

On the virtual automorphism group of a minimal flow

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(Received 4 August 2019 and accepted in revised form 8 January 2020)

Abstract. We introduce the notions ‘virtual automorphism group’ of a minimal flow and ‘semiregular flow’ and investigate the relationship between the virtual and actual group of automorphisms.

Key words: semiregular minimal flow, Ellis group, Morse minimal set, automorphism group

2010 Mathematics Subject Classification: 37B05, 54H20 (Primary)

1. Introduction

The notion of a virtual subgroup was introduced in ergodic theory by Mackey (see, for example, [M]). Here is a brief description of the basic idea behind this highly technical notion, as described by Zimmer [Zi].

Let G be a locally compact group.

If X is an ergodic G -space, one of two mutually exclusive statements holds:

- (i) There is an orbit whose complement is a nullset. In this case, X is called essentially transitive.
- (ii) Every orbit is a nullset. X is then called properly ergodic.

In the first case, the action of G on X is essentially equivalent to the action defined by translation on G/H , where H is a closed subgroup of G ; furthermore, this action is determined up to equivalence by the conjugacy class of H in G . In the second case, no such simple description of the action is available, but it is often useful to think of the action as being defined by a ‘virtual subgroup’ of G . Many concepts defined for a subgroup H , can be expressed in terms of the action of G on G/H ; frequently, this leads to a

natural extension of the concept to the case of an arbitrary virtual subgroup, i.e., to the case of an ergodic G -action that is not necessarily essentially transitive. Perhaps the most fundamental notions that can be extended in this way are those of a homomorphism, and the concomitant ideas of kernel and range. These and other related matters are discussed in [M].

In some sense the concept of the ‘Ellis group associated to a pointed minimal flow’ (see §2) is an analogue of Mackey’s virtual group in topological dynamics. It became a keystone object in the abstract theory of topological dynamics, which was developed by Ellis and collaborators in the 1960s and 1970s (see, for example, [E69, EGS, V]).

If one carries this idea a bit further, and one thinks of A , the Ellis group of a minimal flow (X, T) , as a virtual subgroup, then the group $N_G(A)/A$, where $N_G(A)$ is the normalizer of A in the ambient group G (a subgroup of the enveloping semigroup of (X, T)), can be thought of as the ‘virtual automorphism group’ of the flow (X, T) .

In the present work we make this notion precise (§3) and investigate the question of realization of the virtual automorphism group as an actual group of flow automorphisms.

2. Some notation and basic facts concerning minimal flows

In this work T denotes an arbitrary (discrete) group. A T -flow (X, T) on a compact Hausdorff space X is given by a homomorphism $\rho : T \rightarrow \text{Homeo}(X)$ of T into the group of self-homeomorphisms of X . We usually suppress the homomorphism ρ from our notation of a flow (even when ρ is not an injection) and we write tx for the image of the point $x \in X$ under the homeomorphism $\rho(t)$ ($t \in T$).

In the next few paragraphs we will survey some of the basic definitions and facts from the theory of abstract topological dynamics which will be repeatedly used in this work. This theory started with the classical monograph by Gottschalk and Hedlund [GH] and was then greatly developed by Ellis. For more details we refer to the monographs [E69, GI76, A88, dV93].

The flow (X, T) is *minimal* if every point in X has a dense orbit. A pair of points $x, x' \in X$ is *proximal* if there exist a net $t_i \in T$ and a point $z \in X$ such that $\lim t_i x = \lim t_i x' = z$. We write $P[x]$ for the proximal cell of x (i.e. the set of points proximal to x). A point $x \in X$ is a *distal point* if it is proximal only to itself: $P[x] = \{x\}$. A minimal flow is *point distal* if there is at least one distal point in X , and it is *distal* if every point is distal. Ellis has shown that in a metric minimal flow the existence of one distal point implies that the set $X_0 \subseteq X$ of distal points is a dense G_δ set. A continuous map $\pi : (X, T) \rightarrow (Y, T)$ between two minimal flows is a *homomorphism* (or an *extension*) if it intertwines the T -actions ($t\pi(x) = \pi(tx)$, for all $x \in X, t \in T$). We say that the homomorphism is *proximal* if, for every $y \in Y$, every pair of points in $\pi^{-1}(y)$ is proximal, and that it is *distal* if, for every $y \in Y$, we have $P[x] \cap \pi^{-1}(y) = \{x\}$, for all $x \in \pi^{-1}(y)$.

The *enveloping semigroup* of the flow (X, T) , denoted by $E(X, T)$, is the closure of the set $\{\rho(t) : t \in T\}$ in the compact space X^X . This is indeed a compact subsemigroup of the semigroup (under composition of maps) X^X , and thus, for any fixed $p \in E(X, T)$, right multiplication by p , $R_p : q \mapsto qp$ ($q \in E(X, T)$), is continuous on $E(X, T)$. However, left multiplication $L_p : q \mapsto pq$, $q \in E(X, T)$, is often highly non-continuous (usually not

even measurable) unless p is a continuous map. As the elements of T are continuous maps the homomorphism $t \mapsto L_t$ ($t \in T$) makes $E(X, T)$ a T -flow.

It is well known that the semigroup βT , the Stone–Čech compactification of the discrete T , is the universal point transitive T -flow and therefore also a universal enveloping semigroup. We will use this universality and often consider elements of βT as maps in the enveloping semigroup of each and every T -flow under consideration. The semigroup βT admits many (for infinite T) minimal left ideals (which coincide with the minimal subflows). All these ideals are isomorphic to each other both as compact right topological semigroups and as minimal flows. As usual, we will fix a minimal ideal M of βT . The universality of βT implies that (M, T) is a universal minimal flow. Ellis has shown that as a flow (M, T) is *coalescent*; that is, every endomorphism of (M, T) is an automorphism, and thus up to an automorphism (M, T) is the unique universal minimal flow. Each minimal ideal contains (usually many) idempotents and for convenience we usually fix one such idempotent $u = u^2 \in M$. We denote the collection of idempotents in M by the letter J .

