

Integrable zero-Hopf singularities and three-dimensional centres

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In this paper we show that the well-known Poincaré–Lyapunov non-degenerate analytic centre problem in the plane and its higher-dimensional version, expressed as the three-dimensional centre problem at the zero-Hopf singularity, have a lot of common properties. In both cases the existence of a neighbourhood of the singularity in the phase space completely foliated by periodic orbits (including equilibria) is characterized by the fact that the system is analytically completely integrable. Hence its Poincaré–Dulac normal form is analytically orbitally linearizable. There also exists an analytic Poincaré return map and, when the system is polynomial and parametrized by its coefficients, the set of systems with centres corresponds to an affine variety in the parameter space of coefficients. Some quadratic polynomial families are considered.

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continua of periodic orbits; Poincaré map

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1. Introduction and statement of the main results

In this work, we consider an analytic three-dimensional system

$$\left. \begin{aligned} \dot{x} &= -y + F_1(x, y, z), \\ \dot{y} &= x + F_2(x, y, z), \\ \dot{z} &= F_3(x, y, z), \end{aligned} \right\} \quad (1.1)$$

where $\mathcal{F} = (F_1, F_2, F_3): \mathcal{U} \rightarrow \mathbb{R}^3$ is a real analytic vector field on the neighbourhood of the origin $\mathcal{U} \subset \mathbb{R}^3$ with $\mathcal{F}(0) = 0$ and whose Jacobian matrix $D\mathcal{F}(0) = 0$. The origin is a *zero-Hopf* (also called a *fold-Hopf*) singularity of system (1.1) because its associated eigenvalues are $\{\pm i, 0\}$ with $i^2 = -1$.

Since the linear part of system (1.1) generates a rotation, it makes sense to extend the classical Poincaré–Lyapunov centre problem for planar analytic vector fields to the zero-Hopf singularity. The origin of system (1.1) will be called a *three-dimensional centre* if there is a neighbourhood of it completely foliated by periodic orbits of (1.1), including continua of equilibria as trivial periodic orbits. On the other hand, we say that system (1.1) is *completely analytically integrable* if it admits two independent locally analytic first integrals.

REMARK 1.1. It is well known that there are vector fields without singular points only on odd-dimensional spheres. A consequence of this topological result is that isolated singularities of vector fields in \mathbb{R}^n having a punctured open neighbourhood filled by non-trivial periodic orbits may only exist for even phase space dimension n . Therefore, since $n = 3$ for system (1.1), any three-dimensional centre at the origin of (1.1) is a non-isolated singularity and, consequently, there is an invariant curve Γ filled by equilibria passing through it tangent to the z -axis. For any analytic vector field in \mathbb{R}^3 with an isolated singularity, it was proved in [3] that there is always a solution tending to the singularity (in the future or in the past) with a well-defined tangent.

In fact, applying formal normal-form theory to (1.1) yields the existence of a formally invariant one-dimensional manifold given by the z -axis (see § 4). The reader is also referred to [2], where the existence of the rotation axis in the C^∞ case is proved. Indeed, we will see that the formal invariant one-dimensional manifold really does exist in the analytic category too.

PROPOSITION 1.2. *System (1.1) possesses a one-dimensional local analytic invariant manifold Γ tangent to the z -axis at the origin. In particular, there are local analytic coordinates tangent to the identity $(x, y, z) \mapsto \phi(x, y, z) = (x + \dots, y + \dots, z + \dots)$ stretching the manifold Γ towards the z -axis and transforming (1.1) into*

$$\dot{x} = -y + \tilde{F}_1(x, y, z), \quad \dot{y} = x + \tilde{F}_2(x, y, z), \quad \dot{z} = \tilde{F}_3(x, y, z), \quad (1.2)$$

with $\tilde{F}_j(0, 0, z) = 0$ for $j = 1, 2$.

Proof. In any case, be the origin a three-dimensional centre of (1.1) or not, there is always a one-dimensional local analytic invariant manifold Γ of (1.1) tangent to the z -axis at the origin. This fact is a consequence of the spectrum structure of the linear part of the analytic system (1.1): one eigenvalue is real and the others are not (see [6]). From here, the existence of the local analytic change of coordinates ϕ stretching Γ follows. Clearly, since the linear part of ϕ is the identity, the linear parts of (1.1) and (1.2) are equal. The fact that $\tilde{F}_j(0, 0, z) = 0$ for $j = 1, 2$ comes from the invariance of the z -axis in the new coordinates. \square

Taking cylindrical coordinates $(x, y, z) \mapsto (\varphi, \rho, z)$ with $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, system (1.2) becomes

$$\begin{aligned} \dot{\rho} &= \hat{\mathcal{R}}(\varphi, \rho, z) = \cos \varphi \hat{F}_1(\varphi, \rho, z) + \sin \varphi \hat{F}_2(\varphi, \rho, z), \\ \dot{\varphi} &= 1 + \hat{\Theta}(\varphi, \rho, z) = 1 + \frac{1}{\rho} [\cos \varphi \hat{F}_2(\varphi, \rho, z) - \sin \varphi \hat{F}_1(\varphi, \rho, z)], \\ \dot{z} &= \hat{F}_3(\varphi, \rho, z), \end{aligned}$$

where $\hat{F}_i(\varphi, \rho, z) = \tilde{F}_i(\rho \cos \varphi, \rho \sin \varphi, z)$ for $i = 1, 2, 3$. This system is analytic at $\rho = 0$ since $\tilde{F}_j(0, 0, z) = 0$ for $j = 1, 2$. Indeed, the set $\{\rho = 0\}$ is invariant because $\hat{\mathcal{R}}(\varphi, 0, z) = 0$. We remark that $\dot{\varphi} > 0$ for ρ and $|z|$ sufficiently small. Summarizing, after transforming to cylindrical coordinates, system (1.1) can be written in the

