{K}$ is $(\mathfrak{A} \times \mathfrak{P})$-convex in $\Sigma \times \mathbf{C}^\Lambda$. Therefore, since $K \subseteq \hat{K} \subseteq \Omega$ and $\Gamma = \tilde{K}$, it follows that the $(\mathfrak{A} \times \mathfrak{P})$-convex hull of Γ is contained in the graph of $\hat{K}$ and hence is contained in $\tilde{\Omega}$. In other words, $\tilde{\Omega}$ is $(\mathfrak{A} \times \mathfrak{P})$-convex and the proof is complete.

We also have the following converse to the above theorem.

3.3. THEOREM. *Denote by $\mathfrak{H}$ the subalgebra of $\mathscr{O}_G$ generated by $\mathfrak{A}|G$ plus $\{h_\lambda: \lambda \in \Lambda\}$. Let Ω be a subset of G for which $\tilde{\Omega}$ is $(\mathfrak{A} \times \mathfrak{P})$-convex in $\Sigma \times \mathbf{C}^\Lambda$. Then Ω is $\mathfrak{H}$-convex and hence is also $\mathscr{O}_G$-convex.*

Proof. Let K be a compact subset of Ω. Then $\tilde{K}$ is a compact subset of $\tilde{\Omega}$. Hence the $(\mathfrak{A} \times \mathfrak{P})$-convex hull of $\tilde{K}$ is contained in $\tilde{\Omega}$. Moreover, since $[\Sigma \times \mathbf{C}^\Lambda, \mathfrak{A} \times \mathfrak{P}]$ is natural, the $(\mathfrak{A} \times \mathfrak{P})$-convex hull of $\tilde{K}$ is compact and is therefore equal to $\tilde{K}_1$ for some compact set $K_1 \subseteq \Omega$. Note that $K \subseteq K_1$. Now, if $\delta \in G - K_1$, then $\tilde{\delta} \notin \tilde{K}_1$ so there exists $F \in \mathfrak{A} \times \mathfrak{P}$ such that

$$|F(\delta, \check{h}(\delta))| > |F|_{\tilde{K}} = \max_{\omega \in K} |F(\omega, \check{h}(\omega))|.$$

Define

$$h(\omega) = F(\omega, \check{h}(\omega)), \qquad \omega \in G.$$

Then h is a polynomial in a finite number of the h_λ's with coefficients in $\mathfrak{A}$. Therefore $h \in \mathfrak{H}$ and $|h(\delta)| > |h|_K$, so $\delta \notin \hat{K}(\mathfrak{H})$. Thus we have

$$\hat{K}(\mathfrak{H}) \subseteq K_1 \subseteq \Omega.$$

Since K_1 is compact, it follows that $\hat{K}(\mathfrak{H})$ is also compact and Ω is $\mathfrak{H}$-convex.

3.4. Corollary. *If for each compact set $K \subseteq G$, it is true that $\hat{K}(\{h_\lambda\}) = \hat{K}(\mathcal{O}_G)$, then G is $\mathfrak{A}$-holomorphically convex if and only if $\tilde{G}$ is $(\mathfrak{A} \times \mathfrak{P})$-convex.*

3.5. Theorem. *Let G be an open set in Σ and let $\mathfrak{H}$ be a subalgebra of $\mathcal{O}_G$ that contains $\mathfrak{A}|G$. Then a subset Ω of G is $\mathfrak{H}$-convex if and only if the system $[\Omega, \mathfrak{H}]$ is natural.*

Proof. Assume first that $[\Omega, \mathfrak{H}]$ is natural and let K be a compact subset of Ω. Let σ_0 be an arbitrary point of $\hat{K}(\mathfrak{H})$. Since $K \subseteq \Omega$, we note that if $f, g \in \mathfrak{H}$, then $f|\Omega = g|\Omega$ implies that $f(\sigma_0) = g(\sigma_0)$. Therefore the mapping

$$f|\Omega \to f(\sigma_0), \qquad f \in \mathfrak{H},$$

is a well-defined homomorphism of $\mathfrak{H}|\Omega$ onto C. Moreover, since $|f(\sigma_0)| \leqslant |f|_K$ for all $f \in \mathfrak{H}$, the homomorphism is continuous. It follows, since $[\Omega, \mathfrak{H}]$ is natural, that there exists a point $\omega_0 \in \Omega$ such that $f(\sigma_0) = f(\omega_0)$ for $f \in \mathfrak{H}$. This implies that $\sigma_0 = \omega_0$ and we conclude that $\hat{K}(\mathfrak{H}) \subseteq \Omega$. Observe that $\hat{K}$ is the $\mathfrak{H}|\Omega$-convex hull of the set K in Ω. Again since $[\Omega, \mathfrak{H}]$ is natural, the set $\hat{K}$ is compact. In other words, Ω is $\mathfrak{H}$-convex.