It turns out that the set $G = uM \subset M$ is actually a group and, moreover, via the representation $g \mapsto R_g$, $g \in G$, this group is isomorphic to the group $\text{Aut}(M, T)$ of automorphisms of the flow (M, T) . M is the disjoint union of the collection of groups $\{vM : v \in J\}$ and each member p of M has a unique representation $p = vg$, where $v = v^2$ is an idempotent in M and g is in G . We sometimes write p^{-1} for vg^{-1} ; this is indeed the inverse element of p in the group vG .

If (X, T) is minimal then, for every $x \in X$, there is an idempotent $v \in M$ such that $vx = x$. In other words, $X = \bigcup\{vX : v \in J\}$. However, whereas $M = \bigcup\{vM : v \in J\}$ is a disjoint union, usually the sets vX are not necessarily disjoint. For example, a point x in a minimal flow (X, T) is distal if and only if $vx = x$ for every $v \in J$. In particular, a minimal flow (X, T) is distal if and only if $X = uX = vX$ for all the idempotents $v \in J$. Thus, in a minimal distal flow $E(X, T)$ is a group.

A minimal flow with a distinguished point $x_0 \in X$ is called a *pointed flow* and we usually assume that $ux_0 = x_0$; that is, $x_0 \in uX$. We write

$$\mathfrak{G}(X, x_0) = \{g \in G : gx_0 = x_0\}.$$

This subgroup of G is called the *Ellis group* of the pointed flow (X, x_0, T) . It is easy to check that, for $g \in G$, we have

$$\mathfrak{G}(X, gx_0) = g\mathfrak{G}(X, x_0)g^{-1}.$$

In the sequel we will often use the following fact. A homomorphism $\pi : (X, x_0, T) \rightarrow (Y, y_0, T)$ of pointed minimal flows is a proximal homomorphism if and only if $\mathfrak{G}(X, x_0) = \mathfrak{G}(Y, y_0)$.

Note that when the group T is abelian we have $tp = pt$ for every $t \in T$ and $p \in E(X, T)$.

3. The group of virtual automorphisms of a minimal flow

Given a flow (X, T) , we let $\text{Aut}(X, T)$ denote its group of automorphisms and $\text{End}(X, T)$ its semigroup of endomorphisms. Given a subgroup $A < G$, we write $N_G(A) = \{h \in G : h^{-1}Ah = A\}$ and $N'_G(A) = \{h \in G : h^{-1}Ah \subseteq A\}$.

PROPOSITION 3.1. *Let (X, x_0, T) be a pointed minimal flow, $A = \mathfrak{G}(X, x_0)$, and let ψ be an endomorphism of the flow (X, T) . Then there is an element $h \in N'_G(A)$ such that $\psi = \phi_h$, where, for every $p \in M$,*

$$\phi_h(px_0) = phx_0.$$

If ψ is an automorphism then $h \in N_G(A)$ and $\psi^{-1} = \phi_{h^{-1}}$.

Proof. Because ψ is an endomorphism it commutes with u , hence $u\psi(x_0) = \psi(ux_0) = \psi(x_0) \in uX = Gx_0$, and there exists an element $h \in G$ such that $\psi(x_0) = hx_0$. Now, for every $p \in \beta T$ (or $p \in E(X, T)$), we have

$$\psi(px_0) = p\psi(x_0) = phx_0.$$

If $a \in A$ then $\psi(x_0) = \psi(ax_0) = ahx_0 = hx_0$, hence $h^{-1}ah \in A$, so that $h \in N'_G(A)$.

If ψ is an automorphism then

$$\psi^{-1}(hx_0) = h\psi^{-1}(x_0) = \psi^{-1}(\psi(x_0)) = x_0,$$

hence $\psi^{-1}(x_0) = h^{-1}x_0$, so that, as above, $\psi^{-1} = \phi_{h^{-1}}$, and also $hAh^{-1} \subseteq A$, hence $h^{-1}Ah = A$. □

PROPOSITION 3.2. *Let (X, x_0, T) be a pointed minimal flow, $A = \mathfrak{G}(X, x_0)$ its Ellis group.*

- (1) *Let $h \in G$ be such that the map ϕ_h where, for every $p \in M$,*

$$\phi_h(px_0) = phx_0$$

is well defined. Then $h \in N'_G(A)$ and ϕ_h is a continuous endomorphism; that is, $\phi_h \in \text{End}(X, T)$.

- (2) *If $h \in G$ is such that both maps ϕ_h and $\phi_{h^{-1}}$ are well defined, then $h \in N_G(A)$ and $\phi_h \in \text{Aut}(X, T)$.*

Proof. (1) For $a \in A$, we have $hx_0 = \phi_h(x_0) = \phi_h(ax_0) = ahx_0$, whence $h^{-1}ah \in A$; that is, $h \in N'_G(A)$. Let $\pi : M \rightarrow X$ be the evaluation map $p \mapsto px_0$ ($p \in M$). Let $R_h : M \rightarrow M$ denote right multiplication by h and let

$$L = (\pi \times \pi)(\text{graph}(R_h)) = (\pi \times \pi)\{(p, ph) : p \in M\} = \{(px_0, phx_0) : p \in M\}.$$

By our assumption L is a graph of a map $\phi_h : X \rightarrow X$, $px_0 \mapsto phx_0$ ($p \in M$). Since the graph of R_h is T -invariant (R_h commutes with the elements of T), we deduce that also ϕ_h commutes with the T -action. Since L is a closed subset of $M \times M$, it follows that the map ϕ_h is continuous. Thus $\phi_h \in \text{End}(X, T)$.

- (2) If $\phi_{h^{-1}}$ is also well defined then, as above, $hAh^{-1} \subseteq A$, hence $h \in N_G(A)$, and we have $\phi_{h^{-1}} = (\phi_h)^{-1}$, so that $\phi_h \in \text{Aut}(X, T)$. □

Example 3.3. In [D] Downarowicz constructs a Toeplitz flow (X, T) which is not coalescent; that is, it admits an endomorphism which is not an automorphism. Now, a Toeplitz flow is an almost one-to-one extension of its maximal equicontinuous factor $\pi : X \rightarrow Y$ (which is an adding machine). This implies the following facts. (i) For any choice of a base point $x_0 \in uX$, $\mathfrak{G}(X, x_0) = A$ is a normal subgroup of G (i.e.

