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Sampling analysis in the complex reproducing kernel Hilbert space[1]

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We consider and analyse sampling theories in the reproducing kernel Hilbert space (RKHS) in this paper. The reconstruction of a function in an RKHS from a given set of sampling points and the reproducing kernel of the RKHS is discussed. Firstly, we analyse and give the optimal approximation of any function belonging to the RKHS in detail. Then, a necessary and sufficient condition to perfectly reconstruct the function in the corresponding RKHS of complex-valued functions is investigated. Based on the derived results, another proof of the sampling theorem in the linear canonical transform (LCT) domain is given. Finally, the optimal approximation of any band-limited function in the LCT domain from infinite sampling points is also analysed and discussed.

Key words: reproducing kernel Hilbert space (RKHS); sampling theorem; linear canonical transform (LCT)

1 Introduction

Shannon's sampling theorem [1] given by the formula

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(x-n)}{\pi(x-n)} \tag{1.1}$$

holds for any π-band-limited function with finite energy. It is so important that there are many generalisations and extensions of this theorem such as [2–4]. The Shannon sampling theorem says that if a signal is band-limited signal of finite energy, then it can be completely characterised by its samples. However, in many engineering applications, such as MRI imaging, signals and images are not band-limited [5, 6] in the classical Fourier transform sense, one such example being the shift-invariant spaces. In order to analyse and process non-band-limited functions, many transforms and analysis methods

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are proposed, for example, the fractional Fourier transform and the linear canonical transform [6], the wavelet transform [7], and the methods in shift-invariant spaces [4, 5], spline subspace [8, 9], and reproducing kernel Hilbert space [10, 11].

Among them, Nashed and Walter [10, 11] presented a very important reformulation of the sampling theorem by using the theorems about RKHS. In their work, they discussed the relationship between a given set of kernel functions which are determined by a given set of sampling points and a reproducing kernel, and proposed a perfect reconstruction of any function in the RKHS. Based on these results, Tanaka *et al.* in [12] derived a necessary and sufficient condition to easily check whether a perfect reconstruction of any function can be got in a general and practical case when the kernel functions and the set of sampling points are given. They also give anther proof of the Shannon's sampling theorem in the sense of the RKHS.

However, in paper [12], the functions in the RKHS are real-valued functions, and we can not apply their methods in the RKHS of complex-valued functions. In real applications, we often encounter the analysis of the RKHS of complex-valued functions, for example, the kernel function of the RKHS which consists of band-limited functions in linear canonical transform domain or fractional Fourier transform domain is a complex-valued function [13, 14], therefore the above theorem proposed in [12] can not be applied in an LCT domain. The purpose of this paper is to analyse the sampling problems in the RKHS that consists of complex-valued functions in detail, a necessary and sufficient condition to easily check whether a perfect reconstruction can be got in a general and practical case is derived. Based on the derived results, another proof of the sampling theorem in LCT domain by the RKHS is given. We also investigate the optimal approximation of any band-limited functions in LCT domain from infinite sampling points associated with results of RKHS.

The outline of this paper is as follows. In Section 2, we give some mathematical definitions about the LCT and RKHS. In Section 3, we will study the optimal approximation by orthogonal projection. In Section 4, we discuss the necessary and sufficient condition to perfectly recover the function in the corresponding RKHS. In Section 5, we obtain another proof of the sampling theorem in an LCT domain and the optimal approximation of any band-limited functions in an LCT domain from infinite sampling points.

2 Preliminary

2.1 The linear canonical transform (LCT)

The linear canonical transform (LCT) can be considered as a further generalisation of the classical Fourier transform and the fractional Fourier transform [15], the LCT definition of a signal $f(t)$ being as follows.

Definition 1 *The LCT of a signal $f(t)$ with parameters (a,b,c,d) is defined [15] to be*

$$L_{(a,b,c,d)}[f](u) = \begin{cases} \int_{-\infty}^{\infty} f(t)\sqrt{\frac{1}{j2\pi b}}e^{\frac{j}{2}(\frac{a}{b}t^2-\frac{2}{b}ut+\frac{d}{b}u^2)}dt, & b \neq 0 \\ \sqrt{d}e^{\frac{j}{2}cdu^2}f(du), & b = 0 \end{cases} \tag{2.1}$$

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where a, b, c, d are real numbers satisfying $ad - bc = 1$.

