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# A NEW CLASS OF MAXIMAL TRIANGULAR ALGEBRAS

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*Abstract* Triangular algebras, and maximal triangular algebras in particular, have been objects of interest for over 50 years. Rich families of examples have been studied in the context of many w*- and C*-algebras, but there remains a dearth of concrete examples in $B(\mathcal{H})$. In previous work, we described a family of maximal triangular algebras of finite multiplicity. Here, we investigate a related family of maximal triangular algebras with infinite multiplicity, and unearth a new asymptotic structure exhibited by these algebras.

*Keywords:* triangular algebra; nest algebra; infinite multiplicity

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## 1. Introduction

Triangular algebras have been studied in a variety of contexts for over 50 years, since Kadison and Singer first introduced the concept of triangularity in [**6**]. Their initial study was of algebras $\mathcal{T}$ of bounded operators on a Hilbert space. Such an algebra is said to be *triangular* if its diagonal subalgebra, $\mathcal{T} \cap \mathcal{T}^*$, is a maximal abelian self-adjoint algebra (masa) in $B(\mathcal{H})$. In finite dimensions, a masa is just the set of diagonal matrices with respect to a fixed basis, and any matrix algebra containing the masa consists of a span of matrix units with respect to this basis. The triangularity condition amounts to $\mathcal{T}$ being precisely the span of matrix units $e_{i,j}$, where $i \preceq j$, as determined by some partial ordering $\preceq$ of $\{1, 2, \dots, n\}$. The algebra is then *maximal* as a triangular algebra if and only if the associated partial order is a linear order.

In infinite dimensions, the masa generalizes the algebra of diagonal matrices. It is not, of course, always associated with a basis, but through spectral theory, can always be associated with a compact spectral set, and the goal is to correspondingly associate the triangular algebra with a one-sided action or partial order on the spectral set. This correspondence has been the subject of study in a wide range of contexts. The nest algebras, introduced by Ringrose [**18**] shortly after the triangular algebras, extended the class of hyperreducible triangular algebras studied by Kadison and Singer, and proved more tractable than general triangular algebras. Later authors explored triangular algebras of certain C*-algebras [**9**, **15**, **17**] and of von Neumann algebras [**8**], which stimulated a rich body of results by many mathematicians in these contexts.

 

Little, however, is known in detail about maximal triangular algebras on infinite-dimensional Hilbert spaces, where in general the masa is not a Cartan algebra [**5**]. Kadison and Singer [**6**] showed that the lattice of invariant projections of a maximal triangular algebra must be linearly ordered. They focused on those maximal triangular algebras whose invariant lattice is multiplicity free (i.e. has a cyclic separating vector), and showed that in this case the algebra is determined by its invariant lattice; that is to say, it is a nest algebra. They called such maximal triangular algebras *hyperreducible*. They also showed, in contrast to the finite-dimensional case, that not all maximal triangular algebras are hyperreducible and indeed that there exist maximal triangular algebras that are irreducible (i.e. having no non-trivial invariant projections). Solel [**19**] further investigated irreducible triangular algebras. Poon [**16**] and, independently, the present author [**10**], showed that, in general, maximal triangular algebras need not even be norm closed.

But apart from the hyperreducible case, no concrete examples of maximal triangular algebras in $B(\mathcal{H})$ were known until [**13**]. There, using techniques derived from the similarity theory of nests [**1**], it was possible to describe two classes of non-hyperreducible maximal triangular algebras. The first of these was based on a tensor-product construction proposed in [**4**] (see Theorem 2.1, below). The other was based on a construction of block operator matrices (see Theorem 2.4, below). The goal was to study a family of maximal triangular algebras we termed the *compressible* maximal triangular algebras (see [**13**, Definition 6.1], and Definition 2.2, below), which are defined in analogy with the type I von Neumann algebras, and we succeeded in obtaining detailed descriptions of most finite-multiplicity compressible maximal triangular algebras in [**13**, Theorem 6.1].

The purpose of the present work is to extend the construction of compressible maximal triangular algebras from finite multiplicity to infinite multiplicity. In Theorem 3.8, we present a new construction for infinite-multiplicity triangular algebras, and in Theorem 3.14, we show when this construction yields maximal triangular algebras. In § 4, we explore the range of examples provided by this construction, and in § 5, we present criteria for recognizing maximal triangular algebras that can be represented in this way.

One feature of this construction is that it exposes a new kind of 'asymptotic triangularity' condition that appears in infinite multiplicity but not in finite multiplicity. This is based on the 'liminal seminorms' introduced in Definition 3.3 and the properties of their support sets seen in Definition 3.6. Heuristically, these conditions can be thought of as describing the contributions to the norms of rows and columns that are not localized in individual block matrix entries but rather are residual in the row or column 'at infinity'.

In this paper, we focus on those infinite-multiplicity compressible maximal triangular algebras that are quite uniform with respect to this asymptotic behaviour, which we term *simple* algebras; although we also present examples of the more complex, but still tractable, behaviour of non-simple algebras (e.g. Example 5.7). The general case of compressible algebras with infinite multiplicity is still unclear, but this study illustrates the kind of subtleties that arise when passing from finite to infinite multiplicity. Further work will be needed to understand the infinite-multiplicity case completely.

## 2. Preliminaries

Throughout this paper, the underlying Hilbert spaces are always assumed separable. A *nest* is a linearly ordered set of projections on a Hilbert space which contains 0 and $I$,

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and is weakly closed (equivalently, order-complete). The *nest algebra*, $Alg(\mathcal{N})$, of a nest $\mathcal{N}$ is the set of bounded operators leaving invariant the ranges of $\mathcal{N}$. An *interval* of $\mathcal{N}$ is the difference of two projections $N > M$ in $\mathcal{N}$. Minimal intervals are called *atoms*, and the atoms (if there are any) are pairwise orthogonal. If the join of the atoms is $I$, the nest is called *atomic*; if there are no atoms, it is called *continuous*. See [**2**] for further properties of nest algebras.

Nests have a spectral theory analogous to the spectral theory for self-adjoint operators [**3**]. Each nest is unitarily equivalent to a nest constructed from a triple consisting of a linearly ordered set $X$ which is compact in its order topology, a finite regular Borel measure $m$ and a measurable multiplicity function $d : X \to \mathbb{N} \cup \{+\infty\}$. Briefly, the construction is as follows. For each $i \in \mathbb{N}$ let $X_i := \{x \in X : d(x) \geq i\}$, and for each $x \in X$ write $L_x := \{y \in X : y \preceq x\}$. Let $\mathcal{H}_i := L^2(X_i, m)$, and the nest consists of the projections on $\mathcal{H} := \bigoplus H_i$ corresponding to multiplication by the characteristic functions of $L_x \cap X_i$ on each $\mathcal{H}_i$. If the multiplicity function is constant, then the nest is said to have *uniform multiplicity*, the non-zero $\mathcal{H}_i$ are unitarily equivalent and $\mathcal{N}$ can be represented as a direct sum of copies of a multiplicity-free nest. If the nest is continuous, we can take $X = [0, 1]$ and $m$ to be Lebesgue measure. This representation also provides each nest with an associated projection-valued spectral measure corresponding to multiplication by the characteristic function of a Borel set. When the nest is continuous, we write $E(S)$ for the corresponding spectral measure on the Borel sets of $[0, 1]$.

We now describe in more detail the two previously known constructions for maximal triangular algebras mentioned in the introduction. The first of these realizes the 'triangular tensor product' construction envisioned in [**4**].

**Theorem 2.1 (Orr [13, Theorem 5.1]).** *Let $\mathcal{N}_0$ and $\mathcal{M}_0$ be multiplicity-free nests on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and let $\mathcal{N} := \mathcal{N}_0 \otimes I_{\mathcal{K}}$ and $\mathcal{M} := I_{\mathcal{H}} \otimes \mathcal{M}_0$. Then there is a unique maximal triangular algebra $\mathcal{T}$ satisfying*

$$Alg(\mathcal{N}_0) \otimes Alg(\mathcal{M}_0) \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes B(\mathcal{K}).$$

*Moreover $\mathcal{T}$ is the set of operators $X \in Alg(\mathcal{N})$ such that (i) whenever $M \in \mathcal{M}$ has both an immediate predecessor and successor in $\mathcal{M}$ then*

$$M^{\perp} X M \in \mathcal{R}_{\mathcal{N}}^{\infty}$$

*and (ii) whenever $M > M' > M''$ are in $\mathcal{M}$ then*

$$M^{\perp} X M'' \in \mathcal{R}_{\mathcal{N}}^{\infty}.$$

In the statement of the theorem, $\mathcal{R}_{\mathcal{N}}^{\infty}$ denotes Larson's ideal, introduced in [**7**], which is the largest off-diagonal ideal of $Alg(\mathcal{N})$ [**13**, Theorem 4.1]. See also [**2**, Chapter 15] for details of tensor products; in the theorem the tensor products are weakly closed spatial tensor products of the respective algebras.

While Theorem 2.1 will provide us with useful examples, our main focus in this paper will be on the class of *compressible* maximal triangular algebras, which we introduced in [**13**] in analogy with the type I von Neumann algebras.

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**Definition 2.2.** Let $\mathcal{T}$ be a maximal triangular algebra. Let $\mathcal{N}$ be the lattice of invariant projections of $\mathcal{T}$, which was shown to be a nest in [6]. The commutant $\mathcal{N}'$ is a type I von Neumann algebra and so contains a partition of the identity $E_i$ consisting of abelian projections. If such $E_i$ can be chosen so that $E_i\mathcal{T}|_{E_i\mathcal{H}}$ is maximal triangular for all $i$, we say that $\mathcal{T}$ is *compressible*.

In [13] we saw both that the compression of a maximal triangular algebra to the range of an abelian projection in $\mathcal{N}'$ need not always be maximal, and also that, if such projections can be found, they can provide a basis for completely describing the algebra. More precisely, in [13, Theorem 6.1] we saw that if $\mathcal{T}$ is a compressible maximal triangular algebra and $\mathcal{N}$ has no infinite-multiplicity part and satisfies some other mild regularity conditions on its spectral multiplicity, then $\mathcal{T}$ can be completely described.

In our present study we will go to the other extreme and focus on the case when $\mathcal{N}$ has uniform infinite multiplicity. (Studying the case of mixed finite and infinite multiplicity is premature when the full range of infinite-multiplicity behaviour is not yet understood.) The starting point for our study will therefore be in analogy with the results from [13, Example 6.3], which present an easily visualized construction for uniform *finite* multiplicity compressible maximal triangular algebras as finite block operator matrices. We will give a precise statement of the finite-multiplicity result after first defining the diagonal seminorm function, which will be another key ingredient of our study.