$N_G(A) = G$. (ii) The extension π is proximal, whence $\mathfrak{G}(X, x_0) = \mathfrak{G}(Y, y_0) = A$ (with $y_0 = \pi(x_0)$). Now let ϕ be an endomorphism of (X, T) which is not one-to-one. As in Proposition 3.1 we have $\phi = \phi_h$ for some $h \in G$ and, although here $h^{-1}Ah = A$, yet ϕ_h is not an automorphism. This phenomenon, however, cannot occur for distal flows, as we can see in the following proposition.

PROPOSITION 3.4. *Let (X, T) be a minimal distal flow. If $h \in N_G(A)$ is such that the map $\phi_h(px_0) = phx_0$ is well defined then it is an automorphism.*

Proof. For distal (X, x_0, T) we have $Gx_0 = X$. Suppose $gx_0 \neq g'x_0$ for some $g, g' \in G$ but $\phi_h(gx_0) = \phi(g'x_0)$. Then $gx_0 \neq g'x_0$ implies that $b = g^{-1}g' \notin A$, and $ghx_0 = g'hx_0$, hence $x_0 = h^{-1}g^{-1}g'hx_0$, implies that $h^{-1}bh \in A$. Thus $b \in hAh^{-1} \setminus A$, contradicting our assumption that $h \in N_G(A)$. □

We call $N_G(A)/A$ the group of *virtual automorphisms* of the minimal flow (X, T) .

Remark 3.5. In a recent work [Zu19] Zucker shows, following the previous works [CP] and [GTWZ], that for any countable groups T and H , with T infinite, there is a minimal, free, Cantor T -flow (X, T) such that the group H embeds into $\text{Aut}(X, T)$.

4. Semiregular flows

The notion of regular minimal flows was introduced in [A66]. A minimal flow (X, T) is *regular* if, for any pair of points $x, y \in X$, there is an automorphism $\psi \in \text{Aut}(X, T)$ such that the pair $(x, \psi(y))$ is proximal. Equivalently, we can say that (X, T) is regular if and only if, (for $x_0 \in uX$ and) for every $g \in G$, there is an automorphism $\psi \in \text{Aut}(X, T)$ such that $gx_0 = \psi(x_0)$.

Definition 4.1. We say that a pointed minimal flow (X, x_0, T) (with $\mathfrak{G}(X, x_0) = A$), is *semiregular* (SR) if, for every $h \in N_G(A)$ there is an automorphism $\psi \in \text{Aut}(X, T)$ such that $\psi(x_0) = hx_0$. More generally, given a subgroup $\Gamma < N_G(A)$, we say that (X, x_0, T) is Γ -*semiregular* if for every $h \in \Gamma$ there is an automorphism $\psi \in \text{Aut}(X, T)$ such that $\psi(x_0) = hx_0$. (See also [H].)

Thus a minimal flow is SR if and only if every virtual automorphism of (X, T) is realized, so that $N_G(A)/A \cong \text{Aut}(X, T)$.

Remark 4.2. The dependence on the various choices we made in order to formulate the definition of the SR property (namely the choice of M in βT , the choice of u in J and finally the choice of x_0 in uX) is either immaterial or has the effect of replacing a subgroup of G by some conjugate.

Examples 4.3.

- (1) Every regular flow is SR (clear).
- (2) Every minimal distal flow is SR (see §5).
- (3) Every minimal proximal flow is SR (a minimal proximal flow is regular).
- (4) If (X, T) is SR then so is its maximal highly proximal extension (see §7).
- (5) The Morse minimal set is SR (see §6).
- (6) The Sturmian minimal set is not SR (see §5).

- (7) Toeplitz flows are not SR (see §5).
- (8) More generally, a metrizable almost automorphic flow which is not equicontinuous is never SR (see §5).
- (9) A regular point distal minimal flow is distal (see §5).
- (10) Every minimal flow admits a proximal extension which is SR (see §7).

Remark 4.4. We note that an element $g \in G$ belongs to $N_G(A)$ if and only if $\mathfrak{G}(X, gx_0) = \mathfrak{G}(X, x_0) = A$. For a commutative T we have $\mathfrak{G}(X, tx_0) = A$ for every $t \in T$. Thus for commutative T , $Tu \subset N_G(A)$ and the set $\{x \in X : \mathfrak{G}(X, x) = A\}$ is dense in X .

One can easily relativize the notion of semiregularity.

Definition 4.5. Let $\pi : (X, x_0, T) \rightarrow (Y, y_0, T)$ be a homomorphism of pointed minimal flows. Let $\mathfrak{G}(X, x_0) = A < \mathfrak{G}(Y, y_0) = F$ (where $y_0 = \pi(x_0)$). We say that π is an *SR extension* if, for every $h \in N_F(A) = F \cap N_G(A)$, the map $\phi_h : X \rightarrow X$ defined by $\phi_h(px_0) = phx_0$ is a well-defined automorphism of (X, T) .

5. *Distal minimal flows are semiregular*

THEOREM 5.1. *Every minimal distal flow is SR.*

Proof. Let (X, x_0, T) be a minimal distal flow with $A = \mathfrak{G}(X, x_0)$. We have $X = Gx_0$ and, for $h \in N_G(A)$, we let $\phi_h : X \rightarrow X$ be defined by

$$\phi_h(px_0) = phx_0 \quad (p \in M).$$

To see that this is well defined suppose $px_0 = qx_0$ for some $p, q \in M$. We have to show that $phx_0 = qhx_0$. Now in a distal flow the enveloping semigroup $E(X, T)$ is a group and the image of M under the canonical map from βT onto $E(X, T)$ is surjective. Thus we can consider p, q as elements of the group $G = E(X, T)$. As $px_0 = qx_0$, it follows that $p^{-1}qx_0 = x_0$, hence $a = p^{-1}q \in A$. Thus $q = pa$ and we get

$$\phi_h(qx_0) = qhx_0 = pahx_0 = ph(h^{-1}ah)x_0 = phx_0 = \phi_h(px_0).$$

The continuity of ϕ_h follows from the continuity of right multiplication R_h on M , and it follows that ϕ_h is an endomorphism of (X, T) . Finally, as the same argument applies for $\phi_{h^{-1}}$, we conclude that $\phi_h \in \text{Aut}(X, T)$. □

Example 5.2. In [PW] the authors construct an example of a minimal metric cascade (X, T) which is not coalescent. If ϕ is an endomorphism of (X, T) which is not an automorphism, then $\phi = \phi_h$ for some $h \in G$ for which $h^{-1}Ah \subsetneq A$, where $A = \mathfrak{G}(X, x_0)$ for some choice of $x_0 \in X$. In fact, as we have seen above (Proposition 3.4), in a minimal distal flow $h^{-1}Ah = A$ would imply that $\phi = \phi_h$ is an automorphism.

