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ON MINIMAL LOG DISCREPANCIES AND KOLLÁR COMPONENTS

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Abstract In this article, we prove a local implication of boundedness of Fano varieties. More precisely, we prove that d-dimensional a-log canonical singularities with standard coefficients, which admit an ϵ-plt blow-up, have minimal log discrepancies belonging to a finite set which only depends on d, a and ϵ. This result gives a natural geometric stratification of the possible mld's in a fixed dimension by finite sets. As an application, we prove the ascending chain condition for minimal log discrepancies of exceptional singularities. We also introduce an invariant for klt singularities related to the total discrepancy of Kollár components.

Keywords: log terminal; Kollar components; log discrepancies

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1. Introduction

The development of projective birational geometry has been deeply connected with the understanding of singularities [38]. In particular, global theories shed light on the singularities of the minimal model program [22]. However, singularities of dimension greater than three seem too complicated to have an explicit characterization [23]. Therefore, a more qualitative and intrinsic description of these singularities is desirable.

A common technique to study singularities using birational geometry is to apply certain monoidal transformation to the singularity to extract an exceptional projective divisor over it and then try to deduce some local information of the singularity from the global information of the exceptional divisor. This approach has been successful in many cases: the study of dual complexes of singularities [10, 26], the finiteness of the algebraic fundamental group of a klt singularity [37], the ascending chain condition for log canonical thresholds [9, 16], the study of the normalized volume function on klt singularities [27–29], and the theory of log canonical complements [7, 31, 32], among others.

In this article, we use this approach to study the minimal log discrepancies of klt singularities [3, 36]. We aim to explain how a bound on the singularities of a plt blow-up implies finiteness of the possible minimal log discrepancies. More precisely, we say that a plt blow-up $\pi\colon Y \to X$ at a point x of a klt pair (X, Δ) is ϵ-plt if the log discrepancies of the corresponding plt pair $(Y, \Delta_Y + E)$ are either zero or greater than ϵ (see

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Definition 2.5). Here, E is the exceptional divisor and Δ_Y the strict transform of Δ on Y. We say that the point x of the klt pair (X, Δ) admits an ϵ-plt blow-up if there exists π as above. For ϵ a positive real number, adjunction [13, Theorem 0.1] and the boundedness of Fano varieties [6, Theorem 1.1], allows us to conclude that the normal projective varieties E belong to a bounded family. By [37, Lemma 1], we know that every klt singularity admits a plt blow-up and a simple argument using the resolution of singularities proves that every klt singularity admits an ϵ-plt blow-up for some positive real number ϵ (see Proposition 2.7).

Using the above notation, we can introduce the set of minimal log discrepancies of a-log canonical pairs admitting an ϵ-plt blow-up:

$$\mathcal{M}(d,\mathcal{R})_{a,\epsilon} := \left\{ \mathrm{mld}_x(X,\Delta) \middle| \begin{array}{c} \dim(X)=d, \mathrm{coeff}(\Delta)\in\mathcal{R}, \text{ and } (X,\Delta) \text{ is an } a-\text{lc pair} \\ \text{which admits an } \epsilon-\text{plt blow-up at } x \end{array} \right\}.$$

We recall the conjecture known as the ascending chain condition for minimal log discrepancies:

Conjecture 1 ((cf. [36, ACC])). *Let d be a positive integer and $\mathcal{R}$ be a set of real numbers satisfying the descending chain condition. Then the set $\mathcal{M}(d, \mathcal{R})_{0,0}$ satisfies the ascending chain condition.*

The above conjecture is equivalent to Shokurov's ACC for mld's conjecture by means of Proposition 2.7. The importance of the ACC conjecture is that, together with the semicontinuity conjecture for minimal log discrepancies [36, LSC], they imply the termination of flips. This last problem is one of the main obstacles to complete the Minimal Model Program.

The ascending chain condition for minimal log discrepancies is known for surface singularities [1], for certain terminal threefold singularities [34, Lemma 4.4.1], and for toric singularities [4, 8]. However, in higher dimensions, there is not much that we can say about the possible mld's in a fixed dimension. In this paper, we give a first step towards the understanding of higher dimensional minimal log discrepancies. The following theorem can be understood as a natural geometric stratification of the possible mld's of a fixed dimension by finite sets:

Theorem 1. *Let d be a positive integer and let a and ϵ be positive real numbers, and $\mathcal{S}$ the set of standard rational numbers, i.e., $\mathcal{S} := \{1 - \frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. Then the set $\mathcal{M}(d, \mathcal{S})_{a,\epsilon}$ is finite.*

The theorem implies the following corollary towards the ascending chain condition.

Corollary 1. *Let d be a positive integer, ϵ a positive real number, and $\mathcal{S}$ the set of standard rational numbers. The set $\mathcal{M}(d, \mathcal{S})_{0,\epsilon}$ satisfies the ascending chain condition.*

In the above setting, 0-log canonical means just log canonical. In § 2.5, we give two examples that show that the theorem does not hold if a and ϵ are not positive. We also prove that a-log canonical singularities which admit an ϵ-plt blow-up have bounded Cartier index.

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Theorem 2. *Let d be a positive integer, and let a and ϵ be positive real numbers. There exists p only depending on d, a and ϵ satisfying the following. Let (X, Δ) be a d-dimensional a-log canonical pair with standard coefficients which admits an ϵ-plt blow-up at $x \in X$. Then $p(K_X + \Delta)$ is a Cartier divisor on a neighbourhood of $x \in X$.*

Exceptional singularities are those klt singularities for which any log canonical threshold is computed at a unique divisorial valuation [35, Definition 1.5]. The exceptional Du Val surface singularities are the E_6, E_7 and E_8 singularities. Hypersurfaces exceptional singularities were studied by Ishii and Prokhorov [19]. In dimension 3, Prokhorov and Markusevich proved that there are only finitely many ϵ-log canonical exceptional quotient singularities [30]. However, a classification of exceptional singularities in higher dimensions seems unfeasable. In this direction, we prove the following application of our main theorem:

Corollary 2. *The ascending chain condition for minimal log discrepancies of exceptional singularities with standard coefficients holds.*

It is worth mentioning that Corollary 2 follows almost directly from the proof of [31, Theorem 4.4] and [6, Theorem 1.1]. These two results together with [18] and [12] are the main motivation of Theorem 1. The proof of [31, Theorem 4.4] already contains some of the ideas used in this article.

After completing this project, the author was informed that J. Han, J. Liu and V. Shokurov have obtained the results of this paper with more general coefficients [17]. The proof of this manuscript goes along that of [35] in dimension 3. In the following remark, we give a detailed account of this generalization.

Remark 1.1. Corollary 1 (respectively Corollary 2) is generalized by [17, Theorem 1.3] (respectively [17, Theorem 1.2]), where the authors prove the statement with general coefficients. Theorem 1 and Theorem 2 are implied by Theorem [17, Theorem 1.6], where the authors prove a uniform bound for the regional fundamental group.

2. Preliminaries

All varieties in this paper are quasi-projective over a fixed algebraically closed field of characteristic zero unless stated otherwise. In this section, we collect some definitions and preliminary results which will be used in the proof of the main theorem.

