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On the existence of n but not $n+1$ easy combinators

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Recall that M is easy if it is consistent with every combinator. We say that M is m-easy if there is no proof with $< m+1$ steps that M is inconsistent with any combinator. Obviously, if M is easy, it is m-easy for each m. Here we shall show that for infinitely many m there are m but not $m+1$ easy terms.

1. m-easy combinators

Given two combinators M and N, we define the graph $G(M,N)$ as follows: the points of $G(M,N)$ are the combinators modulo beta conversion, and we make P adjacent to Q if there exists an R such that $P = RM$ and $Q = RN$, or, $P = RN$ and $Q = RM$ where $=$ denotes β conversion. Now the proof theoretic properties of the equation $M = N$ are reflected by the properties of $G(M,N)$. For example, $M = N$ is inconsistent $\Leftrightarrow G(M,N)$ is connected $\Leftrightarrow K$ and K_* lie in the same $G(M,N)$ component. In particular, if we wish to count steps in proofs it is convenient to count edges in $G(M,N)$.

Recall that M is easy if it is consistent with every combinator. We say that M is m-easy if there is no proof with $< m+1$ steps that M is inconsistent with any combinator *i.e.*, if for each N the diameter of $G(M,N)$ is at least m. Obviously, if M is easy, it is m-easy for each m. Here we shall show that for infinitely many m there are m but not $m+1$ easy terms.

Defining terms $E(n), F(n), G(n)$ as follows:

$E(0) := \lambda x.\ K$

$E(n+1) := \lambda x.\ xI\,E(n)x$

$F(n) := \lambda x.\ xxI\,E(n)(xx)\ \lambda x.\ xx\ I\,E(n)(xx)$

$G(n) := \lambda x.\ F(n)(xx)\ \lambda x.\ F(n)(xx),$

we shall show that $G(n)$ is n-easy but not $2n+5$ easy.

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2. Lower bounds on the failure of Church–Rosser

Let M be given. A term X is said to be an F-term of type n, M, k if it has the form

$$F(n) I E(k_1) X_1 \ldots I E(k_t) X_t$$

where $-1 < t$, $k - 1 < k_i$, and each X_i is an F-term of type n, M, k. A term Y is said to be a G-term of type n, M, k if it has the form

(a) $Y_1(\ldots(Y_t(\lambda x.\ Y_{t+1}(xx)\ \lambda x.\ Y_{t+2}(xx)))\ldots)$

where $-1 < t$, and each Y_i is an F-term of type n, M, k or

(b) $Y_1(\ldots(Y_t N)\ldots)$

where $0 < t$, each Y_i is an F-term of type n, M, k, and

$$BT(M) \underline{[}\ BT(N)$$

Observe that the notion of an F-term of type n, M, k does not depend on M but that of a G-term of type n, M, k does.

We define relations $\mapsto_{(k)}$, $\shortmid\!\mapsto_{(k)}$, $\rightarrowtail_{(k)}$ as follows:

— $X \mapsto_{(k)} Y \Leftrightarrow X$ is a G-term of type n, M, k and $Y := M$.
— $X \shortmid\!\mapsto_{(k)} Y \Leftrightarrow X := X[Z_1, \ldots, Z_r]$, where each Z_i is a G-term of type n, M, k and $Y := X[M, \ldots, M]$.
— $X \rightarrowtail_{(k)} Y \Leftrightarrow$ there exists a Z such that $X \twoheadrightarrow Z \shortmid\!\mapsto_{(k)} Y$.
— For $0 < t$ we define the relation $\rightarrowtail\!\!\!\twoheadrightarrow_{(k,t)}$ by
— $X \rightarrowtail\!\!\!\twoheadrightarrow_{(k,t)} Y \Leftrightarrow$ there exists $Z_1, \ldots, Z_{t-1}$ $X \rightarrowtail_{(k)} Z_1 \rightarrowtail_{(k)} \ldots \rightarrowtail_{(k)} Z_{t-1} \rightarrowtail_{(k)} Y$.

We shall prove the following proposition.