neighbourhood \mathcal{U} of the origin as the analytic system

$$\frac{d\rho}{d\varphi} = \hat{R}(\varphi, \rho, z), \quad \frac{dz}{d\varphi} = \hat{Z}(\varphi, \rho, z) \tag{1.3}$$

with invariant set $\{\rho = 0\}$ and defined on the cylinder $\{(\varphi, \rho, z) \in \mathbb{S}^1 \times \mathbb{R}^2\}$ with ρ and $|z|$ sufficiently small taking $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Using the solutions of system (1.3) near $(\rho, z) = (0, 0)$ for $\varphi \in \mathbb{S}^1$, it is easy to construct a Poincaré return map of (1.2), and hence a displacement map $\Delta(\rho_0, z_0)$.

REMARK 1.3. Notice that cylindrical coordinates are well adapted to analyse periodic orbits of system (1.2) near the origin but not for the pullback system (1.1). The reason is that, in general, (1.1) written in cylindrical coordinates gives a system that is non-analytic at $r = 0$. Additionally, another problem arises since, in general, system (1.2) is analytic but not polynomial even when (1.1) is a polynomial system. Although the main difficulty, of course, is that we do not know the analytic diffeomorphism ϕ stretching Γ .

We will solve the problems identified in remark 1.3 by changing cylindrical coordinates to the polar blow-up (1.4), although other difficulties will appear (see remark 1.5). The next theorem is an adaptation of a result stated in [5]. It is based on the polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined by

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = rw. \tag{1.4}$$

Observe that (1.4) explodes the origin $(x, y, z) = (0, 0, 0)$ into $\{r = 0\}$ but there is no image of the z -axis except for the origin. In other words, (1.4) is defined in $\Omega = \mathcal{U} \setminus \{(0, 0, z) \in \mathbb{R}^3 : z \neq 0\}$ and is a diffeomorphism in $\Omega \setminus \{(0, 0, 0)\}$.

THEOREM 1.4. *We consider system (1.1) defined on a neighbourhood $\mathcal{U} \subset \mathbb{R}^3$ of the origin. Let $\delta > 0$ be sufficiently large but fixed and define $\mathcal{C}_\delta = \{(x, y, z) \in \mathcal{U} : z^2 > \delta(x^2 + y^2)\}$, a thin solid cone with vertex at the origin surrounding the z -axis. Performing the polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined by (1.4), system (1.1) can be written in $\mathcal{U} \setminus \mathcal{C}_\delta$, for $|r|$ sufficiently small, as the analytic system*

$$\frac{dr}{d\theta} = R(\theta, r, w), \quad \frac{dw}{d\theta} = W(\theta, r, w) \tag{1.5}$$

with invariant set $\{r = 0\}$ and defined on the cylinder $\{(\theta, r, w) \in \mathbb{S}^1 \times \mathbb{R} \times \mathcal{K}\}$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{K} = \{w \in \mathbb{R} : |w| \leq \delta\}$.

Let us define the Poincaré translation map $\Pi(r_0, w_0)$ associated with (1.5) as $\Pi(r_0, w_0) = \Psi(2\pi; r_0, w_0)$, where $\Psi(\theta; r_0, w_0) = (r(\theta; r_0, w_0), w(\theta; r_0, w_0))$ is the solution of (1.5) having initial condition $\Psi(0; r_0, w_0) = (r_0, w_0) \in \mathbb{R} \times \mathcal{K}$ with $|r_0|$ sufficiently small. We also define the displacement map $d(r_0, w_0) = \Pi(r_0, w_0) - \text{Id}(r_0, w_0)$, where Id denotes the identity map. We emphasize that d is an analytic function at $(r_0, w_0) \in \mathbb{R} \times \mathcal{K}$ with $|r_0| \ll 1$ due to the analyticity of (1.5).

REMARK 1.5. The need to restrict the values of w to the arbitrary but fixed compact set \mathcal{K} containing the origin is clarified in the proof of theorem 1.4. Now, we recall the geometry associated with the polar blow-up (1.4). First, we notice the

key point that $(x, y, z) \in \mathcal{U} \setminus \mathcal{C}_\delta$ when $w \in \mathcal{K}$. Indeed, (1.4) is a diffeomorphism in $\mathcal{U} \setminus \mathcal{C}_\delta$. Once we set the value of δ , the cone \mathcal{C}_δ is fixed and, consequently, in principle we cannot control (via the zeros of the displacement map d) the periodicity of those orbits of (1.1) in \mathcal{U} that intersect \mathcal{C}_δ . Despite this, although the polar blow-up (1.4) does not cover the neighbourhood \mathcal{U} of the origin, the gap \mathcal{C}_δ can be made very thin provided that we take δ large enough. In this direction, we emphasize that any periodic orbit of (1.1) in \mathcal{U} not intersecting the z -axis is contained in $\mathcal{U} \setminus \mathcal{C}_\delta$ for δ sufficiently large. Thus, in the three-dimensional centre case, most of the continuum of periodic orbits of (1.1) in \mathcal{U} are completely contained in $\mathcal{U} \setminus \mathcal{C}_\delta$. Each one of these orbits corresponds to a 2π -periodic solution of system (1.5) with $|r|$ sufficiently small and $w(\theta; r_0, w_0) \in \mathcal{K}$ for all θ . Consequently, the zeros of the displacement map $d(r_0, w_0)$, with $w_0 \in \mathcal{K}$ and $|r_0| \ll 1$, pick up all these periodic orbits. Thus, if the origin is a three-dimensional centre of (1.1), then $d(r_0, w_0)$ should have non-isolated zeros filling subsets of full Lebesgue measure of a neighbourhood of $(r_0, w_0) = (0, 0)$. At this point, we recall that a real analytic function of several variables that vanishes on a set of positive measure must be identically zero. Taking into account the analyticity of d near the origin, it follows that a necessary and sufficient condition in order that the origin of (1.1) becomes a three-dimensional centre is that $d(r_0, w_0) \equiv 0$ for all (r_0, w_0) close to $(0, 0)$. This last assertion has been checked with several polynomial families of the form (1.2) (as, for example, with the forthcoming examples (5.2) and (5.4)), where we can use both cylindrical coordinates to compute a displacement map $\Delta(\rho_0, z_0)$ in \mathcal{U} and the polar blow-up (1.4) to construct the displacement map $d(r_0, w_0)$ in $\mathcal{U} \setminus \mathcal{C}_\delta$. All the computations made show that $\Delta(\rho_0, z_0) \equiv 0$ near the origin if and only if $d(r_0, w_0) \equiv 0$ near the origin.

