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OPEN BOOKS FOR CLOSED NON-ORIENTABLE 3–MANIFOLDS

ABHIJEET GHANWAT
Chennai Mathematical Institute,
H1, SIPCOT IT Park, Siruseri, Kelambakkam, 603103, Tamilnadu, India
e-mail: abhijeet@cmi.ac.in

SUHAS PANDIT and A SELVAKUMAR
Indian Institute of Technology Madras,
IIT PO., Chennai, 600036, Tamilnadu, India
e-mails: suhas@iitm.ac.in; ma15d010@smail.iitm.ac.in

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Abstract. In this note, we give a new proof of the existence of an open book decomposition for a closed non-orientable 3–manifold. This open book decomposition is analogous to a planar open book decomposition for a closed orientable 3–manifold. More precisely, in this note, we give an open book decomposition of a given closed non-orientable 3–manifold with the pages punctured Möbius bands. We also give an algorithm to determine the monodromy of this open book.

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1. Introduction. An open book decomposition for a closed 3–manifold M is a way to express M as a locally trivial fiber bundle over S^1 in the complement of an oriented link B having a trivial normal bundle in M such that the boundary of the closure of each fiber is B. The fibration π in a neighborhood of B looks like the trivial fibration of $(B \times D^2) \setminus (B \times \{0\}) \to S^1$ sending (x, r, θ) to θ, where $x \in B$ and (r, θ) are polar co-ordinates on D^2. The closure of each fiber is called the page of the open book, and the monodromy of the fibration is called the monodromy of the open book. An open book decomposition of M is determined—up to diffeomorphism of M—by the topological surface type of the page Σ and the isotopy class of the monodromy which is an element of the mapping class group of Σ.

Alexander in [1] proved that every closed orientable 3–manifold admits an open book. There are many proofs of this fact, see for example [4]. The study of open books for closed orientable 3–manifolds is a very important topic due to the connection between open books and contact structures, see [5].

In [2], Berstein and Edmonds showed that any closed non-orientable 3–manifold is a branched covering of $\mathbb{R}P^2 \times S^1$ of degree at most 6 with the branch set a knot in $\mathbb{R}P^2 \times S^1$. Using this, they proved an existence of an open book for a closed non-orientable 3–manifold. In their proof, the topological surface type of the page and the monodromy of the open book are not clear. In [6], Klassen exhibited an open book decomposition for $\mathbb{R}P^2 \times S^1$.

In [11], a proof of the existence of a planar open book decomposition of a closed orientable 3–manifold is given. In [10], an algorithm for determining the monodromy of a planar open book of a closed orientable 3–manifold is discussed.

On the similar line, we prove that every closed non-orientable 3–manifold admits an open book decomposition where the pages are the punctured Möbius bands, see Theorem 11 in Section 3. Thus, here the topological surface type is a simpler one.

Further, in Section 4, we give an algorithm to determine the monodromy of the open book with pages the punctured Möbius band. In the last section, we illustrate the algorithm for determining the monodromy described in Section 4.

The open books for non-orientable manifolds may be useful in studying mapping class group of non-orientable surfaces with boundary as open books naturally give Heegaard splittings for non-orientable 3–manifolds.

2. Preliminary. In this section, we will see some important notions which we require in this article.

2.1. Mapping class group. We will start with the following definition.

DEFINITION 1. (The mapping class group of a non-orientable manifold) The mapping class group of a non-orientable manifold M is the group of diffeomorphisms of M up to isotopy. If M has non-empty boundary ∂M, we always assume that all the diffeomorphisms and the isotopies are the identity in a collar neighborhood of the boundary. We will denote it by $\mathcal{M}CG(M)$.

Lickorish in [8] showed that the mapping class group of a non-orientable surface Σ is generated by Dehn twists along two-sided simple closed curves and Y–homeomorphisms. Recall that by a two-sided simple closed curve on a surface, we mean a curve whose regular neighborhood is an embedded annulus and by a one-sided simple closed curve, we mean a curve whose regular neighborhood is an embedded Möbius band. We write the definitions of a Dehn twist and a Y–homeomorphism.