PROPOSITION 5.3. *A minimal point distal regular flow is distal.*

Proof. Let x_0 be a distal point. Given $x \in X$, there is, by regularity, an automorphism $\psi \in \text{Aut}(X)$ such that the points $\psi(x)$ and x_0 are proximal. But x_0 being a distal point, we have $\psi(x) = x_0$, and it follows that x is also a distal point. Thus X is a distal flow. □

COROLLARY 5.4. *A minimal metric almost automorphic flow which is SR is actually equicontinuous. Thus Toeplitz flows and Sturmian like flows are not SR.*

Proof. By definition an almost automorphic flow is a metric minimal flow (X, T) such that the homomorphism $\pi : (X, T) \rightarrow (Y, T)$ from (X, T) onto its maximal equicontinuous factor (Y, T) is an almost one-to-one extension. Such a flow is point distal and it satisfies $\mathfrak{G}(X, x_0) = \mathfrak{G}(Y, y_0) = A \triangleleft G$. Thus $N_G(A) = G$ and semiregularity for (X, T) is the same as regularity. Now the previous proposition applies. \square

It is natural to ask whether a minimal, point distal, flow which is not distal can be SR. We will see in §6 below that the Morse minimal cascade (i.e. a \mathbb{Z} -flow), which is point distal, metric and not distal, is in fact SR.

We also have an analogous statement concerning distal extensions.

PROPOSITION 5.5. *A distal extension $\pi : (X, x_0, T) \rightarrow (Y, y_0, T)$ of pointed minimal flows is an SR extension.*

Proof. Let $\mathfrak{G}(X, x_0) = A < \mathfrak{G}(Y, x_0) = F$, and let $h \in N_F(A)$. We have to show that the map ϕ_h is well defined. Suppose, then, that $px_0 = qx_0$ for $p, q \in M$. Then $a = up^{-1}q \in A$, hence $h^{-1}ah = a' \in A$ and we have

$$phx_0 = pha'x_0 = ph(h^{-1}ah)x_0 = pahx_0 = pp^{-1}qhx_0 = vqhx_0, \tag{5.1}$$

where $v = v^2$ is the unique idempotent in J such that $vp = p$. Thus the points phx_0 and qhx_0 are proximal. On the other hand, we have $\pi(phx_0) = ph\pi(x_0) = phy_0 = py_0$ and also $\pi(qhx_0) = qh\pi(x_0) = qhy_0 = qy_0$. Since by assumption $px_0 = qx_0$ we also have $py_0 = qy_0$, so that phx_0 and qhx_0 are in the same π fiber. Since π is a distal extension we conclude that the points phx_0 and qhx_0 are both proximal and distal, whence equal. Thus ϕ_h is a well-defined element of $\text{End}(X, T)$ and, as the same argument applies to $\phi_{h^{-1}}$, we see that $\phi_h \in \text{Aut}(X, T)$. \square

A similar argument yields the following proposition.

PROPOSITION 5.6. *Let $\pi : (X, x_0, T) \rightarrow (Y, y_0, T)$ be a distal extension, with Y being SR. If, in addition, we assume that $N_G(A) \subset N_G(F)$, then X is also SR. In particular, this is the case when Y is regular.*

Proof. Let $h \in N_G(A)$. We show that $\phi_h(px_0)$ defined by $\phi_h(px_0) = phx_0$ ($p \in M$) is well defined. Assuming $px_0 = qx_0$, we have

$$\begin{aligned} \pi(phx_0) &= phy_0 = \tilde{\phi}_h(py_0), \\ \pi(qhx_0) &= qhy_0 = \tilde{\phi}_h(qy_0), \end{aligned}$$

where $\tilde{\phi}_h$ is the element of $\text{Aut}(Y)$ defined by h (recalling that Y is SR and that $N_G(A) \subset N_G(F)$). Since $px_0 = qx_0$ it follows that $py_0 = qy_0$, whence also $\tilde{\phi}_h(py_0) = \tilde{\phi}_h(qy_0)$. Thus the points phx_0 and qhx_0 lie in the same π -fiber, and therefore are distal. As in the proof of Proposition 5.5 (equation (5.1)), they are also proximal points and hence equal.

The last assertion follows since by regularity $F \triangleleft G$ and $N_G(A) \subset N_G(F) = G$. \square

6. The Morse minimal set is semiregular

We have already raised the question whether a minimal point distal, non-distal flow can be SR. In this section we will show that the classical Morse minimal set provides such an example. This example will also show that there is a minimal metric point distal SR \mathbb{Z} -flow which is not equicontinuous. This should be contrasted with the fact that a regular proximal-isometric (PI) \mathbb{Z} -flow which is not equicontinuous is necessarily non-metrizable (see [G192]). For a description and detailed analysis of the Morse minimal set we refer to [GH, Ch. 12].

THEOREM 6.1. *The Morse minimal set is semiregular.*

Proof. Let (X, S) denote the Morse minimal flow. Here we deviate from our usual notation and use the letter S to denote the shift homeomorphism that generates the \mathbb{Z} -flow on X , a subshift $X \subset \{0, 1\}^{\mathbb{Z}}$. We know that (X, S) has the structure

$$X \xrightarrow{\sigma} Y \xrightarrow{\theta} Z,$$

where (i) $\pi = \theta \circ \sigma$ is the homomorphism of (X, T) onto its maximal equicontinuous factor (a dyadic adding machine); (ii) θ is an almost one-to-one extension, and σ is a \mathbb{Z}_2 group extension. More precisely, there is a point $z_1 \in Z$ such that $\theta^{-1}(z_1) = \{y_1, \bar{y}_1\}$ and, for every point $z \in Z$ which is not in the orbit of z_1 , we have that $\theta^{-1}(z)$ is a singleton. Finally, on X there is an involution κ (a self-homeomorphism satisfying $\kappa^2 = \text{id}$) which commutes with the shift. (It sends the sequence $x \in X \subset \{0, 1\}^{\mathbb{Z}}$ into the ‘flipped’ sequence $\kappa(x) = x'$, where $x'(n) = x(n)'$, and $0' = 1, 1' = 0$.) The map $\sigma : X \rightarrow Y$ is then the quotient map under the action of $\mathbb{Z}_2 = \{\text{id}, \kappa\}$. We let $\pi^{-1}(z_1) = \sigma^{-1}\{y_1, \bar{y}_1\} = \{x_1, x'_1, \bar{x}_1, \bar{x}'_1\}$.