Proposition 2.1. The diagrams

$$Q \twoheadleftarrow P \shortmid\!\mapsto_{(k)} R$$

and

$$Q\ {}_{(r)}\!\leftarrow\!\!\dashv\!\shortmid\ P \shortmid\!\mapsto_{(k)} R$$

can be completed to

$$Q \shortmid\!\mapsto_{(k)} T \twoheadleftarrow R,$$

and

$$Q \twoheadrightarrow Q^* \shortmid\!\mapsto_{(k-1)} T\ {}_{(r-1)}\!\leftarrow\!\!\dashv\!\shortmid\ R^* \twoheadleftarrow R,$$

respectively.

Therefore the diagram

$$T\ {}_{(r)}\!\leftarrow\!\!\dashv\!\shortmid\ Q \twoheadleftarrow P \shortmid\!\mapsto_{(k)} R$$

can be completed to

$$\begin{array}{lllllll} & R \twoheadrightarrow U & & & & & \\ & \quad\ \ ||| & & & & & \\ Q \shortmid\!\mapsto_{(k)} & U & \twoheadrightarrow X \shortmid\!\mapsto_{(r-1)} Y & {}_{(k-1)}\!\twoheadleftarrow\!\!\dashv\!\shortmid & Z & \twoheadleftarrow & T \end{array}$$

when $0 < k$ and $0 < k$.

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This then leads to the diamond property.

Corollary 2.1. $Q\ {}_{(r)}\!\leftarrowtail P \rightarrowtail_{(k)} R$ can be completed to $Q \rightarrowtail_{(k-1)} T\ {}_{(r-1)}\!\leftarrowtail R$ when $0 < r$ and $0 < k$.

We define the notion of head positions as follows: if X is an F-term of type n, M, k and $X := F(n)IE(k_1)X_1 \dots IE(k_t)X_t$, then X has for its head positions the head positions of the X_i together with the subterm occurrence $F(n)I$ above.

Lemma 2.1. (Matching Lemma) Suppose that Y is a G-term of type n, M, k. If Y is of the form (a) then all its G-subterms of type n, M, r have the form (a) and are among the $Y_i(\dots(Y_t(\lambda x.\ Y_{t+1}(xx)\ \lambda x.\ Y_{t+2}xx)))\dots)$ except in the case $M = I$, when they can have the form (b), the shape $F(n)I$, and occur at the head positions of the Y_i.

Proof. First let X be an F-term of type n, M, k. We will show that X has no G-subterms of type n, M, r except when $M = I$ and these are at the head positions of X. This is proved by induction on the definition and toward this end we let Y be a G-term of type n, M, r of the form (a) or (b) above. Then

(i) The last component of Y_1 is either $\lambda x.\ xxIE(n)(xx)$ or has no normal form; therefore it $= / = I$ or $E(r)$ for any r.
(ii) If $Y := Y_1L$, then L has no normal form, so it $= / = I$ or $E(r)$ for any r.
(iii)If Y is of the form (a), then $\lambda x.\ Y_{t+j}(xx)$ has no normal form, so it $= / = I$ or $E(r)$ for any r when $j = 1, 2$.
(iv)If Y is of the form (a), then $\lambda x.\ Y_{t+j}(xx)$ has order 1 (Barendregt 1984, page 446), so it is not an F-term or a G-term of type n, M, s when $j = 1, 2$.

Now suppose that Y is a subterm of X. First suppose that Y has the form (a). If $t = 0$, then by (iii) and (iv), Y is a subterm of X_i for some i. If $0 < t$, then by (i) and (ii), Y is a subterm of some X_i. Next suppose that Y has the form (b). If $t = 1$, then by (i), Y is a subterm of X_i for some i unless $Y := F(n)I$ and Y occupies the leftmost head position of X. But in this case we have $N := I = M$. If $1 < t$, then by (i) and (ii), Y is a subterm of some X_i. Our claim then follows by induction.