From (2.1) we can see that, when $b = 0$, the LCT of a signal is essentially a chirp multiplication and it is of no particular interest to us here. Therefore, we set $b > 0$ in the following sections of the paper. The LCT has been shown to be a powerful tool and more flexible than the fractional Fourier transform and the Fourier transform, and it has been applied in many real situations, such as in filter design, phase retrieval, pattern recognition, encryption, modulation, and multiplexing in communication [16, 17].

A signal $f(t)$ is said to be σ-band-limited in the LCT domain with parameters (a, b, c, d) , if

$$L_{(a,b,c,d)}[f](u) = 0, \text{ for } |u| > \sigma,$$

where σ is the bandwidth of the signal $f(t)$ in the LCT domain. It is proved in [18, 19] that the non-band-limited signal of the classical Fourier transform domain can be band-limited in the fractional Fourier transform domain and the linear canonical transform domain. Based on this fact, the sampling theories about the uniform or nonuniform sampling associated with the LCT domain have been investigated in detail in recent years [20–24].

The discrete realisation methods and the other important concepts, for example, the uncertainty principle, the convolution and product theorem and the poisson sum formulae and the eigenfunctions, associated with the LCT and the fractional Fourier transform are also studied and investigated in [25–30]. In this paper, we let $H^{\sigma}_{(a,b,c,d)}$ denote the class of signals which are σ-band-limited in the LCT domain with parameters (a, b, c, d).

2.2 Reproducing kernel Hilbert space (RKHS)

As an efficient processing method, the reproducing kernel Hilbert spaces (RKHS) methods are widely used in many areas, and their definition can be summarised as follows [10, 11, 31].

Definition 2 *Let H be a class of functions which form a Hilbert space defined on a set X. If there exists $g(x,y)$ defined on $X \times X$, and satisfying*

$$g(x,y) \in H, y \in X \tag{2.2}$$

$$f(y) = \langle f(\cdot), g(\cdot, y)\rangle_H, f \in H \tag{2.3}$$

then $g(x,y)$ is a reproducing kernel for H, and H is called a reproducing kernel Hilbert space (RKHS), where the $\langle\cdot,\cdot\rangle_H$ denotes the inner product of the Hilbert space H.

If a kernel of $g(x, y)$ is the reproducing kernel, then from the well known Aronszajn-Moore theorem [32], the kernel $g(x, y)$ determines a unique Hilbert space H, and from the definition of the inner product of Hilbert space, the following property holds for the kernel $g(x, y)$

$$g(x, y) = \overline{g(y, x)}.$$

It is shown in [20] that the $H^{\sigma}_{(a,b,c,d)}$ is a RKHS, and the reproducing kernel is

$$G_{(a,b,c,d)}(t,x) = \frac{\sigma}{\pi b} e^{\frac{ia}{2b}(x^2-t^2)} \frac{\sin[\sigma(t-x)/b]}{\sigma(t-x)/b}.$$

Obviously, the kernel function is a complex-valued function.

2.3 Schatten product

The following Schatten product definition will be used in the following Sections of this paper.

Definition 3 *Let H_1 and H_2 be Hilbert space. The Schatten product of $h_2 \in H_2$ and $h_1 \in H_1$ is defined by*

$$(h_2 \otimes h_1)f = h_2\langle f, h_1\rangle_{H_1}, f \in H_1. \tag{2.4}$$

From this definition, it is easy to show that $(h_2 \otimes h_1)$ is a linear operator from H_1 onto H_2, and the following relations hold.

$$(h_1 \otimes h_2)^* = (h_2 \otimes h_1), \tag{2.5}$$

$$(h_1 \otimes h_2)(u \otimes v) = \langle u, h_2\rangle_{H_2}(h_1 \otimes v), \tag{2.6}$$

where $h_1, v \in H_1, \quad h_2, u \in H_2$ and the superscript * denotes the adjoint operator.