**Definition 2.3.** Let $\mathcal{N} = \{N_t : t \in [0,1]\}$ be a continuous nest where the indexing is compatible with the spectral measure (i.e. $N_t = E([0,t])$). For $X \in Alg(\mathcal{N})$ and $x \in [0,1]$ we define the *diagonal seminorm function* $i_x(X)$ by the formula

$$i_x(X) := \inf\{\|(N_t - N_s)X(N_t - N_s)\| \ : \ s < x < t\}.$$

**Theorem 2.4 (Orr [13, Theorem 6.3]).** *Suppose that $\mathcal{N}_0$ is a continuous nest on $\mathcal{H}$ and $\mathcal{M}_0$ is a finite nest of length $n$ on $\mathcal{K} = \mathbb{C}^n$, and let $\mathcal{N} := \mathcal{N}_0 \otimes I_{\mathcal{K}}$ and $\mathcal{M} := I_{\mathcal{H}} \otimes \mathcal{M}_0$. Write $E_i$ for the minimal intervals (atoms) of $\mathcal{M}$. Then every maximal triangular algebra $\mathcal{T}$ satisfying*

$$Alg(\mathcal{N}_0) \otimes \mathcal{M}_0' \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes B(\mathcal{K})$$

*is of the form*

$$\{X \in Alg(\mathcal{N}) \ : \text{for each } 1 \le i,j \le n \text{ and almost every } x \notin S_{i,j}, i_x(E_iXE_j) = 0\}$$

*where the sets $S_{i,j}$ $(1 \le i,j \le n)$ are Borel subsets of $[0,1]$ satisfying*

$$\begin{aligned} S_{i,i} &= [0,1], \\ S_{i,j} &= S_{j,i}^c \quad \text{for } i \ne j \quad \text{and} \\ S_{i,j} \cap S_{j,k} &\subseteq S_{i,k} \quad \text{for all } i,j,k. \end{aligned}$$

Note the inclusion condition on $\mathcal{T}$ in the last result holds for any compressible algebra with finite uniform multiplicity nest, and so there is no loss of generality involved, just a selection of a fixed representation of the nest.

The key technical result involved in proving the last two theorems was the *Interpolation theorem*, which is also a crucial tool in the present work:

**Theorem 2.5 (Interpolation theorem; [13, Theorem 3.1]).** *Let $\mathcal{N}$ be a continuous nest indexed as above, let $X \in Alg(\mathcal{N})$ and, for $a > 0$, let $S := \{t \in [0,1] : i_t(X) \geq a\}$. Then there are operators $A, B \in Alg(\mathcal{N})$ such that $AXB = E(S)$.*

Although the substance of this result was proved in [13, Theorem 3.1], it should be noted that the proof there made use of a slightly different diagonal seminorm function (the $i_x^+$ of Ringrose's [18]) and that the Interpolation theorem based on our present $i_x$ seminorms was given in [12, Theorem 1.2].

We will use the diagonal seminorm function throughout our results. The following lemma is routine to prove and captures the key technical properties of the function.

**Lemma 2.6.** *For fixed $x \in [0,1]$, $i_x(X)$ is a submultiplicative seminorm on $Alg(\mathcal{N})$. For fixed $X \in Alg(\mathcal{N})$, $i_x(X)$ is an upper semicontinuous function on $[0,1]$.*

## 3. The simple uniform algebras

In this section, we will see the precise definition of the new class of infinite-multiplicity maximal triangular algebras which will be our main object of study in this paper, and which we term the *simple* uniform algebras (see Definition 3.9).

For the rest of this paper, fix $\mathcal{H}$ and $\mathcal{K}$ as separable infinite-dimensional Hilbert spaces. Let $\mathcal{N}_0$ be a multiplicity-free continuous nest on $\mathcal{H}$ and $\mathcal{D}_0$ an atomic masa on $\mathcal{K}$. Let $\mathcal{N} := \mathcal{N}_0 \otimes I_{\mathcal{K}}$ and $\mathcal{D} := I_{\mathcal{H}} \otimes \mathcal{D}_0$. We naturally visualize the elements of $Alg(\mathcal{N})$ as infinite block operator matrices with entries in the continuous nest algebra $Alg(\mathcal{N}_0)$.

The atoms of $\mathcal{D}_0$ are one-dimensional so pick a basis $\epsilon_i$ of $\mathcal{K}$ consisting of unit vectors in the atoms of $\mathcal{D}_0$. Let $E_{i,j} := I \otimes (\epsilon_i \otimes \epsilon_j^*)$, where we adopt the notation $\alpha \otimes \beta^*$ for the rank-1 operator $\langle\, .\, , \beta\rangle\alpha$. Also write $E_i := E_{i,i}$ and $N_x$ ($x \in [0,1]$) for the nest projections of $\mathcal{N}$, where the indexing is compatible with the spectral measure (i.e. $N_t = E([0,t])$).

Note that any triangular algebra $\mathcal{T}$ satisfying the inclusion relation

$$Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes \mathcal{B}(\mathcal{K})$$

is compressible and so throughout the remainder of this paper, and especially in § 5, we shall focus on triangular algebras satisfying this relation.

The following definition is just the direct analogue of the sets used in Theorem 2.4, except with infinite multiplicity. In the proposition that follows it, we see that these properties alone are not enough to specify a triangular algebra in the infinite-multiplicity case.

**Definition 3.1.** Let $\mathbf{S} = (S_{i,j})_{i,j \in \mathbb{N}}$ be a collection of Borel subsets of $[0,1]$ satisfying:

(1) $S_{i,i} = [0,1]$ for all $i \in \mathbb{N}$;

(2) $S_{i,j} \cap S_{j,i} = \emptyset$ for all $i \neq j$ in $\mathbb{N}$;

(3) $S_{i,j} \cap S_{j,k} \subseteq S_{i,k}$ for all $i,j,k \in \mathbb{N}$.

Then $\mathbf{S}$ is called a *triangular system.*

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**Proposition 3.2.** *Let* $\boldsymbol{S}$ *be a triangular system and let* $\mathcal{T}(\boldsymbol{S})$ *be the set of all* $X \in Alg(\mathcal{N})$ *such that* $i_x(E_iXE_j) = 0$ *for each* $1 \le i, j < \infty$ *and almost every* $x \notin S_{i,j}$*. Then* $\mathcal{T}(\boldsymbol{S})$ *is a triangular space but is never an algebra; that is to say, it is a linear space and* $\mathcal{T}(\boldsymbol{S}) \cap \mathcal{T}(\boldsymbol{S})^*$ *is a masa, but it is not closed under multiplication.*

**Proof.** For each $i, j \in \mathbb{N}$ and $x \in [0, 1]$, the function $X \mapsto i_x(E_iXE_j)$ is a norm-bounded seminorm. From this it is routine to see that $\mathcal{T}(\mathbf{S})$ is a norm-closed linear space. If $T \in \mathcal{T}(\mathbf{S}) \cap \mathcal{T}(\mathbf{S})^*$ then $T \in Alg(\mathcal{N}) \cap (Alg(\mathcal{N}))^* = \mathcal{N}'$. We must show $E_iTE_j = 0$ for all $i \neq j$, for then $T$ commutes with all $E_i$ and so $T \in \mathcal{N}_0' \otimes \mathcal{D}_0$, the diagonal masa.

Suppose $E_iTE_j \neq 0$ for some $i \neq j$. Then $i_x(E_iTE_j)$ is zero almost everywhere (a.e.) outside $S_{i,j}$ and $i_x(E_jT^*E_i)$ is zero a.e. outside $S_{j,i}$. Since these two quantities are equal, and $S_{i,j} \cap S_{j,i} = \emptyset$, it follows that $i_x(E_iTE_j)$ is zero a.e. By Theorem 2.1 of [**13**], $E_iTE_j \in \mathcal{R}_{\mathcal{N}}^{\infty}$, which is a diagonal-disjoint ideal of $Alg(\mathcal{N})$, and yet $E_iTE_j$ belongs to $\mathcal{N}'$, the diagonal of $Alg(\mathcal{N})$, hence $E_iTE_j = 0$.

To see that $\mathcal{T}(\mathbf{S})$ is not an algebra, we shall fix arbitrary $i, j$ and construct operators $X = E_iX$ and $Y = YE_j$ in $\mathcal{T}(\mathbf{S})$ that satisfy $i_x(E_iXYE_j) \ge 1$ for all $x \in [0, 1]$. Let $(s_n, t_n)$ be an enumeration of all the open intervals with rational endpoints in $[0, 1]$. For each $n$ pick $s_n < x < y < t_n$ and set $X_n := \alpha_n \otimes \beta_n^*$ and $Y_n := \beta_n \otimes \gamma_n^*$, where $\alpha_n$, $\beta_n$, and $\gamma_n$ are unit vectors in, respectively, the range of $(N_x - N_{s_n})E_i$, $(N_y - N_x)E_n$, and $(N_{t_n} - N_y)E_j$. Since each of these ranges is infinite dimensional, we can choose the $\alpha_n$, $\beta_n$, and $\gamma_n$ inductively to be pairwise orthogonal sequences. Each $X_n$ and $Y_n$ is in $Alg(\mathcal{N})$ since $X_n = N_xX_nN_x^{\perp}$ and $Y_n = N_yY_nN_y^{\perp}$, and so $X := \sum_n X_n$ and $Y := \sum_n Y_n$ converge strongly to operators in $Alg(\mathcal{N})$. For any $m, n$ we have $E_mXE_n$ is zero unless $m = i$, in which case $i_x(E_iXE_n) = i_x(X_n) = 0$, since $X_n$ is finite rank. Thus $i_x(E_mXE_n)$ is zero for all $m, n$ and $x \in [0, 1]$ so $X \in \mathcal{T}(\mathbf{S})$ and, similarly, $Y \in \mathcal{T}(\mathbf{S})$. Conversely, $XY = E_iXYE_j = \sum_n \alpha_n \otimes \gamma_n^*$. Since $\|(N_{t_n} - N_{s_n})XY(N_{t_n} - N_{s_n})\| \ge 1$ for all $n$ it follows that $i_x(E_iXYE_j) \ge 1$ for all $x$ and so $\mathcal{T}(\mathbf{S})$ cannot be closed under multiplication. □

Moreover, as we shall see in Proposition 5.5, every maximal triangular algebra, $\mathcal{T}$, satisfying $Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes \mathcal{B}(\mathcal{K})$ is contained in $\mathcal{T}(\mathbf{S})$ for some triangular system $\mathbf{S}$. Thus it makes sense to seek additional constraints on the elements of $\mathcal{T}(\mathbf{S})$ that will determine a maximal triangular algebra. In the following two definitions, we introduce the properties related to 'behaviour at infinity' of block operator matrices, which enable us to specify triangular algebras.

**Definition 3.3.** Let $M_i := \sum_{i=1}^{n} E_i$. For $X \in Alg(\mathcal{N})$, $t \in [0, 1]$, and $i, j \in \mathbb{N}$, define the *liminal row seminorm*

$$r_{i,t}(X) := \lim_{k\to\infty} i_t(E_iXM_k^{\perp})$$

and the corresponding *liminal column seminorm*

$$c_{j,t}(X) := \lim_{k\to\infty} i_t(M_k^{\perp}XE_j).$$

**Remark 3.4.** Despite superficial appearances, the values of $r_{i,t}$ and $c_{j,t}$ do not depend on the ordering of the atoms of $\mathcal{D}$.

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The proof of the following basic properties of the liminal seminorms is routine, and left to the reader.

**Lemma 3.5.** *For fixed $i, j$ and $t \in [0,1]$, the functions $r_{i,t}$ and $c_{j,t}$ are seminorms on $Alg(\mathcal{N})$. For fixed $i, j$ and $X \in Alg(\mathcal{N})$, the functions $r_{i,t}(X)$ and $c_{j,t}(X)$ are upper semicontinuous functions of $t \in [0,1]$.*

We now add extra properties to the definition of a triangular system, which will enable us to specify a triangular algebra, as seen in Theorem 3.8.

**Definition 3.6.** Let $\mathbf{S} = (S_{i,j})_{i,j\in\mathbb{N}}$, $\mathbf{R} = (R_i)_{i\in\mathbb{N}}$, and $\mathbf{C} = (C_j)_{j\in\mathbb{N}}$ be collections of Borel subsets of $[0,1]$ satisfying:

(1) $S_{i,i} = [0,1]$ for all $i \in \mathbb{N}$;

(2) $S_{i,j} \cap S_{j,i} = \emptyset$ for all $i \neq j$ in $\mathbb{N}$;

(3) $S_{i,j} \cap S_{j,k} \subseteq S_{i,k}$ for all $i, j, k \in \mathbb{N}$;

(4) $C_i \cap S_{i,j} \subseteq C_j$ for all $i, j \in \mathbb{N}$;

(5) $S_{i,j} \cap R_j \subseteq R_i$ for all $i, j \in \mathbb{N}$;

(6) $R_i \cap C_j \subseteq S_{i,j}$ for all $i, j \in \mathbb{N}$.