In summary, the above discussion leads to the following result.

THEOREM 1.6. *The origin of system (1.1) is a three-dimensional centre if and only if $d(r_0, w_0) \equiv 0$ in a neighbourhood of the origin.*

From now on, we shall assume that (1.1) is a family of polynomial differential systems parametrized by its coefficients, which we collect in the real vector parameter $\lambda \in \Lambda \subseteq \mathbb{R}^p$. Hence, its associated displacement map $d(r_0, w_0; \lambda)$ can be expanded in a Taylor series at $(r_0, w_0) = (0, 0)$ of the form $d(r_0, w_0; \lambda) = \sum_{i+j \geq 1} d_{i,j}(\lambda) r_0^i w_0^j$. In analogy with the theory of the classical two-dimensional centres (see, for example, [9]) we call the two-dimensional vector functions $d_{i,j}(\lambda) = (d_{i,j}^{[r]}(\lambda), d_{i,j}^{[w]}(\lambda))$ *Poincaré–Lyapunov constants*. Clearly, $d_{i,j}(\lambda^*) = 0$ for all admissible (i, j) is a necessary condition for family (1.1) with $\lambda = \lambda^*$ to have a three-dimensional centre at the origin.

COROLLARY 1.7. *Let (1.1) be a polynomial family having as parameters its coefficients λ . Then the components of the Poincaré–Lyapunov constants $d_{i,j}(\lambda)$ are polynomials in $\mathbb{R}[\lambda]$.*

Therefore, for polynomial families (1.1), the characterization of its three-dimensional centres leads to a collection of polynomials in the coefficients λ of (1.1) whose simultaneous vanishing picks out those systems for which the singularity is a three-dimensional centre. This implies that each member of family (1.1) having a centre at

the origin corresponds with a point $\lambda = \lambda^*$ of an affine variety V_C in the parameter space of coefficients called the *centre variety*.

Let $\mathcal{B} = \langle d_{i,j}^{[r]}(\lambda), d_{i,j}^{[w]}(\lambda) : (i, j) \in \mathbb{N}^2 \rangle$ be the ideal in the polynomial ring $\mathbb{R}[\lambda]$ generated by all the components of the Poincaré–Lyapunov constants. Again by analogy with the two-dimensional centre theory, \mathcal{B} will be termed the *Bautin ideal* at the origin of (1.1). Of course the variety $V(\mathcal{B})$ associated with the ideal \mathcal{B} is just V_C . The ideal \mathcal{B} is Noetherian and so it is generated by a finite number of polynomials by the Hilbert basis theorem but, unfortunately, we do not know this basis *a priori*.

The second aim of this work is to characterize three-dimensional centres of (1.1) via integrability and normal-form theory.

THEOREM 1.8. *The origin of system (1.1) is a three-dimensional centre if and only if (1.1) is completely analytically integrable.*

A consequence of the proof of theorem 1.8 is the following.

COROLLARY 1.9. *The origin of system (1.1) is a three-dimensional centre if and only if (1.1) is analytically orbitally linearizable.*

The structure of the paper is as follows. In §§2 and 3 we give the proof of theorem 1.4 and corollary 1.7, respectively. We devote §4 to proving theorem 1.8, and the final section focuses on examples.

2. Proof of theorem 1.4

We perform the polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined by (1.4) that converts system (1.1) into a system of the form

$$\left. \begin{aligned} \dot{r} &= \mathcal{R}(\theta, r, w), \\ \dot{\theta} &= 1 + \Theta(\theta, r, w), \\ \dot{w} &= \mathcal{W}(\theta, r, w), \end{aligned} \right\} \tag{2.1}$$

where

$$\begin{aligned} \mathcal{R}(\theta, r, w) &= \cos \theta \hat{F}_1(\theta, r, w) + \sin \theta \hat{F}_2(\theta, r, w), \\ \Theta(\theta, r, w) &= \frac{1}{r} [\cos \theta \hat{F}_2(\theta, r, w) - \sin \theta \hat{F}_1(\theta, r, w)], \\ \mathcal{W}(\theta, r, w) &= \frac{1}{r} [\hat{F}_3(\theta, r, w) - w \mathcal{R}(\theta, r, w)]. \end{aligned}$$

Here, we have defined $\hat{F}_i(\theta, r, w) = F_i(r \cos \theta, r \sin \theta, rw)$ for $i = 1, 2, 3$. We remark that system (2.1) is analytic around $r = 0$ since $\hat{F}_i(\theta, r, w) = \mathcal{O}(r^2)$ and that $\{r = 0\}$ is an invariant set of (2.1) because $\mathcal{R}(\theta, 0, w) = 0$. Also, we observe that $\dot{\theta} > 0$ for $|r|$ sufficiently small and w in an arbitrary fixed compact set \mathcal{K} since $\Theta(\theta, 0, w) = 0$. Therefore, under these conditions we can write system (2.1) as system (1.5).