2.4 Some symbols

In this paper, we use the following symbols $H_K, K(x,\tilde{x}), S, H_C, H_G, G, M, A, P$.

$$S = \overline{span[\{K(\cdot,x_i) \mid i \in N\}]},$$

$$H_C = \overline{\{\boldsymbol{\alpha} \in \ \mathbf{C}^\infty \mid \overline{\boldsymbol{\alpha}}' G\boldsymbol{\alpha} < \infty\}},$$

$$H_G = \{\boldsymbol{g} \in \mathbf{C}^\infty \mid \overline{\boldsymbol{g}}' M\boldsymbol{g} < \infty\},$$

$$A = \left(\sum_{i=1}^{\infty}[\mathbf{e}_i \otimes K(\cdot,x_i)]\right),$$

$$P = A^*A,$$

$$G = (K(x_i,x_j)) \in \mathbf{C}^{\infty\times\infty},$$

$$G = GMG,$$

where $\mathbf{e}_i$ is the unit vector in $\mathbf{R}^\infty$ with only the ith component being unity, and the Hilbert H space that has a reproducing kernel K is called a RKHS, denoted by H_K.

From the above definition, H_K consists of complex-valued functions defined on $D \subset \mathbf{C}^n$ and $K(x, \tilde{x})$ is a complex-valued function which is the reproducing kernel of H_K and $\boldsymbol{\alpha}, \boldsymbol{g} \in \mathbf{C}^{\infty}$. It is easy to show

$$K(x, \tilde{x}) = \overline{K(\tilde{x}, x)}, G^{'} = \overline{G}, \overline{G}^{'} = G$$

and S is a closed linear subspace in H_K spanned by the basis function $\{K(\cdot, x_i) \mid i \in N\}$, defined as

$$S = \overline{span[\{K(\cdot, x_i) \mid i \in N\}]}.$$

Hence any function$f(\cdot) \in S$ can be represented by

$$f(\cdot) = \sum_{i=1}^{\infty} \alpha_i K(\cdot, x_i)$$

with coefficients $\alpha_i \in \mathbf{C}$. For any $f(\cdot) \in S$,

$$\begin{aligned}
\| f(\cdot) \|_{H_K}^2 &= \langle f(\cdot), f(\cdot) \rangle_{H_K} \\
&= \left\langle \sum_{i=1}^{\infty} \alpha_i K(\cdot, \mathbf{x}_i), \sum_{j=1}^{\infty} \alpha_j K(\cdot, \mathbf{x}_j) \right\rangle_{H_K} \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i \overline{\alpha}_j \langle K(\cdot, \mathbf{x}_i), K(\cdot, \mathbf{x}_j) \rangle_{H_K} \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i \overline{\alpha}_j K(x_j, x_i) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{\alpha}_j K(x_j, x_i) \alpha_i \\
&= \overline{\boldsymbol{\alpha}}^{'} G \boldsymbol{\alpha} < \infty,
\end{aligned}$$

where $\| \cdot \|_{H_K}$ denotes the inducing norm in H_K, $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_l, \ldots]^{'} \in \boldsymbol{C}^{\infty}$ and $G = (K(x_i, x_j)) \in \mathbf{C}^{\infty \times \infty}$ denotes the Gramian matrix of the kernel K with sampling points X.

We intend to use S as a linear subspace to which a reconstructed function belongs.

$$H_C = \overline{\{\boldsymbol{\alpha} \in \mathbf{C}^{\infty} \mid \overline{\boldsymbol{\alpha}}^{'} G \boldsymbol{\alpha} < \infty\}},$$

is also a Hilbert space which is homeomorphic with S, since S is closed.

It is easy to show that G is also a reproducing kernel [12]. Thus, G has the unique corresponding RKHS denoted by H_G. Since H_G is complete and closed, there exist a symmetric and non-negative matrix M that specifies the metric of H_G .Thus, H_G is characterised as

$$H_G = \{\boldsymbol{g} \in \mathbf{C}^{\infty} \mid \overline{\boldsymbol{g}}^{'} M \boldsymbol{g} < \infty\}.$$

According to (2.2) in Definition 2,

$$G\mathbf{e}_i \in H_G$$

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holds for any $i \in \mathbf{N}$, which implies that each column of G belongs to H_G; and (2.3) in Definition 2 yields

$$
\begin{aligned}
g_k &= \langle \boldsymbol{g}, G\mathbf{e}_k \rangle_{H_G} \\
&= \mathbf{e}_k^{'} \overline{G}^{'} M \boldsymbol{g} \\
&= \mathbf{e}_k^{'} G M \boldsymbol{g}
\end{aligned}
$$

for any $\boldsymbol{g} = [g_1, \ldots, g_l, \ldots]^{'} \in H_G$. The summation premultiplied by $\mathbf{e}_k$, with respect to k produces

$$\boldsymbol{g} = GM\boldsymbol{g}$$

for any $\boldsymbol{g} \in H_G$. Therefore,since $G\mathbf{e}_i \in H_G$

$$G\mathbf{e}_i = GMG\mathbf{e}_i$$

holds for any $i \in \mathbf{N}$ and the summation postmultiplied by $\mathbf{e}_i^{'}$, with respect to i yields

$$G = GMG.$$

The last equation implies that M is a 1-inverse [33] of G. When G is a Matrixoid, $M = G^{-1}$.