Then the triple $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is called an *extended triangular system*.

**Definition 3.7.** Given collections of Borel sets, $\mathbf{S} = (S_{i,j})_{i,j\in\mathbb{N}}$, $\mathbf{R} = (R_i)_{i\in\mathbb{N}}$, and $\mathbf{C} = (C_j)_{j\in\mathbb{N}}$, we shall write $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ for the set of all $X \in Alg(\mathcal{N})$ such that:

(1) $i_t(E_i X E_j) = 0$ for each $1 \leq i, j < \infty$ and almost every $t \notin S_{i,j}$;

(2) $r_{i,t}(X) = 0$ for each $1 \leq i < \infty$ and almost every $t \notin R_i$;

(3) $c_{j,t}(X) = 0$ for each $1 \leq j < \infty$ and almost every $t \notin C_j$.

**Theorem 3.8.** *If $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ is an extended triangular system then $\mathcal{T}(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ is a triangular algebra.*

**Definition 3.9.** The algebras $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ described in the last theorem are called the *simple uniform triangular algebras*.

**Proof of Theorem 3.8.** By the same techniques as Proposition 3.2, and since $r_{i,t}$ and $c_{j,t}$ are seminorms, $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is a triangular space. It remains to show it is closed under multiplication. Let $X, Y \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ and verify criteria (1)–(3) for $XY$.

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To verify (1), fix $i$ and $j$ and consider

$$\begin{aligned} i_t(E_iXYE_j) &\leq \sum_{k=1}^{r} i_t(E_iXE_kYE_j) + i_t(E_iXM_r^{\perp}YE_j) \\ &\leq \sum_{k=1}^{r} i_t(E_iXE_k)i_t(E_kYE_j) + i_t(E_iXM_r^{\perp})i_t(M_r^{\perp}YE_j). \end{aligned}$$

The terms in the sum are zero a.e. outside $S_{i,j}$ and the remainder term converges to zero (as $r \to \infty$) for almost every $t$ outside $R_i \cap C_j \subseteq S_{i,j}$. Integrate (wrt $t$) over $S_{i,j}^c$ and apply the Dominated Convergence theorem to the limit as $r \to \infty$ to see that $i_t(E_iXYE_j) = 0$ for almost all $t \notin S_{i,j}$.

Verify (2) in the same way by considering the inequality

$$\begin{aligned} i_t(E_iXYM_j^{\perp}) &\leq \sum_{k=1}^{r} i_t(E_iXE_kYM_j^{\perp}) + i_t(E_iXM_r^{\perp}YM_j^{\perp}) \\ &\leq \sum_{k=1}^{r} i_t(E_iXE_k)i_t(E_kYM_j^{\perp}) + i_t(E_iXM_r^{\perp})\|Y\| \end{aligned}$$

from which, taking $j \to \infty$,

$$r_{i,t}(XY) \leq \sum_{k=1}^{r} i_t(E_iXE_k)r_{k,t}(Y) + i_t(E_iXM_r^{\perp})\|Y\|.$$

The terms in the sum are zero a.e. outside $S_{i,k} \cap R_k \subseteq R_i$, and the remainder term converges to zero (as $r \to \infty$) for almost every $t$ outside $R_i$. Thus, similarly by the Dominated Convergence theorem, $r_{i,t}(XY) = 0$ on $R_i^c$. The case of criterion (3) is analogous. □

We now present a key observation, which relates our construction, with families of measurable sets, to partial orders, something to be expected in the context of triangular algebras. Note that an extended triangular system induces a set of partial orders and 'Dedekind cuts' on $\mathbb{N}$. More precisely, for each fixed $x \in [0, 1]$, we define a partial order on $\mathbb{N}$ by $i \preceq j$ if $x \in S_{i,j}$ and let $A = \{i \in \mathbb{N} : x \in C_i\}$ and $B = \{i \in \mathbb{N} : x \in R_i\}$. Note that $A$ is an increasing set, since if $i \in A$ and $i \preceq j$ then $x \in C_i$ and $x \in S_{i,j}$, so that $x \in C_i \cap S_{i,j} \subseteq C_j$ so that $j \in A$. In the same way, one sees that $B$ is a decreasing set and that every element in $A$ dominates every element in $B$ (i.e. $b \preceq a$). Although the pair $(A, B)$ is not exactly a Dedekind cut – most importantly it does not always partition $\mathbb{N}$ – we shall continue to employ the terminology because of the unmistakable similarities and the fact that, like a true Dedekind cut, this pair of sets does indicate the behaviour at a missing or virtual point, in our case the asymptotic point at infinity.

**Definition 3.10.** In our context, a Dedekind cut on a partially ordered set is a pair of subsets $(A, B)$ such that: $A$ is increasing, $B$ is decreasing, and every element of $A$ dominates every element of $B$ (though not strictly).

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We shall see below (Theorem 3.14) that if this induced set of partial orders and Dedekind cuts is maximal, then $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is a maximal triangular algebra. We collect a few simple facts about this ordering/cuts viewpoint in the following lemmas.

**Lemma 3.11.** *Let* $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ *be an extended triangular system and for each* $x \in [0,1]$ *let* $\preceq_x$ *and* $(A_x, B_x)$ *be the partial order and Dedekind cut induced on* $\mathbb{N}$ *in the context of* $x$*, as described above. Then* $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ *is maximal (in the sense that none of the sets in* $\boldsymbol{S}$, $\boldsymbol{R}$, *or* $\boldsymbol{C}$ *can be enlarged without violating the requirements of an extended triangular system) if and only if for each* $x \in [0,1]$, $\preceq_x$ *is a linear order,* $A_x \cup B_x = \mathbb{N}$, *and either (a)* $A_x$ *has no smallest element and* $B_x$ *has no greatest element, or (b)* $\min A_x$ *and* $\max B_x$ *both exist, and are equal.*

**Proof.** First suppose $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is maximal.

Fix an arbitrary $x \in [0,1]$. Even if $\preceq_x$ were not linear, it could at least be extended to a linear order $\preceq'_x$ on $\mathbb{N}$. Enlarge $A_x$ and $B_x$ to

$$A'_x := \{i \ : \ \exists a \in A_x \text{ with } a \preceq'_x i\} \quad \text{and} \quad B'_x := \{i \ : \ \exists b \in B_x \text{ with } i \preceq'_x b\}.$$

Clearly $A'_x$ is increasing, $B'_x$ is decreasing, and if $a' \in A'_x$ and $b' \in B'_x$ then there are $a \in A_x$ and $b \in B_x$ with

$$b' \preceq'_x b \preceq_x a \preceq'_x a'$$

and so $(A'_x, B'_x)$ is a Dedekind cut. Enlarge $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ accordingly (i.e. put $x$ in $S'_{i,j}$ whenever $i \preceq'_x j$, etc.), and so by maximality each $S'_{i,j} = S_{i,j}$, and so $\preceq_x$ is equal to $\preceq'_x$, and so it is linear.

Now suppose there is a $c \notin A_x \cup B_x$ for some $x$. Since $\preceq_x$ is a linear order, it follows that $b \prec_x c \prec_x a$ for every $a \in A_x$ and $b \in B_x$. Take

$$A'_x := \{i \ : \ i \succeq_x c\} \quad \text{and} \quad B'_x := \{i \ : \ i \preceq_x c\}.$$

Clearly $(A'_x, B'_x)$ is a Dedekind cut and $A_x \subseteq A'_x$ and $B_x \subseteq B'_x$. But, having enlarged $(A_x, B_x)$ we can correspondingly enlarge $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ by putting $x$ in some of the $R_i$ and $C_j$, contrary to supposition. Thus $A_x \cup B_x = \mathbb{N}$ for all $x$.

Finally fix $x$ and consider two cases based on whether or not $A_x \cap B_x$ is empty. If it is empty then $A_x$ cannot have a smallest element, for if it did then by maximality $B_x$ would have to be $\{i \ : \ i \preceq_x \min A_x\}$, which would meet $A_x$. Likewise, $B_x$ cannot have a greatest element. Conversely, if $c \in A_x \cap B_x$ then by maximality $A_x = \{i \ : \ i \succeq_x c\}$ and $B_x = \{i \ : \ i \preceq_x c\}$, hence $\min A_x = \max B_x = c$.

To prove the converse, now suppose that $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is not maximal. Suppose further that for all $x \in [0,1]$, both $\preceq_x$ is linear (so that $S_{i,j} = S^c_{j,i}$ for all $i, j$) and $A_x \cup B_x = \mathbb{N}$. Since $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is not maximal, find $(\mathbf{S}', \mathbf{R}', \mathbf{C}')$ that strictly extends $(\mathbf{S}, \mathbf{R}, \mathbf{C})$. Since $\mathbf{S}$ is maximal, one of $\mathbf{R}', \mathbf{C}'$ must be bigger. Without loss, assume $C'_i$ is a proper superset of $C_i$, and let $x \in C'_i \setminus C_i$. Thus $i \in A'_x \setminus A_x$ and so $b \preceq_x i \prec_x a$ for all $a \in A_x$ and $b \in B_x$. Thus, on the one hand, case (b) is impossible. On the other hand, $i$ must belong to one of $A_x$ or $B_x$ by supposition, and clearly $i \notin A_x$, so $i \in B_x$ and so $B_x$ does have a greatest element. Thus case (a) fails too. By contrapositive, if for all $x$, the ordering $\preceq_x$ is linear, $A_x \cup B_c = \mathbb{N}$, and one of case (a) or case (b) holds, then $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ must be maximal. □

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**Lemma 3.12.** *Every extended triangular system can be enlarged to a maximal extended triangular system.*

**Proof.** A routine Zorn's lemma argument would be enough to see the result, except that we must maintain measurability of the sets. For that we will need to enlarge the sets in a series of deterministic steps.

First we shall extend $\preceq_x$ to a linear order for all $x$. Enumerate all the pairs $(i_0, j_0)$ in a fixed order and run through them. Whenever we come to a pair $(i_0, j_0)$ for which there are $x$ with $i_0 \not\preceq_x j_0$ and $j_0 \not\preceq_x i_0$, we extend $\preceq_x$ to $\preceq'_x$ by declaring $i_0 \preceq'_x j_0$ and, consequently, $i \preceq'_x j_0$ for all $i \preceq_x i_0$ and $j \succeq'_x i_0$ for all $j \succeq_x j_0$. This translates to enlarging $S_{i_0,j_0}$ to $S'_{i_0,j_0} := S_{i_0,j_0} \cup (S_{i_0,j_0} \cup S_{j_0,i_0})^c = S^c_{j_0,i_0}$ and

$$S'_{i,j_0} := S_{i,j_0} \cup (S_{i,i_0} \cap S'_{i_0,j_0}),$$
$$S'_{i_0,j} := S_{i_0,j} \cup (S'_{i_0,j_0} \cap S_{j_0,j}).$$

Likewise, enlarge $A_x$ and $B_x$ to

$$\{i \;:\; a \preceq'_x i \text{ for some } a \in A_x\} \quad \text{and} \quad \{i \;:\; i \preceq'_x b \text{ for some } b \in B_x\}$$

respectively. This translates to

$$R'_i := \bigcup_{b\in\mathbb{N}} S'_{i,b} \cap R_b,$$
$$C'_i := \bigcup_{a\in\mathbb{N}} C_a \cap S'_{a,i}$$

for each $i$. At each stage all the sets $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ grow measurably, and continue to be an extended triangular system. So finally replace each set with the union of all the intermediate versions and we obtain an extended triangular system in which each $\preceq_x$ is linear.