To prove the lemma, simply apply the above claim to the Y_i after using (iv), and the unsolvability of G-terms of type n, M, s. This completes the proof of the lemma. □

Lemma 2.2. (Replacement Lemma) Let X be a G-term of type n, M, k. Then the replacement of any proper G-subterm of type n, M, r by M results in a G-term of type n, M, k except in the case $M = I$ when if $0 < k$ it results in a term that $\twoheadrightarrow$ to a G-term of type $n, M, k - 1$.

Proof. Suppose first that $M = / = I$. Let Y be a G-term of type n, M, k with a proper G-subterm Z of type n, M, r. By the Matching Lemma, if Y has the form (a), the replacement of Z by M is a G-term of type n, M, k and of the form (b). If Y is of the form (b), the result of replacing Z by M remains of the form (b) since Z is unsolvable and its replacement in M yields a term whose Bohm tree still] the Bohm tree of M. Now, if $M = I$, then Z can occur at the head positions of the Y_i if $Z := F(n)I$. These Y_i are

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F-terms of type n, M, k and of the form

$$F(n)IE(k_1)U_1 \dots IE(k_t)U_t$$

and the replacement of Z yields

$$\begin{array}{rcl} IE(k_1)U_1 & \dots & IE(k_t)U_t \to \\ E(k_1)U_1 & \dots & IE(k_t)U_t \to \\ U_1IE(k_{i-1})U_1 & \dots & IE(k_t)U_t \end{array}$$

since $0 < k$ and $k < k_{i+1}$, which is an F-term of type $n, M, k-1$. This proves the lemma. □

We can now proceed with the proof of the proposition. We remark here that in the case $M = I$, $k-1$ and $r-1$ can be replaced in the corollary by k and r. In this case $\rightarrowtail\!\!\twoheadrightarrow_{(k)}$ is Church–Rosser. However, this already follows from the Replacement Lemma by the theorem of Mitchke.

Proof of Proposition. First suppose that $X := X[Z_1, \dots, Z_r]$ where the Z_i are G-term occurrences of type n, M, k that are pairwise disjoint. We can follow each Z_i in a reduction $X \twoheadrightarrow Y$. It can be copied, projected (deleted), and beta reduced internally. Thus, we can write

$$Y := Y[Z_{11}, \dots, Z_{1s(1)}, \dots, Z_{r1}, \dots Z_{rs(r)}]$$

where $Z_i \twoheadrightarrow Z_{ij}$ for $-1 < j < s(i) + 1$, so that

$$X[x_1, \dots, x_r] \twoheadrightarrow Y[x_1, \dots, x_1, \dots, x_r, \dots, x_r].$$

Thus we have

$$Y[M, \dots, M, \dots, M, \dots, M] \leftarrow\!\shortmid Y \twoheadleftarrow X \shortmid\!\!\rightarrow X[M, \dots, M] \twoheadrightarrow Y[M, \dots, M, \dots, M, \dots, M]$$

Next suppose $X'[U_1, \dots, U_p] := X := X''[V_1, \dots, V_s]$ where the U_i are G-subterm occurrences of type n, M, k and pairwise disjoint, and the V_j are G-subterm occurrences of type n, M, r and also pairwise disjoint. Each U_i can contain one or more V_j, say $U_i := U_i[V_{i1}, \dots, V_{it(i)}]$, and by the Replacement Lemma $U_i[M, \dots, M]$ is a G-term of type n, M, k unless $M = I$, in which case $U_i[I, \dots, I] \twoheadrightarrow$ to a G-term of type $n, M, k-1$. Similar remarks hold for the V_j. Let $Z_1, \dots, Z_q$ be the maximal occurrences in the union of the two sets $\{U_1, \dots, U_p\}$ and $\{V_1, \dots, V_s\}$. Then we have $X := X'''[Z_1, \dots, Z_q]$ and

$$X'''[M, \dots, M] \;_{(r-1)}\!\!\leftarrowtail X'[M, \dots, M] \leftarrow\!\shortmid X \shortmid\!\!\rightarrow X''[M, \dots, M] \rightarrowtail_{(k-1)} X'''[M, \dots, M].$$

This completes the proof of the proposition. □

Corollary 2.2. (Strip Lemma) The diagram $Z_{(k)} \leftarrowtail X \rightarrowtail\!\!\twoheadrightarrow_{(k,t)} Y$ can be completed to $Z \rightarrowtail\!\!\twoheadrightarrow_{(k-1,t)} U \;_{(k-t)}\!\!\leftarrowtail Y$ provided $0 < t < k+1$.