Next fix a point $x_0 \in uX$ which is not on the orbits of the four points $\{x_1, x'_1, \bar{x}_1, \bar{x}'_1\}$. We let $y_0 = \sigma(x_0)$ and $z_0 = \theta(y_0)$. Also set $\mathfrak{G}(X, x_0) = A$ and $\mathfrak{G}(Y, y_0) = \mathfrak{G}(Z, z_0) = F$. We then have $F \triangleleft G$ and $Z \cong G/F$, and $A \triangleleft F$ and $\mathbb{Z}_2 \cong F/A$. It is also shown in [GH] that the group $\text{Aut}(X, S)$ is the group $\{S^n : n \in \mathbb{Z}\} \oplus \{\text{id}, \kappa\} \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Since $\theta^{-1}(z_0) = \{y_0\}$, it follows that $\pi^{-1}(z_0) = \sigma^{-1}(y_0) = \{x_0, x'_0\}$ and both x_0 and x'_0 are distal points.

Now let x be an arbitrary point in uX and set $W = \overline{\mathcal{O}(x_0, x)}$, the orbit closure of the point $(x_0, x) \in X \times X$. Because $x \in uX$, the point (x_0, x) is an almost periodic point of the product flow $X \times X$, so that W is a minimal flow. Also, there exists $g \in G$ with $x = gx_0$. Let $y = \sigma(x)$ and $z = \pi(x) = \theta(y)$.

We claim that, for every $(a, b) \in W$, we have

$$W[a] := \{c \in X : (a, c) \in W\} \subseteq \{a\} \times \pi^{-1}(\pi(b)).$$

To see this note that if $(x_0, c) \in W$ then $(x_0, c) = p(x_0, x)$ for some $p \in M$ and

$$(\pi \times \pi)(x_0, c) = (\pi \times \pi)(p(x_0, x)) = (pz_0, p\pi(x)) = (z_0, pz).$$

Since Z is equicontinuous, $pz_0 = z_0$ implies that $pz = z$, so that $\pi(c) = z$. Thus

$$W[x_0] = \{c \in X : (x_0, c) \in W\} \subseteq \{x_0\} \times \pi^{-1}(z),$$

and therefore $W[p x_0] \subseteq \pi^{-1}(\pi(p x))$ for every $p \in M$, as claimed.

We now consider two possible cases.

Case 1: Suppose that $(x_0, x') = (x_0, \kappa(x)) \in W$. (Note that this implies that $(\text{id} \times \kappa)(W) \cap W \neq \emptyset$, whence $(\text{id} \times \kappa)(W) = W$.) Since $(x_0, x') \in uW = G(x_0, x)$, there exists $h \in G$ such that $h(x_0, x) = (x_0, x')$. In particular, $hx_0 = x_0$, whence $h \in A$. On the other hand, $x' = \kappa(x) = \kappa(gx_0) = gx'_0 = hx = hgx_0$, hence $x'_0 = g^{-1}hgx_0$, hence $g^{-1}hg \notin A$; that is, $g \notin N_G(A)$.

Case 2: Now assume that $(x_0, x') \notin W$. As in case 1, we deduce that

$$\text{if } (a, b) \in W \text{ then } (a, \kappa(b)) \notin W. \tag{6.1}$$

Let P denote the projection map from W onto X (as its first coordinate). We have $P^{-1}(a) = W[a] \subseteq \{a\} \times \sigma^{-1}(\theta^{-1}(z))$.

Claim. For every $a \in X$, we have $P^{-1}(a) = \{(a, \phi(a))\}$ for a surjection $\phi : X \rightarrow X$.

We recall here that the enveloping semigroup $E(X, S)$ of the Morse flow has exactly two minimal ideals, say I_1, I_2 , each containing exactly two idempotents, say $J_1 = \{u, v\} \subset I_1$ and $J_2 = \{\tilde{u}, \tilde{v}\} \subset I_2$ (see [HJ] and [S]). Now for the point a we have at least one of the possibilities $ua = a$ or $va = a$. We will assume that $ua = a$ and the other case is treated in the same way, with v replacing our usual u .

Fix some $b \in X$ with $(a, b) \in W$ and $ub = b$. We write $\eta = \sigma(b)$ and $\zeta = \pi(b) = \theta(\eta)$.

To prove our claim we again consider two cases.

Case 2a: The point η is a non-split point; that is, $\theta^{-1}(\zeta) = \{\eta\}$.

In this case we have $P^{-1}(a) \subseteq \{a\} \times \sigma^{-1}(\eta) = \{a\} \times \{b, b'\}$, and, since by assumption $(a, b') \notin W$, we indeed have $P^{-1}(a) = \{(a, b)\}$, as claimed (putting $b = \phi(a)$).

Case 2b: The point η is a split point; that is, $\theta^{-1}(\zeta) = \{\eta, \bar{\eta}\}$, with $\eta \neq \bar{\eta}$.

In this case we have $\pi^{-1}(\zeta) = \{b, b', \bar{b}, \bar{b}'\}$. Suppose $(a, \bar{b}) \in W$ (the case $(a, \bar{b}') \in W$ is symmetric). By the general theory of enveloping semigroups there is in I_2 a unique idempotent \tilde{u} equivalent to u ; that is, $u\tilde{u} = \tilde{u}$ and $\tilde{u}u = u$ (see, for example, [GI76, I, Proposition 2.5]). Both u and \tilde{u} act as the identity on uX . However, as they are distinct elements of $E(X, S)$, we must have

$$\begin{aligned} u(b, b', \bar{b}, \bar{b}') &= (b, b', b, b'), \\ \tilde{u}(b, b', \bar{b}, \bar{b}') &= (b, b', b', b) \end{aligned} \tag{6.2}$$

(or vice versa). (In fact, if $u\bar{b} = \tilde{u}\bar{b}$ then also $uS^n\bar{b} = \tilde{u}S^n\bar{b}$ and $uS^n\bar{b}' = \tilde{u}S^n\bar{b}'$ for every $n \in \mathbb{Z}$, whence $u = \tilde{u}$, which is impossible.) Thus we have $\tilde{u}(a, \bar{b}) = (a, b') \in W$ (or $u(a, \bar{b}) = (a, b') \in W$), contradicting our assumption that $(a, b') \notin W$. We have shown, in view of (6.1), that indeed $P^{-1}(a) = \{(a, b)\}$, and we let $\phi(a) = b$.