3. Proof of corollary 1.7

We assume that (1.1) is a family of polynomial differential systems with parameters λ , that is,

$$\dot{x} = -y + F_1(x, y, z; \lambda), \quad \dot{y} = x + F_2(x, y, z; \lambda), \quad \dot{z} = F_3(x, y, z; \lambda), \quad (3.1)$$

where all the F_i depend in a polynomial way on λ . The associated Poincaré–Lyapunov quantities $d_{i,j}(\lambda)$ can be determined in a recursive way, although many computations are involved.

Following the proof of theorem 1.4, we can check that after the polar blow-up (1.4), family (3.1) is transformed into the family

$$\dot{r} = \mathcal{R}(\theta, r, w; \lambda), \quad \dot{\theta} = 1 + \Theta(\theta, r, w; \lambda), \quad \dot{w} = \mathcal{W}(\theta, r, w; \lambda), \quad (3.2)$$

where \mathcal{R} , Θ and \mathcal{W} are polynomials in the parameters λ of the family. Therefore, the corresponding system (1.5) now becomes the family

$$\frac{dr}{d\theta} = R(\theta, r, w; \lambda), \quad \frac{dw}{d\theta} = W(\theta, r, w; \lambda), \quad (3.3)$$

where

$$R(\theta, r, w; \lambda) = \frac{\mathcal{R}(\theta, r, w; \lambda)}{1 + \Theta(\theta, r, w; \lambda)}, \quad W(\theta, r, w; \lambda) = \frac{\mathcal{W}(\theta, r, w; \lambda)}{1 + \Theta(\theta, r, w; \lambda)}$$

and $\Theta(\theta, 0, w; \lambda) = 0$. Hence we can write the Taylor series

$$R(\theta, r, w; \lambda) = r^2 \sum_{i+j \geq 0} R_{ij}(\theta; \lambda) r^i w^j, \quad W(\theta, r, w; \lambda) = r \sum_{i+j \geq 1} W_{ij}(\theta; \lambda) r^i w^j,$$

where $R_{ij}(\theta; \lambda)$ and $W_{ij}(\theta; \lambda)$ are 2π -periodic functions in the variable θ and are polynomials in λ .

Let $\Psi(\theta; r_0, w_0; \lambda) = (r(\theta; r_0, w_0; \lambda), w(\theta; r_0, w_0; \lambda))$ be the solution of (3.3) with initial condition $\Psi(0; r_0, w_0; \lambda) = (r_0, w_0)$. We can expand it as

$$\Psi(\theta; r_0, w_0; \lambda) = \left(r_0 \sum_{i+j \geq 0} \Psi_{ij}^r(\theta; \lambda) r_0^i w_0^j, \sum_{i+j \geq 1} \Psi_{ij}^w(\theta; \lambda) r_0^i w_0^j \right)$$

with associated initial conditions

$$\Psi_{00}^r(0; \lambda) = \Psi_{01}^w(0; \lambda) = 1 \quad \text{and} \quad \Psi_{ij}^r(0; \lambda) = \Psi_{pq}^w(0; \lambda) = 0$$

for all $(i, j) \neq (0, 0)$ and $(p, q) \neq (0, 1)$. Notice that with this notation we have that the Poincaré–Lyapunov constants are

$$\left. \begin{aligned} d_{1,0}(\lambda) &= \begin{pmatrix} \Psi_{0,0}^r(2\pi; \lambda) - 1 \\ \Psi_{1,0}^w(2\pi; \lambda) \end{pmatrix}, & d_{0,1}(\lambda) &= \begin{pmatrix} 0 \\ \Psi_{0,1}^w(2\pi; \lambda) - 1 \end{pmatrix}, \\ d_{i,j}(\lambda) &= \begin{pmatrix} \Psi_{i-1,j}^r(2\pi; \lambda) \\ \Psi_{i,j}^w(2\pi; \lambda) \end{pmatrix} & \text{for all } i + j \geq 2. \end{aligned} \right\} \quad (3.4)$$

Differentiating the former series Ψ with respect to θ and inserting into (3.3) yields

$$\begin{aligned}
 r_0 \sum_{i+j \geq 0} \frac{d\Psi_{ij}^r}{d\theta}(\theta; \lambda) r_0^i w_0^j &= \sum_{i+j \geq 0} R_{ij}(\theta; \lambda) \left(r_0 \sum_{p+q \geq 0} \Psi_{pq}^r(\theta; \lambda) r_0^p w_0^q \right)^{i+2} \\
 &\quad \times \left(\sum_{p+q \geq 1} \Psi_{pq}^w(\theta; \lambda) r_0^p w_0^q \right)^j, \\
 \sum_{i+j \geq 1} \frac{d\Psi_{ij}^w}{d\theta}(\theta; \lambda) r_0^i w_0^j &= \sum_{i+j \geq 1} W_{ij}(\theta; \lambda) \left(r_0 \sum_{p+q \geq 0} \Psi_{pq}^r(\theta; \lambda) r_0^p w_0^q \right)^{i+1} \\
 &\quad \times \left(\sum_{p+q \geq 1} \Psi_{pq}^w(\theta; \lambda) r_0^p w_0^q \right)^j,
 \end{aligned}$$