In a similar way, we shall enlarge $\mathbf{R}$ and $\mathbf{C}$ to be maximal. For each $x$, either $(A_x, B_x)$ is already maximal, or else there is an $i_0$ such that

$$A'_x := \{i \;:\; i \succeq_x i_0\} \supseteq A_x \tag{3.1}$$

and

$$B'_x := \{i \;:\; i \preceq_x i_0\} \supseteq B_x \tag{3.2}$$

and $(A'_x, B'_x)$ is a maximal Dedekind cut. Turning this around, we shall take each value of $i_0$ in turn and enlarge $\mathbf{R}$ and $\mathbf{C}$ by those $x$ for which (3.1) and (3.2) hold. Thus, fix $i_0$ and for each $i$, replace $R_i$ with $R_i \cup (S_{i,i_0} \cap (R_{i_0} \cup C_{i_0})^c)$ and $C_i$ with $C_i \cup (S_{i_0,i} \cap (R_{i_0} \cup C_{i_0})^c)$. Repeat this process for each $i_0$, and take the union of the successively enlarged sets. □

The next lemma provides technical results necessary to establish Theorem 3.14, that the triangular algebras associated with maximal extended triangular systems are themselves *maximal* triangular algebras. The lemma will enable us to see that the presence of operators that violate the constraints of $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ leads to violations of triangularity.

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**Lemma 3.13.** *Suppose $X \in Alg(\mathcal{N})$ and for some $i \in \mathbb{N}$, $a > 0$, and a fixed closed $K \subseteq [0,1]$, we have $r_{i,t}(X) \geq a$ for all $t \in K$. Let $j \in \mathbb{N}$. Then there are $A, B \in Alg(\mathcal{N})$ satisfying $A = E_i A E_i$ and $B = BE_j$, and such that:*

(1) $AXB = E(K)E_{i,j}$;

(2) $i_t(B) = 0$ *for all* $t \notin K$;

(3) $i_t(E_m B) = 0$ *for all* $t \in [0,1]$ *and* $m \in \mathbb{N}$.

**Proof.** Consider the intervals of the form $((p-1)/q, (p+1)/q)$ for natural numbers $p < q$ and let $(s_n, t_n)$ be an enumeration of all such intervals that contain a point of $K$. Observe that $t_n - s_n \to 0$. For each $n$, choose $x_n$, $y_n$ and $z_n$ with $s_n < x_n < y_n < z_n < t_n$ and $y_n \in K$. Taking $Y_n := E_i(N_{z_n} - N_{x_n})X(N_{z_n} - N_{x_n})M_n^{\perp}$, note that since $\|Y_n M_k^{\perp}\| \geq a$ for all $k$, in particular the essential norm of each $Y_n$ is at least $a$. Thus, there are orthonormal sequences of vectors $\beta_n = E_i(N_{z_n} - N_{x_n})\beta_n$, and $\gamma_n = M_n^{\perp}(N_{z_n} - N_{x_n})\gamma_n$ such that $\langle \beta_m, X\gamma_n \rangle = 0$ for all $m \neq n$, and $\langle \beta_n, X\gamma_n \rangle > a/2$ for all $n$ (by e.g. [**14**, Lemma 2.2]). In addition, pick orthonormal sequences $\alpha_n = E_i(N_{x_n} - N_{s_n})\alpha_n$, and $\delta_n = E_j(N_{t_n} - N_{z_n})\delta_n$, and let $A := \sum_n \alpha_n \otimes \beta_n^*$ and $B := \sum_n \gamma_n \otimes \delta_n^*$. Note that each summand of $A$ satisfies $\alpha_n \otimes \beta_n^* = N_{x_n}\alpha_n \otimes \beta_n^* N_{x_n}^{\perp}$ so that $A \in Alg(\mathcal{N})$. Similarly $B \in Alg(\mathcal{N})$.

Fix $x \notin K$ and $s < x < t$, and choose $s < s' < x < t' < t$ such that $(s', t')$ is a positive distance from $K$. Then since $t_n - s_n \to 0$, $(s', t')$ is disjoint from all but finitely many $(s_n, t_n)$ and so $(N_{t'} - N_{s'})B(N_{t'} - N_{s'})$ is finite rank. Hence $i_x(B) = 0$. Likewise for any fixed $m$, $E_m\gamma_n = 0$ for all but finitely many $n$ so that $E_m B$ is finite rank and $i_x(E_m B) = 0$ for all $x \in [0,1]$ and $m \in \mathbb{N}$.

Conversely, $AXB = E_i AXBE_j = \sum_n c_n \alpha_n \otimes \delta_n^*$ where $c_n > a/2$. Fix $x \in K$ and $s < x < t$ and note there is an $n$ such that $s < s_n < x < t_n < t$ so that $\|(N_t - N_s)AXB(N_t - N_s)\| > a/2$, and so $i_x(AXB) \geq a/2$ for all $x \in K$. By the Interpolation theorem (Theorem 2.5), there are $A'$ and $B'$ in $Alg(\mathcal{N})$ such that $(A'A)X(BB') = E(K)$. Then $(E_i A'A)X(BB'E_{i,j}) = E(K)E_{i,j}$ and the two other conditions hold for $E_i A'A$ and $BB'E_{i,j}$ by submultiplicativity of the diagonal seminorm. □

**Theorem 3.14.** *Let $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ be a maximal extended triangular system. Then $\mathcal{T}(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ is a maximal triangular algebra.*

**Proof.** Suppose $X \notin \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ and show that the algebra $\mathcal{A}$ generated by $X$ and $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is not triangular. Of course if $X \notin Alg(\mathcal{N})$, there is an $N \in \mathcal{N}$ such that $N^{\perp}XN \neq 0$. Since $NX^*N^{\perp} \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ this would yield the desired result, so assume $X \in Alg(\mathcal{N})$.

Since $X \notin \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$, $X$ must fail to satisfy one of the three conditions of membership. If it fails the first one then there are $i, j$ such that $i_t(E_i X E_j) \neq 0$ on a non-null subset of $S_{i,j}^c = S_{j,i}$. By upper semicontinuity of $i_t$ there is a closed non-null subset $K$ of $S_{j,i}$ and $a > 0$ such that $i_t(E_i X E_j) \geq a$ for all $t \in K$. Thus by the Interpolation theorem (Theorem 2.5), there are $A = E_i A E_i$ and $B = E_j B E_j$ in $Alg(\mathcal{N})$ such that $AXB = E(K)E_{i,j}$ and, of course, $E(K)E_{j,i} \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ since $K \subseteq S_{j,i}$, contradicting triangularity of $\mathcal{A}$.

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Next suppose that $X$ fails the second condition. (The case where it fails the third condition is handled analogously.) Then there is an $i$ such that $r_{i,t}(X) \neq 0$ on a non-null subset of $R_i^c$. By upper semicontinuity of $r_{i,t}(X)$ as a function of $t$, there is a non-null closed subset $K$ of $R_i^c$ and $a > 0$ such that $r_{i,t}(X) \geq a$ for all $t \in K$. There are now two distinct cases to be considered.

**Case 1.** Suppose that $K$ meets $\bigcup_{j\in\mathbb{N}} R_j \cap C_j$ in a non-null set. In this case, replacing $K$ with a smaller non-null closed set we may assume that $K \subseteq R_j \cap C_j$ for some $j$. Of course, since $K$ is disjoint from $R_i$, we know $i \neq j$. By Lemma 3.13 there are $A = E_i A E_i$ and $B = BE_j$ in $Alg(\mathcal{N})$ such that $AXB = E(K)E_{i,j}$ and in addition $c_{j,t}(B) \leq i_t(B) = 0$ for all $t \notin K$ (in particular, for all $t \notin C_j$) and $i_t(E_a B E_b) = 0$ for all $t \in [0,1]$ and all $a, b \in \mathbb{N}$. Thus $A, B \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ and so $E(K)E_{i,j} \in \mathcal{A}$. Conversely, $K \subseteq R_i^c \subseteq C_i$ (since by Lemma 3.11, $R_i \cup C_i = [0,1]$) and so $K \subseteq R_j \cap C_i \subseteq S_{j,i}$ by the properties of extended triangular systems, and so $E(K)E_{j,i} \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$. Thus $E(K)E_{i,j} \in \mathcal{A} \cap \mathcal{A}^*$ but, since $i \neq j$, it does not belong to the diagonal masa $\mathcal{N}_0' \otimes \mathcal{D}_0$, contradicting triangularity of $\mathcal{A}$.

**Case 2.** Suppose that $K \cap \bigcup_{j\in\mathbb{N}} R_j \cap C_j = \emptyset$. (Possibly replacing $K$ with a subset to make this intersection empty and not just null.) For each $t \in K$ the induced Dedekind cut $(A_t, B_t)$ satisfies $A_t \cap B_t = \emptyset$ and so by Lemma 3.11, $A_t$ has no least element and $B_t$ has no greatest element. Since $t \notin R_i$, this means $i \notin B_t$ and so $i \in A_t$. Since $A_t$ has no least element there is a $j \in A_t$ with $j \preceq_t i$. Of course this $j$ depends on $t$ but by decomposing $K$ into a countable union over candidate values of $j$ we can find a non-null subset on which the same $j \in A_t$ satisfies $j \preceq_t i$ for all $t$. Replacing $K$ with a closed non-null subset of this, we end up with $K \subseteq S_{j,i}$ and $K \subseteq C_j$.

By Lemma 3.13 there are $A = E_i A E_i$ and $B = BE_j$ in $Alg(\mathcal{N})$ such that $AXB = E(K)E_{i,j}$ and in addition $c_{j,t}(B) \leq i_t(B) = 0$ for all $t \notin K$ and $i_t(E_a B E_b) = 0$ for all $t \in [0,1]$ and all $a, b \in \mathbb{N}$. Thus $A, B \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ and so $E(K)E_{i,j} \in \mathcal{A}$. However, we have arranged that $K \subseteq S_{j,i}$ so that $E(K)E_{j,i} \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$, again contradicting triangularity for $\mathcal{A}$. □

## 4. Examples

In this section we will focus on the case where the induced order $\preceq_x$ and Dedekind cuts $(A_x, B_x)$ are constant on $[0,1]$; in other words, the case where each $S_{i,j}$, $R_i$, and $C_j$ is either $[0,1]$ or $\emptyset$. It should be borne in mind throughout that all the behaviours described here can in general be mixed together when non-constant components are used.

To simplify things further, since there will be only one induced order, rather than indexing the atoms by $\mathbb{N}$ and adopting a secondary ordering $\preceq_x$, we will index the countable set of atoms by some other ordered set (e.g. $\mathbb{Z}$, $\mathbb{Q}$, etc.) and work with the natural ordering from the indexing. Note that if $E_i$ are indexed by $i \in I$,

$$r_{i,t}(X) = \inf\{i_t(E_i X M_F^{\perp}) \ : \ F \subseteq I \text{ is finite}\}$$

and

$$c_{j,t}(X) = \inf\{i_t(M_F^{\perp} X E_j) \ : \ F \subseteq I \text{ is finite}\}$$

where $M_F := \sum_{i\in F} E_i$, so that the values of $r_{i,t}$ and $c_{j,t}$ do not depend on the ordering of the index set $I$.