Given $c \in X$, there is a point $a \in X$ such that $(a, c) \in W$, and as also $(a, \phi(a)) \in W$, we have $\phi(a) = c$. This shows that ϕ is surjective and our claim is proven.

We now have $W = \{(a, \phi(a)) : a \in X\}$, and it follows that $\phi \in \text{End}(X, S)$. Since $g^{-1}(x_0, x) = g^{-1}(x_0, gx_0) = (g^{-1}x_0, x_0) \in W$ we conclude, by symmetry, that $\phi \in \text{Aut}(X)$.

To sum up, we have shown that, for every $h \in G$, either $h \notin N_G(A)$ or $\phi_h \in \text{Aut}(X, S)$; in other words, we have shown that (X, S) is SR. Thus the group of virtual automorphisms $N_G(A)/A$ is realized as $\text{Aut}(X, S) = \{S^n : n \in \mathbb{Z}\} \oplus \{\text{id}, \kappa\} \cong \mathbb{Z} \oplus \mathbb{Z}_2$. □

Remark 6.2. Using the notation of [GH], we can write $\{x_1, x'_1, \bar{x}_1, \bar{x}'_1\} = \{\mu, \mu', \nu, \nu'\}$, where

$$\begin{aligned} \mu &= \check{Q}Q, & \mu' &= \check{Q}'Q', \\ \nu &= \check{Q}'Q, & \nu' &= \check{Q}Q'. \end{aligned}$$

From this description it follows immediately that the pairs $\{\nu, \mu\} = \{\bar{x}_1, x_1\}$ and $\{\nu, \mu'\} = \{\bar{x}'_1, x'_1\}$ are positively asymptotic and negatively asymptotic pairs, respectively, hence proximal. This directly implies (6.2). In fact, this argument can be used to prove that indeed $E(X, T)$ has exactly two minimal ideals, each having exactly two idempotents.

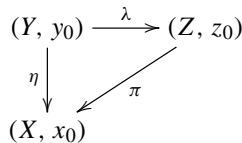
Remark 6.3. We note that the proof of Theorem 6.1 shows also that the Morse flow is coalescent.

Remark 6.4. After a first version of the present work was posted on the arXiv, a paper by Kellendonk and Yassawi appeared in the arXiv in which the authors generalize Theorem 6.1 and show that the dynamical system corresponding to a primitive aperiodic bijective substitution is SR [KY, Corollary 5.9]. See also the recent work of P. Staynova [S] from which the same result can be deduced.

7. *Every minimal flow admits a proximal extension which is SR*

THEOREM 7.1. *Let (X, x_0, T) be a minimal flow with $\mathfrak{G}(X, x_0) = A$, and let $\Gamma \leq N_G(A)$ be a subgroup. Then there exist a minimal pointed flow $Z_\Gamma = (Z, z_0, T)$ and a homomorphism $\pi : Z \rightarrow X$ such that:*

- (1) $\mathfrak{G}(Z, z_0) = A$ (so that π is a proximal extension);
- (2) $\Gamma A/A \leq \text{Aut}(Z)$;
- (3) for every minimal flow (Y, y_0, T) which is a proximal extension $\eta : (Y, y_0, T) \rightarrow (X, x_0, T)$ such that $\Gamma A/A < \text{Aut}(Y)$, there is a commutative diagram:



Proof. Let $C = \{hx_0 : h \in \Gamma\} \subset X$. Let $z_0 \in X^C$ be the point

$$z_0(hx_0) = hx_0, \quad (h \in \Gamma).$$

Let $Z = \overline{\mathcal{O}_T(z_0)} \subset X^C$ (alternatively, $Z = \bigvee_{h \in \Gamma} (X, hx_0)$). Because $C \subseteq uX$, it follows that the point z_0 is an almost periodic point of the product flow X^C , hence Z is a minimal flow. The projection on the $x_0 \in C$ coordinate is a homomorphism $\pi : (Z, z_0, T) \rightarrow (X, x_0, T)$.

Clearly $az_0 = z_0$ for every $a \in A$, so that $A \subseteq \mathfrak{G}(Z, z_0)$. Conversely, if $bz_0 = z_0$ for some $b \in G$ then

$$(bz_0)(x_0) = bz_0(x_0) = bx_0 = z_0(x_0) = x_0,$$

hence $b \in A$, $\mathfrak{G}(Z, z_0) \subseteq A$, hence $\mathfrak{G}(Z, z_0) = A$.

For $h \in \Gamma$, the map

$$\phi_h : Z \rightarrow Z, \quad \phi_h(pz_0) = phz_0 \quad (p \in M)$$

is well defined. In fact if $pz_0 = qz_0$ for $p, q \in M$, then, for all $h' \in \Gamma$,

$$\begin{aligned} (phz_0)(h'x_0) &= ph(z_0(h'x_0)) = phh'x_0 = pz_0(hh'x_0), \\ (qhz_0)(h'x_0) &= qh(z_0(h'x_0)) = qhh'x_0 = qz_0(hh'x_0). \end{aligned}$$

As Γ is a group, $hh' \in \Gamma$ and $hh'x_0 \in C$. Therefore, by our assumption, $phz_0(hh'x_0) = qhz_0(hh'x_0)$, whence $phz_0 = qhz_0$. We conclude, as in §3, that $\phi_h \in \text{Aut}(Z)$. We thus get $\Gamma A/A \leq \text{Aut}(Z)$.

