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SOME APPLICATIONS OF MODULAR UNITS

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Abstract We show that a weakly holomorphic modular function can be written as a sum of modular units of higher level. Furthermore, we find a necessary and sufficient condition for a meromorphic Siegel modular function of degree g to have neither a zero nor a pole on a certain subset of the Siegel upper half-space $\mathbb{H}_g$.

Keywords: modular units; modular functions; Siegel modular forms

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1. Introduction

Let g be a positive integer. We let

$$\mathbb{H}_g = \{Z \in \mathrm{Mat}_g(\mathbb{C}) \mid Z^{\mathrm{T}} = Z,\ \mathrm{Im}(Z) \text{ is positive definite}\}$$

be the Siegel upper half-space of degree g on which the symplectic group

$$\mathrm{Sp}_g(\mathbb{Z}) = \{\gamma \in \mathrm{GL}_{2g}(\mathbb{Z}) \mid \gamma^{\mathrm{T}} J \gamma = J\} \quad \text{with } J = \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}$$

acts by the rule

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}(Z) = (AZ + B)(CZ + D)^{-1},$$

where A, B, C and D are $g \times g$ block matrices. For a positive integer N we furthermore let

$$\Gamma(N) = \{\gamma \in \mathrm{Sp}_g(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod N\}$$

be the principal congruence subgroup of level N of the group $\mathrm{Sp}_g(\mathbb{Z})$. In particular, when $g = 1$, $\mathbb{H}_g$ becomes the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ and $\mathrm{Sp}_g(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ acts on it by fractional linear transformations.

Define a subset $\mathbb{H}_g^{\rm diag}$ of $\mathbb{H}_g$ by

$$\mathbb{H}_g^{\rm diag} = \{\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g) \mid \tau_1, \tau_2, \ldots, \tau_g \in \mathbb{H}\},$$

where $\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g)$ stands for the $g \times g$ diagonal matrix whose diagonal entries are $\tau_1, \tau_2, \ldots, \tau_g$. If $g = 1$, then $\mathbb{H}_g^{\rm diag}$ is nothing but $\mathbb{H}$. Let $f(Z)$ be a meromorphic Siegel modular function of degree g and level N (over $\mathbb{C}$); $f(Z)$ is the quotient of two Siegel modular forms of degree g and the same weight so that it is invariant under $\Gamma(N)$. When $g = 1$, f becomes a usual meromorphic modular function of level N. We shall mainly consider the case in which f has neither a zero nor a pole on $\mathbb{H}_g^{\rm diag}$.

Let $X(N) = \bar{\Gamma}(N)\backslash\mathbb{H}^*$ be the modular curve of level N that is a compact Riemann surface, where $\bar{\Gamma}(N) = \pm\Gamma(N)/\{\pm I_2\}$ and $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\mathrm{i}\infty\}$. We denote its function field by $\mathbb{C}(X(N))$. As is well known, $X(1)$ is of genus zero and $\mathbb{C}(X(1)) = \mathbb{C}(j)$, where

$$j = j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots \quad (q = \mathrm{e}^{2\pi \mathrm{i}\tau},\ \mathrm{i} = \sqrt{-1})$$

is the elliptic modular function [**9**, Theorem 2.9]. Furthermore, $\mathbb{C}(X(N))$ is a Galois extension of $\mathbb{C}(X(1))$ whose Galois group is naturally isomorphic to $\bar{\Gamma}(1)/\bar{\Gamma}(N)$. Let $\mathcal{O}_N$ be the integral closure of $\mathbb{C}[j]$ in $\mathbb{C}(X(N))$. We call the invertible elements in $\mathcal{O}_N$ *modular units* of level N (over $\mathbb{C}$), and these are precisely those functions in $\mathbb{C}(X(N))$ having neither zeros nor poles on $\mathbb{H}$ [**7**, p. 36]. Kubert and Lang [**7**] developed the theory of modular units in terms of Siegel functions, which will be defined in § 2. (In addition, they require that the Fourier coefficients of a modular unit of level N lie in the Nth cyclotomic field.) In this paper we first describe $\mathcal{O}_N$ in view of modular units as follows. If $N \equiv 0 \pmod 4$, then the ring $\mathcal{O}_N$ is generated over $\mathbb{C}$ by the following five modular units:

$$g_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(4\tau)^{-8} g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(4\tau)^{8}, \qquad g_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(4\tau)^{8} g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(4\tau)^{-8},$$
$$\left(g_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(4\tau)^{-8} g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(4\tau)^{8} - 16\right)^{-1},$$
$$\left(\wp_{\left[\begin{smallmatrix}0\\1/N\end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)\right) \Big/ \left(\wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau)\right),$$
$$\left(\wp_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)\right) \Big/ \left(\wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau)\right),$$

where $g_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(\tau)$ is a Siegel function and $\wp_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(\tau)$ is a Weierstrass $\wp$-function for $\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right] \in \mathbb{Q}^2 - \mathbb{Z}^2$ (see Theorem 3.3). We then conclude that any weakly holomorphic modular function can be expressed as a sum of modular units of higher level (see Corollary 3.5). Here, a function is said to be weakly holomorphic if it is holomorphic on $\mathbb{H}$.

On the other hand, suppose that g and N are two positive integers greater than or equal to 2 and let $f(Z)$ be a meromorphic Siegel modular function of degree g and level N. We furthermore prove that $f(Z)$ has neither a zero nor a pole on $\mathbb{H}_g^{\rm diag}$ if and only if $f(\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g))$ is a product of g modular units of variables $\tau_1, \tau_2, \ldots, \tau_g \in \mathbb{H}$ (see Theorem 4.2). To this end, we examine some necessary basic properties of modular units in § 2 and we show that a certain quotient of theta constants of degree g on $\mathbb{H}_g^{\rm diag}$ is a product of modular units (see Example 4.3).

2. Properties of modular units

For a positive integer N we denote the group of all modular units of level N by V_N (that is, $V_N = \mathcal{O}_N^\times$), which contains $\mathbb{C}^\times$ as a subgroup. In this section we develop some necessary properties about modular units that will be used in later sections.