so that equating coefficients of like powers of r_0 and w_0 we obtain

$$\begin{aligned}
 \frac{d\Psi_{0,0}^r}{d\theta}(\theta; \lambda) &= \frac{d\Psi_{0,1}^r}{d\theta}(\theta; \lambda) = \frac{d\Psi_{1,0}^w}{d\theta}(\theta; \lambda) = 0, \\
 \frac{d\Psi_{1,0}^r}{d\theta}(\theta; \lambda) &= [\Psi_{0,0}^r(\theta; \lambda)]^2 [R_{0,0}(\theta; \lambda) + R_{0,1}(\theta; \lambda) \Psi_{0,0}^w(\theta; \lambda)], \\
 \frac{d\Psi_{1,0}^w}{d\theta}(\theta; \lambda) &= W_{0,1}(\theta; \lambda) \Psi_{0,0}^r(\theta; \lambda) \Psi_{0,0}^w(\theta; \lambda), \\
 &\vdots
 \end{aligned}$$

and in general for all admissible (i, j) we get that

$$\begin{aligned}
 \frac{d\Psi_{i,j}^r}{d\theta}(\theta; \lambda) &= P_{i,j}^r(R_{p,q}(\theta; \lambda), \Psi_{k,\ell}^r, \Psi_{k,\ell}^w), \\
 \frac{d\Psi_{i,j}^w}{d\theta}(\theta; \lambda) &= Q_{i,j}^r(W_{p,q}(\theta; \lambda), \Psi_{k,\ell}^r, \Psi_{k,\ell}^w),
 \end{aligned}$$

where $p + q \leq i + j$, $k + \ell \leq i + j$ and $P_{i,j}^r$ and $Q_{i,j}^r$ are polynomial functions of their arguments. This fact together with (3.4) proves the corollary.

4. Proof of theorem 1.8

Poincaré–Dulac normal forms of system (1.1) have been studied in several works. The underlying idea is to write the vector field as a sum of homogeneous polynomials and search for near-identity changes of variables that iteratively simplify the homogeneous parts of the vector field; see, for example, [1, 4].

It is known (see, for example, [8]) that a formal normal form of system (1.1) is

$$\left. \begin{aligned}
 \dot{x} &= -y[1 + h(z, r^2)] + xg(z, r^2), \\
 \dot{y} &= x[1 + h(z, r^2)] + yg(z, r^2), \\
 \dot{z} &= f(z, r^2),
 \end{aligned} \right\} \tag{4.1}$$

where $r^2 = x^2 + y^2$ and $f, g, h \in \mathbb{R}[[z, r^2]]$ are formal power series in the variables z and r^2 . More precisely,

$$\left. \begin{aligned} f(z, r^2) &= \sum_{i,j} a_{ij} z^i r^{2j} = a_{20} z^2 + a_{01} r^2 + \dots, \\ g(z, r^2) &= \sum_{i,j} b_{ij} z^i r^{2j} = b_{10} z + \dots, \\ h(z, r^2) &= \sum_{i,j} c_{ij} z^i r^{2j} = c_{10} z + \dots \end{aligned} \right\} \tag{4.2}$$

for some real coefficients a_{ij} , b_{ij} and c_{ij} .

First we shall assume the origin of system (1.1) to be a three-dimensional centre. Hence, the formal displacement map $\hat{d}(r_0, w_0)$ associated with its formal normal form (4.1) must be identically zero. To compute $\hat{d}(r_0, w_0)$ we will use the formal version of theorem 1.4. Actually, following the ideas of [5] we will first carry out the rescaling $(x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)$, introducing a small parameter ε . Next we perform the polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined in (1.4), and the formal system (4.1) becomes a formal (autonomous) system

$$\left. \begin{aligned} \frac{dr}{d\theta} &= \varepsilon R(r, w; \varepsilon) = \frac{rg(\varepsilon rw, \varepsilon^2 r^2)}{1 + h(\varepsilon rw, \varepsilon^2 r^2)}, \\ \frac{dw}{d\theta} &= \varepsilon W(r, w; \varepsilon) = \frac{f(\varepsilon rw, \varepsilon^2 r^2) - \varepsilon rwg(\varepsilon rw, \varepsilon^2 r^2)}{r\varepsilon(1 + h(\varepsilon rw, \varepsilon^2 r^2))} \end{aligned} \right\} \tag{4.3}$$

with $R, W \in \mathbb{R}[[r, w, \varepsilon]]$. The formal Poincaré translation map $\hat{\Pi}(r_0, w_0; \varepsilon)$ associated with (4.3) will be $\hat{\Pi}(r_0, w_0; \varepsilon) = \hat{\Psi}(2\pi; r_0, w_0; \varepsilon)$, where $\hat{\Psi}(\theta; r_0, w_0; \varepsilon)$ is the formal solution of (4.3) with $\hat{\Psi}(0; r_0, w_0; \varepsilon) = (r_0, w_0)$. Actually we can formally write

$$\hat{\Psi}(\theta; r_0, w_0; \varepsilon) = (r_0, w_0) + \sum_{i \geq 1} \Psi_i(\theta; r_0, w_0) \varepsilon^i.$$

Then the formal displacement map is constructed as

$$\hat{d}(r_0, w_0; \varepsilon) = \hat{\Pi}(r_0, w_0; \varepsilon) - (r_0, w_0) \in \mathbb{R}[[r_0, w_0, \varepsilon]].$$

Clearly, the origin of (1.1) is a three-dimensional centre if and only if $\hat{d}(r_0, w_0; \varepsilon) \equiv 0$. An equivalent condition characterizing three-dimensional centres is the vanishing of all the coefficients $d_j(r_0, w_0)$ (formal Melnikov functions) with $j \geq 1$ of the formal power series

$$\hat{d}(r_0, w_0; \varepsilon) = \sum_{j \geq 1} d_j(r_0, w_0) \varepsilon^j. \tag{4.4}$$