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**Example 4.1.** Let $E_i$ be indexed by $\mathbb{N}$ so that $\mathcal{T}(\mathbf{S})$ is the set of bounded infinite block matrices with entries from $Alg(\mathcal{N}_0)$ on and above the diagonal, and entries from $\mathcal{R}^\infty_{\mathcal{N}_0}$ below the diagonal. The only possible maximal Dedekind cuts on $\mathbb{N}$ are the pairs $A = [n, \infty)$, $B = [1, n]$ for $n = 1, 2, \ldots$, together with $A = \emptyset$, $B = \mathbb{N}$. The latter case corresponds to $C_i = \emptyset$ and $R_i = [0, 1]$ for all $i$. Thus there is no asymptotic restriction on the rows in $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$, but $c_{j,t}(X) = 0$ for all $j$ and almost all $t$. It is easy to see that in fact each $X \in \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ satisfies $M_j^\perp X E_j \in \mathcal{R}^\infty_{\mathcal{N}}$ for all $j \in \mathbb{N}$, since

$$i_t(M_j^\perp X E_j) \le i_t(M_r^\perp X E_j) + \sum_{k=j+1}^{r} i_t(E_k X E_j)$$

and integrating with respect to $t$ and applying the dominated convergence theorem as $r \to \infty$ shows that $i_t(M_j^\perp X E_j) = 0$ a.e. Thus, taking finite sums of columns we see that in this case $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ coincides with the maximal triangular algebra of Theorem 2.1.

The other cases, however, are new, and consist of bounded infinite block matrices as before, with entries from $Alg(\mathcal{N}_0)$ on and above the diagonal and entries from $\mathcal{R}^\infty_{\mathcal{N}_0}$ below. However, for some fixed $n \ge 1$, the first $n$ rows have no other restrictions, but all rows after that have asymptotically a.e. zero-diagonal support (i.e. $r_{i,t}(x) = 0$). The first $n-1$ columns must be in $\mathcal{R}^\infty_{\mathcal{N}}$ but the rest have no asymptotic constraint.

**Example 4.2.** Let $E_i$ be indexed by $\mathbb{Z}$. In this case $\mathcal{T}(\mathbf{S})$ is the set of bounded doubly infinite block operator matrices with entries from $Alg(\mathcal{N}_0)$ on and above the diagonal, and entries from $\mathcal{R}^\infty_{\mathcal{N}_0}$ below the diagonal. The following maximal Dedekind cuts are possible: (i) $A = \emptyset$ and $B = \mathbb{Z}$; (ii) $A = \mathbb{Z}$ and $B = \emptyset$; and (iii) $A = [n, \infty)$ and $B = (-\infty, n]$ for $n \in \mathbb{Z}$.

The first two cases bear a deceptive similarity to the algebras obtained by Theorem 2.1 and yet they are not the same. Theorem 2.1 gives us the algebra $\mathcal{T}$ of all doubly infinite block operator matrices satisfying $M_i^\perp X M_i \in \mathcal{R}^\infty_{\mathcal{N}}$ for all $i \in \mathbb{Z}$. However, in our construction of $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$, in case (i), the lower half of each block column is in $\mathcal{R}^\infty_{\mathcal{N}}$ (by a similar argument to Example 4.1) but the left-hand half of each row need not be. In case (ii) the situation is reversed.

Moreover in $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ the asymptotic condition on the rows and the columns is two-sided, so that

$$c_{j,t}(X) = 0 \Longleftrightarrow \lim_{i \to +\infty} i_t(M_i^\perp X E_j) = \lim_{i \to -\infty} i_t(M_i X E_j) = 0$$

and similarly for $r_{i,t}(X)$. Thus in case (i) each column is asymptotically zero a.e. on the diagonal, both approaching $-\infty$ and approaching $+\infty$, and case (ii) is the same with the roles of rows and columns reversed. Case (iii) is a blend of the two, in which for a fixed $n \in \mathbb{Z}$ the block matrices are asymptotically zero-diagonal (a.e.) on the columns for $i < n$ and on the rows for $i > n$. Of course up to re-indexing this is really just a single case and we may as well take $n = 0$.

**Example 4.3.** Let $E_i$ be indexed by any well-ordered set $S$. If $(A, B)$ is any Dedekind cut where $A$ is non-empty, then $A$ has a smallest element and the cut is of the form $A = \{i : i \ge a\}$, $B = \{i : i \le a\}$. The only other case is $A = \emptyset$, $B = S$.

**Example 4.4.** Let $E_q$ be indexed by $q \in \mathbb{Q}$. This corresponds to the so-called Cantor nest, studied in [**7**]. In this case, the maximal Dedekind cuts very naturally are either $A = [q, +\infty)$ and $B = (-\infty, q]$ for some $q \in \mathbb{Q}$, or else $A = (\gamma, +\infty)$ and $B = (-\infty, \gamma)$ for an irrational $\gamma$. In addition the cases $A = \emptyset$ and $B = \mathbb{Q}$, and $A = \mathbb{Q}$, $B = \emptyset$ are possible.

As observed at the start of this section, the behaviours of these examples, and indeed of any other linear orderings of the index set of the atoms $E_i$, can be blended together at different values of $x \in [0, 1]$. Purely for illustrative purposes, we close this section with the construction of a maximal triangular algebra that mixes the behaviours of the previous examples in a complex fashion.

**Example 4.5.** Let $F_i$ ($i \in \mathbb{N}$) be a sequence of pairwise disjoint measure-dense subsets of $[0, 1]$. (Measure-dense means that the set meets every non-empty open interval in a non-null set; see [**13**, Lemma 3.1] for a construction.) Re-index the $F_i$ as $F_i^j$ for $i \in \mathbb{N}$ and $j \in \{1, 2, 3, 4\}$. For $x \in F_i^j$ and $j = 1, 2, 3, 4$, let $\preceq_x$ be the ordering of the four examples, 4.1, 4.2, 4.3 and 4.4, respectively. Now suppose that for each of the examples (indexed by $j$) we have enumerated the countable family of maximal Dedekind cuts described in that example as $(A_i^j, B_i^j)$ ($i \in \mathbb{N}$) and adopt that cut for $x \in F_i^j$. By Lemma 3.11, this induces a maximal extended triangular system $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ and by Theorem 3.14, $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is a maximal triangular algebra, with extremely complex internal ordering structure.

## 5. Characterizing simple uniform algebras

In this section we shall study maximal triangular algebras satisfying

$$Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes B(\mathcal{K})$$

and will identify conditions under which $\mathcal{T}$ is equal to $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ for some extended triangular system. We shall see in Proposition 5.5 that every triangular algebra satisfying this condition is associated with a *nearly triangular system*, that is to say, a family of sets satisfying all the properties of Definition 3.6 except the last one. From this, to show that $\mathcal{T} = \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$, it will be enough to find conditions that guarantee that the last property (i.e. $R_i \cap C_j \subseteq S_{i,j}$ for all $i, j$) is satisfied. In Theorems 5.6 and 5.9, we shall present two necessary and sufficient criteria for $\mathcal{T} = \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$.

First, however, we observe that all maximal triangular algebras lying between $Alg(\mathcal{N}_0) \otimes \mathcal{D}_0$ and $Alg(\mathcal{N}_0) \otimes B(\mathcal{K})$, whether they are of the form $\mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$ or not, must contain $\mathcal{R}_{\mathcal{N}}^{\infty}$. This shows that, for maximal triangular algebras of this type, all of the complexity of behaviour is to be found in the asymptotics at the boundary of the block matrix entries.

**Proposition 5.1.** *Let $\mathcal{T}$ be a maximal triangular algebra satisfying*

$$Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes B(\mathcal{K}).$$

*Then $\mathcal{R}_{\mathcal{N}}^{\infty} \subseteq \mathcal{T}$.*

**Proof.** Since $\mathcal{T}$ is a subalgebra of $Alg(\mathcal{N})$ and $\mathcal{R}_{\mathcal{N}}^{\infty}$ is an ideal of $Alg(\mathcal{N})$, $\mathcal{T} + \mathcal{R}_{\mathcal{N}}^{\infty} \subseteq Alg(\mathcal{N})$ and is an algebra. We shall prove that $\mathcal{T} + \mathcal{R}_{\mathcal{N}}^{\infty}$ is triangular and then by maximality $\mathcal{T} + \mathcal{R}_{\mathcal{N}}^{\infty} = \mathcal{T}$ and the result follows.

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Suppose on the contrary that there is a self-adjoint operator $T+R$ for $T \in \mathcal{T}$ and $R \in \mathcal{R}^{\infty}_{\mathcal{N}}$ that is not in the masa $\mathcal{N}_0'' \otimes \mathcal{D}_0$. Nevertheless, $T+R$ is in the diagonal of $Alg(\mathcal{N})$, which is $\mathcal{N}' = \mathcal{N}_0'' \otimes \mathcal{B}(\mathcal{K})$, and so there must be some $i \neq j$ such that $E_i(T+R)E_j \neq 0$.

Now $i_t(E_i(T+R)E_j)$ is not a.e. zero, because if it were, then by [**13**, Theorem 2.1], $E_i(T+R)E_j$ would be in $\mathcal{R}^{\infty}_{\mathcal{N}}$, which is a diagonal-disjoint ideal of $Alg(\mathcal{N})$ and does not have any non-zero elements of $\mathcal{N}'$. But $T+R$ is self-adjoint, so that, again by [**13**, Theorem 2.1], $i_t(R) \overset{\text{a.e.}}{=} 0$, and so

$$i_t(E_iTE_j) \overset{\text{a.e.}}{=} i_t(E_i(T+R)E_j) = i_t(E_j(T+R)E_i) \overset{\text{a.e.}}{=} i_t(E_jTE_i).$$

Thus there is an $a > 0$ and a non-null set $K$ such that $i_t(E_iTE_j)$ and $i_t(E_jTE_i)$ are both at least $a$ on $K$. By the Interpolation theorem (Theorem 2.5), there are $A, B, C$, and $D$ in $Alg(\mathcal{N})$ such that

$$AE_iTE_jB = CE_jTE_iD = E(K) \neq 0.$$

Thus $E_iAE_iTE_jBE_{i,j} = E(K)E_{i,j}$ and $E_jCE_jTE_iDE_{j,i} = E(K)E_{j,i}$. Since the operators $E_iAE_i$, $E_jBE_{i,j}$, $E_jCE_j$, and $E_iDE_{j,i}$ all belong to $Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T}$, it follows that $E(K)E_{i,j}$ and $E(K)E_{j,i}$ belong to $\mathcal{T}$, contradicting triangularity. □

The following two lemmas provide necessary technical tools for the theorems of this section. Remark 5.4 below describes how these lemmas are used subsequently.

**Lemma 5.2.** *Let $A_i$ be a bounded sequence of operators such that for each $i$ $\lim_{j\to\infty} \|A_iA_j^*\| = \lim_{j\to\infty} \|A_i^*A_j\| = 0$. Then there is a subsequence $k(i)$ such that $\sum_{i=1}^{\infty} A_{k(i)}$ converges strongly. Moreover, given a sequence of infinite subsets $S_i$ of $\mathbb{N}$, we can choose the subsequence so that each $k(i) \in S_i$.*

**Proof.** Fix $\alpha_i > 0$ such that $\sum_i \alpha_i < \infty$. The result clearly follows if we can construct $k(i)$ and two sequences of pairwise orthogonal projections $P_i$, $Q_i$ such that

$$\|A_{k(i)} - P_iA_{k(i)}Q_i\| \leq \alpha_i$$

for all $i$. We shall do this inductively and, to ease the induction step, we shall add the hypothesis that each $P_i$ and $Q_i$ satisfies $\lim_{j\to\infty} \|P_iA_j\| = \lim_{j\to\infty} \|A_jQ_i\| = 0$.