Lemma 2.1. *If f is a weakly holomorphic modular function of level 1, then it is a polynomial in j over $\mathbb{C}$. That is, we have $f \in \mathbb{C}[j]$.*

Proof. See [**8**, Chapter 5, Theorem 2]. □

Remark 2.2. Note that j gives rise to a bijection $j\colon \bar{\Gamma}(1)\backslash\mathbb{H} \to \mathbb{C}$ [**8**, Chapter 3, Theorem 4].

Proposition 2.3. *Let $h \in \mathbb{C}(X(N))$. Then h is weakly holomorphic if and only if h is integral over $\mathbb{C}[j]$.*

Proof. Assume that $h = h(\tau)$ is weakly holomorphic. We consider the following monic polynomial in X,

$$P(X) = \prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} (X - h \circ \gamma).$$

Since $\mathrm{Gal}(\mathbb{C}(X(N))/\mathbb{C}(X(1))) \simeq \bar{\Gamma}(1)/\bar{\Gamma}(N)$, every coefficient of $P(X)$ belongs to $\mathbb{C}(X(1))$ and is holomorphic on $\mathbb{H}$. So, it is a polynomial in j over $\mathbb{C}$ by Lemma 2.1. This shows that h is integral over $\mathbb{C}[j]$.

Conversely, assume that h is integral over $\mathbb{C}[j]$. Then h is a zero of a monic polynomial

$$X^n + P_{n-1}(j)X^{n-1} + \cdots + P_1(j)X + P_0(j),$$

where $n \geqslant 1$ and $P_{n-1}(j), \dots, P_1(j), P_0(j) \in \mathbb{C}[j]$. Suppose on the contrary that h has a pole at $\tau_0 \in \mathbb{H}$ (so, $h \neq 0$). Since h satisfies

$$h^n + P_{n-1}(j)h^{n-1} + \cdots + P_1(j)h + P_0(j) = 0,$$

we obtain, by dividing both sides by h^n and substituting $\tau = \tau_0$,

$$1 + P_{n-1}(j(\tau_0))(1/h(\tau_0)) + \cdots + P_1(j(\tau_0))(1/h(\tau_0))^{n-1} + P_0(j(\tau_0))(1/h(\tau_0))^n = 0.$$

This yields the contradiction $1 = 0$ because $j(\tau_0) \in \mathbb{C}$ and $1/h(\tau_0) = 0$. Therefore, h must be weakly holomorphic. □

Remark 2.4. By definition, $h \in \mathbb{C}(X(N))$ is a modular unit if and only if both h and h^{-1} are integral over $\mathbb{C}[j]$. Hence, Proposition 2.3 gives an elementary proof of the well-known fact that h is a modular unit if and only if it has neither a zero nor a pole on $\mathbb{H}$ [**7**, p. 36].

Given a vector $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ for $N \geqslant 2$, the Siegel function $g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau)$ is defined on $\mathbb{H}$ by the infinite product

$$g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau) = -q^{(1/2)(r^2-r+1/6)}\mathrm{e}^{\pi \mathrm{i} s(r-1)}(1-q^r\mathrm{e}^{2\pi \mathrm{i} s})\prod_{n=1}^{\infty}(1-q^{n+r}\mathrm{e}^{2\pi \mathrm{i} s})(1-q^{n-r}\mathrm{e}^{-2\pi \mathrm{i} s}), \quad (2.1)$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$. It is a weakly holomorphic modular function of level $12N^2$ [**7**, Chapter 3, Theorem 5.2].

Lemma 2.5. *Suppose that $N \geqslant 2$ and let n be the number of inequivalent cusps of $X(N)$. Then the rank of the subgroup of $V_N/\mathbb{C}^\times$ generated by $g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau)^{12N}$ for $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ is $n-1$.*

Proof. See [**7**, Chapter 2, Theorem 3.1]. □

Remark 2.6. We have the formula

$$n = |\bar{\Gamma}(1)/\bar{\Gamma}(N)|/N = \begin{cases} 3 & \text{if } N = 2, \\ \dfrac{N^2}{2}\displaystyle\prod_{p|N}(1-p^{-2}) & \text{if } N > 2 \end{cases}$$

(see [**9**, pp. 22–23]).

Proposition 2.7. *With the same assumption and notation as in Lemma 2.5, $V_N/\mathbb{C}^\times$ is a free abelian group of rank $n-1$.*

Proof. Let $\infty_1, \infty_2, \ldots, \infty_n$ be the inequivalent cusps of $X(N)$ and let $\mathcal{D}_N$ be the free abelian group of rank n generated by these cusps. An element of $\mathcal{D}_N$ is then uniquely written as

$$m_1(\infty_1) + m_2(\infty_2) + \cdots + m_n(\infty_n) \quad \text{for some integers } m_1, m_2, \ldots, m_n.$$

Now, we consider a (well-defined) injective homomorphism

$$\begin{aligned} V_N/\mathbb{C}^\times &\to \mathcal{D}_N \\ h &\mapsto \operatorname{div}(h). \end{aligned}$$

If $h \in V_N/\mathbb{C}^\times$ has $\operatorname{div}(h) = \sum_{k=1}^{n} m_k(\infty_k)$, then we get the relation $\sum_{k=1}^{n} m_k = 0$. Hence, $V_N/\mathbb{C}^\times$ is a free abelian group of rank less than or equal to $n-1$. Thus, it follows from Lemma 2.5 that the rank of $V_N/\mathbb{C}^\times$ is exactly $n-1$. □

Remark 2.8. Since every cusp of $X(1)$ is equivalent to $\mathrm{i}\infty$ [**9**, p. 14], if $h \in V_1$, then $\operatorname{div}(h) = m(\mathrm{i}\infty)$ for some integer m. On the other hand, now that the sum of the orders of zeros and poles of h is zero, we get $m = 0$. This yields $V_1 = \mathbb{C}^\times$.

Lemma 2.9. *Let $N \geqslant 2$ and let $h \in V_N - \mathbb{C}^\times$. There is a finite subset S of $\mathbb{C}^\times$ such that the map*

$$\begin{aligned} \varphi\colon \mathbb{H} &\to \mathbb{C}^\times - S \\ \tau &\mapsto h(\tau) \end{aligned}$$

is surjective.