Expanding in formal power series in ε as

$$R(r, w; \varepsilon) = \sum_{i \geq 0} R_i(r, w) \varepsilon^i, \quad W(r, w; \varepsilon) = \sum_{i \geq 0} W_i(r, w) \varepsilon^i,$$

we can recursively compute the formal Melnikov functions, thereby obtaining

$$\begin{aligned} d_1(r_0, w_0) &= 2\pi(R_0(r_0, w_0), W_0(r_0, w_0)), \\ d_2(r_0, w_0) &= 2\pi(R_1(r_0, w_0), W_1(r_0, w_0)) + D_2(R_0(r_0, w_0), W_0(r_0, w_0)), \\ &\vdots \\ d_j(r_0, w_0) &= 2\pi(R_{j-1}(r_0, w_0), W_{j-1}(r_0, w_0)) + D_j(\bar{R}_j(r_0, w_0), \bar{W}_j(r_0, w_0)), \end{aligned}$$

where $\bar{R}_j(r_0, w_0) = (R_0(r_0, w_0), R_1(r_0, w_0), \dots, R_j(r_0, w_0))$ and also $\bar{W}_j(r_0, w_0) = (W_0(r_0, w_0), W_1(r_0, w_0), \dots, W_j(r_0, w_0))$ and the functions D_j vanish at the origin. In summary, the origin of (1.1) is a three-dimensional centre if and only if $d_j(r_0, w_0) \equiv 0$ for all $j \geq 1$, which is true if and only if $(R_i(r_0, w_0), W_i(r_0, w_0)) \equiv (0, 0)$ for any $i \geq 0$. Therefore, one has $R(r, w; \varepsilon) = W(r, w; \varepsilon) \equiv 0$ or equivalently $f(\varepsilon r w, \varepsilon^2 r^2) = g(\varepsilon r w, \varepsilon^2 r^2) \equiv 0$. Clearly, this implies that $f(z, r^2) = g(z, r^2) \equiv 0$, as we wanted to prove.

Thus, in the three-dimensional centre case the formal normal form of system (1.1) is

$$\dot{x} = -y[1 + h(z, r^2)], \quad \dot{y} = x[1 + h(z, r^2)], \quad \dot{z} = 0, \tag{4.5}$$

admitting therefore two independent polynomial first integrals z and $x^2 + y^2$. Clearly, this means that system (1.1) has two formal independent first integrals $z + \dots$ and $x^2 + y^2 + \dots$. Using results of Zhang [10, 11] (see also [7, theorem 9]), we know that in fact (1.1) has two independent analytic first integrals $z + \dots$ and $x^2 + y^2 + \dots$, and moreover there is a convergent normalizing transformation. This proves one part of the theorem.

The converse is easy to prove. Assume now that (1.1) is completely analytically integrable. This implies that (1.1) has two independent analytical first integrals $H_1(x, y, z)$ and $H_2(x, y, z)$. Since the linearization $\dot{x} = -y, \dot{y} = x, \dot{z} = 0$ of (1.1) has the first integrals z and $x^2 + y^2$, it is clear that H_1 and H_2 can be chosen in the form $H_1(x, y, z) = x^2 + y^2 + \dots$ and $H_2(x, y, z) = z + \dots$. Therefore (1.1) possesses a three-dimensional centre at the origin. This can be checked just by intersecting the level sets (topological cylinders and planes) of H_1 and H_2 in a neighbourhood of the origin.

REMARK 4.1. Since the z -axis is a formal invariant rotation axis of the normal form (4.1), we can use cylindrical coordinates $(x, y, z) \mapsto (\varphi, \rho, z)$ so that (4.1) can be reduced to

$$\frac{d\rho}{d\varphi} = \hat{\mathcal{R}}(\rho, z) = \frac{\rho g(z, \rho^2)}{1 + h(z, \rho^2)}, \quad \frac{dz}{d\varphi} = \hat{\mathcal{Z}}(\rho, z) = \frac{f(z, \rho^2)}{1 + h(z, \rho^2)}, \tag{4.6}$$

with $\hat{\mathcal{R}}, \hat{\mathcal{Z}} \in \mathbb{R}[[\rho, z]]$. Continuing in this direction, the proof of theorem 1.8 can be made simpler (there is no need even to introduce the small parameter ε) when we show that the normal form (4.3) of a three-dimensional centre must satisfy $f = g \equiv 0$. Anyway, we retain that proof since it introduces Melnikov functions (see (4.4)) and this is an appropriate way to calculate necessary three-dimensional centre parameter conditions. We will check the above in the forthcoming examples.

5. Examples

The computation of the necessary three-dimensional centre conditions are performed using the technique of the proof of theorem 1.8 developed in [5]. Hence we first do the rescaling $(x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)$ with a small parameter ε and next we perform the polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined in (1.4) in order to compute the Melnikov functions of family (1.1). Then, for $|\varepsilon|$ sufficiently small, (1.1) can be written as the analytic system

$$\frac{dr}{d\theta} = \varepsilon R(\theta, r, w; \varepsilon), \quad \frac{dw}{d\theta} = \varepsilon W(\theta, r, w; \varepsilon), \quad (5.1)$$

near the invariant set $\{r = 0\}$. Clearly, (5.1) is defined on the cylinder because $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Then we can define the displacement map $d(r_0, w_0; \varepsilon) = \Pi(r_0, w_0; \varepsilon) - \text{Id}(r_0, w_0)$, where Id is the identity map and $\Pi(r_0, w_0; \varepsilon)$ is the Poincaré translation map given by the flow of (5.1) at the angle 2π . The origin of (1.1) is a three-dimensional centre if and only if $d(r_0, w_0; \varepsilon) \equiv 0$ or equivalently when $d_j(r_0, w_0) \equiv 0$ with $j \geq 1$, where $d(r_0, w_0; \varepsilon) = \sum_{j \geq 1} d_j(r_0, w_0) \varepsilon^j$ is the Taylor series of d near $\varepsilon = 0$. We call the analytic function $d_j: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the *j*th Melnikov function.