DEFINITION 2 (Dehn twist). A Dehn twist along a circle $S^1 \times \{\frac{3}{2}\}$ in $S^1 \times [0, 3]$ is a diffeomorphism of $S^1 \times [0, 3]$ given by $(e^{i\theta}, t) \to (e^{i(\theta \pm 2\pi f(t))}, t)$, where $f : [0, 3] \to [0, 1]$ is a smooth function that is identically 0 on the interval $[0, 1]$, on the interval $[1, 2]$ it increases monotonically from 0 to 1 and f is identically 1 on the interval $[2, 3]$. Note that the Dehn twist fixes a collar of both the boundary components of $S^1 \times [0, 3]$. Hence, given a two sided embedded circle c in a surface Σ, we can define the Dehn twist of an annular neighborhood $N(c) = S^1 \times [0, 3]$ of $c = S^1 \times \{\frac{3}{2}\}$ in Σ which is the identity when restricted to the boundary of this annular neighborhood. Clearly, we can extend this diffeomorphism by the identity in the complement of the annular neighborhood to produce a diffeomorphism of Σ. This diffeomorphism is called a Dehn twist d_c along the embedded circle c in the surface Σ.

Let M_D be a Möbius band with interior $\mathring{D}$ of a disc D removed. We attach a Möbius band M along ∂D to get a Klein bottle K with a hole, which we call a Y–space.

The precise definition of the Y–homeomorphisms is as follows:

DEFINITION 3. (Y–homeomorphism) Consider a Y–space as described above. Then, move the Möbius band M once along the core of M_D to get a homeomorphism called the Y–homeomorphism of K which fixes the boundary of K, see Figure 1. We can consider an embedding of K in a non-orientable surface N and extend this homeomorphism by the identity on $N \setminus K$, to a Y–homeomorphism of N about K.

Now, we will discuss the notion of open book decomposition of a 3–manifold.

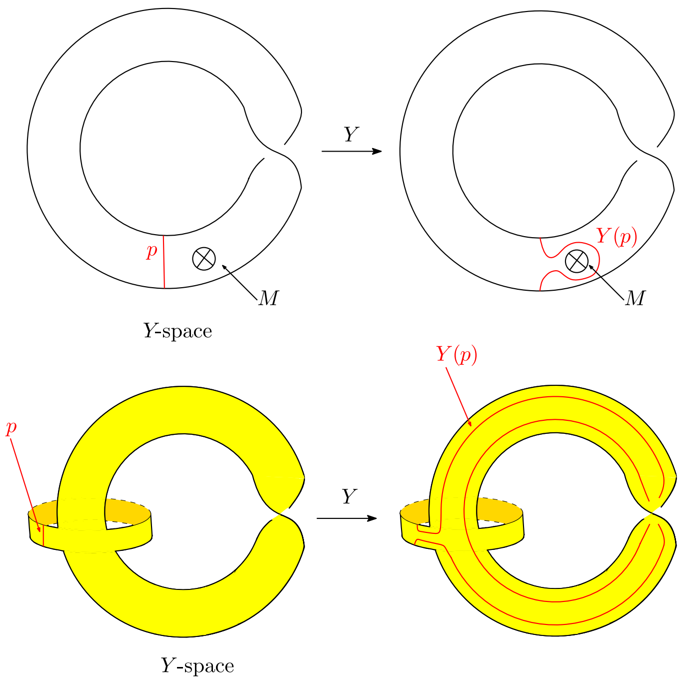

Figure 1. Two pictures of Y–space with an action of Y–homeomorphism. Here, M is a Möbius band attached to M_D.

2.2. Open books.

DEFINITION 4. (Open book decomposition)
Let M be a closed 3–manifold. An open book decomposition of M is a pair (B, π), where B is an oriented link in M with a trivial normal bundle in M called the binding of the open book and $\pi : M \setminus B \to S^1$ is a locally trivial fibration of the complement of B such that the fibration π in a neighborhood of B looks like the trivial fibration of $(B \times D^2) \setminus B \times \{0\} \to S^1$ sending (x, r, θ) to θ, where $x \in B$ and (r, θ) are polar coordinates on D^2 and $\pi^{-1}(\theta)$ is the interior of a compact surface $N_\theta \subset M$ and $\partial N_\theta = B$, for all $\theta \in S^1$. The compact surface $N = N_\theta$, for any θ, is called the page of the open book. We will denote M with an open book (B, π) by $(M, (B, \pi))$.

There is an equivalent notion associated with a 3–manifold, namely, an *abstract open book*. An abstract open book is a pair (Σ, ϕ), where Σ is a compact surface with non-empty boundary $\partial\Sigma$ and φ is a diffeomorphism of Σ which is the identity in a collar neighborhood of $\partial\Sigma$ such that

$$M = (\Sigma \times [0, 1]/\sim) \cup_{\mathrm{Id}} \partial\Sigma \times D^2.$$

Here, we identify the boundary $\partial\Sigma \times S^1$ of $\Sigma \times [0, 1]/\sim$ with the boundary of $\partial\Sigma \times D^2$ using the identity map, and $\sim$ is the equivalence relation identifying $(x, 0)$ with $(\varphi(x), 1)$. The map ϕ is called the monodromy of the abstract open book.