Proof. Consider the following holomorphic map between compact Riemann surfaces

$$\begin{aligned} X(N) &\to \mathbb{P}^1(\mathbb{C}) \\ \tau &\mapsto [h(\tau) : 1]. \end{aligned}$$

Since h is not a constant, the above map is surjective. Take a subset S of $\mathbb{C}^\times$ as

$$S = \{h(\tau) \mid \tau \text{ is a cusp of } X(N)\} - \{0, \infty, h(\tau) \mid \tau \in \mathbb{H}\}.$$

Since there are only finitely many inequivalent cusps of $X(N)$, it is a finite set, and therefore the map φ becomes surjective. □

Let h be a non-zero modular function. Considering h as a Laurent series with respect to $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, we denote its smallest exponent by $\mathrm{ord}_q h$ (in $\mathbb{Q}$).

Proposition 2.10. *Let h be a modular unit. Suppose that*

$$\mathrm{ord}_q h \circ \gamma \neq 0 \quad \textit{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.2}$$

Then $h - c$ is not a modular unit for any $c \in \mathbb{C}^\times$.

Proof. Let us consider the holomorphic map between two compact Riemann surfaces

$$\begin{aligned} \varphi\colon X(N) &\to \mathbb{P}^1(\mathbb{C}) \\ \tau &\mapsto [h(\tau) : 1]. \end{aligned}$$

Since h is not a constant by (2.2), φ is surjective.

Now, let $c \in \mathbb{C}^\times$. Since φ is surjective and the values of φ at the cusps of $X(N)$ are either $[0 : 1]$ or $[\infty : 1] = [1 : 0]$ by (2.2), there exists $\tau_0 \in \mathbb{H}$ such that $\varphi(\tau_0) = [c : 1]$. This implies that $h(\tau) - c$ has a zero at $\tau = \tau_0$, and hence $h - c$ is not a modular unit. □

Example 2.11. Let $N \geqslant 2$ and $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$. Consider the Siegel function

$$h(\tau) = g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau)^{12N},$$

which is a modular unit of level N by Lemma 2.5. We then have the following properties.

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(i) $h \circ \gamma = g_{\gamma^{\mathrm{T}}\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau)^{12N}$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where γ^{T} indicates the transpose of γ [**7**, Chapter 2, Proposition 1.3].

(ii) $\mathrm{ord}_q h = 6N\mathrm{B}_2(\langle r \rangle)$, where $\mathrm{B}_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial and $\langle x \rangle$ is the fractional part of x such that $0 \leqslant \langle x \rangle < 1$ for $x \in \mathbb{R}$ [**7**, p. 31].

(iii) $\mathrm{B}_2(x) \neq 0$ for all $x \in \mathbb{Q}$.

Thus, h satisfies the assumption (2.2) in Proposition 2.10.

Remark 2.12. If h does not satisfy (2.2), then $h - c$ could be a modular unit for some constant $c \in \mathbb{C}^\times$ (see Remark 3.4).

3. Integral closures in modular function fields

In this section, when $N \equiv 0 \pmod 4$ we investigate explicit generators of the integral closure $\mathcal{O}_N$ of $\mathbb{C}[j]$ in $\mathbb{C}(X(N))$ by using Weierstrass units.

For a lattice $L = [\omega_1, \omega_2] = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$, the Weierstrass $\wp$-function is defined by

$$\wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \quad (z \in \mathbb{C}).$$

Lemma 3.1. *Let $z, w \in \mathbb{C} - L$. Then, $\wp(z; L) = \wp(w; L)$ if and only if $z \equiv \pm w \pmod L$.*

Proof. See [**11**, Chaper IV, § 3]. □

Let $N \geqslant 2$. For a vector $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ we define

$$\wp_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau) = \wp(r\tau + s; [\tau, 1]) \quad (\tau \in \mathbb{H}),$$

which is a weakly holomorphic modular form of level N and weight 2 [**8**, Chapter 6]. More precisely, it satisfies the transformation formula

$$\left(\wp_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]} \circ \gamma\right)(\tau) = (c\tau + d)^2 \wp_{\gamma^{\mathrm{T}}\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau) \quad \text{for any } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{3.1}$$

Hence, the function

$$\left(\wp_{\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} c_1 \\ d_1 \end{smallmatrix}\right]}(\tau)\right) \Big/ \left(\wp_{\left[\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} c_2 \\ d_2 \end{smallmatrix}\right]}(\tau)\right)$$

for $\left[\begin{smallmatrix} a_k \\ b_k \end{smallmatrix}\right], \left[\begin{smallmatrix} c_k \\ d_k \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ with $\left[\begin{smallmatrix} a_k \\ b_k \end{smallmatrix}\right] \not\equiv \pm\left[\begin{smallmatrix} c_k \\ d_k \end{smallmatrix}\right] \pmod{\mathbb{Z}^2}$ $(k = 1, 2)$ is a modular unit of level N by Lemma 3.1, which is called a Weierstrass unit of level N.

We further define three functions on $\mathbb{H}$,

$$g_2(\tau) = 60 \sum_{\omega \in [\tau, 1] - \{0\}} \omega^{-4},$$

$$g_3(\tau) = 140 \sum_{\omega \in [\tau, 1] - \{0\}} \omega^{-6},$$

$$\Delta(\tau) = g_2(\tau)^3 - 27 g_3(\tau)^2,$$

which are modular forms of level 1 and weight 4, 6 and 12, respectively [**8**, Chapter 3, Theorem 3].

For a positive integer N, let

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod N \right\},$$

and let $X_1(N) = \bar{\Gamma}_1(N)\backslash\mathbb{H}^*$ be the corresponding modular curve, where $\bar{\Gamma}_1(N) = \pm\Gamma_1(N)/\{\pm I_2\}$.

Lemma 3.2.

(i) *If* $N \geqslant 2$, *then* $\mathbb{C}(X_1(N)) = \mathbb{C}\Big(j, (g_2g_3/\Delta)\wp_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}\Big)$.

(ii) *If* $N \geqslant 2$, *then* $\mathbb{C}(X(N)) = \mathbb{C}(X_1(N))\Big((g_2g_3/\Delta)\wp_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}\Big)$.

(iii) $\mathbb{C}(X_1(4)) = \mathbb{C}\Big(g_{\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(4\tau)^{-8} g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(4\tau)^{8}\Big)$.