3 Optimal approximation by orthogonal projection

In this section we try to prove that P is an orthogonal projector onto the closed linear subspace S in H_K. Firstly, it is easy to show that $G^{'} = \overline{G}, \overline{G}^{'} = G$, and then the following Lemmas 1 and 2 can be obtained by a similar proof as proposed in [12].

Lemma 1 *G is a closed linear operator from H_C onto H_G.*

Lemma 2 *Let $A = \left(\sum_{i=1}^{\infty} [\boldsymbol{e}_i \otimes K(\cdot, x_i)]\right)$, then $A \in \ell(H_K, H_G)$, where $\ell(H_K, H_G)$ denotes the set of bounded linear operators from H_K onto H_G.*

According to Lemma 2, it immediately follows that:

$$A^* \in \ell(H_G, H_K),$$

$$A^* A \in \ell(H_K, H_K).$$

Based on these lemmas, we obtain main theorem of this paper as follows.

Theorem 1 *$P = A^* A$ is the orthogonal projector onto the closed linear subspace S in H_K.*

Proof Let $f(y) = \sum_{i=1}^{\infty} \alpha_i K(y, x_i)$ be an arbitrary function in S with respect to y, then

$$\begin{aligned} Pf(y) &= A^* A f(y) = A^* G\boldsymbol{\alpha} \\ &= \left(\sum_{i=1}^{\infty} K(y, x_i) \otimes \boldsymbol{e}_i \right) G\boldsymbol{\alpha} \\ &= \sum_{i=1}^{\infty} \overline{\boldsymbol{\alpha}}' \overline{GM} \boldsymbol{e}_i K(y, x_i) \\ &= \overline{\boldsymbol{\alpha}}' \overline{GM} \boldsymbol{k} \end{aligned}$$

where

$$\boldsymbol{k} = \sum_{i=1}^{\infty} \boldsymbol{e}_i K(y, x_i) = [K(y, x_1), \ldots, (y, x_l), \ldots]' \in \boldsymbol{C}^{\infty}.$$

Since $A \in \ell(H_K, H_G)$ and $K(\cdot, y) \in H_K$ for any $y \in D$

$$\begin{aligned} AK(\cdot, y) &= \left(\sum_{i=1}^{\infty} [\boldsymbol{e}_i \otimes K(\cdot, x_i)] \right) K(\cdot, y) \\ &= \overline{\boldsymbol{k}} \in H_G \end{aligned}$$

is followed with any fixed $y \in D$. Then

$$\overline{\boldsymbol{k}}_k = \langle \overline{\boldsymbol{k}}, G\boldsymbol{e}_k \rangle_{H_G}$$

so

$$\begin{aligned} \boldsymbol{k}_k &= \langle G\boldsymbol{e}_k, \overline{\boldsymbol{k}} \rangle_{H_G} \\ &= \boldsymbol{e}_k' \overline{GM} \boldsymbol{k}. \end{aligned} \tag{3.1}$$

The summation of (3.1), premultiplied by $\boldsymbol{e}_k$, with respect to k produces

$$\boldsymbol{k} = \overline{GM} \boldsymbol{k}. \tag{3.2}$$

Thus, from (3.2),

$$\begin{aligned} Pf(y) &= \boldsymbol{\alpha}' \overline{GM} \boldsymbol{k} \\ &= \boldsymbol{\alpha}' \boldsymbol{k} \\ &= \sum_{i=1}^{\infty} \alpha_i K(y, x_i) \\ &= f(y) \end{aligned}$$

is obtained for any $y \in D$.