One can easily see that given an abstract open book decomposition of M, we can clearly associate an open book decomposition of M with the pages Σ and the monodromy

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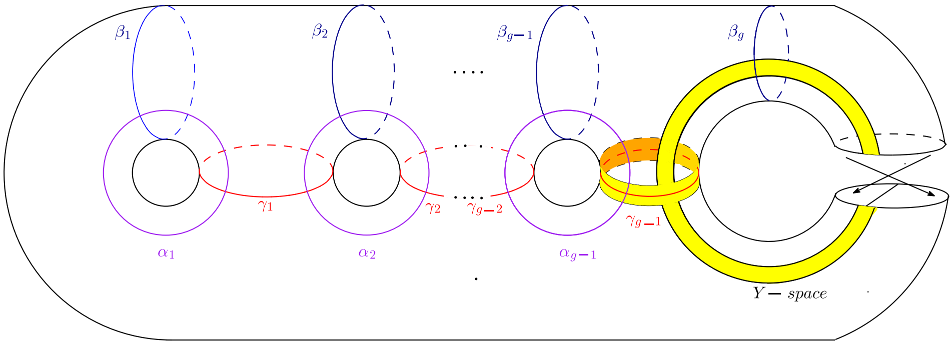

Figure 2. Non-orientable handle body H_g with generators of the mapping class group of the boundary of H_g. Here, the generators are the Dehn twists along the curves shown in the figure and the Y–homeomorphism of the Y–space. The two arrows in the figure denote the antipode identification which corresponds to a non-orientable handle.

ϕ of the fibration and vice versa. For more details on open books, refer [4]. Now, we will see some examples of open books.

EXAMPLES 5.

(1) The 3–sphere S^3 admits an open book with pages a Hopf annulus and the monodromy the Dehn twist along the central curve of the Hopf annulus.
(2) The non-orientable 3-manifold $S^2 \widetilde{\times} S^1$ has an open book decomposition with pages the Möbius band and the monodromy the identity.

3. Existence of open book decomposition for closed non-orientable 3–manifolds. In this section, we will give a proof of the existence of an open book decomposition for a closed non-orientable 3–manifold.

First, recall that a closed non-orientable surface is a connected sum of the real projective planes, $\mathbb{R}P^2$'s. The number of $\mathbb{R}P^2$'s occurring in the connected sum is called the *genus* of the non-orientable surface. By a handlebody of genus g, we mean a 3–ball D^3 with g number of 3-dimensional 1–handles attached. By a non-orientable handlebody H_g of genus g, we mean a handlebody of genus g such that ∂H_g is a non-orientable surface of genus $2g$. We can think of a non-orientable handlebody of genus g as a connected sum of an orientable handlebody of genus $g-1$ with a solid Klein bottle.

Lickorish in [9] proved that the mapping class group of a non-orientable surface is generated by finitely many Dehn twists together with only one Y–homeomorphism. Chillingworth in [3] showed that the mapping class group of a non-orientable surface can be generated by finitely many Dehn twists $\{d_{\alpha_i}, d_{\beta_j}, d_{\gamma_i} : 1 \leq i \leq g-1,\ 1 \leq j \leq g\}$ and a Y–homeomorphism corresponding to the Y–space, as shown in Figure 2.

In order to prove the existence of an open book for non-orientable 3–manifolds, first we require the following Lemma 6. A proof of this lemma is given in [8]. We will write a proof of this here, as we require the content of the proof later.

LEMMA 6. *Let $f : \partial H_g \to \partial H'_g$ be a diffeomorphism, where H_g and H'_g are the genus g non-orientable handlebodies. Then, there exist disjoint solid tori $\tau_1, \tau_2, \ldots, \tau_n$ in H_g and disjoint solid tori $\tau'_1, \tau'_2, \ldots, \tau'_n$ in H'_g such that f can be extended to a diffeomorphism $\widetilde{f} : H_g \setminus \{\mathring{\tau}_1, \mathring{\tau}_2, \ldots, \mathring{\tau}_n\} \to H'_g \setminus \{\mathring{\tau}'_1, \mathring{\tau}'_2, \ldots, \mathring{\tau}'_n\}$, where $\mathring{\tau}_i$ and $\mathring{\tau}'_i$ are the interiors of the tori τ_i and τ'_i, respectively.*

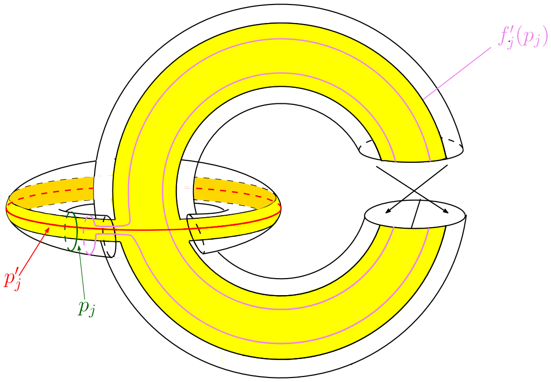

Figure 3. Solid Klein bottle with a handle attached.