5.1. Example 1

We study the three-dimensional centre problem for the quadratic family

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = a_1 x^2 + a_2 y^2 + a_3 xy. \quad (5.2)$$

PROPOSITION 5.1. *The origin is a three-dimensional centre of family (5.2) if and only if $a_1 + a_2 = 0$.*

Proof. Performing the described algorithm on family (5.2) and after some computations we get the first Melnikov function

$$d_1(r_0, w_0) = (0, (a_1 + a_2)\pi r_0).$$

Then $d_1(r_0, w_0) \equiv 0$ if and only if $a_1 + a_2 = 0$. Finally, taking $a_2 = -a_1$ it is easy to check that family (5.2) has two polynomial first integrals

$$H_1(x, y) = x^2 + y^2, \quad H_2(x, y, z) = z + \frac{a_3}{2}x^2 - a_1xy,$$

thereby finishing the proof. \square

REMARK 5.2. Of course, instead of computing the Melnikov function, we could use reduction to normal form. If we perform the first step of such a computation on family (5.2), we get

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = (a_1 + a_2)(x^2 + y^2) + \dots, \quad (5.3)$$

and hence the condition $a_1 + a_2 = 0$ is obtained from corollary 1.9.

5.2. Example 2

We consider a quadratic linear-like system

$$\dot{x} = -y + f(z)x, \quad \dot{y} = x + g(z)y, \quad \dot{z} = F(x, y, z), \tag{5.4}$$

with f and g linear and F any homogeneous quadratic polynomial. Introducing parameters, one has $f(z) = az$, $g(z) = bz$ and $F(x, y, z) = c_1x^2 + c_2y^2 + c_3z^2 + c_4xy + c_5xz + c_6yz$.

PROPOSITION 5.3. *The following parameter restrictions are necessary conditions for the origin to be a three-dimensional centre of family (5.4):*

- (i) $a = b = c_3 = 0$, $c_2 = -c_1$ and $c_1c_5^2 + c_4c_5c_6 - c_1c_6^2 = 0$;
- (ii) $c_3 = c_4 = 0$, $b = -a$, $c_2 = -c_1$ and $c_5^2 = c_6^2$.

Proof. Performing computations, we get the first Melnikov function

$$d_1(r_0, w_0) = \pi r_0((a + b)r_0w_0, c_1 + c_2 - (a + b - 2c_3)w_0^2).$$

Then $d_1(r_0, w_0) \equiv 0$ if and only if $b = -a$, $c_2 = -c_1$ and $c_3 = 0$. Next, the second Melnikov function is given by

$$d_2(r_0, w_0) = \frac{\pi r_0^2}{4}ac_4(-r_0, 5w_0).$$

For $d_2(r_0, w_0)$ to vanish, we have $ac_4 = 0$, giving two different cases.

In case (i) we assume that $a = 0$ and obtain that $d_j(r_0, w_0) \equiv 0$ for $j = 3, 4, 5, 6$ if and only if $c_1c_5^2 + c_4c_5c_6 - c_1c_6^2 = 0$.

In case (ii) we take $c_4 = 0$ and $a \neq 0$. The computations show that now $d_j(r_0, w_0) \equiv 0$ for $j = 3, 4, 5, 6$ if and only if $c_5^2 = c_6^2$. □

REMARK 5.4. We conjecture that conditions (i) and (ii) are also sufficient for having a three-dimensional centre at the origin in family (5.4). It is interesting to notice that in case (i) if we first impose the parameter constrains $a = b = c_3 = 0$ and $c_2 = -c_1$ and next perform the polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined in (1.4), family (5.4) is transformed into a system (1.5) of the form

$$\frac{dr}{d\theta} = 0, \quad \frac{dw}{d\theta} = A(\theta, r)w + B(\theta, r), \tag{5.5}$$

with $A(\theta, r) = r(c_5 \cos \theta + c_6 \sin \theta)$ and $B(\theta, r) = r[c_1(\cos^2 \theta - \sin^2 \theta) + c_4 \cos \theta \sin \theta]$. Therefore, $r(\theta; r_0, w_0) = r_0$ and $w(\theta; r_0, w_0)$ is the solution of the 2π -periodic linear differential equation $dw/d\theta = A(\theta, r_0)w + B(\theta, r_0)$. Clearly, the origin is a three-dimensional centre of family (5.4) if and only if all the solutions $w(\theta; r_0, w_0)$ of the former equation are 2π -periodic. Using the classical theory for such equations, we have from the variation of constants formula

$$w(2\pi; r_0, w_0) = \Phi(2\pi, r_0)w_0 + \Phi(2\pi, r_0) \int_0^{2\pi} \Phi^{-1}(\theta, r_0)B(\theta, r_0)d\theta,$$

where $\Phi(\theta, r_0) = \exp(\int_0^\theta A(\sigma, r_0)d\sigma)$. We then deduce that the periodicity condition $w(2\pi; r_0, w_0) = w_0$ holds if and only if the initial condition w_0 is a solution of the algebraic linear system

$$(1 - \Phi(2\pi, r_0))w_0 = \Phi(2\pi, r_0) \int_0^{2\pi} \Phi^{-1}(\theta, r_0)B(\theta, r_0)d\theta. \tag{5.6}$$