To start the induction, pick $k = k(1)$ in $S_1$ and take

$$P_1 := E_{|A_k^*|}([\alpha_1/2, \infty)) \quad \text{and} \quad Q_1 := E_{|A_k|}([\alpha_1/2, \infty))$$

where $E_H$ denotes the spectral measure on $\mathbb{R}$ for a self-adjoint operator $H$. Then clearly $\|P_1^{\perp}A_k\| = \|P_1^{\perp}|A_k^*|\,\| \leq \alpha_1/2$ and $\|A_kQ_1^{\perp}\| = \|\,|A_k|Q_1^{\perp}\| \leq \alpha_1/2$, so that

$$\|A_k - P_1A_kQ_1\| \leq \|P_1^{\perp}A_k\| + \|A_kQ_1^{\perp}\| \leq \alpha_1. \tag{5.1}$$

Notice also that $A_jQ_1 = A_jA_k^*A_kf(|A_k|)$ where $f(t) = 1/t^2$ on $[\alpha_1/2, \infty)$ and zero elsewhere. Thus $\lim_{j\to\infty} \|A_jQ_1\| = 0$ and by a similar argument $\lim_{j\to\infty} \|P_1A_j\| = 0$, so the induction hypotheses hold for $k = k(1)$.

Next suppose that $k(i) \in S_i$, along with pairwise orthogonal $P_i, Q_i$, have been found to satisfy the induction hypotheses for $1 \leq i < n$. Let $P := (P_1 + \cdots + P_{n-1})^{\perp}$ and $Q := (Q_1 + \cdots + Q_{n-1})^{\perp}$. For each $j$ write $A_j' := PA_jQ$. Clearly $P^{\perp}A_j$ and $A_jQ^{\perp}$ converge to

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zero in norm as $j \to \infty$, so $\|A_k - A'_k\| = \|A_k - PA_kQ\| < \alpha_n/2$ for all sufficiently large $k$. Pick one such $k := k(n) \in S_n$. Let

$$P_n := E_{|A'_k{}^*|}([\alpha_n/4, \infty)) \quad \text{and} \quad Q_n := E_{|A'_k|}([\alpha_n/4, \infty))$$

where clearly $P_n \le P$, $Q_n \le Q$ and, as with (5.1),

$$\|A'_k - P_nA_kQ_n\| = \|A'_k - P_nA'_kQ_n\| \le \|P_n^\perp A'_k\| + \|A'_kQ_n^\perp\| \le \alpha_n/2.$$

Thus $\|A_k - P_nA_kQ_n\| \le \alpha_n$.

Also, $\|A_jQ_n\| \le \|P^\perp A_j\| + \|PA_jQ\,Q_n\|$. The first term converges to zero so we must show $\|A'_jQ_n\|$ converges to zero. As before, $A'_jQ_n = A'_jA'_k{}^*A'_kf(|A'_k|)$ where $f(t) = 1/t^2$ on $[\alpha_n/4, \infty)$ and zero elsewhere. Since

$$\|A'_jA'_k{}^*\| \le \|A_jQA_k^*\| \le \|A_jA_k^*\| + \sum_{i=1}^{n-1} \|A_jQ_i\|\|A_k\|$$

and all the terms on the right converge to zero, it follows that $\lim_{j\to\infty} \|A'_jQ_n\| = 0$ and so $\lim_{j\to\infty} \|A_jQ_n\| = 0$. By similar reasoning, $\lim_{j\to\infty} \|P_nA_j\| = 0$, which completes the induction. □

**Lemma 5.3.** *Let $a > 0$ and let $(A_i)$, $(B_i)$ and $(D_i)$ be sequences of operators satisfying $\|A_iD_iB_i\| > a$ for all $i \in \mathbb{N}$. Suppose further that $(A_i)$ converges strongly to zero, $(B_i)$ is bounded and the $(D_i)$ are compact and converge strong-$^*$ to zero. Then given $\epsilon > 0$ we can pick a subsequence $k(i)$ such that $D := \sum_{i=1}^{\infty} D_{k(i)}$ converges strongly, and*

$$\|A_{k(i)}DB_{k(i)}\| > (1 - \epsilon)a$$

*for all $i \in \mathbb{N}$. Moreover, given a sequence of infinite subsets $S_i$ of $\mathbb{N}$, we can choose the subsequence so that each $k(i) \in S_i$.*

**Proof.** We shall choose the subsequence $k(i)$ inductively. To meet the constraints on $k(i)$ and $S_i$, we fix a sequence $m(i)$ of natural numbers that takes every value in $\mathbb{N}$ infinitely many times. When choosing $k(i)$, we shall ensure that each $k(i) \in S_{m(i)}$ so that ultimately each $k^{-1}(S_i)$ will be infinite. After the sequence $k(i)$ has been chosen we shall apply Lemma 5.2 to $D_{k(i)}$ and $k^{-1}(S_i)$ to get a subsequence $D_{k(l(i))}$ such that $\sum_i D_{k(l(i))}$ converges strongly and each $l(i) \in k^{-1}(S_i)$, so that $k(l(i)) \in S_i$.

By Banach–Steinhaus, the sequences are all bounded in norm; let $K$ bound all of them. To choose $k(i)$, for each $i$ we pick a unit vector $\xi_i$ such that $\|A_iD_iB_i\xi_i\| > a$. We start the induction with an arbitrary $k(1)$ in $S_{m(1)}$ and then suppose the first $n - 1$ values have already been chosen. Because the $D_i$ are compact, each $A_kD_{k(j)}$ converges to zero in norm as $k \to \infty$ and so

$$\sum_{j=1}^{n-1} \|A_kD_{k(j)}B_k\| \le K \sum_{j=1}^{n-1} \|A_kD_{k(j)}\| < \frac{\epsilon a}{2} \tag{5.2}$$

for all sufficiently large $k$. Likewise, for all sufficiently large $k$,

$$\max_{1\le j<n} \|D_kB_{k(j)}\xi_{k(j)}\| < \frac{\epsilon a}{2^{n+1}K}. \tag{5.3}$$

Choose $k(n) \in S_{m(n)}$ to satisfy both of these.

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With the subsequence $k(i)$ chosen in this way, observe that $(D_i)$ satisfies the hypotheses of Lemma 5.2 and so we can find a subsequence $D_{k(l(i))}$ of $D_{k(n)}$ such that $D := \sum_i D_{k(l(i))}$ converges strongly and, as outlined in the first paragraph, $k(l(i)) \in S_i$ for all $i$. Now write $k' := k \circ l$ and observe that

$$\begin{aligned}\|A_{k'(n)}DB_{k'(n)}\| &\geq \|A_{k'(n)}DB_{k'(n)}\xi_{k'(n)}\| \\ &\geq \|A_{k'(n)}D_{k'(n)}B_{k'(n)}\xi_{k'(n)}\| \\ &\quad - K\sum_{j=1}^{n-1}\|A_{k'(n)}D_{k'(j)}\| \\ &\quad - K\sum_{j=n+1}^{\infty}\|D_{k'(j)}B_{k'(n)}\xi_{k'(n)}\|.\end{aligned}$$

Recall we chose $k(n)$ so that

$$\sum_{j=1}^{n-1}\|A_{k(n)}D_{k(j)}\| < \frac{\epsilon a}{2K} \quad \text{and} \quad \max_{1\leq j<n}\|D_{k(n)}B_{k(j)}\xi_{k(j)}\| < \frac{\epsilon a}{2^{n+1}K}$$

and so, substituting $l(n)$ for $n$,

$$\sum_{j=1}^{l(n)-1}\|A_{k'(n)}D_{k(j)}\| < \frac{\epsilon a}{2K} \quad \text{and} \quad \max_{1\leq j<l(n)}\|D_{k'(n)}B_{k(j)}\xi_{k(j)}\| < \frac{\epsilon a}{2^{l(n)+1}K}.$$

Since clearly $l(1), \ldots, l(n-1)$ are found among $1, 2, \ldots, l(n) - 1$ it follows that

$$\sum_{j=1}^{n-1}\|A_{k'(n)}D_{k'(j)}\| < \frac{\epsilon a}{2K}$$

and

$$\max_{1\leq j<n}\|D_{k'(n)}B_{k'(j)}\xi_{k'(j)}\| < \frac{\epsilon a}{2^{l(n)+1}K} \leq \frac{\epsilon a}{2^{n+1}K}.$$

From this it is clear that $\|A_{k'(n)}DB_{k'(n)}\| > a - \epsilon a/2 - \epsilon a/2 = (1-\epsilon)a$ (exchanging the roles of $j$ and $n$ in the last term). □

**Remark 5.4.** In subsequent results, we will apply the following technique when we make use of Lemma 5.3. Suppose that $K$ is a non-null closed subset of $(0, 1)$. By the Cantor–Bendixson theorem, we can find a perfect subset $K'$ of $K$ that differs from $K$ by a countable, and hence null, set. Consider all the intervals of the form $((p-1)/q, (p+1)/q)$ for natural numbers $p < q$ and let $(s_n, t_n)$ be an enumeration of all such intervals that contain a point of $K'$. Every element of $K'$ (and almost every element of the original $K$) belongs to an interval $(s_n, t_n)$ and every interval contains infinitely many points of $K$. For any $\epsilon > 0$, only finitely many $t_n - s_n$ are greater than $\epsilon$ and so $t_n - s_n \to 0$ and $N_{t_n} - N_{s_n}$ converges to zero strongly. Let $S_n := \{m : (s_m, t_m) \subseteq (s_n, t_n)\}$ and observe each $S_n$ is infinite since $(s_n, t_n)$ contains infinitely many points of $K$ and so, given any

$N$, we can choose $q$ large enough that there are at least $N$ pairwise disjoint intervals of the form $((p-1)/q,(p+1)/q)$ that contain elements of $(s_n,t_n)\cap K$. In arguments below we will apply Lemma 5.3, using operators $A_n$ and $B_n$, which are derived from expressions involving $N_{t_n}-N_{s_n}$, and we will obtain subsequences $k(n)$ satisfying $k(n)\in S_n$ for all $n$. We will then focus on a fixed $x\in K'$ and interval $(s,t)$ containing $x$. Then, clearly we can find $n$ such that $(s_n,t_n)\subseteq(s,t)$ so that also $(s_{k(n)},t_{k(n)})\subseteq(s,t)$ and $N_{t_{k(n)}}-N_{s_{k(n)}}\leq N_t-N_s$. This will enable us to apply norm estimates involving $k(n)$ obtained from Lemma 5.3 to general intervals containing $x\in K'$.

**Proposition 5.5.** *Let $\mathcal{T}$ be a triangular algebra satisfying*

$$Alg(\mathcal{N}_0)\otimes\mathcal{D}_0\subseteq\mathcal{T}\subseteq Alg(\mathcal{N}_0)\otimes B(\mathcal{K}).$$

*Then there are families of Borel sets $\boldsymbol{S}=(S_{i,j})$, $\boldsymbol{R}=(R_i)$ and $\boldsymbol{C}=(C_j)$ satisfying properties (1) to (5) of Definition 3.6 such that $\mathcal{T}\subseteq\mathcal{T}(\boldsymbol{S},\boldsymbol{R},\boldsymbol{C})$. Such a collection of sets will be called a* nearly triangular system *for $\mathcal{T}$.*

**Proof.** Note that it is enough to prove each of the relations of Definition 3.6 to within a null set. For, after sets have been found to satisfy the relations to within null sets, simply form the union of all the excess sets $S_{i,j}\cap S_{j,i}$, $(S_{i,j}\cap S_{j,k})\setminus S_{i,k}$, $(C_i\cap S_{i,j})\setminus C_j$ and $(S_{i,j}\cap R_j)\setminus R_i$ for $i,j,k\in\mathbb{N}$ and remove this null set from each of the individual sets $S_{i,j}$, $R_i$ and $C_j$. Now all the required properties hold exactly, except possibly $S_{i,i}=[0,1]$, and so finally enlarge the $S_{i,i}$ by a null set to equal $[0,1]$, which does not alter the validity of the other relations.