On the other hand, for any $f(y) \in S^{\perp}$

$$Pf(y) = 0$$

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trivially holds for any $y \in D$. Thus, we know that $P = A^*A \in \ell(H_K, H_K)$ is the orthogonal projector onto the closed linear subspace S. This concludes the proof. □

From the definition of P, the closed form of P is written as,

$$\begin{aligned} P &= A^*A \\ &= \left(\sum_{j=1}^{\infty}[K(\cdot,x_j)\otimes \boldsymbol{e}_j]\right)\left(\sum_{i=1}^{\infty}[\boldsymbol{e}_i\otimes K(\cdot,x_i)]\right) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M_{i,j}[K(\cdot,x_i)\otimes K(\cdot,x_j)] \end{aligned} \tag{3.3}$$

and that of $Pf(\cdot)$ is written as

$$\begin{aligned} Pf(\cdot) &= \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M_{i,j}[K(\cdot,x_i)\otimes K(\cdot,x_j)]\right) f(\cdot) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(x_j) M_{i,j} K(\cdot,x_i). \end{aligned} \tag{3.4}$$

From Theorem 1, P is the orthogonal projector onto the closed linear subspace S, then $Pf(\cdot)$, which is in S, is the optimal approximation of any $f(\cdot) \in H_K$. Thus, the above discussion is an extension of the framework shown in [12] to complex-valued functions.

4 Sampling theories analysis in the RKHS

From the discussions in the previous sections, in orderto perfectly reconstruct any function $f(\cdot) \in H_K$ by (11), $H_K = S$ must hold. In this section, we obtain a necessary and sufficient condition for a reproducing kernel and a set of sampling points to perfectly reconstruct any function in the corresponding RKHS of complex-valued functions. The main result of this section can be represented as following Theorem 2.

Theorem 2 $H_K = S$ *if and only if*

$$K(y,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \overline{K(y,x_j)} M_{i,j} K(y,x_i) \tag{4.1}$$

holds for any $y \in D$.

Proof Since $K(\cdot,y) \in H_K$ for any $y \in D$, if $H_K = S$ holds

$$K(\cdot,y) - PK(\cdot,y) = 0 \tag{4.2}$$

that is

$$\begin{aligned} K(\cdot, y) &= PK(\cdot, y) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \overline{K(y, x_j)} M_{i,j} K(\cdot, x_i) \end{aligned}$$

must hold for any $y \in D$ at least. So (4.1) holds.

On the other hand, if we assume that (4.2) holds, then

$$\begin{aligned} f(y) &= \langle f(\cdot), K(\cdot, y)\rangle H_K \\ &= \langle f(\cdot), PK(\cdot, y)\rangle H_K \\ &= \langle f(\cdot), P^* K(\cdot, y)\rangle H_K \\ &= \langle Pf(\cdot), K(\cdot, y)\rangle H_K \\ &= Pf(y) \end{aligned}$$

is obtained for any $f(\cdot) \in H_K$ and any $y \in D$, since P is an orthogonal projection, which implies $H_K = S$.

It is easy to show that (4.2) identical to

$$\| K(\cdot, y) - PK(\cdot, y) \|_{H_K}^2 = 0.$$

By applying the Pythagorean theorem and a similar method to [12], the above equation can be written as

$$\begin{aligned} &\| K(\cdot, y) - PK(\cdot, y) \|_{H_K}^2 \\ &= \| K(\cdot, y) \|_{H_K}^2 - \| PK(\cdot, y) \|_{H_K}^2 \\ &= K(y, y) - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \overline{K(y, x_j)} M_{i,j} K(y, x_i) \\ &= 0. \end{aligned}$$

This concludes the proof. □

5 Analysis of sampling theories associated with LCT

In this section, we will apply the derived results in previous section to the sampling theories associated with an LCT. From the sampling theories of an LCT [20], we know that the $H^{\sigma}_{(a,b,c,d)}$ is an RKHS, and the reproducing kernel is

$$G_{(a,b,c,d)}(t, x) = \frac{\sigma}{\pi b} e^{\frac{ia}{2b}(x^2 - t^2)} \frac{\sin[\sigma(t-x)/b]}{\sigma(t-x)/b}.$$

Obviously, the kernel function $G_{(a,b,c,d)}(t, x)$ is a complex-valued function, and we cannot apply the results in [12] to obtain the desired results.

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5.1 The proof of sampling theorem in LCT domain

Now, we give the proof of the sampling theorem in the LCT domain. Let $X_b = \{\ldots, -b, 0, b, \ldots\}(b > 0)$ be the set of the uniform sampling points with the Nyquist interval for π-band-limited functions in the $H^{\pi}_{(a,b,c,d)}$. It is trivial that

$$G_{(a,b,c,d)}(t,t) = \frac{1}{b}$$

for any $t \in R$, in the left hand-side of (4.1).