Now let us assume that Ω is $\mathfrak{H}$-convex and take $\{h_\lambda \colon \lambda \in \Lambda\} = \mathfrak{H}$. Then by Theorem 3.2, the graph $\tilde{\Omega}$ is $(\mathfrak{A} \times \mathfrak{P})$-convex in $\Sigma \times \mathrm{C}^\Lambda$. Hence, by Proposition 1.3, the system $[\tilde{\Omega}, \mathfrak{A} \times \mathfrak{P}]$ is natural. Next let F be an arbitrary element of $\mathfrak{A} \times \mathfrak{P}$ and define

$$h_F(\sigma) = F(\sigma, \check{h}(\sigma)) = F(\tilde{\sigma}), \qquad \sigma \in G.$$

Since F is a polynomial in a finite number of the variables ζ_λ with coefficients in $\mathfrak{A}$, it follows that $h_F \in \mathfrak{H}$. Moreover, the mapping $F \to h_F$ is a homomorphism of $\mathfrak{A} \times \mathfrak{P}$ into $\mathfrak{H}$. Furthermore, if we define

$$Z_\lambda(\sigma, \check{\zeta}) = \zeta_\lambda, \qquad (\sigma, \check{\zeta}) \in \Sigma \times \mathrm{C}^\Lambda,$$

then $Z_\lambda \in \mathfrak{A} \times \mathfrak{P}$ and $h_{Z_\lambda} = h_\lambda$. Therefore the homomorphism $F \to h_F$ is onto all of $\mathfrak{H}$.

Consider an arbitrary continuous homomorphism ϕ of $\mathfrak{H}|\Omega$ onto C. Then there exists a compact set $K \subseteq \Omega$ such that

$$|\widehat{h|\Omega}(\phi)| \leqslant |h|_K, \qquad h \in \mathfrak{H}.$$

The mapping

$$F \to h_F \to h_F|\Omega \to \widehat{h_F|\Omega}(\phi), \qquad F \in \mathfrak{A} \times \mathfrak{P},$$

defines a homomorphism of $\mathfrak{A} \times \mathfrak{P}$ onto C. Moreover,

$$\begin{aligned}|\widehat{h_F|\Omega}(\phi)| &\leqslant |h_F|_K = \max_{\omega \in K} |h_F(\omega)| \\ &\leqslant \max_{\tilde{\omega} \in \tilde{K}} |F(\tilde{\omega})| = |F|_{\tilde{K}},\end{aligned}$$

for every $F \in \mathfrak{A} \times \mathfrak{P}$. Since $\tilde{K}$ is a compact set in $\Sigma \times C^{\Lambda}$, it follows that the homomorphism is continuous. Hence there exists $(\delta, \check{\xi}) \in \Sigma \times C^{\Lambda}$ such that

$$\widehat{h_F|\Omega}(\phi) = F(\delta, \check{\xi}), \qquad F \in \mathfrak{A} \times \mathfrak{P}.$$

Furthermore, since

$$|F(\delta, \check{\xi})| \leqslant |F|_{\tilde{K}}, \qquad F \in \mathfrak{A} \times \mathfrak{P},$$

it follows that the point $(\delta, \check{\xi})$ belongs to the $(\mathfrak{A} \times \mathfrak{P})$-convex hull of $\tilde{K}$ in $\Sigma \times C^{\Lambda}$.

Finally, since $\tilde{\Omega}$ is $(\mathfrak{A} \times \mathfrak{P})$-convex and $K \subseteq \Omega$, the $(\mathfrak{A} \times \mathfrak{P})$-convex hull of $\tilde{K}$ is contained in $\tilde{\Omega}$. In particular, $(\delta, \check{\xi}) \in \tilde{\Omega}$, so $\delta \in \Omega$ and $(\delta, \check{\xi}) = (\delta, \check{h}(\delta)) = \tilde{\delta}$. Therefore

$$F(\delta, \check{\xi}) = F(\tilde{\delta}) = h_F(\delta).$$

Thus ϕ is given by evaluation of elements of $\mathfrak{H}$ at the point $\delta \in \Omega$. In other words, $[\Omega, \mathfrak{H}]$ is natural.

3.6. COROLLARY. *The open set G is $\mathfrak{A}$-holomorphically convex if and only if the system $[G, \mathcal{O}_G]$ is natural.**

In the presence of certain countability assumptions, we can show that an open $\mathfrak{A}$-holomorphically convex set G is actually a region of holomorphy. In other words G is the domain of definition of an $\mathfrak{A}$-holomorphic function which cannot be holomorphically extended to a larger region. The set G is said to be *separable* if it is the union of a sequence of open sets $\{G_n\}$ such that $\bar{G}_n$ is compact and contained in G_{n+1} for each n. We say that G is *strongly separable* if it is separable and there exists a sequence $\{V_n\}$ of open sets in Σ such that $V_n \cap (\text{bdry } G) \neq \emptyset$ for each n and, if U is any open set that intersects bdry G, then there is a V_n with $V_n \cap G \subset U$.

It is obvious that every open set in C^n is strongly separable. The proof of the following theorem is similar to the proof of an analogous result for C^n (**1**, p. 84). We are indebted for this remark to M. Shauck who has also obtained a similar abstract theorem (**10**).