Next assume that (Y, y_0, T) is as in (3). By assumption $\Gamma A/A \leq \text{Aut}(Y)$ and it follows that, for each $h \in \Gamma$, the function $\phi_h(py_0) = phy_0$ is a well-defined element of $\text{Aut}(Y, T)$. Thus the map $\eta \circ \phi_h : Y \rightarrow X$ satisfies

$$(\eta \circ \phi_h)(y_0) = \eta(hy_0) = hx_0,$$

and $\eta \circ \phi_h : (Y, y_0, T) \rightarrow (X, hx_0, T)$ is a homomorphism of pointed flows. It then follows that there is a homomorphism

$$\lambda : (Y, y_0, T) \rightarrow \bigvee_{h \in \Gamma} (X, hx_0, T) \cong Z_\Gamma$$

which satisfies $\eta = \pi \circ \lambda$. □

Remark 7.2. Note that when (X, T) is metrizable and the group $\Gamma A/A$ is countable, the SR flow Z_Γ is also metrizable. In particular, this is the case for $\Gamma = \langle \gamma_0 \rangle = \{\gamma_0^n : n \in \mathbb{Z}\}$ for some $\gamma_0 \in N_G(A)$.

COROLLARY 7.3. *Every minimal flow admits a proximal extension which is SR.*

Proof. Take $\Gamma = N_G(A)$. □

Definition 7.4. We write X_{SR} for the flow $Z_{N_G(A)}$. With this notation it is easy to check that (X, T) is SR if and only if $X = X_{\text{SR}}$.

Example 7.5. Let (Z, R_α) denote the rotation by $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ on the circle $Z = \mathbb{R}/\mathbb{Z}$, $R_\alpha(x) = z + \alpha \pmod{1}$. Let (X, S) be the Sturmian flow, with $\pi : X \rightarrow Z$ being its maximal equicontinuous factor. Then the flow Z is regular and the extension π is almost one-to-one (so that X is almost automorphic). The regularizer of (X, S) , which is the same as X_{SR} , is the Ellis two-circles flow \tilde{X} , a non-metrizable flow (see, for example, [GMe, Example 14.10] for more details on this). We have $\mathfrak{G}(Z, z_0) = \mathfrak{G}(X, x_0) = \mathfrak{G}(\tilde{X}, \tilde{x}_0) = A$, and we observe that the virtual automorphism group $N_G(A)/A$ is realized on Z as the compact group $\text{Aut}(Z) \cong \mathbb{T} = \mathbb{R}/\mathbb{Z}$, on \tilde{X} again as $\text{Aut}(\tilde{X}) \cong \mathbb{R}/\mathbb{Z}$, but with the discrete topology, and it is mostly non-realizable on X , where $\text{Aut}(X) = \{S^n : n \in \mathbb{Z}\}$.

Given any minimal flow (X, T) , we will next describe another natural construction that yields an SR flow X^{SR} which is a proximal extension of X . In the collection of all the minimal flows which are SR and are proximal extensions of X , the flow X_{SR} is the minimum and the flow X^{SR} is the maximum (with respect to being a factor).

Let (X, x_0, T) be a minimal flow with $\mathfrak{G}(X, x_0) = A$. Let $X^{\text{SR}} = \Pi(A)$ denote the minimal flow which is the maximal proximal extension of X . As described in [GI76], this flow can be presented as a quasifactor of M , as follows:

$$\Pi(A) = \{p \circ A : p \in M\}.$$

Moreover, the map $\pi : M \rightarrow \Pi(A)$ is such that $\pi^{-1}(p \circ (u \circ A)) = p \circ A$, for all $p \in M$. In particular, the elements of $\Pi(A)$ form a partition of M . (For more details on the quasifactor $\Pi(A)$ and the circle operation see [GI76, Ch. IX].)

LEMMA 7.6. For $h \in N_G(A)$ and $p = \lim t_i \in M$,

$$ph \circ A = p \circ hA = (p \circ A)h.$$

Proof. We have

$$\begin{aligned} ph \circ A &= p \circ h \circ A \supseteq p \circ hA \\ &= \{\lim t_i h a_i : a_i \in A\} = \{\lim t_i a'_i h : a'_i \in A\} \\ &= p \circ Ah = (p \circ A)h \in \Pi(A). \end{aligned}$$

However, as the elements of $\Pi(A)$ form a partition of M , we get $ph \circ A = p \circ hA = (p \circ A)h$. □

PROPOSITION 7.7. For every pointed minimal flow (X, x_0, T) with $\mathfrak{G}(X, x_0) = A$, the universal minimal proximal extension $\Pi(A)$ of X (which depends only on A) is semiregular.

Proof. Given $h \in N_G(A)$ and $p \in M$, we set $\phi_h(p \circ A) = ph \circ A$. This is well defined since if $p \circ A = q \circ A$ then, by Lemma 7.6,

$$\begin{aligned} ph \circ A &= p \circ hA = p \circ Ah = (p \circ A)h \\ &= (q \circ A)h = q \circ Ah = q \circ hA = qh \circ A. \end{aligned} \quad \square$$

Remark 7.8. Since every τ -closed subgroup A of G (see, for example, [GI76, Ch. IX]) is the Ellis group of a minimal flow, namely $A = \mathfrak{G}(\Pi(A), u \circ A)$, we get, in view of Proposition 7.7, that $N_G(A)/A \cong \text{Aut}(\Pi(A))$, hence we conclude that for every τ -closed subgroup $A < G$, the group $N_G(A)/A$ is realized as an actual automorphism group of some minimal flow.

We end this section with the following proposition (for information on the maximal highly proximal extension of a minimal flow see [AG]).

PROPOSITION 7.9. If (X, T) is SR then so is its maximal highly proximal extension.

Proof. One way to describe the maximal highly proximal extension X^* of X is as the quasifactor of M obtained from a homomorphism $\pi : M \rightarrow X$, as follows:

$$X^* = \{p \circ \pi^{-1}(x_0) : p \in M\},$$

where $x_0 \in uX$ and $\pi(p_0) = px_0$, ($p \in M$). As with the quasifactor $\Pi(A)$, it can be shown that $\{p \circ \pi^{-1}(x_0) : p \in M\}$ is a partition of M . As X is SR we have $N_G(A)/A \cong \text{Aut}(X)$

and it suffices to show that every $\psi \in \text{Aut}(X)$ lifts to an automorphism ψ^* of X^* . We define $\psi^*(p \circ \pi^{-1}(x_0)) = p \circ \pi^{-1}(\psi(x_0))$, $p \in M$. If $p \circ \pi^{-1}(x_0) = q \circ \pi^{-1}(x_0)$, for $p, q \in M$, then also $px_0 = qx_0$, whence

$$\psi(px_0) = \psi(qx_0) \in p \circ \pi^{-1}(\psi(x_0)) \cap q \circ \pi^{-1}(\psi(x_0)),$$

hence $p \circ \pi^{-1}(\psi(x_0)) = q \circ \pi^{-1}(\psi(x_0))$. (It is not hard to check that $p \circ \pi^{-1}(x) \in X^*$ for every $p \in M$ and every $x \in X$.)