2.1. Singularities

In this subsection, we recall the singularities of the minimal model program, the set of standard coefficients and exceptional singularities. We also prove some basic properties about singularities.

Definition 2.1. In this paper, a *sub-pair* (X, Δ) consists of a normal quasi-projective variety X and a $\mathbb{Q}$-divisor Δ so that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. If the coefficients of Δ are non-negative then we say that (X, Δ) is a *log pair*, or simply a *pair*.

Let $\pi\colon W \to X$ be a log resolution of the pair (X, Δ) and denote by $K_W + \Delta_W + F_W$ the log pull-back of $K_X + \Delta$, where Δ_W is the strict transform of Δ on W and F_W is an

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exceptional divisor. The *discrepancy* of a prime divisor E on W with respect to the pair $(X,\, \Delta)$ is

$$a_E(X, \Delta) := -\mathrm{coeff}_E(\Delta_W + F_W).$$

The *log discrepancy* of a prime divisor E on W with respect to the pair $(X,\, \Delta)$ is the value

$$a_E(X, \Delta) + 1.$$

The *center* of E on X is its image on X via the morphism π. We denote by $c_X(E)$ the centre of the prime divisor E on the variety X. A *prime divisor* over X is a prime effective divisor E on a normal quasi-projective variety W which admits a projective birational morphism $W \to X$.

We say that the sub-pair $(X,\, \Delta)$ is *sub-ϵ-log canonical* if

$$a_E(X, \Delta) \geq -1 + \epsilon,$$

for every prime divisor E on W. If $(X,\, \Delta)$ is a pair then we say that is ϵ-log canonical in the above situation. If $\epsilon > 0$ is arbitrary, we may also say that $(X,\, \Delta)$ is *Kawamata log terminal* (or klt) and if $\epsilon = 0$ we just say that the pair is log canonical, equivalently, the centre of a log canonical place. The *total discrepancy* $a(X,\, \Delta)$ of the pair $(X,\, \Delta)$ is the infimum among all discrepancies $a_E(X,\, \Delta)$ with E a prime divisor over X. Thus, $(X,\, \Delta)$ is a-log canonical if and only if $a(X,\, \Delta) + 1 \geq a$.

Let $(X,\, \Delta)$ be a log canonical pair. A *log canonical place* of $(X,\, \Delta)$ is a prime divisor E on a birational model of X so that $a_E(X,\, \Delta) = -1$. A *log canonical center* of $(X,\, \Delta)$ is the image on X of a log canonical place.

Definition 2.2. Let $(X,\, \Delta)$ be a log pair and $x \in X$. The *minimal log discrepancy* of $(X,\, \Delta)$ at x is

$$\mathrm{mld}_x(X, \Delta) := \inf \{ a_E(X, \Delta) + 1 \mid E \text{ is a prime divisor over } X \text{ so that } c_X(E) = x \}.$$

If $(X,\, \Delta)$ is a log canonical pair, taking a log resolution and using [25, Lemma 2.29], we can see that the above infimum is indeed a minimum. Observe that $\mathrm{mld}_x(X,\, \Delta) \geq 0$ if and only if the pair $(X,\, \Delta)$ is log canonical at $x \in X$. Moreover, $\mathrm{mld}_x(X,\, \Delta) > 0$ if and only if the pair $(X,\, \Delta)$ is klt at $x \in X$. On the other hand, if $(X,\, \Delta)$ is not log canonical at $x \in X$, then $\mathrm{mld}_x(X,\, \Delta) = -\infty$ (see, e.g. [25, Corollary 2.32]).

Definition 2.3. We say that the pair $(X,\, \Delta)$ is *divisorial log terminal* (or dlt) if the following conditions hold:

(1) there exists a closed subset $Z \subset X$ so that $X \setminus Z$ is smooth,

(2) $\Delta|_{X \setminus Z}$ has simple normal crossing support, and

(3) every divisor E over X with centre in Z has positive log discrepancy with respect to $(X,\, \Delta)$.

A pair $(X,\, \Delta)$ is called *purely log terminal* (or plt) if the log discrepancy of every exceptional prime divisor over X is strictly positive. In this case, $\lfloor \Delta \rfloor$ is a disjoint union of normal prime divisors.

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Definition 2.4. Given a finite set $\mathcal{R}$ of rational numbers, we denote by

$$\mathcal{H}(\mathcal{R}) := \{1\} \cup \left\{1 - \frac{r}{m} \,\middle|\, r \in \mathcal{R} \text{ and } m \in \mathbb{Z}_{>0}\right\},$$

and call $\mathcal{H}(\mathcal{R})$ the *set of hyperstandard coefficients* associated with $\mathcal{R}$. We denote $\mathcal{S} := \mathcal{H}(\{1\})$ and call this the set of *standard coefficients*.

Definition 2.5. A *plt blow-up* of a log pair (X, Δ) at a point $x \in X$ is a projective birational morphism $\pi\colon Y \to X$ with the following properties:

(1) Y is a quasi-projective normal variety,

(2) the exceptional locus of π is an irreducible divisor E whose image on X is x,

(3) the pair $(Y, \Delta_Y + E)$ is purely log terminal, where Δ_Y is the strict transform of Δ on Y, and

(4) $-E$ is ample over X.

We say that the pair (X, Δ) admits a plt blow-up at $x \in X$ if there exists π *with the above conditions. Moreover, we say that the plt blow-up is an ϵ-plt blow-up* if any log discrepancy of $(Y, \Delta_Y + E)$ is either zero or greater than ϵ. Analogously, we say that the pair (X, Δ) admits an ϵ-plt blow-up at $x \in X$ if there exists π with the above conditions.

Definition 2.6. The exceptional divisor of a plt blow-up is often called *a Kollár component* of the singularity [28, Definition 1.1]. We may use both acceptions on this paper. We may also call the log Fano pair (E, Δ_E), where

$$K_E + \Delta_E := (K_Y + \Delta_Y + E)|_E$$

is obtained by adjunction to E, a *Kollár component* over the klt pair (X, Δ). Observe that $-(K_E + \Delta_E)$ is ample and (E, Δ_E) is klt. Hence, (E, Δ_E) is a log Fano pair [7, 2.10].

Proposition 2.7. *Let (X, Δ) be a klt pair and $x \in X$. There exists an ϵ-plt blow-up of (X, Δ) at x for some positive ϵ.*

Proof. By [37, Lemma 1], we can construct a plt blow-up $\pi\colon Y \to X$ of (X, Δ) over x. Let $\pi_Y\colon W \to Y$ be a log resolution of $(Y, \Delta_Y + E)$, so we can write

$$K_W + \Delta_W + F_W = \pi_Y^*(K_Y + \Delta_Y + E),$$

where Δ_W is the strict transform of Δ_Y on W, and F_W is an exceptional divisor. We can take ϵ small enough so that

$$\operatorname{coeff}_D(\Delta_W + F_W - E_W) < 1 - \epsilon$$

for all exceptional prime divisors D, where E_W is the strict transform of E on W. By [25, Lemma 2.29], we conclude that every exceptional divisor over W has log discrepancy either zero or greater than ϵ with respect to the purely log terminal pair $(Y, \Delta_Y + E)$. $\square$

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Theorem 1 motivates the following natural invariant of klt singularities.