Recall from elementary measure theory that in any $\sigma$-finite measure space, given a (not necessarily countable) collection of sets $S_\alpha$ $(\alpha\in A)$, we can find a measurable set $S:=\bigvee_{\alpha\in A}S_\alpha$ called the *essential union* of the family, having the property that $S_\alpha\setminus S$ is null for all $\alpha$, and for any measurable $K$, if $K\cap S_\alpha$ is null for all $\alpha$, then $K\cap S$ is also null. The set is not unique, but is unique to within a null set, and we shall assume an arbitrary choice has been made to assign a concrete value to $\bigvee_{\alpha\in A}S_\alpha$.

For each $i,j\in\mathbb{N}$ let

$$S_{i,j}:=\bigvee_{X\in\mathcal{T},a>0}\{t\ :\ i_t(E_iXE_j)\geq a\},$$

$$R_i:=\bigvee_{X\in\mathcal{T},a>0}\{t\ :\ r_{i,t}(X)\geq a\},$$

$$C_j:=\bigvee_{X\in\mathcal{T},a>0}\{t\ :\ c_{j,t}(X)\geq a\}.$$

Clearly $\mathcal{T}\subseteq\mathcal{T}(\mathbf{S},\mathbf{R},\mathbf{C})$; it remains to show $\mathbf{S},\mathbf{R},\mathbf{C}$ is a nearly triangular system, at least to within null sets.

Property (1) of Definition 3.6 is trivial, since $\mathcal{T}$ is unital.

Suppose property (2) does not hold. Then for some $i\neq j$, $S_{i,j}\cap S_{j,i}$ is non-null and so there must be $X,Y\in\mathcal{T}$ and $a>0$ such that $i_t(E_iXE_j)$ and $i_t(E_jYE_i)$ are at least $a$ on a non-null set $K$. Thus $i_t(E_iXE_{j,i})\geq a$ on $K$ (since $i_t(E_iXE_{j,i})\geq i_t(E_iXE_{j,i}E_{i,j})=i_t(E_iXE_j)$), and so by the Interpolation theorem (Theorem 2.5), there are $A,B$ in $Alg(\mathcal{N})$

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such that $AE_iXE_{j,i}B = E(K)$. Thus $E_iAE_iXE_{j,i}BE_{i,j} = E(K)E_{i,j}$, which must be in $\mathcal{T}$, since $E_iAE_i$ and $E_{j,i}BE_{i,j}$ are in $Alg(\mathcal{N}_0) \otimes \mathcal{D}_0$. But by the same argument applied to $Y$, $E(K)E_{j,i}$ must be $\mathcal{T}$, which would contradict triangularity.

Suppose property (3) of Definition 3.6 does not hold. Then for some $i, j, k$, $S_{i,j} \cap S_{j,k} \setminus S_{i,k}$ is non-null and there must be $X, Y \in \mathcal{T}$ and $a > 0$ such that $i_t(E_iXE_j)$ and $i_t(E_jYE_k)$ are greater than $a$ on a non-null set $K$ which is disjoint from $S_{i,k}$. Choose intervals $(s_n, t_n)$ and subsets $S_n$ of $\mathbb{N}$ as in Remark 5.4. Take $A_n := (N_{t_n} - N_{s_n})E_iXE_j$ and $B_n := E_jYE_k(N_{t_n} - N_{s_n})$. Note that $t_n - s_n \to 0$, so that $A_n$ and $B_n$ converge strongly to 0. For each $n$ pick $s_n < x < y < z < t_n$ where $x$ and $z$ are in $K$. Since $\|(N_y - N_{s_n})E_iXE_j(N_y - N_{s_n})\|$ and $\|(N_{t_n} - N_y)E_jYE_k(N_{t_n} - N_y)\|$ are both greater than $a$, we can find a rank-1 contraction $D_n = (N_y - N_{s_n})E_jD_nE_j(N_{t_n} - N_y)$ such that $\|A_nD_nB_n\| > a^2$. Thus by Lemma 5.3 there is a subsequence $k(n) \in S_n$ such that $D := \sum_{n=1}^{\infty} D_{k(n)}$ converges strongly, and

$$\|(N_{t_{k(n)}} - N_{s_{k(n)}})E_iXDYE_k(N_{t_{k(n)}} - N_{s_{k(n)}})\| \geq a^2/2.$$

By Remark 5.4, for almost every $x \in K$, if $(s, t)$ contains $x$ then we can find an $n$ such that $(s_{k(n)}, t_{k(n)}) \subseteq (s, t)$ so that also $\|(N_t - N_s)E_iXDYE_k(N_t - N_s)\| \geq a^2/2$. Thus $i_x(E_iXDYE_k) \geq a^2/2$ for almost all $x \in K$. However, each $D_n \in E_j\, Alg(\mathcal{N})E_j$, so that $D \in E_j\, Alg(\mathcal{N})E_j \subseteq \mathcal{T}$ and so $XDY \in \mathcal{T}$. This contradicts the fact that $K$ is disjoint from $S_{i,k}$ and so property (3) of Definition 3.6 must hold.

The proofs of properties (4) and (5) of Definition 3.6 are similar to each other, so we present only the first. Suppose property (4) does not hold. Then there are $i$ and $j$ such that $C_i \cap S_{i,j} \setminus C_j$ is non-null. As before, find $X, Y \in \mathcal{T}$, $a > 0$, and a non-null set $K$ disjoint from $C_j$ on which $c_{i,t}(X)$ and $i_t(E_iYE_j)$ are greater than $a$. As before, choose intervals $(s_n, t_n)$ and subsets $S_n$ of $\mathbb{N}$ according to Remark 5.4. As usual let $M_n := E_1 + \cdots + E_n$. Let $A_n := M_n^{\perp}(N_{t_n} - N_{s_n})XE_i$ and $B_n := E_iYE_j(N_{t_n} - N_{s_n})$. Note that $A_n, B_n \to 0$ strongly. For each $n$, as in the previous case, we can find $s_n < x < y < z < t_n$ with $x, z \in K$. Thus

$$\|(N_y - N_{s_n})A_n(N_y - N_{s_n})\| \quad \text{and} \quad \|(N_{t_n} - N_y)B_n(N_{t_n} - N_y)\|$$

are both greater than $a$ and so there is a finite-rank contraction $D_n = (N_y - N_{s_n})E_iD_nE_i(N_{t_n} - N_y)$ such that $\|A_nD_nB_n\| > a^2$. Clearly $D_n$ converges strong-* to zero. By Lemma 5.3 we find $k(n) \in S_n$ such that $D := \sum_{n=1}^{\infty} D_{k(n)}$ converges strongly, and

$$\|M_{k(n)}^{\perp}(N_{t_{k(n)}} - N_{s_{k(n)}})XDYE_j(N_{t_{k(n)}} - N_{s_{k(n)}})\| \geq a^2/2$$

for all $n$. Now, for almost any $x \in K$, given any open interval $(s, t)$ that contains $x$, and any $n_0 \in \mathbb{N}$, we can find $n \geq n_0$ such that $(s_n, t_n) \subseteq (s, t)$. Then, since $n_0 \leq n \leq k(n) \in S_n$ we know $(s_{k(n)}, t_{k(n)}) \subseteq (s, t)$ and $M_{k(n)}^{\perp} \leq M_{n_0}^{\perp}$ so that also

$$\|M_{n_0}^{\perp}(N_t - N_s)XDYE_j(N_t - N_s)\| \geq a^2/2.$$

Thus $c_{j,x}(XDY) \geq a^2/2$ for almost every $x \in K$ and yet, since $D \in E_i\, Alg(\mathcal{N})E_i \subseteq \mathcal{T}$, we have $XDY \in \mathcal{T}$, contradicting the fact that $K$ is disjoint from $C_j$. So property (4) of Definition 3.6 must hold. $\square$

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**Theorem 5.6.** *Let $\mathcal{T}$ be a maximal triangular algebra satisfying*

$$Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes B(\mathcal{K})$$

*and let $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ be a nearly triangular system for $\mathcal{T}$. Then $\mathcal{T} = \mathcal{T}(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ and $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ is an extended triangular system if and only if $\mathcal{T}$ contains all $X \in \mathcal{T}(\boldsymbol{S})$ such that for each $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $E_m X M_n^\perp = 0$ and $M_n^\perp X E_m = 0$.*

**Proof.** Necessity is trivial, since all $X \in \mathcal{T}(\mathbf{S})$ that satisfy the condition must satisfy $r_{i,t}(X) = c_{j,t}(X) = 0$ for all $t$. We focus now on the converse. By the maximality of $\mathcal{T}$ it suffices, in view of Theorem 3.8 and Proposition 5.5, to show that in this case property (6) of Definition 3.6 also holds. Suppose for a contradiction that property (6) does not hold. Then for some $i$ and $j$, $R_i \cap C_j \setminus S_{i,j}$ is non-null. Thus we can find operators $X, Y \in \mathcal{T}$ and $a > 0$ such that $r_{i,t}(X) > a$ and $c_{j,t}(Y) > a$ on a non-null set $K$ which is disjoint from $S_{i,j}$. Let the intervals $(s_n, t_n)$ and sets $S_n$ be chosen as in Remark 5.4.

For each fixed $n$, pick $s_n < x < y < z < t_n$ where $x$ and $z$ are in $K$. Let $A_n := E_i(N_{t_n} - N_{s_n})X$ and $B_n := Y(N_{t_n} - N_{s_n})E_j$. Clearly $A_n$ and $B_n$ converge strongly to zero. Also, $\|A_n M_n^\perp N_y\| \geq \|E_i(N_y - N_{s_n})X(N_y - N_{s_n})M_n^\perp\| \geq r_{i,x}(X) > a$ and, similarly $\|M_n^\perp N_y^\perp B_n\| > a$. Thus we can find a finite-rank contraction $D_n$ such that $\|A_n D_n B_n\| > a^2$, which satisfies $D_n = M_n^\perp N_y D_n N_y^\perp M_n^\perp$, and consequently belongs to $Alg(\mathcal{N})$. By weak lower semicontinuity of the norm, we can also stipulate that $D_n = M_k D_n M_k$ for some sufficiently large $k$.

By Lemma 5.3, there is a subsequence $k(n) \in S_n$ such that $D := \sum_n D_{k(n)}$ converges strongly and

$$\|E_i(N_{t_{k(n)}} - N_{s_{k(n)}})XDY(N_{t_{k(n)}} - N_{s_{k(n)}})E_j\| > a^2/2.$$

Thus by Remark 5.4, for almost every $x \in K$, if the open interval $(s,t)$ contains $x$ then there is an $n$ such that $(s_{k(n)}, t_{k(n)}) \subseteq (s,t)$ and so $\|(N_t - N_s)E_i XDYE_j(N_t - N_s)\| > a^2/2$. Thus $i_x(E_i XDYE_j) \geq a^2/2$ for almost every $x \in K$. Furthermore, $D$ is in $\mathcal{T}$ since for each $m$, $DE_m$ and $E_m D$ are finite sums of operators $D_k E_m$ and $E_m D_k$, respectively. So for sufficiently large $n$, $DE_m = M_n DE_m$ and $E_m D = E_m DM_n$. Since also each $DE_m$ and $E_m D$ are finite rank, $D$ is in $\mathcal{T}(\mathbf{S})$ and hence also in $\mathcal{T}$ by hypothesis.