An alternative proof is as follows. If (X, T) is a minimal flow then, as a topological space, the maximal highly proximal extension of X , say X^* , is the Stone space of the Boolean algebra of regular open sets in X (see, for example, [Zu]). It thus follows that every self-homeomorphism of X lifts to X^* , and clearly an automorphism of (X, T) lifts to an automorphism of (X^*, T) . □

Remark 7.10. With a given minimal dynamical system (X, T) one can always associate its unique regularizer as follows. Let u be a minimal idempotent and let $x_0 \in X$ satisfy $ux_0 = x_0$. Let $A = \mathfrak{G}(X, x_0)$ and set

$$\text{Reg}(X) = \bigvee \{(X, gx_0) : g \in G\} \subset X^G.$$

Then the minimal flow $\text{Reg}(X)$ is a regular flow which extends X and such that, for any other regular flow Y which admits X as a factor, there is a factor map $Y \rightarrow \text{Reg}(X)$. Denoting

$$A_0 = \bigcap \{gAg^{-1} : g \in G\},$$

we have $\mathfrak{G}(\text{Reg}(X), z_0) = A_0$. Here $z_0 \in \text{Reg}(X)$ is the point of X^G whose g -coordinate is gx_0 , so that $\text{Reg}(X)$ is the orbit closure of z_0 in X^G . (One can easily show that, up to isomorphism, this construction does not depend on the choices of M and u .)

Note, however, that whereas the extension $X_{\text{SR}} \rightarrow X$ is always a proximal extension, the extension $\text{Reg}(X) \rightarrow X$ will be a proximal one when and only when the group A is normal in G .

8. A Koopman representation of the virtual automorphism group

When a minimal flow (X, T) is strictly ergodic (i.e. it admits a T -invariant probability measure and this measure is unique), the group $\text{Aut}(X, T)$ also preserves this measure. In fact, if μ is the T -invariant measure on X and $\psi \in \text{Aut}(X, T)$, then clearly the pushforward measure $\psi_*(\mu)$ is T -invariant as well, and thus $\psi_*(\mu) = \mu$ by uniqueness.

The *Koopman representation* of T associated to the measure-preserving system (X, \mathcal{X}, μ, T) is the representation on the Hilbert space $L_2(\mu)$ given by $t \mapsto U_t$, where for $t \in T$ the unitary operator U_t is defined by $U_t(f) = f \circ t^{-1}$.

Now in a special case we are able to show that also the virtual automorphism group $N_G(A)/A$ admits such a faithful representation.

THEOREM 8.1. *Let (X, T) be a minimal, metrizable, point distal, uniquely ergodic flow with T -invariant probability measure μ , and suppose that X is measure-regular, where the latter property means that the T -invariant G_δ set $X_0 \subseteq X$ consisting of distal points has measure 1. Then each element of the virtual group automorphisms of the flow defines an*

automorphism of the measure space (X, \mathcal{X}, μ) and this correspondence defines a unitary representation of the virtual group of automorphisms as a group of unitary operators on the separable Hilbert space $L_2(\mu)$.

Proof. As usual, we pick a point $x_0 \in X_0$, so necessarily $ux_0 = x_0$, then let $A = \mathfrak{G}(X, x_0)$. Our virtual automorphism group is the group $N_G(A)/A$. Clearly $uX = Gx_0 \supseteq X_0$. Next define, for $h \in N_G(A)$, $\phi_h : uX \rightarrow uX$ by

$$\phi_h(gx_0) = ghx_0 \quad (g \in G).$$

Then the map ϕ_h is well defined and it is a homeomorphism of the (usually not even measurable) set uX . In fact, if $gx_0 = g'x_0$ then $g^{-1}g'x_0 = x_0$, hence $a = g^{-1}g' \in A$ and

$$g'hx_0 = g(g^{-1}g')hx_0 = gahx_0 = gh(h^{-1}ah)x_0 = ghx_0.$$

The continuity of ϕ_h follows from the continuity of right multiplication on G . Moreover, we have $t\phi_h = \phi_h t$ for all $t \in T$.

Now $\phi_h : X_0 \rightarrow \phi_h(X_0)$ is a homeomorphism and it pushes the measure μ on X_0 to a measure $(\phi_h)_*(\mu)$ on $\phi_h(X_0)$. By uniqueness $(\phi_h)_*(\mu) = \mu$.

In particular, $\mu(X_0 \cap \phi_h(X_0)) = 1$. Similarly, $\mu(X_0 \cap \phi_h^n(X_0)) = 1$ for every $n \in \mathbb{Z}$ and we conclude that the T -invariant dense G_δ set $X_\infty = \bigcap_{n \in \mathbb{Z}} \phi_h^n(X_0)$ has measure 1. Since it is also ϕ_h -invariant, this shows that ϕ_h is an automorphism of the measure space (X, \mathcal{X}, μ) .

Now the composition map $U_h : f \mapsto f \circ \phi_h^{-1}$ defines a unitary operator on $L_2(\mu)$ and the map $h \mapsto U_h$ from $N_G(A)/A \rightarrow \mathcal{U}(L_2(\mu))$ is the desired unitary representation. \square

9. Questions

We conclude with the following short list of related questions.

- (1) Given a τ -closed subgroup A of G (see, for example, [G176, Ch. IX]), when is there a minimal metric pointed flow (X, x_0, T) with $\mathfrak{G}(X, x_0) = A$?
- (2) Given a minimal metric pointed flow (X, x_0, T) when is there a metric SR proximal extension of X ?
- (3) When is there a unitary representation of $N_G(A)/A$ on a separable Hilbert space?

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