Definition 2.8. Let (X, Δ) be a klt pair and $x \in X$. We define the *mildest component*, or $\mathcal{MC}$ for simplicity, of (X, Δ) at the point $x \in X$ to be:

$$\mathcal{MC}_x(X,\Delta) := \sup\{a(E,\Delta_E)+1 \mid (E,\Delta_E) \text{ is a Kollár component of } (X,\Delta) \text{ over } x\in X\}.$$

In Proposition 2.25, we will prove that the $\mathcal{MC}$ is indeed attained by some Kollár component over $x \in X$. In this setting, the conditions of Theorem 1 and Theorem 2 can be abbreviated as (X, Δ) is a d-dimensional pair which is a-log canonical at $x \in X$ and $\mathcal{MC}_x(X, \Delta) \geq \epsilon$.

Remark 2.9. In general, one could define the mildest component of (X, Δ) at $x \in X$ for every Grothendieck point of the variety X. However, by cutting down with general hyperplanes, the study of minimal log discrepancies on a variety X of dimension d at a point $x \in X$ of codimension k is equivalent to the study of minimal log discrepancies on a variety X of dimension $d - k$ at a closed point.

Definition 2.10. A *dlt modification* of a log canonical pair (X, Δ) is a projective birational morphism $\pi\colon Y \to X$ so that $\pi^*(K_X + \Delta) = K_Y + \Delta_Y + E_Y$, where Δ_Y is the strict transform of Δ on X and E_Y is an exceptional divisor over X so that the sub-pair $(Y, \Delta_Y + E_Y)$ is a dlt pair. It is known that every log canonical pair admits a dlt modification [24, Theorem 3.1].

Definition 2.11. We say that a klt pair (X, Δ) is *exceptional* at $x \in X$ if for every boundary $\Gamma \geq 0$ on X so that $(X, \Delta + \Gamma)$ is a log canonical pair, any dlt modification of $(X, \Delta + \Gamma)$ is indeed plt.

Remark 2.12. It is known that if (X, Δ) is exceptional at x and X is $\mathbb{Q}$-factorial, then there exists a unique prime divisor over X which determines all log canonical thresholds over $x \in X$ [30, Proposition 2.7].

Definition 2.13. Let (X, Δ) be a log canonical pair and $X \to Z$ be a contraction of normal quasi-projective varieties. We say that $\Gamma \geq 0$ is a *strong* (δ, n)*-complement* over $z \in Z$ of (X, Δ) if the following conditions hold:

- $(X, \Delta + \Gamma)$ is an δ-log canonical pair, and
- $n(K_X + \Delta + \Gamma) \sim 0$ over a neighbourhood of z.

In the case that $X \to Z$ is the identity, then we say that Γ is a *local* strong (δ, n)-complement around $x \in X$ for (X, Δ). On the other hand, if $Z = \mathrm{Spec}(k)$ for some field k, then we say that Γ is a *global* strong (δ, n)-complement for the pair (X, Δ). A *strong* n*-complement* is a strong $(0, n)$-complement.

Lemma 2.14. *Let (X, Δ) be a klt pair which is exceptional at $x \in X$. Assume that the coefficients of Δ are standard. There exists an ϵ_d-blow-up of (X, Δ) at x, for some positive real number ϵ_d which only depends on the dimension d of X.*

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Proof. By Lemma 2.7, we can extract an ϵ-plt blow-up over $x \in X$ for some real number ϵ. By [7, Theorem 1.8] there exists a strong n-complement $K_Y + \Delta_Y + \Gamma_Y + E$ for $K_Y + \Delta_Y + E$ over x, for n only depending on d and $\mathcal{S}$. Since $\mathcal{S}$ is a fixed set then n only depends on d. Since $K_Y + \Delta_Y + \Gamma_Y + E$ is $\mathbb{Q}$-linearly trivial over $x \in X$ then we can write

$$K_Y + \Delta_Y + \Gamma_Y + E = \pi^*(K_X + \Delta + \Gamma)$$

for some boundary Γ on X so that $(X, \Delta + \Gamma)$ is a log canonical pair. By the exceptionality of (X, Δ) at x we deduce that $(X, \Delta + \Gamma)$ has a unique log canonical place and hence $(Y, \Delta_Y + \Gamma_Y + E)$ has a unique log canonical place E. Moreover, since $n(K_Y + \Delta_Y + \Gamma_Y + E)$ is Cartier over x, we conclude that

$$a_F(Y, \Delta_Y + E) + 1 \geq a_F(Y, \Delta_Y + \Gamma_Y + E) + 1 \geq \frac{1}{n}$$

for every divisor F over Y which is not equal to E. Thus, it suffices to take $\epsilon_d = \frac{1}{n}$. □

2.2. Bounded families

In this subsection, we recall the definition of a log bounded family and prove some properties of such families.

Definition 2.15. We say that a set of pairs $\mathcal{P}$ is *log bounded* if there exists a projective morphism $\mathcal{X} \to T$ of possibly reducible varieties and a divisor $\mathcal{B}$ on $\mathcal{X}$ so that for every pair $(X, \Delta) \in \mathcal{P}$ there exists a closed point $t \in T$ and an isomorphism $\phi\colon \mathcal{X}_t \to X$ so that $(\mathcal{X}_t, \mathcal{B}_t)$ is a pair and $\phi^{-1}_* \Delta \leq \mathcal{B}_t$. If, moreover, ϕ induces an isomorphism of pairs between (X, Δ) and $(\mathcal{X}_t, \mathcal{B}_t)$, meaning that for any prime divisor D on X, we have that

$$\mathrm{coeff}_D(\Delta) = \mathrm{coeff}_{\phi^{-1}_* D}(\mathcal{B}_t),$$

we will say that the set of pairs $\mathcal{P}$ is *strictly log bounded*. We say that $\mathcal{X} \to T$ is a *bounding family* for the varieties X and that $\mathcal{B}$ is a *bounding divisor* for the set of divisors $\{\Delta \mid (X, \Delta) \in \mathcal{P}\}$.

The following lemma follows from the definition of strictly log bounded family.

Lemma 2.16. *Let $\mathcal{P}$ be a log bounded family of pairs so that the set $\{\mathrm{coeff}(\Delta) \mid (X, \Delta) \in \mathcal{P}\}$ is finite. Then the family $\mathcal{P}$ is strictly log bounded.*

Lemma 2.17. *Let $\mathcal{P}$ be a bounded family of d-dimensional projective varieties so that $-K_X$ is pseudo-effective for every $X \in \mathcal{P}$. Then, there exists a positive constant C, only depending on $\mathcal{P}$, satisfying the following: for every $X \in \mathcal{P}$, we can find a very ample Cartier divisor A with $A^d \leq C$ and $A^{d-1}(-K_X) \leq C$.*

Proof. Let $\mathcal{X} \to T$ be the bounding family. By Notherian induction, we can stratify the base T into finitely many locally closed subvarieties $T_i \subset T$, satisfying the following: For each fibre over $t \in T_i$ of $\mathcal{X}_i \to T_i$ we have that $K_{\mathcal{X}_i}|_{\mathcal{X}_t} = K_{\mathcal{X}_t}$. We may assume that the T_i's are disjoint by successively replacing T_i, T_j with $T_i \setminus \mathrm{supp}(T_i \cap T_j)$, $T_j \setminus \mathrm{supp}(T_i \cap T_j)$ and $T_i \cap T_j$. For each i, we can consider a very ample effective Cartier divisor $\mathcal{A}_i$ on $\mathcal{X}_i$.