Proof. See [**2**, Proposition 7.5.1] and [**6**, Table 2]. □

The modular curve $X_1(4)$ is of genus 0 and has three inequivalent cusps, namely, 0, 1/2 and i∞ [**5**, p. 131]. Set

$$g_{1,4}(\tau) = g_{\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(4\tau)^{-8} g_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(4\tau)^{8},$$

which is a primitive generator of $\mathbb{C}(X_1(4))$ over $\mathbb{C}$ by Lemma 3.2 (iii). It then follows from [**6**, Theorem 6.5] that the map

$$\begin{aligned} X_1(4) = \bar{\Gamma}_1(4)\backslash\mathbb{H}^* &\to \mathbb{P}^1(\mathbb{C}) \\ \tau &\mapsto [g_{1,4}(\tau) : 1] \end{aligned}$$

is an isomorphism between compact Riemann surfaces. Moreover, $g_{1,4}(\tau)$ has values 16, 0 and ∞ at the cusps $\tau = 0$, 1/2 and i∞, respectively (see [**5**, Theorem 3 (ii)] and [**6**, Table 3]). Thus, we claim that

$$g_{1,4} - c \text{ for } c \in \mathbb{C} \text{ is a modular unit (for } \Gamma_1(4)) \iff c = 16 \text{ or } 0. \tag{3.2}$$

Theorem 3.3. *Let* $\mathcal{O}_{1,N}$ *and* $\mathcal{O}_N$ *be the integral closures of* $\mathbb{C}[j]$ *in* $\mathbb{C}(X_1(N))$ *and* $\mathbb{C}(X(N))$, *respectively. Assume that* $N \equiv 0 \pmod 4$.

(i) $\mathcal{O}_{1,4} = \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4} - 16)^{-1}]$.

(ii) $\mathcal{O}_{1,N} = \mathcal{O}_{1,4}[h_{1,N}]$, *where*

$$h_{1,N}(\tau) = \Big(\wp_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)\Big) \Big/ \Big(\wp_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} 0 \\ 1/4 \end{smallmatrix}\right]}(\tau)\Big).$$

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(iii) $\mathcal{O}_N = \mathcal{O}_{1,N}[h_N]$, *where*

$$h_N(\tau) = \Big(\wp_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)\Big)\Big/\Big(\wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau)\Big).$$

Proof. (i) Since $g_{1,4}$ and $g_{1,4}-16$ are modular units in $\mathbb{C}(X_1(4))$ by Lemma 3.2 (iii) and (3.2), we get the inclusion $\mathcal{O}_{1,4} \supseteq \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4}-16)^{-1}]$. Conversely, let $h \in \mathcal{O}_{1,4}$. It is then a rational function of $g_{1,4}$ by Lemma 3.2 (iii), namely, $h = P(g_{1,4})/Q(g_{1,4})$ for some polynomials $P(X), Q(X) \in \mathbb{C}[X]$ that are relatively prime. If $Q(X)$ has a linear factor other than $g_{1,4}$ and $g_{1,4}-16$, then h has a pole on $\mathbb{H}$ by (3.2). Hence, we obtain the reverse inclusion $\mathcal{O}_{1,4} \subseteq \mathbb{C}[g_{1,4}, g_{1,4}^{-1}, (g_{1,4}-16)^{-1}]$. This proves (i).

(ii) Since $h_{1,N} \in \mathcal{O}_{1,N}$ by Lemma 3.2 (i) and the paragraph below Lemma 3.1, we have the inclusion $\mathcal{O}_{1,N} \supseteq \mathcal{O}_{1,4}[h_{1,N}]$.

We find that

$$\begin{aligned}\mathbb{C}(X_1(N)) &= \mathbb{C}\Big(j, (g_2g_3/\Delta)\wp_{\left[\begin{smallmatrix}0\\1/N\end{smallmatrix}\right]}\Big) \quad \text{by Lemma 3.2 (i)}\\ &= \mathbb{C}(X_1(4))\Big((g_2g_3/\Delta)\wp_{\left[\begin{smallmatrix}0\\1/N\end{smallmatrix}\right]}\Big) \quad \text{because } j \in \mathbb{C}(X_1(4))\\ &= \mathbb{C}(X_1(4))\Big((g_2g_3/\Delta)\Big(\Big(\wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]} - \wp_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}\Big)h_{1,N} + \wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}\Big)\Big)\\ &= \mathbb{C}(X_1(4))(h_{1,N}),\end{aligned}$$

since

$$(g_2g_3/\Delta)\wp_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}, (g_2g_3/\Delta)\wp_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]} \in \mathbb{C}(X_1(4)),$$

by Lemma 3.2 (i). So, if $f \in \mathcal{O}_{1,N}$, then it can be written in the form

$$f = r_0 + r_1h + r_2h^2 + \cdots + r_{d-1}h^{d-1}, \tag{3.3}$$

where $h = h_{1,N}$, $d = [\mathbb{C}(X_1(N)) : \mathbb{C}(X_1(4))]$ and $r_0, r_2, \ldots, r_{d-1} \in \mathbb{C}(X_1(4))$. Multiplying both sides of (3.3) by $1, h, \ldots, h^{d-1}$, we obtain a linear system (with unknowns $r_0, r_1, \ldots, r_{d-1}$)

$$\begin{bmatrix} 1 & h & \cdots & h^{d-1}\\ h & h^2 & \cdots & h^d\\ \vdots & \vdots & \ddots & \vdots\\ h^{d-1} & h^d & \cdots & h^{2d-2}\end{bmatrix}\begin{bmatrix} r_0\\ r_1\\ \vdots\\ r_{d-1}\end{bmatrix} = \begin{bmatrix} f\\ hf\\ \vdots\\ h^{d-1}f\end{bmatrix}.$$

Taking the trace Tr (equal to $\mathrm{Tr}_{\mathbb{C}(X_1(N))/\mathbb{C}(X_1(4))}$) on both sides, we achieve

$$\begin{bmatrix} \mathrm{Tr}(1) & \mathrm{Tr}(h) & \cdots & \mathrm{Tr}(h^{d-1})\\ \mathrm{Tr}(h) & \mathrm{Tr}(h^2) & \cdots & \mathrm{Tr}(h^d)\\ \vdots & \vdots & \ddots & \vdots\\ \mathrm{Tr}(h^{d-1}) & \mathrm{Tr}(h^d) & \cdots & \mathrm{Tr}(h^{2d-2})\end{bmatrix}\begin{bmatrix} r_0\\ r_1\\ \vdots\\ r_{d-1}\end{bmatrix} = \begin{bmatrix} \mathrm{Tr}(f)\\ \mathrm{Tr}(hf)\\ \vdots\\ \mathrm{Tr}(h^{d-1}f)\end{bmatrix}. \tag{3.4}$$