The result follows, since the fact $XDY \in \mathcal{T}$ and $i_x(E_i XDYE_j) \geq a^2/2$ a.e. on $K$ together contradict the assumption that $K$ is disjoint from $S_{i,j}$. □

One way to interpret the last result is to start with a maximal triangular algebra $\mathcal{T}$ satisfying the inclusion relation and first calculate the triangular system $\mathbf{S}$ such that $\mathcal{T} \subseteq \mathcal{T}(\mathbf{S})$ and then form the extended triangular system $(\mathbf{S}, \mathbf{R}_0, \mathbf{C}_0)$ where $\mathbf{R}_0 = \mathbf{C}_0 = \emptyset$. Then $\mathcal{T}(\mathbf{S}, \mathbf{R}_0, \mathbf{C}_0)$ is a triangular algebra contained in $\mathcal{T}(\mathbf{S})$. We have just seen that $\mathcal{T}$ is simple if and only if it contains $\mathcal{T}(\mathbf{S}, \mathbf{R}_0, \mathbf{C}_0)$.

The following example is of a maximal triangular that is *not* simple, and illustrates one way simplicity can fail: if the algebra has different asymptotic behaviour at infinity on different infinite subsets of the indexing set.

**Example 5.7.** Let $E_i$ be indexed by $i \in \mathbb{Z}$ and $M_n := \sum_{m \leq n} E_m$. By Theorem 2.1 the set, $\mathcal{T}$, of $X \in Alg(\mathcal{N})$ such that $M_n^\perp X M_n \in \mathcal{R}_{\mathcal{N}}^\infty$ for all $n$ is a maximal triangular

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algebra. Note $R_i = C_j = [0,1]$ for all $i, j$ and so $R_i \cap C_j \not\subseteq S_{i,j}$ for $i > j$, and $(\mathbf{S}, \mathbf{R}, \mathbf{C})$ is not an extended triangular system. Note also, however, that every $X \in \mathcal{T}$ satisfies $r_{i,x}(XM_0) = c_{j,x}(M_0^\perp X) = 0$ a.e., so that the support sets along rows and columns are very different when localized to this projection $M_0$ or to its complement:

$$\bigvee_{X\in\mathcal{T},a>0} \{t \,:\, r_{i,t}(XM_0) \geq a\} \neq \bigvee_{X\in\mathcal{T},a>0} \{t \,:\, r_{i,t}(XM_0^\perp) \geq a\}$$

and

$$\bigvee_{X\in\mathcal{T},a>0} \{t \,:\, c_{j,t}(M_0^\perp X) \geq a\} \neq \bigvee_{X\in\mathcal{T},a>0} \{t \,:\, c_{j,t}(M_0 X) \geq a\}.$$

In the following result we establish the converse: if the diagonal seminorms asymptotically have the same support sets when localized to any infinite set along the rows and columns then the algebra is a simple uniform algebra.

**Definition 5.8.** For any $S \subseteq \mathbb{N}$ write $M_S := \sum_{i\in S} E_i$ and define

$$r_{i,t}^\infty(X) := \inf\{i_t(E_i X M_S) \,:\, S \subseteq \mathbb{N} \text{ is infinite}\}$$

and

$$c_{j,t}^\infty(X) := \inf\{i_t(M_S X E_j) \,:\, S \subseteq \mathbb{N} \text{ is infinite}\}.$$

Then for each $i, j \in \mathbb{N}$ let

$$R_i^\infty := \bigvee_{X\in\mathcal{T},a>0} \{t \,:\, r_{i,t}^\infty(X) \geq a\},$$

$$C_j^\infty := \bigvee_{X\in\mathcal{T},a>0} \{t \,:\, c_{j,t}^\infty(X) \geq a\}.$$

**Theorem 5.9.** *Let $\mathcal{T}$ be a maximal triangular algebra satisfying*

$$Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T} \subseteq Alg(\mathcal{N}_0) \otimes B(\mathcal{K})$$

*and let $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ be a nearly triangular system for $\mathcal{T}$. Then $\mathcal{T} = \mathcal{T}(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ and $(\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{C})$ is an extended triangular system if and only if*

$$C_j = C_j^\infty \quad \text{and} \quad R_i = R_i^\infty$$

*to within a null set for all $i$ and $j$.*

This theorem relies on the main result of [11] in which we used techniques of infinite Ramsey theory to prove the following:

**Theorem 5.10 (Orr [11, Theorem 1.2]).** *Let $X, Y \in B(\mathcal{H})$ and suppose that $\|XM_S\| > 1$ and $\|M_S Y\| > 1$ for all infinite $S \subseteq \mathbb{N}$. Then there is a block diagonal contraction $D$ such that $\|XDY\| \geq 1/5$.*

**Proof of Theorem 5.9.** We shall first prove necessity, so suppose $\mathcal{T} = \mathcal{T}(\mathbf{S}, \mathbf{R}, \mathbf{C})$. Clearly $C_j \supseteq C_j^\infty$ and $R_i \supseteq R_i^\infty$ for all $i$ and $j$. Suppose if possible that there is a non-null

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closed set $K \subseteq C_j \setminus C_j^\infty$. Find a sequence of countably many pairwise disjoint measure-dense subsets of $[0,1]$ (see [**13**, Lemma 3.1] for a construction) and index them as $F_{m,n}$ for $m,n \in \mathbb{N}$. Let $(s_n,t_n)$ be an enumeration of all the intervals with rational endpoints that contain a point of $K$. For each fixed $m$, let $n$ run through $\mathbb{N}$ and pick a rank-1 operator $R_{m,n}$ of unit norm satisfying

$$R_{m,n} = E_m(N_x - N_{s_n})E(F_{m,n})R_{m,n}E(F_{m,n})(N_{t_n} - N_x)E_j$$

for some $x \in K$ with $s_n < x < t_n$. Let $T := \sum_{m=1}^{\infty}\sum_{n=1}^{m} R_{m,n}$, which converges strongly since the ranges and domains of the $R_{m,n}$ are pairwise orthogonal. Clearly $T = TE_j \in Alg(\mathcal{N})$ and for any $m$, $E_mTE_j$ is finite rank, so $i_x(E_mTE_n) = 0$ for all $m,n$ and all $x \in [0,1]$. Likewise, $r_{m,x}(T) \le i_x(E_mTE_j) = 0$ for all $m$ and $x$. If $x \notin K$, then there is $s < x < t$ such that $(s,t)$ is disjoint from $K$, and so $(N_t - N_s)R_{m,n}(N_t - N_s) = 0$ for all $m,n$ and hence $(N_t - N_s)T(N_t - N_s) = 0$. Thus $c_{j,x}(T) \le i_x(TE_j) = 0$ for $x \notin K$, so that $T \in \mathcal{T}(\mathbf{S},\mathbf{R},\mathbf{C}) = \mathcal{T}$. However, for any fixed $x \in K$ and $s < x < t$, find $n$ such that $(s_n,t_n) \subseteq (s,t)$ and observe that, for any $m > n$,

$$\begin{aligned}
&\|E_m(N_t - N_s)T(N_t - N_s)E_j\| \\
&\quad \ge \|E_m(N_{t_n} - N_{s_n})E(F_{m,n})TE(F_{m,n})(N_{t_n} - N_{s_n})E_j\| \\
&\quad \ge \|E_m(N_{t_n} - N_{s_n})E(F_{m,n})R_{m,n}E(F_{m,n})(N_{t_n} - N_{s_n})E_j\| \\
&\quad = \|R_{m,n}\| = 1.
\end{aligned}$$

Thus if $S \subseteq \mathbb{N}$ is infinite then $\|M_S(N_t - N_s)T(N_t - N_s)E_j\| \ge 1$ and so $c_{j,x}^\infty(T) \ge 1$ on $K$, contradicting the assumption that $K$ was disjoint from $C_j^\infty$. It follows by contradiction that $C_j = C_j^\infty$, and the fact that $R_i = R_i^\infty$ follows similarly.

We now prove sufficiency. As in Theorem 5.6, it is enough to prove that property (6) of Definition 3.6 holds. Suppose for a contradiction that property (6) does not hold. Then for some $i$ and $j$, $R_i^\infty \cap C_j^\infty \setminus S_{i,j}$ is non-null. Thus we can find operators $X, Y \in \mathcal{T}$ and $a > 0$ such that $r_{i,t}^\infty(X) > a$ and $c_{j,t}^\infty(Y) > a$ on a non-null set $K$ that is disjoint from $S_{i,j}$. Let the intervals $(s_n,t_n)$ and sets $S_n$ be chosen as in Remark 5.4.

As usual, for each fixed $n$, pick $s_n < x < y < z < t_n$ where $x$ and $z$ are in $K$. Let $A_n := E_i(N_{t_n} - N_{s_n})XM_n^\perp$ and $B_n := M_n^\perp Y(N_{t_n} - N_{s_n})E_j$. Clearly $A_n$ and $B_n$ converge strongly to zero. Also, for any infinite $S \subseteq \mathbb{N}$,

$$\|A_nN_yM_S\| \ge \|E_i(N_y - N_{s_n})X(N_y - N_{s_n})M_{S\cap(n,\infty)}\| \ge r_{i,x}^\infty(X) > a$$

and, similarly, $\|M_SN_y^\perp B_n\| > a$. Thus, by Theorem 5.10 there is an infinite block diagonal contraction $D_n$ such that $\|A_nN_yD_nN_y^\perp B_n\| \ge a' := a^2/5$. Finite-rank operators are weakly dense in the set of infinite block diagonals, so by weak lower semicontinuity of the norm, $D_n$ can be assumed to be finite rank. Without loss, also take $D_n = M_n^\perp N_yD_nN_y^\perp M_n^\perp$ so that $D_n$ is in $Alg(\mathcal{N}_0) \otimes \mathcal{D}_0$ and converges strong-* to zero.

The proof now completes exactly as Theorem 5.6. By Lemma 5.3, there is a subsequence $k(n) \in S_n$ such that $D := \sum_n D_{k(n)}$ converges strongly and

$$\|E_i(N_{t_{k(n)}} - N_{s_{k(n)}})XDY(N_{t_{k(n)}} - N_{s_{k(n)}})E_j\| > a'/2.$$

(Here we use $A_n' := E_i(N_{t_n} - N_{s_n})X$ and $B_n' := Y(N_{t_n} - N_{s_n})E_j$.) Thus by Remark 5.4, for almost every $x \in K$, if the open interval $(s,t)$ contains $x$, then there is an $n$

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such that $(s_{k(n)}, t_{k(n)}) \subseteq (s, t)$ and so $\|(N_t - N_s)E_i XDYE_j(N_t - N_s)\| > a'/2$. Thus $i_x(E_i XDYE_j) \geq a'/2$ for almost every $x \in K$. Furthermore, since each $D_n$ is in $Alg(\mathcal{N}_0) \otimes \mathcal{D}_0$, which is weakly closed, it follows that $D \in Alg(\mathcal{N}_0) \otimes \mathcal{D}_0 \subseteq \mathcal{T}$.

The result then follows, since the fact that $XDY \in \mathcal{T}$ and $i_x(E_i XDYE_j) \geq a/2$ a.e. on $K$ together contradict the assumption that $K$ is disjoint from $S_{i,j}$. □

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