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For each fibre, we define $\mathcal{A}_{\mathcal{X}_t} := \mathcal{A}_i|_{\mathcal{X}_t}$. Then, we have that $\mathcal{A}^d_{\mathcal{X}_t}(-K_{\mathcal{X}_t}) = \mathcal{A}^{d-1}(-K_{\mathcal{X}})|_{\mathcal{X}_t}$ for each $t \in T_i$. Thus, the intersection $\mathcal{A}^d_{\mathcal{X}_t}(-K_{\mathcal{X}_t})$ only takes finitely many values over T_i. Analogously, $\mathcal{A}^d_{\mathcal{X}_t}$ only takes finitely many values over T_i. We define C to be the maximum of all these intersection numbers among all the T_i's. Then, for each $X \in \mathcal{P}$ it suffices to consider A to be the pull-back of $\mathcal{A}_{\mathcal{X}_t}$ to X via the isomorphism $X \to \mathcal{X}_t$. □

Lemma 2.18. *Let $\mathcal{P}$ be a bounded family of d-dimensional projective varieties and $\mathcal{Q}$ a set of pairs $\{(X, \Delta) \mid X \in \mathcal{P}\}$, so that coeff($\Delta$) satisfies the descending chain condition and for every $(X, \Delta) \in \mathcal{Q}$ we have that $-(K_X + \Delta)$ is pseudo-effective. Then $\mathcal{Q}$ is a log bounded set of pairs.*

Proof. By Lemma 2.17, we can find a positive constant C so that for each $X \in \mathcal{P}$ there is a very ample Cartier divisor A with $A^d \leq C$ and $A^{d-1}(-K_X) \leq C$. The set coeff(Δ) satisfies the descending chain condition, so there exists $\delta > 0$ small enough so that $\delta < \operatorname{coeff}(\Delta)$ for every boundary Δ of a pair (X, Δ) on $\mathcal{Q}$. Since $-(K_X + \Delta)$ is pseudo-effective we have that $A^{d-1} \cdot (-K_X - \Delta) \geq 0$, hence we conclude that

$$A^{d-1} \cdot (\delta \Delta_{\text{red}}) \leq A^{d-1} \cdot \Delta \leq A^{d-1} \cdot (-K_X) \leq C,$$

where Δ_{red} is Δ with the reduced structure. Thus, we get that $A^{d-1} \cdot \Delta_{\text{red}} \leq \delta^{-1} C$, so by [2, Lemma 3.7.(2)], we conclude that the set of pairs $\mathcal{Q}$ is log bounded. □

Lemma 2.19. *Let $\mathcal{P}$ be a strictly log bounded family of log Fano pairs. Then, there exists m only depending on $\mathcal{P}$, so that $|-m(K_X + \Delta)|$ is base point free for $(X, \Delta) \in \mathcal{P}$.*

Proof. Note that the dimension of the varieties of $\mathcal{P}$ is bounded. Let $(\mathcal{X}, \mathcal{B}) \to T$ be a log bounding family for $\mathcal{P}$. We can stratify the base T into finitely many locally closed subvarieties $T_i \subset T$, satisfying the following: For each fibre over $t \in T_i$ of $\mathcal{X}_i \to T_i$, we have that $(K_{\mathcal{X}_i} + \mathcal{B}_i)|_{\mathcal{X}_t} = K_{\mathcal{X}_t} + \mathcal{B}_t$. In particular, we conclude that there exists a constant c, only depending on $\mathcal{P}$, so that $c(K_X + \Delta)$ is Cartier for every $(X, \Delta) \in \mathcal{P}$. Then, applying Kollár's effective base point freeness [21, Theorem 1.1] with $a = 1$ and $L = -c(K_X + \Delta)$, we conclude that there exists a constant m, only depending on $\mathcal{P}$, so that $|-m(K_X + \Delta)|$ is base point free. □

Definition 2.20. Let X be an irreducible projective variety of dimension d and let D be a $\mathbb{Q}$-Cartier divisor on X. The *volume* of D is

$$\operatorname{vol}(D) := \limsup_{m \to \infty} \frac{d! h^0(X, \mathcal{O}_X(mD))}{m^d}.$$

In particular, a big divisor has positive volume.

Lemma 2.21. *Let $\mathcal{P}$ be a strictly log bounded family of d-dimensional klt pairs. Then there exist positive real numbers v_1 and v_2, only depending on $\mathcal{P}$, so that for every $(X, \Delta) \in \mathcal{P}$ with $-(K_X + \Delta)$ ample we have that*

$$v_1 < \operatorname{vol}(-(K_X + \Delta)) < v_2.$$

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Proof. Since ampleness is open in families, we may restrict to an open set $U \subset T$ so that $-(K_{\mathcal{X}_t} + \mathcal{B}_t)$ is ample for every $t \in U$. Furthermore, by Noetherian induction, we may assume that the induced morphism $\mathcal{X}_U \to U$ is a projective smooth morphism of relative dimension d with normal fibres. In this case, we have that

$$\mathrm{vol}(-(K_{\mathcal{X}_t} + \mathcal{B}_t)) = (-(K_{\mathcal{X}_t} + \mathcal{B}_t))^d$$

is an upper-semicontinuous function on $t \in U$. Hence, it takes finitely many values on U. □

Lemma 2.22. *Let $\mathcal{P}$ be a strictly log bounded family of log Fano pairs. We can find a positive real number M, only depending on $\mathcal{P}$, so that for every $(X, \Delta) \in \mathcal{P}$ there exists an ample curve C on X so that*

$$-(K_X + \Delta) \cdot C \leq M.$$

Proof. By Lemma 2.19, we can find a positive natural number m, only depending on $\mathcal{P}$, so that $|-m(K_X + \Delta)|$ is a base point free linear system for any $(X, \Delta) \in \mathcal{P}$. Hence, a general curve C on the rational equivalence class of $(-m(K_X + \Delta))^{d-1}$ will be an ample curve. Moreover, by Lemma 2.21, we have that

$$-(K_X + \Delta) \cdot C = m^{d-1}(-(K_X + \Delta))^d = m^{d-1}\,\mathrm{vol}(-(K_X + \Delta)) < m^{d-1}v_2.$$

Thus, it suffices to take $M = m^{d-1}v_2$. □

Lemma 2.23. *Let d be a positive integer and ϵ a positive real number. There exists a constant l only depending on d and ϵ satisfying the following. Let X be a d-dimensional klt variety so that K_X is Cartier at $x \in X$. Assume that X admits an ϵ-plt blow-up at $x \in X$ extracting a divisor E. Then we have that*

$$a_E(X, 0) \leq l.$$

In particular, there are finitely many possible values for $a_E(X, 0)$.