Let T be the $d \times d$ matrix on the left-hand side of (3.4) and let $c_1, c_2, \dots, c_d$ be the conjugates of $h \in \mathbb{C}(X_1(N))$ over $\mathbb{C}(X_1(4))$. We then obtain that

$$\begin{aligned}\det(T) &= \begin{vmatrix} \sum_{k=1}^{d} c_k^0 & \sum_{k=1}^{d} c_k^1 & \cdots & \sum_{k=1}^{d} c_k^{d-1} \\ \sum_{k=1}^{d} c_k^1 & \sum_{k=1}^{d} c_k^2 & \cdots & \sum_{k=1}^{d} c_k^{d} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{d} c_k^{d-1} & \sum_{k=1}^{d} c_k^{d} & \cdots & \sum_{k=1}^{d} c_k^{2d-2} \end{vmatrix} \\ &= \begin{vmatrix} c_1^0 & c_2^0 & \cdots & c_d^0 \\ c_1^1 & c_2^1 & \cdots & c_d^1 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{d-1} & c_2^{d-1} & \cdots & c_d^{d-1} \end{vmatrix} \begin{vmatrix} c_1^0 & c_1^1 & \cdots & c_1^{d-1} \\ c_2^0 & c_2^1 & \cdots & c_2^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_d^0 & c_d^1 & \cdots & c_d^{d-1} \end{vmatrix} \\ &= \prod_{1 \leqslant m < n \leqslant d} (c_m - c_n)^2 \quad \text{by the Vandermonde determinant formula.}\end{aligned}$$

On the other hand, any conjugate of $h \in \mathbb{C}(X_1(N))$ over $\mathbb{C}(X_1(4))$ is of the form

$$\Big(\wp_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)\Big) \Big/ \Big(\wp_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} 0 \\ 1/4 \end{smallmatrix}\right]}(\tau)\Big) \quad \text{for some } \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{Z}^2 - N\mathbb{Z}^2$$

owing to the fact that $\mathrm{Gal}(\mathbb{C}(X_1(N))/\mathbb{C}(X_1(4))) \simeq \bar{\Gamma}_1(N)/\bar{\Gamma}_1(4)$, the transformation formula (3.1) and Lemma 3.1. Moreover, we see that the function

$$\Big(\wp_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} c/N \\ d/N \end{smallmatrix}\right]}(\tau)\Big) \Big/ \Big(\wp_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau) - \wp_{\left[\begin{smallmatrix} 0 \\ 1/4 \end{smallmatrix}\right]}(\tau)\Big)$$

for $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] \in \mathbb{Z}^2 - N\mathbb{Z}^2$ with $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \not\equiv \pm\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] \pmod{N\mathbb{Z}^2}$ has neither a zero nor a pole on $\mathbb{H}$ by Lemma 3.1. This implies that $\det(T)$ becomes a modular unit in $\mathbb{C}(X_1(4))$. In particular, $\det(T)$ belongs to $\mathcal{O}_{1,4}^{\times}$. It then follows that $r_0, r_1, \dots, r_{d-1} \in \mathcal{O}_{1,4}$, and hence we deduce the inclusion $\mathcal{O}_{1,N} \subseteq \mathcal{O}_{1,4}[h_{1,N}]$. This completes the proof of (ii).

(iii) One can readily prove (iii) through the use of Lemma 3.2 (ii) and the fact that $\mathrm{Gal}(\mathbb{C}(X(N))/\mathbb{C}(X_1(N))) \simeq \bar{\Gamma}(N)/\bar{\Gamma}_1(N)$. □

Remark 3.4. Let

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{\pi \mathrm{i}(n+1/2)^2 \tau}, \quad \theta_3(\tau) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} n^2 \tau} \quad \text{and} \quad \theta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{e}^{\pi \mathrm{i} n^2 \tau}$$

be the classical Jacobi theta functions and let

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{3.5}$$

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be the Dedekind eta function. They then satisfy the relations

$$\theta_2(\tau)^4 + \theta_4(\tau)^4 = \theta_3(\tau)^4 \tag{3.6}$$

and

$$\theta_2(2\tau) = 2\eta(4\tau)^2/\eta(2\tau) \quad \text{and} \quad \theta_4(2\tau) = \eta(\tau)^2/\eta(2\tau), \tag{3.7}$$

due to Jacobi [**1**, pp. 27–29]. Furthermore, we have

$$g_{1,4}(\tau) = 16\theta_3(2\tau)^4/\theta_2(2\tau)^4$$

as a modular unit with $\mathrm{ord}_q(g_{1,4} \circ [\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]) = 0$ [**6**, Table 3 and Theorem 6.2]. Hence, we derive that

$$\begin{aligned} g_{1,4}(\tau) - 16 &= 16\theta_3(2\tau)^4/\theta_2(2\tau)^4 - 16 \\ &= 16\theta_4(2\tau)^4/\theta_2(2\tau)^4 && \text{by (3.6)} \\ &= \eta(\tau)^8/\eta(4\tau)^8 && \text{by (3.7)} \\ &= q^{-1}\prod_{n=1}^{\infty}(1+q^n)^{-8}(1+q^{2n})^{-8} && \text{by the definition (3.5).} \end{aligned}$$

Therefore, $g_{1,4} - 16$ is indeed a modular unit.

Corollary 3.5. *Every weakly holomorphic modular function can be expressed as a sum of modular units (of higher level).*

Proof. Let h be a weakly holomorphic modular function of level N. Since it belongs to $\mathcal{O}_{4N/\gcd(4,N)}$ by Proposition 2.3, h can be written as a sum of modular units of level $4N/\gcd(4,N)$ by Theorem 3.3. This completes the proof. □

Let k and N (greater than or equal to 1) be integers. We denote the vector space of all weakly holomorphic modular forms of level N and weight k by $\mathcal{M}_k^!(\Gamma(N))$. We then have a graded algebra

$$\mathcal{M}^!(\Gamma(N)) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k^!(\Gamma(N))$$

with respect to weight k.