Proof. Without loss of generality, we can assume that $H_g = H_g'$. Notice from [3] that $f : \partial H_g \to \partial H_g$ is isotopic to a product of Dehn twists and a Y–homeomorphism, i.e., $f = \eta \circ f_n \circ f_{n-1} \circ \ldots \circ f_1$, where $\eta \sim Id$ and f_i's are the Dehn twists along the curves or the Y–homeomorphism along the Y–space, as shown in Figure 2.

Suppose $\eta \sim Id$ by an isotopy η_t with $\eta_0 = \eta$ and $\eta_1 = Id$. We consider $\partial H_g \times [0, 1]$ as a collar of ∂H_g in H_g with $\partial H_g = \partial H_g \times \{0\}$. Then, the diffeomorphism $\eta_t \times Id$ of $\partial H_g \times [0, 1]$ and the identity on $H_g \setminus \partial H_g \times [0, 1]$ will give an extension $\widetilde{\eta} : H_g \to H_g$ of η.

Consider $\partial H_g \times [0, 2n]$ as a collar of ∂H_g in H_g with $\partial H_g = \partial H_g \times \{0\}$. If f_j is a Dehn twist along a two-sided closed curve c_j which is the identity outside an annular neighborhood A_j of c_j, then a diffeomorphism $f_j \times Id$ on $A_j \times [0, 2j-1]$ and the identity on $H_g \setminus \mathring{A}_j \times (0, 2j)$ will give an extension $\widetilde{f_j} : H_g \setminus \mathring{A}_j \times (2j-1, 2j) \to H_g \setminus \mathring{A}_j \times (2j-1, 2j)$ of f_j.

If all f_j's are the Dehn twists, then

$$\widetilde{\eta} \circ \widetilde{f_n} \circ \widetilde{f}_{n-1} \circ \ldots \circ \widetilde{f_1} : H_g \setminus \{\mathring{\tau}_1, \mathring{\tau}_2, \ldots, \mathring{\tau}_n\} \to H_g' \setminus \{\mathring{\tau}_1', \mathring{\tau}_2', \ldots, \mathring{\tau}_n'\}$$

is an extension of f, where τ_j's are the solid tori given by $A_j \times [2j-1, 2j]$ and τ_j''s are the images of τ_j's under the map $\widetilde{\eta} \circ \widetilde{f_n} \circ \widetilde{f}_{n-1} \circ \ldots \circ \widetilde{f}_{j+1}$.

If f_j is the Y–homeomorphism which is the identity outside the Y–space in ∂H_g, then a diffeomorphism $f_j \times Id$ on $Y \times [0, 2j-1]$ and the identity on $H_g \setminus \mathring{Y} \times (0, 2j)$ will give an extension

$$f_j' : H_g \setminus \mathring{\kappa}_j \to H_g \setminus \mathring{\kappa}_j,$$

where $\kappa_j = Y \times [2j-1, 2j]$ is a solid Klein bottle with a 1–handle attached. Our aim is to extend f_j on H_g with certain solid tori removed from H_g. This can be achieved as given in [8] as follows:

Consider the curve p_j on the boundary of the 1–handle of κ_j such that p_j bounds a disc, as shown in Figure 3. The image $f_j'(p_j)$ of p_j is as shown in Figure 3. Now, consider the simple closed two-sided curve p_j', as shown in Figure 3. Note that the curve p_j' intersects each of the curves p_j and $f_j'(p_j)$ in exactly one point.