Proof. Let $\pi \colon Y \to X$ be the ϵ-plt blow-up at $x \in X$ and write

$$K_Y - a_E(X, 0)E = \pi^* K_X.$$

By assumption of X being klt and K_X being Cartier at x we know that $a_E(X, 0)$ is a non-negative integer. We write

$$K_E + \Delta_E = (K_Y + E)|_E.$$

Thus, the pair (E, Δ_E) is ϵ-lc and $-(K_E + \Delta_E)$ is ample. By boundedness of Fano varieties [6, Theorem 1.1], we conclude that the projective varieties E belong to a bounded family. By [15, Theorem 3.34], we know that the coefficients of Δ_E are standard, therefore the pairs (E, Δ_E) are log bounded by Lemma 2.18. Moreover, since (E, Δ_E) is ϵ-log canonical then the coefficients of Δ_E are at most $1 - \epsilon$, hence belong to a finite set of rational numbers. By Lemma 2.16, conclude that the pairs (E, Δ_E) belong to a strictly

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log bounded family $\mathcal{P}$, which only depends on the dimension d and the positive real number ϵ.

By [33, Proposition 3.9], we know that at codimension 2 points of Y, every Weil divisor has Cartier index bounded by p, for some constant p which only depends on ϵ. Hence, there exists a closed subset Z on Y of codimension at least 3, so that the Cartier index of E is a divisor of p outside Z. By Lemma 2.22, we may find an ample curve C so that

$$-(K_E + \Delta_E) \cdot C \leq M,$$

for some constant M which only depends on $\mathcal{P}$. Thus, up to replacing C with a rationally equivalent curve, we may assume that C does not intersect Z, so we have $pE \cdot C$ is a negative integer, or equivalently,

$$-E \cdot C \in \mathbb{Z}_{>0}\left[\frac{1}{p}\right].$$

Moreover, since (E, Δ_E) belongs to a strictly log bounded family, we conclude that the Cartier index of $-(K_E + \Delta_E)$ is bounded by a constant which only depends on $\mathcal{P}$ [7, Lemma 2.25]. In particular,

$$-(K_Y + E) \cdot C = -(K_E + \Delta_E) \cdot C$$

belongs to a finite set $\mathcal{F}$ of positive rational numbers, which only depends on $\mathcal{P}$. Finally, observe that from the relation

$$(K_Y + E - (a_E(X, 0) + 1)E) \cdot C = \pi^*(K_X) \cdot C = 0,$$

we conclude that

$$\frac{(a_E(X, 0) + 1)}{p} \leq (a_E(X, 0) + 1)(-E \cdot C) \in \mathcal{F},$$

so

$$a_E(X, 0) \leq p \max\{\mathcal{F}\} - 1,$$

where the right-hand side only depends on ϵ and $\mathcal{F}$. Since $\mathcal{F}$ only depends on $\mathcal{P}$, and $\mathcal{P}$ only depends on d and ϵ, we deduce that $l = p \max\{\mathcal{F}\} - 1$ only depends on d and ϵ. This proves the first statement. Since $a_E(X, 0)$ is a non-negative integer, we conclude that there are finitely many possible values for it, proving the second statement. □

Lemma 2.24. *Let $\mathcal{P}$ be a strictly log bounded family of log Fano pairs. Then, the total discrepancies of pairs $(X, \Delta) \in \mathcal{P}$ take only finitely many values.*

Proof. Let $\mathcal{X} \to T$ be a bounding family. Let $\mathcal{B} \subset \mathcal{X}$ be an effective divisor so that $(\mathcal{X}, \mathcal{B})$ strictly log bound the family $\mathcal{P}$. By Noetherian induction, we can stratify the base T into finitely many locally closed subvarieties $T_i \subset T$, satisfying the following: For each induced family $\mathcal{X}_i \to T_i$, we can find a log resolution of $(\mathcal{X}_i, \mathcal{B}_i)$ so that admits a log resolution over T_i and each blow-up is horizontal over T_i. By [25, Corollary 2.32.(2)], we conclude that the total discrepancy of each fibre of $(\mathcal{X}_i, \mathcal{B}_i) \to T_i$ belongs to a finite set.

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Hence, the total discrepancy of each fibre of $(\mathcal{X}, \mathcal{B}) \to T$ belongs to a finite set. Then, the same holds for pairs $(X, \Delta) \in \mathcal{P}$. □

To conclude this subsection, we will prove that the mildest component of a klt singularity is indeed attained by a Kollár component, or equivalently, the infimum in Definition 2.8 is a minimum.

Proposition 2.25. *Let (X, Δ) be a pair which is klt at x. Assume that the coefficients of Δ are rational numbers. There exists a Kollár component $\pi\colon Y \to X$ extracting a divisor $E \subset Y$ so that*

$$\mathcal{MC}_x(X,\Delta) = a(Y, \Delta_Y + E).$$

Proof. We proceed by contradiction. Recall that

$$\begin{aligned}\mathcal{MC}_x(X,\Delta) := \sup\{a(E,\Delta_E)+1 \mid (E,\Delta_E) \text{ is a Kollár component of}\\ (X,\Delta) \text{ over } x \in X\}.\end{aligned}$$

We want to prove that the supremum on the right is actually a maximum. Assume it is not. Then, we can find a sequence of Kollár components $\pi_i\colon Y_i \to X$ over $x \in X$ for the pair (X, Δ) so that the total discrepancies of the pairs (E_i, Δ_{E_i}) are in an infinite increasing sequence. Since each (E_i, Δ_{E_i}) is klt, its total log discrepancy is positive. We are assuming that the total discrepancy of the sequence (E_i, Δ_{E_i}) is infinite and strictly increasing. Hence, there exists $\epsilon > 0$ so that each (E_i, Δ_{E_i}) is ϵ-log canonical. Thus, by [7, Theorem 1.1], we conclude that the varieties E_i belong to a bounded family.

Since the coefficients of Δ_{Y_i} are fixed by coeff(Δ), we conclude that the coefficients of Δ_{E_i} belong to a finite set of rational numbers which only depend on coeff(Δ) [11, Lemma 5.3]. By Lemma 2.18, we conclude that the pairs (E_i, Δ_{E_i}) are log bounded. By Lemma 2.16, we conclude that the pairs (E_i, Δ_{E_i}) are strictly log bounded. Since the pairs (E_i, Δ_{E_i}) belong to a strictly log bounded family, then their total discrepancies can take only finitely many values (see Lemma 2.24). This leads to a contradiction. We conclude that there is no such infinite sequence, hence the supremum is a maximum. □

2.3. Finite morphisms

In this subsection, we recall the index one cover of a log canonical singularity and the behaviour of log discrepancies under finite dominant morphisms.

Definition 2.26. Let (X, Δ) be a pair and write $\Delta = \sum_i d_i \Delta_i$ where the Δ_i's are pairwise different prime divisors on X. Given a quasi-finite morphism $\phi\colon X' \to X$ between normal varieties, we can write

$$\phi^*(K_X + \Delta) = K_{X'} + \Delta',$$

where

$$\Delta' := \sum_i \sum_{f(E_j)=\Delta_i} (d_i(r_j+1) - r_j)E_j$$

and r_j is the ramification index at the generic point of E_j [33, 2.1]. The above formula is called the *pull-back formula* for quasi-finite morphisms.