For the right-hand side of (4.1), we must first get the M_{ij}.

$$G_{(a,b,c,d)}(t_i,t_j) = \frac{1}{b} e^{\frac{ia}{2b}(t_j^2 - t_i^2)} \frac{\sin \pi(t_i - t_j)/b}{\pi(t_i - t_j)/b} \tag{5.1}$$

where $t_i = ib, i \in N$.

When $t_i = t_j$, (5.1) equals $1/b$, when $t_i \neq t_j$, (5.1) equals to 0. Thus, we know that G is a diagonal matrix and the diagonal element is $1/b$, so M is a diagonal matrix and the diagonal element is b.

Then the right-hand side of (4.1) reduces to

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \overline{K(y,x_j)} M_{i,j} K(y,x_i) \\
&= b \sum_{i=1}^{\infty} \overline{G_{(a,b,c,d)}(t,t_i)} G_{(a,b,c,d)}(t,t_i) \\
&= \frac{1}{b} \sum_{i=1}^{\infty} e^{-\frac{ia}{2b}(t_i^2 - t^2)} \frac{\sin[\pi(t-t_i)/b]}{\pi(t-t_i)/b} e^{\frac{ia}{2b}(t_i^2 - t^2)} \frac{\sin[\pi(t-t_i)/b]}{\pi(t-t_i)/b} \\
&= \frac{1}{b} \sum_{i=1}^{\infty} \left[\frac{\sin[\pi(t-t_i)/b]}{\pi(t-t_i)/b} \right]^2 .
\end{aligned} \tag{5.2}$$

When $t \in X_b$, it is easy to show that (5.2) is equal to $1/b$. On the other hand, when $t \notin X_b$,

$$\begin{aligned}
&\frac{1}{b} \sum_{i=1}^{\infty} \left[\frac{\sin[\pi(t-t_i)/b]}{\pi(t-t_i)/b} \right]^2 \\
&= \frac{1}{b} \sum_{i=1}^{\infty} \left[\frac{\sin[\pi(t/b - i)]}{\pi(t/b - i)} \right]^2 \\
&= \frac{1}{b} \sum_{i=1}^{\infty} \left[\frac{\sin(\pi t/b)\cos \pi i - \cos(\pi t/b) \sin \pi i}{\pi(t/b - i)} \right]^2 \\
&= \frac{1}{b} \sin^2(\pi t/b) \sum_{i=1}^{\infty} \left[\frac{1}{\pi(t/b - i)} \right]^2 \\
&= \frac{1}{b} \sin^2(\pi t/b) \csc^2(\pi t/b) = \frac{1}{b}.
\end{aligned}$$

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Thus, it is concluded that (4.1) holds for any $t \in R$, which gives another proof of the sampling theorem in the LCT domain.

5.2 The optimal approximation of function in LCT domain

Let $X_t = \{t_i \in R \mid i \in N\}$ be a infinite set of sampling points. We will get a subspace V in $H^{\sigma}_{(a,b,c,d)}$, written as

$$V = \overline{span[\{G_{(a,b,c,d)}(t, t_i)\}]}.$$

Thus, according to (3.4), for every $f(t) \in H^{\sigma}_{(a,b,c,d)}$, we will get the optimal approximation of $f(t)$, denoted as

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(t_j) M_{ij} G_{(a,b,c,d)}(t, t_i)$$

where M_{ij} is the element of M, $M = G^{-1}$, and $G = (G_{(a,b,c,d)}(t_i, t_j))$.

6 Conclusion

In this paper, we investigate and analyse of the sampling theories in the reproducing kernel Hilbert space, a necessary and sufficient condition for the pair of a reproducing kernel and a set of sampling points to perfectly reconstruct any function in the complex-valued reproducing kernel Hilbert space is proposed. We also give the optimal approximation of any function of RKHS in a subspace which is determined by the corresponding reproducing kernel and the set of uniform or nonuniform sampling points. Finally, the derived results are applied to the sampling theories associated with LCT, and another proof of the well known sampling theorem of an LCT domain is given. Future working directions will be the analysis of the reproducing kernel method in an abstract setting and applications of the results in real practical situations.

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