Since $\Phi(2\pi, r_0) = 1$ we deduce that all the solutions $w(\theta; r_0, w_0)$ are 2π -periodic if and only if $\alpha(r_0) := \int_0^{2\pi} \Phi^{-1}(\theta, r_0)B(\theta, r_0)d\theta \equiv 0$. Since $\alpha(r_0)$ is analytic near $r_0 = 0$, the above centre condition is equivalent to the vanishing of all its derivatives $d^n\alpha/dr_0^n(0)$ for any $n \in \mathbb{N}$. Using induction it is straightforward to check that

$$\begin{aligned} \frac{\partial^n}{\partial r_0^n}[\Phi^{-1}(\theta, r_0)B(\theta, r_0)] &= \Phi^{-1}(\theta, r_0) \frac{B(\theta, r_0)}{r_0} [c_6(\cos \theta - 1) - c_5 \sin \theta]^{n-1} \\ &\quad \times [n + c_6r_0(\cos \theta - 1) - c_5r_0 \sin \theta] \end{aligned}$$

so that

$$\frac{d^n \alpha}{dr_0^n}(0) = n \int_0^{2\pi} [c_1(\cos^2 \theta - \sin^2 \theta) + c_4 \cos \theta \sin \theta][c_6(\cos \theta - 1) - c_5 \sin \theta]^{n-1} d\theta.$$

Observe that, for any $n \in \mathbb{N}$, $(d^n\alpha/dr_0^n)(0) \in \mathbb{R}[c]$ with $c = (c_1, c_4, c_5, c_6) \in \mathbb{R}^4$. Now if we consider the polynomial ideal $\mathcal{J} = \langle (d^n\alpha/dr_0^n)(0) : n \in \mathbb{N} \rangle$ in the ring $\mathbb{R}[c]$, our conjecture is that its associated affine variety is given by $\mathbf{V}(\mathcal{J}) = \{c \in \mathbb{R}^4 : c_1c_5^2 + c_4c_5c_6 - c_1c_6^2 = 0\}$.

5.3. Example 3

We consider the following quadratic six-parameter family

$$\dot{x} = -y + az^2 + bxz, \quad \dot{y} = x + cxy + dz^2, \quad \dot{z} = ex^2 + fzy. \tag{5.7}$$

Notice that (5.7) is not of the form (1.2) since the z -axis is not invariant provided that $a^2 + d^2 \neq 0$.

PROPOSITION 5.5. *The origin is a three-dimensional centre of family (5.7) if and only if $b = e = af = cd = 0$.*

Proof. The first Melnikov function becomes

$$d_1(r_0, w_0) = \pi r_0(br_0w_0, e - bw_0^2).$$

From $d_1(r_0, w_0) \equiv 0$ we obtain $b = e = 0$. Given this, the second Melnikov function is given by

$$d_2(r_0, w_0) = \pi r_0^2 w_0^2 (-(cd + 2af)r_0, (cd + 4af)w_0).$$

Hence, $d_2(r_0, w_0) \equiv 0$ if and only if $af = cd = 0$. We remark that with these parameter constraints one has $d_j(r_0, w_0) \equiv 0$ for $j = 3, 4, 5, 6$, making it probable that the origin becomes a three-dimensional centre of family (5.7). We are going to prove that, actually, this is the case.

CASE 1. If $a = d = 0$, then family (5.7) becomes

$$\dot{x} = -y, \quad \dot{y} = x(1 + cy), \quad \dot{z} = fzy.$$

The planar subsystem $\dot{x} = -y, \dot{y} = x(1 + cy)$ possesses a centre at $(x, y) = (0, 0)$ because it is time reversible with respect to the involution $(x, y, t) \mapsto (-x, y, -t)$, and hence it has an analytic first integral $H_1(x, y) = x^2 + y^2 + \dots$ defined in a neighbourhood of the origin that is also a first integral of the full family in \mathbb{R}^3 . Additionally, that family has the first integral

$$H_2(x, y, z) = z \exp(fx) = z + \dots, \tag{5.8}$$

which is analytic in a neighbourhood of the origin. Hence the origin is a three-dimensional centre.

CASE 2. If $a = c = 0$, then family (5.7) becomes

$$\dot{x} = -y, \quad \dot{y} = x + dz^2, \quad \dot{z} = fzy.$$

This family also has the first integral H_2 already given in (5.8). Moreover, the system restricted to the level sets $\{H_2(x, y, z) = h\} \subset \mathbb{R}^3$, with $|h|$ sufficiently small, becomes the planar analytic Hamiltonian system

$$\dot{x} = -y, \quad \dot{y} = x + dh^2 \exp(-2fx).$$

This planar system has the first integral $H(x, y) = f(x^2 + y^2) - dh^2 \exp(-2fx)$. We emphasize that, when $|h| \ll 1$, the level sets of H near $(x, y) = (0, 0)$ are ovals. This proves that the zero-Hopf singularity at the origin of the full family in \mathbb{R}^3 is a three-dimensional centre.

CASE 3. If $f = d = 0$, then family (5.7) becomes

$$\dot{x} = -y + az^2, \quad \dot{y} = x + cxy, \quad \dot{z} = 0.$$

In this case we have the trivial first integral z . We claim that the restricted planar system to each horizontal invariant plane $\{z = h\}$ with $|h|$ sufficiently small has an elementary time-reversible centre at the singularity $(0, ah^2)$ so that the origin of the full family is a three-dimensional centre. The claim follows by translating the singularity at the origin, computing the associated purely imaginary eigenvalues $\pm\sqrt{-1 - ach^2}$ and checking the time reversibility with respect to $(x, y, t) \mapsto (-x, y, -t)$.

CASE 4. If $f = d = 0$, then family (5.7) becomes

$$\dot{x} = -y + az^2, \quad \dot{y} = x + dz^2, \quad \dot{z} = 0.$$

The analysis of this case is totally analogous to that of case 3. □

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