Now, define a Klein form

$$\mathfrak{k}_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau) = (1/2\pi\mathrm{i})g_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)/\eta(\tau)^2,$$

which belongs to $\mathcal{M}_{-1}^!(\Gamma(8))$ [**7**, Chapter 3, Theorem 4.1]. It has neither a zero nor a pole on $\mathbb{H}$ by the expansion formulae (2.1) and (3.5).

Theorem 3.6. *For* $N \equiv 0 \pmod 8$, *we get*

$$\mathcal{M}^!(\Gamma(N)) = \mathcal{O}_N\Big[\mathfrak{k}_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}, \mathfrak{k}^{-1}_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}\Big] = \mathbb{C}\Big[g_{1,4}, g_{1,4}^{-1}, (g_{1,4}-16)^{-1}, h_{1,N}, h_N, \mathfrak{k}_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}, \mathfrak{k}^{-1}_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}\Big],$$

where $g_{1,4}$, $h_{1,N}$ *and* h_N *are functions described in Theorem 3.3.*

Proof. It is obvious that $\mathcal{M}_0^!(\Gamma(N)) = \mathcal{O}_N$.
If $k \neq 0$, then the linear map

$$\begin{aligned}\varphi \colon \mathcal{O}_N &\to \mathcal{M}_k^!(\Gamma(N))\\ h &\mapsto \mathfrak{k}^{-k}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]} h\end{aligned}$$

is an isomorphism because

$$\mathfrak{k}^{-1}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]} \in \mathcal{M}_1^!(\Gamma(8)) \quad \text{and} \quad \mathfrak{k}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]} \in \mathcal{M}_{-1}^!(\Gamma(8)).$$

Thus, $\mathcal{M}_k^!(\Gamma(N)) = \mathfrak{k}^{-k}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]} \mathcal{O}_N$ as an $\mathcal{O}_N$-module. Therefore, we attain from Theorem 3.3

$$\begin{aligned}\mathcal{M}^!(\Gamma(N)) &= \bigoplus_{k\in\mathbb{Z}} \mathfrak{k}^{-k}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]} \mathcal{O}_N\\ &= \mathcal{O}_N\left[\mathfrak{k}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}, \mathfrak{k}^{-1}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}\right]\\ &= \mathbb{C}\left[g_{1,4}, g_{1,4}^{-1}, (g_{1,4}-16)^{-1}, h_{1,N}, h_N, \mathfrak{k}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}, \mathfrak{k}^{-1}_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}\right].\end{aligned}$$

□

4. Meromorphic Siegel modular functions

In this section we show that if $f(Z)$ is a meromorphic Siegel modular function of degree g (greater than or equal to 2) that has neither a zero nor a pole on $\mathbb{H}_g^{\mathrm{diag}}$, then $f(Z)$ is a product of g modular units.

Lemma 4.1. *Let $g, N \geqslant 2$ be integers. If $f(Z)$ is a meromorphic Siegel modular function of degree g and level N, then the function*

$$f(\mathrm{diag}(\tau_1, \tau_2, \dots, \tau_g)) \quad (\mathrm{diag}(\tau_1, \tau_2, \dots, \tau_g) \in \mathbb{H}_g^{\mathrm{diag}}),$$

as a function of τ_k ($k = 1, 2, \dots, g$), is a meromorphic modular function of level N.

Proof. Let

$$\gamma_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \quad (k = 1, 2, \dots, g)$$

and set

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathrm{diag}(a_1, a_2, \dots, a_g) & \mathrm{diag}(b_1, b_2, \dots, b_g) \\ \mathrm{diag}(c_1, c_2, \dots, c_g) & \mathrm{diag}(d_1, d_2, \dots, d_g) \end{bmatrix},$$

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where A, B, C and D are $g \times g$ block matrices. We then derive that

$$\begin{aligned}
\gamma^{\mathrm{T}} J \gamma &= \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{because } A,\, B,\, C \text{ and } D \text{ are diagonal} \\
&= \begin{bmatrix} CA - AC & CB - AD \\ DA - BC & DB - BD \end{bmatrix} \\
&= \begin{bmatrix} 0 & \mathrm{diag}(c_1 b_1 - a_1 d_1, \ldots, c_g b_g - a_g d_g) \\ \mathrm{diag}(d_1 a_1 - b_1 c_1, \ldots, d_g a_g - b_g c_g) & 0 \end{bmatrix} \\
&= J \quad \text{due to } \det(\gamma_k) = a_k d_k - b_k c_k = 1 \ (k = 1, 2, \ldots, g),
\end{aligned}$$

from which we see that γ belongs to the group $\mathrm{Sp}_g(\mathbb{Z})$.

Also, for $Z = \mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g) \in \mathbb{H}_g^{\mathrm{diag}}$, we achieve that

$$\begin{aligned}
\gamma(Z) &= (AZ + B)(CZ + D)^{-1} \\
&= \mathrm{diag}(a_1\tau_1 + b_1, \ldots, a_g\tau_g + b_g)\mathrm{diag}(c_1\tau_1 + d_1, \ldots, c_g\tau_g + d_g)^{-1} \\
&= \mathrm{diag}((a_1\tau_1 + b_1)(c_1\tau_1 + d_1)^{-1}, \ldots, (a_g\tau_g + b_g)(c_g\tau_g + d_g)^{-1}) \\
&= \mathrm{diag}(\gamma_1(\tau_1), \gamma_2(\tau_2), \ldots, \gamma_g(\tau_g)). \qquad (4.1)
\end{aligned}$$

On the other hand, assume that $\gamma_k \equiv I_2 \pmod N$ for all $k = 1, 2, \ldots, g$. Then $\gamma \equiv I_{2g} \pmod N$ and for $Z = \mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g) \in \mathbb{H}_g^{\mathrm{diag}}$ we have

$$\begin{aligned}
f(\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g)) &= f(Z) \\
&= f(\gamma(Z)) && \text{since } f \text{ is of level } N \\
&= f(\mathrm{diag}(\gamma_1(\tau_1), \gamma_2(\tau_2), \ldots, \gamma_g(\tau_g))) && \text{by (4.1).}
\end{aligned}$$

In particular, when k is fixed ($k = 1, 2, \ldots, g$) and $\gamma_n = I_2$ for all $n \neq k$, we conclude that $f(Z)$, as a function of τ_k, is a meromorphic modular function of level N. ☐

Theorem 4.2. *Let $g, N \geqslant 2$ be integers, and let $f(Z)$ be a meromorphic Siegel modular function of degree g and level N. Then, $f(Z)$ has neither a zero nor a pole on $\mathbb{H}_g^{\mathrm{diag}}$ if and only if there exist modular units $v_1(\tau), v_2(\tau), \ldots, v_g(\tau) \in V_N$ such that*

$$f(\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g)) = \prod_{k=1}^{g} v_k(\tau_k).$$

Proof. The proof of the 'if' part is clear.