3. Reduction Lemma

Lemma 3.1. (Reduction Lemma) Suppose that P and Q are connected in $G(G(n), I)$ by a path of length $k < n+1$. Then, there exists an R such that $P \rightarrowtail\!\!\twoheadrightarrow_{(n-k,k+1)} R \;_{(n-k,k+1)}\!\!\twoheadleftarrow\!\!\!\leftarrowtail Q$ where $M := I$.

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Proof. The proof is by induction on k. When $k = 0$ the lemma follows from the Church–Rosser theorem. Suppose now that we have a path $P := P(0), P(1), \ldots, P(k) := Q$ where $k < n + 1$. We have by our induction hypothesis that there exists an R such that $P \rightarrowtail\!\!\!\twoheadrightarrow_{(n-(k-1),k)} R \;_{(n-(k-1),k)}\!\twoheadleftarrow\!\!\!\leftarrowtail P(k-1)$. We distinguish two cases.

Case 1: $P(k-1) = TI$ and $Q = TG(n)$. We are assuming that $k > 0$, so we have $TG(n) \rightarrowtail_{(n-(k-1))} TI$. By the Proposition, there exists an R^* such that $P \rightarrowtail\!\!\!\twoheadrightarrow_{(n-(k-1),k)} R^* \;_{(n-(k-1),k)}\!\twoheadleftarrow\!\!\!\leftarrowtail TI$. Again by the Proposition, there exists an R^{**} such that

$$P \rightarrowtail\!\!\!\twoheadrightarrow_{(n-(k-1),k+1)} R^{**} \;_{(n-(k-1),k+1)}\!\twoheadleftarrow\!\!\!\leftarrowtail Q.$$

This suffices to complete the proof for this case.

Case 2: $P(k-1) = TG(n)$ and $Q = TI$. By the Proposition there exists an R^* such that

$$P \rightarrowtail\!\!\!\twoheadrightarrow_{(n-(k-1),k)} R^* \;_{(n-(k-1),k)}\!\twoheadleftarrow\!\!\!\leftarrowtail TG(n).$$

By the Strip Lemma corollary to the Proposition, there exists an R^{**} for the following diagram

$$TG(n) \rightarrowtail_{(n)} TI \rightarrowtail\!\!\!\twoheadrightarrow_{(n-k,k)} R^{**} \;_{(n-k)}\!\leftarrow\!\!\!\leftarrowtail R^* \;_{(n-(k-1),k)}\!\twoheadleftarrow\!\!\!\leftarrowtail P.$$

Finally, by the Proposition, there exists R^{***} such that

$$Q \rightarrowtail\!\!\!\twoheadrightarrow_{(n-k,k+1)} R^{***} \;_{(n-k,k+1)}\!\twoheadleftarrow\!\!\!\leftarrowtail P.$$

and this completes the proof of the lemma. □

Corollary 3.1. Suppose that $k < n + 1$. Then there is no path in $G(G(n), I)$ connecting the combinators K and K^* of length $< k + 1$.

Proof. K and K^* are $\rightarrowtail_{(n-k)}$ normal. □

We can now prove the following theorem.

Theorem 3.1. $G(n)$ is n-easy but not $2n + 5$ easy.

Proof. Suppose that K and K^* are connected in $G(G(n), M)$ by a path. If $M = / = I$, then by the Replacement Lemma and the theorem of Mitchke, $\rightarrowtail\!\!\!\twoheadrightarrow_{(n)}$ is Church–Rosser, so this is impossible. Thus $M = I$. But by the Corollary to the Reduction Lemma such a path must be longer than n. Thus $G(n)$ is n-easy. Clearly there is such a path of length $2n + 5$, so $G(n)$ is not $2n + 5$ easy. □

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