**Added in proof.* A special case of this result has been obtained by Jan-Erik Bjork (written communication).

3.7. THEOREM. *Let G be an open $\mathfrak{A}$-holomorphically convex subset of Σ. If G is strongly separable, then there exists a function $g \in \mathcal{O}_G$ such that $|g|_{U\cap G} = \infty$ for every open set U that intersects* bdry G.

Proof. Let $\{G_n\}$ and $\{V_n\}$ be the sequences of open sets given in the definition of strong separability. Since G is $\mathfrak{A}$-holomorphically convex and $\bar{G}_n$ is compact, the $\mathcal{O}_G$-convex hull of $\bar{G}_n$ is a compact subset of G.

We construct by an induction a subsequence $\{H_n\}$ of $\{G_n\}$ and a sequence $\{\omega_n\}$ of points of G with the property

$$\omega_n \in (H_{n+1} - \Gamma_n) \cap V_n,$$

where Γ_n denotes the $\mathcal{O}_G$-convex hull of $\bar{H}_n$ for each n. First $H_1 = G_1$. Then, since Γ_1 is a compact subset of G and V_1 intersects bdry G, there exists $\omega_1 \in (V_1 - \Gamma_1) \cap G$. Let H_2 denote the first element of $\{G_n\}$ that contains the point ω_1. With this start, assume that

$$H_1, \ldots, H_m \quad \text{and} \quad \omega_1, \ldots, \omega_{m-1}$$

are already defined with the desired property. Again since Γ_m is a compact subset of G and V_m intersects bdry G, there exists $\omega_m \in (V_m - \Gamma_m) \cap G$. Choosing H_{m+1} as the first element of $\{G_n\}$ that contains ω_m, we have

$$\omega_m \in (H_{m+1} - \Gamma_m) \cap V_m$$

so the desired sequences exist by induction. Note also that

$$G = \bigcup_{n=1}^{\infty} H_n.$$

Now, since Γ_n is $\mathcal{O}_G$-convex, there exists $h_n \in \mathcal{O}_G$ such that

$$|h_n|_{\Gamma_n} < 1 < |h_n(\omega_n)|.$$

Again by an induction, we may define an increasing sequence $\{m_n\}$ of positive integers such that

$$|h_n^{m_n}|_{\Gamma_n} < 1/2^n$$

and

$$|h_n^{m_n}(\omega_n)| > \left| \sum_{k=1}^{n-1} h_k^{m_k} \right|_{\bar{H}_{n+1}} + n,$$

for each n. Next define

$$g(\omega) = \sum_{n=1}^{\infty} h_n^{m_n}(\omega), \qquad \omega \in G.$$

Since $\bar{H}_k \subset H_{k+1}$ for each k we have, for all $k \geqslant n$,

$$|h_k^{m_k}|_{\bar{H}_n} \leqslant |h_k^{m_k}|_{\bar{H}_k} < 1/2^k.$$

Therefore the series for g converges uniformly on each of the sets H_n. In particular, the series converges locally uniformly in G so $g \in \mathcal{O}_G$.

Next let p denote an arbitrary positive integer and let $n > p$. Then, since $\omega_n \in H_{n+1} - \Gamma_n$,

$$\left| \sum_{k=1}^{n-1} h_k^{m_k}(\omega_n) \right| \leqslant \left| \sum_{k=1}^{n-1} h_k^{m_k} \right|_{\bar{H}_{n+1}}.$$

Also $\omega_n \in H_k$ for $k > n$ so

$$|h_k^{m_k}(\omega_n)| \leqslant |h_k^{m_k}|_{\bar{H}_k} < 1/2^k$$

and we have

$$\sum_{k=n+1}^{\infty} |h_k^{m_k}(\omega_n)| < \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

Therefore

$$|g(\omega_n)| \geqslant |h_n^{m_n}(\omega_n)| - \left| \sum_{k=1}^{n-1} h_k^{m_k} \right|_{\bar{H}_{n+1}} - \frac{1}{2^n}$$

$$> n - \frac{1}{2^n} \geqslant p + 1 - \frac{1}{2^n} > p.$$

Now let U be an arbitrary open set that intersects bdry G. For arbitrary p, let $U_p = U - \bar{H}_p$. Then U_p is also an open set that intersects bdry G. Hence there exists V_n with $V_n \cap G \subset U_p$. In particular, $\omega_n \in U_p \cap G$ and, since $\omega_n \notin \bar{H}_p$, $n > p$. Therefore $|g(\omega_n)| > p$ and hence

$$|g|_{U \cap G} \geqslant |g|_{U_p \cap G} \geqslant |g(\omega_n)| > p.$$

Since p is arbitrary, it follows that $|g|_{U \cap G} = \infty$ and the proof is complete.

The condition that G be strongly separable may be formulated in other ways. For example, the following weaker version will suffice for the above proof: G is separable and there exists a sequence $\{B_n\}$ of subsets of G such that $\bar{B}_n \cap G$ is not compact for any n and for an arbitrary open set U that intersects bdry G there is a B_n with $B_n \subset U$.

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