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The following lemma follows from the pull-back formula for quasi-finite morphisms.

Lemma 2.27. *Let (X, Δ) be a pair with standard coefficients and $\phi\colon X' \to X$ be a finite morphism of normal varieties. Assume that for every prime divisor E on X', we have that $r_E + 1$ divides $(1 - d_i)^{-1}$, where r_E is the ramification index of ϕ at E and d_i is the coefficient of $\phi(E)$ at Δ. Then Δ' is an effective divisor whose coefficients are standard.*

The following is a theorem of Zariski that is often used instead of resolution of singularities [39].

Theorem 2.28. *Let Y' and X be two integral schemes of finite type over a field over $\mathbb{Z}$, and $f\colon Y' \to X$ a dominant morphism. Let $D \subset Y'$ be a prime divisor and $\eta \in D$ the generic point. Assume that Y' is normal at η. We can define a sequence of schemes and rational maps as follows:*

(1) $X_0 = X$ and $f_0 = f$,

(2) If $f_i\colon Y' \dashrightarrow X_i$ is defined, then let $Z_i \subset X_i$ the closure of $f_i(\eta)$. We define X_{i+1} to be the blow-up of X_i at Z_i and $f_{i+1}\colon Y' \dashrightarrow X_{i+1}$ the induced rational map.

For j large enough $\dim(Z_j) \geq \dim(X) - 1$ and X_j is regular at the generic point of Z_j.

Lemma 2.29. *With the assumptions of Lemma 2.27, the following conditions hold:*

(1) For ϵ a non-negative real number, the pair (X', Δ') is ϵ-log canonical if and only if (X, Δ) is ϵ-log canonical, and

(2) if (X, Δ) is a log canonical pair with a unique log canonical centre $x \in X$, (X, Δ) has a unique log canonical place, and $x' = \phi^{-1}(x)$ is a point, then (X', Δ') has a unique log canonical place.

Proof. Let $\pi\colon Y \to X$ be a projective birational morphism and $Y' \to Y \times_X X'$ the normalization of the main component of the fibre product, then we have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\phi_Y} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\phi} & X \end{array} \tag{2.1}$$

and we can write

$$\pi'^*(K_{X'} + \Delta') = K_{Y'} + \Delta'_{Y'} + E' \text{ and } \pi^*(K_X + \Delta) = K_Y + \Delta_Y + E.$$

Here $\Delta'_{Y'}$ is the strict transform of Δ' on Y'. By Lemma 2.27, we know that (X', Δ') is a log pair. Thus, we have the relation

$$\phi_Y^*(K_Y + \Delta_Y + E) = K_{Y'} + \Delta'_{Y'} + E'.$$

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Therefore, by the pull-back formula for quasi-finite morphisms we have that

$$a_E(X', \Delta') + 1 = r_E(a_E(X, \Delta) + 1),$$

where r_E is the ramification index of ϕ_Y at the generic point of E. By Theorem 2.28, we know that for every divisorial valuation on X' we can find $\pi\colon Y \to X$ so that the centre of this valuation on Y' is a divisor, where Y' is as in the commutative diagram (2.1). This proves the first statement.

Now we turn to prove the second statement by contradiction. By the proof of the first statement, we know that the log canonical pair (X', Δ') has at least one log canonical place, and all its log canonical places map onto $x' \in X'$. Assume that (X', Δ') has more than one log canonical place. Let $\pi\colon Y' \to X'$ be a dlt modification of (X', Δ') and write

$$\pi'^*(K_{X'} + \Delta') = K_{Y'} + \Delta'_{Y'}.$$

Applying [25, Theorem 5.48] to a log resolution of (Y', Δ'_Y), we deduce that $\lfloor \Delta'_{Y'} \rfloor$ is connected. Moreover, since (X', Δ') has more than one log canonical place, the divisor $\lfloor \Delta'_{Y'} \rfloor$ is connected and has at least two irreducible components. Since the intersection of two log canonical centres is a union of log canonical centres [5, Theorem 1.1], we deduce that $(Y', \Delta'_{Y'})$ has infinitely many log canonical places. Thus, (X', Δ') has infinitely many log canonical places. By Theorem 2.28, we conclude that each of those log canonical places appears in a commutative diagram as in (2.1). Moreover, at least one log canonical place of (X', Δ') is exceptional over the dlt model (Y, Δ_Y) of (X, Δ). Therefore, (X, Δ) has at least two log canonical places. This provides the needed contradiction and the claim follows. □

Definition 2.30. Let (X, Δ) be a klt pair with standard coefficients and $x \in X$ a point. Consider a the smallest positive integer so that $a(K_X + \Delta) \sim 0$ on a neighborhood of $x \in X$, or equivalently, a is the Cartier index of $K_X + \Delta$ at $x \in X$. Therefore, we have an isomorphism $\mathcal{O}_X(a(K_X + \Delta)) \simeq \mathcal{O}_X$, we can choose a nowhere zero section

$$s \in H^0(X, \mathcal{O}_X(a(K_X + \Delta))),$$

and consider $\phi\colon X' \to X$ the corresponding cyclic cover. The ramification index of ϕ at a prime divisor E which maps onto Δ_i is exactly $d_i - 1$, where $d_i = \mathrm{coeff}_{\Delta_i}(\Delta)$. Moreover, ϕ is ramified only at the support of Δ. Therefore, from the pull-back formula for quasi-finite morphisms, we know that

$$K_{X'} = \phi^*(K_X + \Delta).$$

We call ϕ the *index one cover* of the klt pair (X, Δ) locally at $x \in X$ [12, Notation 4.1].

2.4. Complements

In this subsection, we prove that klt singularities with plt blow-ups admit local complements with a unique log canonical place.

Remark 2.31. In this notation, a pair (X, Δ) is exceptional at $x \in X$ if and only if every local complement at $x \in X$ has a unique log canonical place.

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Lemma 2.32. *Let (X, Δ) be a d-dimensional klt pair with standard coefficients. Assume that (X, Δ) admits an ϵ-plt blow-up at $x \in X$ extracting the exceptional divisor E. There exists a natural number n, only depending on d and ϵ, and a boundary Γ on X so that the following conditions hold:*

- $n(K_X + \Delta + \Gamma) \sim 0$ *on a neighbourhood of* $x \in X$, *and*
- $K_Y + \Delta_Y + \Gamma_Y + E = \pi^*(K_X + \Delta + \Gamma)$ *is an ϵ-plt pair on a neighbourhood of* $x \in X$.

Moreover, we may assume that the boundary divisors Γ and Δ do not share prime components.

Proof. We will construct a strong $(0, n)$-complement for the divisor $-(K_Y + \Delta_Y + E)$ with respect to the morphism $\pi\colon Y \to X$ around $x \in X$. This complement will push-forward to a $(0, n)$-complement for (X, Δ) locally around $x \in X$. In order to do so, we will do adjunction to E, produce a global complement on E and then pull-back to a neighbourhood of E.