Conversely, assume that $f(Z)$ has neither a zero nor a pole on $\mathbb{H}_g^{\mathrm{diag}}$. Let n (greater than or equal to 2) be the number of inequivalent cusps of $X(N)$. Since $V_N/\mathbb{C}^\times$ is a free abelian group of rank $n-1$ by Proposition 2.7, there exist $g_1(\tau), g_2(\tau), \ldots, g_{n-1}(\tau) \in V_N$ such that $V_N = \langle \mathbb{C}^\times, g_1, g_2, \ldots, g_{n-1} \rangle$. Thus, $f(\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g))$, as a function of τ_g, can be written as

$$f(\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_g)) = c(\tau_1, \tau_2, \ldots, \tau_{g-1}) \prod_{t=1}^{n-1} g_t(\tau_g)^{m_t(\tau_1, \tau_2, \ldots, \tau_{g-1})} \qquad (4.2)$$

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by Lemma 4.1 and on the assumption that $c\colon \mathbb{H}^{g-1} \to \mathbb{C}^\times$ and $m_t\colon \mathbb{H}^{g-1} \to \mathbb{Z}$ are functions of $\tau_1, \tau_2, \dots, \tau_{g-1}$.

We then deduce that

$$\begin{aligned}
&\prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} f(\operatorname{diag}(\tau_1, \tau_2, \dots, \tau_{g-1}, \gamma(\tau_g))) \\
&\quad = \prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} \left(c(\tau_1, \tau_2, \dots, \tau_{g-1}) \prod_{t=1}^{n-1} g_t(\gamma(\tau_g))^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \right) \quad \text{by (4.2)} \\
&\quad = c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} \left(\prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} g_t(\gamma(\tau_g)) \right)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \\
&\qquad\qquad \text{(where } d = |\bar{\Gamma}(1)/\bar{\Gamma}(N)|) \\
&\quad = c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} \mathrm{N}_{\mathbb{C}(X(N))/\mathbb{C}(X(1))}(g_t(\tau_g))^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \\
&\qquad\qquad \text{due to the fact } \mathrm{Gal}(\mathbb{C}(X(N))/\mathbb{C}(X(1))) \simeq \bar{\Gamma}(1)/\bar{\Gamma}(N) \\
&\quad = c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} c_t^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \\
&\qquad\qquad \text{for some } c_1, c_2, \dots, c_{n-1} \in \mathbb{C}^\times \text{ by Remark 2.8,}
\end{aligned}$$

which is a modular unit of level N as a function of each τ_k $(k = 1, 2, \dots, g-1)$ by Lemma 4.1. It follows from (4.2) that

$$\begin{aligned}
&f(\operatorname{diag}(\tau_1, \tau_2, \dots, \tau_g))^d \Big/ \prod_{\gamma \in \bar{\Gamma}(1)/\bar{\Gamma}(N)} f(\operatorname{diag}(\tau_1, \tau_2, \dots, \tau_{g-1}, \gamma(\tau_g))) \\
&\qquad = \left(c(\tau_1, \tau_2, \dots, \tau_{g-1}) \prod_{t=1}^{n-1} g_t(\tau_g)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \right)^d \\
&\qquad\qquad \Big/ \left(c(\tau_1, \tau_2, \dots, \tau_{g-1})^d \prod_{t=1}^{n-1} c_t^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})} \right) \\
&\qquad = \prod_{t=1}^{n-1} (c_t^{-1} g_t(\tau_g)^d)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})}.
\end{aligned}$$

Now, set this function to be $h(\tau_1, \tau_2, \dots, \tau_g)$, which is a modular unit as a function of each τ_k $(k = 1, 2, \dots, g)$.

On the other hand, when $\tau_g \in \mathbb{H}$ is fixed, the image of the holomorphic function

$$\left.\begin{aligned}
\varphi\colon \mathbb{H}^{g-1} &\to \mathbb{C}^\times \\
(\tau_1, \tau_2, \dots, \tau_{g-1}) &\mapsto h(\tau_1, \tau_2, \dots, \tau_g) = \prod_{t=1}^{n-1} (c_t^{-1} g_t(\tau_g)^d)^{m_t(\tau_1, \tau_2, \dots, \tau_{g-1})}
\end{aligned}\right\} \tag{4.3}$$

is a countable set because $m_t(\tau_1, \tau_2, \dots, \tau_{g-1})$ $(t = 1, 2, \dots, n-1)$ are integer-valued functions. Let $\ell \in \{1, 2, \dots, g-1\}$ and suppose that $\tau_1, \tau_2, \dots, \tau_{g-1}$ are fixed except for τ_ℓ. Then φ can be viewed as a holomorphic map from $\mathbb{H}$ to $\mathbb{C}^\times$ with respect to τ_ℓ. Since its image is a countable set as mentioned above, the modular unit $h(\tau_1, \tau_2, \dots, \tau_g)$, as a function of τ_ℓ, must be a constant by Lemma 2.9. This observation essentially indicates that the map φ defined on $\mathbb{H}^{g-1}$ in (4.3) is in fact a constant, and hence the function $h(\tau_1, \tau_2, \dots, \tau_g)$ of g variables is a function of τ_g. Moreover, since $g_1(\tau), g_2(\tau), \dots, g_{n-1}(\tau)$ form a basis for the free abelian group $V_N/\mathbb{C}^\times$, the integer-valued functions $m_t(\tau_1, \tau_2, \dots, \tau_{g-1})$ $(t = 1, 2, \dots, n-1)$ should be fixed integers, say m_t. Thus, if we set $v_g(\tau) = \prod_{t=1}^{n-1} g_t(\tau)^{m_t} \in V_N$, then we derive from (4.2) that

$$f(\operatorname{diag}(\tau_1, \tau_2, \dots, \tau_g)) = c(\tau_1, \tau_2, \dots, \tau_{g-1}) v_g(\tau_g). \tag{4.4}$$