By adjunction, we can write

$$(K_Y + \Delta_Y + E)|_E = K_E + \Delta_E,$$

where (E, Δ_E) is an ϵ-log canonical pair and the coefficients of Δ_E belong to a set of rational numbers satisfying the descending chain condition with rational accumulation points (see, e.g. [11, Lemma 5.3]). Hence, by [11, Theorem 1.3.2], we can find n only depending on d and ϵ, and a global strong (ϵ, n)-complement Γ_E for the pair (E, Δ_E). From the proof of [11, Theorem 1.3], we may assume that Γ_E does not share prime components with Δ_E. Without loss of generality, we may assume that $n\Delta_Y$ is a Weil divisor. Indeed, since $(Y, \Delta_Y + E)$ is ϵ-plt and the coefficients of Δ_Y are standard, then they belong to a finite set.

Let $\pi_Y : W \to Y$ be a log resolution of $(Y, \Delta_Y + E)$, and write

$$-N_W := \pi_Y^*(K_Y + \Delta_Y + E) = K_W + \Delta_W + E_W,$$

where E_W is the strict transform of E on W. We define

$$L_W := -nK_W - nE_W - \lfloor (n+1)\Delta_W \rfloor.$$

Let P_W be the unique integral effective divisor on W so that

$$\Lambda_W := E_W + (n+1)\Delta_W - \lfloor (n+1)\Delta_W \rfloor + P_W$$

is a boundary on W so that (W, Λ_W) is plt and $\lfloor \Lambda_W \rfloor = E_W$. We claim that P_W is an exceptional divisor over Y. Assume it is not. Indeed, if D is a prime divisor on W which

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is not contracted on Y then, we have that

$$\mathrm{coeff}_D(\lfloor (n+1)\Delta_W \rfloor) = \mathrm{coeff}_D(n\Delta_W),$$

because $n\Delta_Y$ is integral. Therefore

$$\mathrm{coeff}_D((n+1)\Delta_W - \lfloor (n+1)\Delta_W \rfloor) = \mathrm{coeff}_D(\Delta_W) = \mathrm{coeff}_{\pi_Y(D)}(\Delta_Y) \in (0,1).$$

Hence, if $\mathrm{coeff}_D(P_W) > 0$, then $\mathrm{coeff}_P(P_W) \geq 1$, so $\mathrm{coeff}_P(\Lambda_W) > 1$. In particular, (W, Λ_W) is not plt. This leads to a contradiction. We conclude that P_W is exceptional over Y.

By definition, we have that

$$L_W + P_W - E_W = K_W + \Lambda_W + (n+1)N_W,$$

is the sum of the klt pair $(W, \Lambda_W - E_W)$ and the nef and big divisor $(n+1)N_W$ over a neighbourhood of $x \in X$. Note that $(W, \Lambda_W - E_W)$ is klt given that (W, Λ_W) is plt with $\lfloor \Lambda_W \rfloor = E_W$. Shrinking around $x \in X$ we may assume that X is affine, then $(n+1)N_W$ is nef and big over X. By the relative version of Kawamata–Viehweg theorem [20, Theorem 1-2-5], we have a surjection

$$H^0(L_W + P_W) \to H^0((L_W + P_W)|_{E_W}). \tag{2.2}$$

We denote by Γ_{E_W} the pull-back of Γ_E to E_W. Observe that we have

$$(L_W + P_W)|_{E_W} \sim G_{E_W} := n\Gamma_{E_W} + n\Delta_{E_W} - \lfloor (n+1)\Delta_{E_W} \rfloor + P_{E_W},$$

where $P_{E_W} := P_W|_{E_W}$ and $\Delta_{E_W} := \Delta_W|_{E_W}$. The divisor G_{E_W} is integral and its coefficients are strictly greater than -1, therefore it is indeed effective. By the surjectivity of (2.2), there exists $0 \leq G_W \sim L_W + P_W$ which restricts to G_{E_W}. We denote by G_Y the push-forward of G_W to Y. By pushing-forward the linear equivalence $L_W + P_W \sim G_W$ to Y, and using the fact that P_W is Y-exceptional, we get that

$$0 \leq G_Y \sim -n(K_Y + \Delta_Y + E).$$

We define $\Gamma_Y := \frac{G_Y}{n}$. Observe that by construction, we have

$$n(K_Y + \Delta_Y + \Gamma_Y + E) \sim 0$$

on a neighbourhood of $x \in X$. We claim that $\Gamma_Y|_E = \Gamma_E$. Indeed, observe that we can define

$$n\Gamma_W := G_W - P_W + \lfloor (n+1)\Delta_W \rfloor - n\Delta_W \sim nN_W \sim_{\mathbb{Q},Y} 0,$$

and $n\Gamma_W$ pushes-forward to G_Y on Y, hence $\Gamma_W = \pi_Y^*(\Gamma_Y)$. On the other hand, we have $n\Gamma_{E_W} = n\Gamma_W|_{E_W}$, which means that $\Gamma_{E_W} = \Gamma_W|_{E_W}$. Thus, we have $\Gamma_Y|_E = \Gamma_E$ as claimed.

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Finally, observe that

$$(K_Y + \Delta_Y + \Gamma_Y + E)|_E = K_W + \Delta_E + \Gamma_E,$$

so by inversion of adjunction, we conclude that $(Y, \Delta_Y + \Gamma_Y + E)$ is ϵ-plt. Moreover, since Γ_E and Δ_E prime components, then Δ_Y and Γ_Y do not share prime components as well. Define $\Gamma = \pi_*(\Gamma_Y)$, and observe that

$$\pi^*(K_X + \Delta + \Gamma) = K_Y + \Delta_Y + \Gamma_Y + E.$$

Thus, $(X, \Delta + \Gamma)$ is a log canonical pair with a unique log canonical place so that $n(K_X + \Delta + \Gamma) \sim 0$ on a neighbourhood of $x \in X$. Moreover, Δ and Γ do not share prime components. □

2.5. Examples

In this subsection, we give two examples to show that the a-log canonical and ϵ-plt blow-up conditions of Theorem 1 and Theorem 2 are indeed necessary.

Example 2.33. Let X_n be the cone over a rational curve of degree n. Blowing-up the vertex $\pi_n : Y_n \to X_n$ gives a log resolution so that the pair (Y_n, E_n) is log smooth. Hence, π_n is a 1-plt blow-up. However, $a_{E_n}(X_n, 0) = -1 + \frac{2}{n}$, and the Cartier index of X_n at the vertex depends on n. In this case, the condition of Theorem 1 that is violated, is the fact that the minimal log discrepancy is bounded away from zero. Indeed, the discrepancies $a_{E_n}(X_n, 0)$ are converging to -1.

Example 2.34. By [14, Proposition 5.1], we can construct terminal threefold singularities X_m of index m and extract two different divisors with discrepancies $1/m$ and $2/m$, respectively. Hence, if there is any plt blow-up of X_m it is an ϵ-plt blow-up for some $\epsilon \leq 2/m$. In this case, the condition of Theorem 1 that is violated, is the existence of a plt blow-up with total discrepancy bounded away from zero.