The only property of $f(\operatorname{diag}(\tau_1, \tau_2, \dots, \tau_g))$ necessary to have (4.4) is that it is a meromorphic modular function of level N as a function of each τ_k $(k = 1, 2, \dots, g)$. Now that $c(\tau_1, \tau_2, \dots, \tau_{g-1})$ retains this property, if we apply the same argument to $c(\tau_1, \tau_2, \dots, \tau_{g-1})$ instead of $f(\operatorname{diag}(\tau_1, \tau_2, \dots, \tau_g))$ and repeat this process, then we eventually reach the conclusion after $(g-1)$ steps. □

Example 4.3. Let $g, N \geqslant 1$. For

$$\boldsymbol{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_g \end{bmatrix}, \boldsymbol{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_g \end{bmatrix} \in (1/N)\mathbb{Z}^g,$$

we define a theta constant by

$$\Theta_{\left[\begin{smallmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{smallmatrix}\right]}(Z) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} e((\boldsymbol{n}+\boldsymbol{r})^{\mathrm{T}} Z (\boldsymbol{n}+\boldsymbol{r})/2 + (\boldsymbol{n}+\boldsymbol{r})^{\mathrm{T}} \boldsymbol{s}) \quad (Z \in \mathbb{H}_g),$$

where $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$ for $z \in \mathbb{C}$. We further set

$$\Phi_{\left[\begin{smallmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{smallmatrix}\right]}(Z) = \Theta_{\left[\begin{smallmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{smallmatrix}\right]}(Z) / \Theta_{\left[\begin{smallmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{smallmatrix}\right]}(Z) \quad (Z \in \mathbb{H}_g),$$

which is a Siegel modular function of level $2N^2$ [**10**, Proposition 7].

Now, we assume that $g \geqslant 2$, $Z' \in \mathbb{H}_{g-1}$ and $\tau \in \mathbb{H}$. We then derive that

$$\begin{aligned} &\Theta_{\left[\begin{smallmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{smallmatrix}\right]}\left(\begin{bmatrix} Z' & 0 \\ 0 & \tau \end{bmatrix} \right) \\ &\quad = \sum_{\boldsymbol{n} = [n_1 \cdots n_g]^{\mathrm{T}} \in \mathbb{Z}^g} e\left(\frac{1}{2} \begin{bmatrix} \boldsymbol{n}' + \boldsymbol{r}' \\ n_g + r_g \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z' & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} \boldsymbol{n}' + \boldsymbol{r}' \\ n_g + r_g \end{bmatrix} + \begin{bmatrix} \boldsymbol{n}' + \boldsymbol{r}' \\ n_g + r_g \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{s}' \\ s_g \end{bmatrix} \right) \\ &\qquad\qquad \left(\text{where } \boldsymbol{n}' = \begin{bmatrix} n_1 \\ \vdots \\ n_{g-1} \end{bmatrix} \right) \end{aligned}$$

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$$= \sum_{\boldsymbol{n}' \in \mathbb{Z}^{g-1}} \sum_{n_g \in \mathbb{Z}} e((\boldsymbol{n}'+\boldsymbol{r}')^{\mathrm{T}} Z'(\boldsymbol{r}'+\boldsymbol{s}')/2 + (n_g+r_g)\tau(n_g+r_g)/2 + (\boldsymbol{n}'+\boldsymbol{r}')^{\mathrm{T}}\boldsymbol{s}' + (n_g+r_g)s_g)$$

$$= \left(\sum_{\boldsymbol{n}' \in \mathbb{Z}^{g-1}} e((\boldsymbol{n}'+\boldsymbol{r}')^{\mathrm{T}} Z'(\boldsymbol{r}'+\boldsymbol{s}')/2 + (\boldsymbol{n}'+\boldsymbol{r}')^{\mathrm{T}}\boldsymbol{s}')\right) \times \left(\sum_{n_g \in \mathbb{Z}} e((n_g+r_g)\tau(n_g+r_g)/2 + (n_g+r_g)s_g)\right)$$

$$= \Theta_{\left[\begin{smallmatrix}\boldsymbol{r}'\\ \boldsymbol{s}'\end{smallmatrix}\right]}(Z')\Theta_{\left[\begin{smallmatrix}r_g\\ s_g\end{smallmatrix}\right]}(\tau). \tag{4.5}$$

Applying this argument inductively, we obtain

$$\Phi_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{s}\end{smallmatrix}\right]}(\mathrm{diag}(\tau_1,\tau_2,\ldots,\tau_g)) = \prod_{k=1}^{g} \Phi_{\left[\begin{smallmatrix}r_k\\ s_k\end{smallmatrix}\right]}(\tau_k) \quad (\mathrm{diag}(\tau_1,\tau_2,\ldots,\tau_g) \in \mathbb{H}_g^{\mathrm{diag}}).$$

On the other hand, it follows from the Jacobi triple product identity [**3**, (17.3)], the definition (2.1) in § 2 and [**4**, Theorem 2] that

$$\Phi_{\left[\begin{smallmatrix}r\\ s\end{smallmatrix}\right]}(\tau) = \begin{cases} e((2rs+r-s)/4)g_{\left[\begin{smallmatrix}1/2-r\\ 1/2-s\end{smallmatrix}\right]}(\tau)/g_{\left[\begin{smallmatrix}1/2\\ 1/2\end{smallmatrix}\right]}(\tau) & \text{if } \left[\begin{smallmatrix}r\\ s\end{smallmatrix}\right] \in \mathbb{Q}^2 - (1/2+\mathbb{Z})^2, \\ 0 & \text{if } \left[\begin{smallmatrix}r\\ s\end{smallmatrix}\right] \in (1/2+\mathbb{Z})^2. \end{cases}$$

Therefore, we conclude that $\Phi_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{s}\end{smallmatrix}\right]}(\mathrm{diag}(\tau_1,\tau_2,\ldots,\tau_g))$ has neither a zero nor a pole on $\mathbb{H}_g^{\mathrm{diag}}$, or is identically zero.

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