3. Proof of the main Theorem

Proof of Theorem 1. Let (X, Δ) be a log pair which admits an ϵ-plt blow-up $\pi : Y \to X$ at $x \in X$. By assumption (X, Δ) is a-lc at $x \in X$. In particular, the minimal log discrepancy $\mathrm{mld}_x(X, \Delta)$ is strictly positive.

Let $\phi : X' \to X$ be the index one cover of the klt pair (X, Δ) locally at x so that $\phi^*(K_X + \Delta) = K_{X'}$ (see [12, Notation 4.1] or Definition 2.30). We denote by Y' the normalization of the main component of $X' \times_X Y$. Hence, we have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\phi_Y} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\phi} & X \end{array}$$

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where π' is birational and ϕ_Y is finite with the same degree as ϕ. We can write

$$\phi^*(K_X + \Delta) = K_{X'} \quad \text{and} \quad \phi_Y^*(K_Y + \Delta_Y + E) = K_{Y'} + E',$$

where E' is the reduced exceptional divisor contracted by π'. We claim that the pair (Y', E') is ϵ-plt. Indeed, by Lemma 2.32, we may find an effective divisor Γ on X so that

$$\pi^*(K_X + \Delta + \Gamma) = K_Y + \Delta_Y + \Gamma_Y + E \tag{3.1}$$

is an ϵ-plt pair, where Γ_Y is the strict transform of Γ on Y. By the equality (3.1), we know that log canonical places of $(X, \Delta + \Gamma)$ are the same as log canonical places of $(Y, \Delta_Y + \Gamma_Y + E)$, i.e., the morphism is crepant for these pairs. The pair $(Y, \Delta_Y + \Gamma_Y + E)$ has a unique log canonical place which corresponds to E. Hence, $(X, \Delta + \Gamma)$ has a unique log canonical place which corresponds to E. By construction, the divisors Δ and Γ do not share prime components. Therefore, by Lemma 2.27 and Lemma 2.29, we conclude that

$$K_{X'} + \Gamma_{X'} = \phi^*(K_X + \Delta + \Gamma)$$

is indeed a pair which has a unique log canonical place and its log discrepancies are either zero or greater than ϵ. By the commutativy of the diagram, we have that

$$\pi'^*(K_{X'} + \Gamma_{X'}) = K_{Y'} + \Gamma_{Y'} + E'$$

is an ϵ-plt pair. Hence (Y', E') is ϵ-plt as well. In particular, $\pi' \colon Y' \to X'$ is an ϵ-plt blow-up at $x' \in X'$.

Observe that by construction the $\mathbb{Q}$-divisors $K_Y + \Delta_Y + E$ and $K_{Y'} + E'$ are both ϵ-plt and anti-ample over X and X', respectively. We define the following pairs by adjunction:

$$K_{E'} + \Delta_{E'} = (K_{Y'} + E')|_{E'} \text{ and } K_E + \Delta_E = (K_Y + \Delta_Y + E)|_E.$$

By the above considerations, we know that both pairs $(E', \Delta_{E'})$ and (E, Δ_E) are log Fano and ϵ-lc. By [6, Theorem 1.1], we know that the algebraic varieties E and E' belong to a bounded family which only depends on $d-1$ and ϵ. Moreover, the boundary divisors Δ_E and $\Delta_{E'}$ have coefficients that belong to a set with the descending chain condition. By Lemma 2.18, we conclude that the log pairs (E, Δ_E) and $(E', \Delta_{E'})$ belong to a log bounded family which only depends on $d-1$, ϵ, and the derived set of standard coefficients. Furthermore, by [7, Lemma 3.3], we know that the coefficients of Δ_E and $\Delta_{E'}$ belong to a set of hyperstandard coefficients $\mathcal{H}(\mathcal{R})$ corresponding to a finite set of rational numbers $\mathcal{R}$, which only depends on $\mathcal{S}$. By the ϵ-log canonical condition of the pairs (E, Δ_E) and $(E', \Delta_{E'})$ and the fact that the only accumulation point of $\mathcal{H}(\mathcal{R})$ is 1, we conclude that Δ_E and $\Delta_{E'}$ have coefficients in the finite set $\mathcal{H}(\mathcal{R}) \cap [0, 1-\epsilon)$ which only depends on ϵ. By Lemma 2.16, we deduce that the pairs (E, Δ_E) and $(E', \Delta_{E'})$ belong to a strictly log bounded family. Denote by ϕ_E the restriction of ϕ_Y to E' and observe that

$$\deg(\phi) = \deg(\phi_Y) = \deg(\phi_E) r_E,$$

where r_E denotes the ramification index of ϕ_Y at the generic point of E'.

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Now we turn to prove that $\deg(\phi)$ has an upper bound which only depends on d, a and ϵ. In order to do so, we just need to provide that there is an upper bound for ϕ_E and r_E. Observe that we have

$$\deg(\phi_E) = \frac{\mathrm{vol}\,(-(K_{E'} + \Delta_{E'}))}{\mathrm{vol}\,(-(K_E + \Delta_E))}.$$

Therefore, by Lemma 2.21, there is an upper bound $C(d, \epsilon)$ for $\deg(\phi_E)$, which only depends on $d-1$ and ϵ. On the other hand, we have the relation

$$a_{E'}(X', 0) + 1 = r_E(a_E(X, \Delta) + 1) \geq r_E a.$$

By Lemma 2.23, we know that $a_{E'}(X',\, 0) + 1$ has an upper bound $C'(d,\, \epsilon)$ which only depends on d and ϵ. We conclude that $deg(\phi)$ has an upper bound which only depends on d, a and ϵ. Indeed, we have that

$$\deg(\phi) = \deg(\phi_E) r_E \leq C(d, \epsilon) \frac{a_{E'}(X', 0) + 1}{a} \leq C(d, \epsilon) C'(d, \epsilon) a^{-1}. \tag{3.2}$$

This proves the claim that $\deg(\phi)$ is bounded above by a constant which only depends on d, a and ϵ. Hence, the index of $K_X + \Delta$ around x is bounded by a constant which only depends on d, a and ϵ (see, e.g. [22, 2.49]).

Thus, $\mathrm{mld}_x(X,\, \Delta)$ belongs to a discrete set which only depends on d, a and ϵ. Finally, by Lemma 2.29, we have that

$$\deg(\phi)\mathrm{mld}_x(X, \Delta) \leq \mathrm{mld}_{x'}(X', 0) \leq a_{E'}(X', 0) + 1,$$

therefore $\mathrm{mld}_x(X,\, \Delta)$ belongs to a finite set which only depends on d, a and ϵ. □

Proof of Theorem 2. It follows from the bound on $\deg(\phi)$ given in the proof of Theorem 1. See equation (3.2). □

Proof of Corollary 1. If there exists a sequence in $\mathcal{M}(d,\, \mathcal{S})_{0,\epsilon}$ which contradicts the ascending chain condition, passing to a subsequence we may assume the sequence is strictly increasing. Therefore, such infinite sequence belongs to $\mathcal{M}(d,\, \mathcal{S})_{a,\epsilon}$ for some positive real number a. This contradicts Theorem 1. □

Proof of Corollary 2. The proof follows from Theorem 1 and Lemma 2.14. □

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