Now, we recall a notion of an equivalence of simple closed two-sided paths given in [7] and in [8]. Recall that two paths γ_1 and γ_2 are said to be equivalent ($\gamma_1 \sim_D \gamma_2$) if there exists Dehn twists $h_1, h_2, \cdots, h_n$ and a diffeomorphism θ isotopic to the identity such that $\gamma_2 = \theta \circ h_n \circ \cdots \circ h_1(\gamma_1)$. In [7], it is shown that if γ_1 and γ_2 are two-sided simple closed curves in a 2–manifold such that γ_1 and γ_2 intersect in one point, then $\gamma_1 \sim_D \gamma_2$. Therefore, we can see that $p_j \sim_D p'_j \sim_D f'_j(p_j)$. Thus, there exist a product h_j of Dehn twists and a diffeomorphism n_j isotopic to the identity on $\partial\kappa_j$ such that $n_j \circ h_j \circ f'_j(p_j) = p_j$. In fact, here, h_j is isotopic to a product of Dehn twists along the curves p_j, p'_j and $f_j(p_j)$.

We choose a collar neighborhood of $\partial\kappa_j$ in $H_g \setminus \kappa_j$ such that it does not intersect the tori which we have removed from H_g while extending f_i's to $\widetilde{f_i}$ for $i < j$. Now, as described above for the Dehn twists, there exist solid tori $\tau_{j_1}, \dots, \tau_{j_l}$ and $\tau'_{j_1}, \dots, \tau'_{j_l}$ in $H_g \setminus \kappa_j$ such that $n_j \circ h_j$ can be extended to

$$(n_j \circ h_j)' : H_g \setminus \{\mathring{\kappa}_j, \mathring{\tau}_{j_1}, \mathring{\tau}_{j_2}, \dots, \mathring{\tau}_{j_l}\} \to H_g \setminus \{\mathring{\kappa}_j, \mathring{\tau}'_{j_1}, \mathring{\tau}'_{j_2}, \dots, \mathring{\tau}'_{j_l}\}.$$

So, we get

$$(n_j \circ h_j)' \circ f'_j : H_g \setminus \{\mathring{\kappa}_j, \mathring{\tau}_{j_1}, \mathring{\tau}_{j_2}, \dots, \mathring{\tau}_{j_l}\} \to H_g \setminus \{\mathring{\kappa}_j, \mathring{\tau}'_{j_1}, \mathring{\tau}'_{j_2}, \dots, \mathring{\tau}'_{j_l}\}$$

and it maps p_j to p_j and hence $(n_j \circ h_j)' \circ f'_j$ can be extended to the 1–handle attached to the solid Klein bottle in κ_j. Thus, we can extend f_j on H_g with some solid tori, and a solid Klein bottle K_j removed from H_g. Here, $\kappa_j = K_j \cup$ the 1–handle. We will use the same notation for the extension. So, we get

$$(n_j \circ h_j)' \circ f'_j : H_g \setminus \{K_j, \mathring{\tau}_{j_1}, \mathring{\tau}_{j_2}, \dots, \mathring{\tau}_{j_l}\} \to H_g \setminus \{K_j, \mathring{\tau}'_{j_1}, \mathring{\tau}'_{j_2}, \dots, \mathring{\tau}'_{j_l}\},$$

where K_j is the solid Klein bottle.

Note that any diffeomorphism of Klein bottle can be extended to the solid Klein bottle, refer [8].

Hence, we have a diffeomorphism

$$(n_j \circ h_j)' \circ f'_j : H_g \setminus \{\mathring{\tau}_{j_1}, \mathring{\tau}_{j_2}, \dots, \mathring{\tau}_{j_l}\} \to H_g \setminus \{\mathring{\tau}'_{j_1}, \mathring{\tau}'_{j_2}, \dots, \mathring{\tau}'_{j_l}\}.$$

We denote this diffeomorphism by $\widetilde{f_j}$.

Now, we can see that the diffeomorphism

$$\widetilde{\eta} \circ \widetilde{f_n} \circ \widetilde{f}_{n-1} \circ \dots \circ \widetilde{f_1} : H_g \setminus \{\mathring{\tau}_1, \mathring{\tau}_2 \dots, \mathring{\tau}_k\} \to H'_g \setminus \{\mathring{\tau}'_1, \mathring{\tau}'_2 \dots, \mathring{\tau}'_k\}$$

is a required extension of f. □

Now, we recall the notion of Heegaard splitting.

DEFINITION 7. (Heegaard splitting) Let M be a 3–manifold. A Heegaard splitting of M consists of two handlebodies H_1 and H_2 of the same genus g and a diffeomorphism $h : \partial H_1 \to \partial H_2$ such that M is diffeomorphic to the identification space $H_1 \cup_h H_2$. In this case, we write $M = H_1 \sqcup_h H_2$.

It is well known that any closed 3–manifold (both orientable and non-orientable) has a Heegaard splitting.

Now, we prove the following well-known Theorem given in [8] for the sake of completeness.

THEOREM 8. *Every closed non-orientable 3–manifold M can be obtained by removing finitely many solid tori from $S^2 \widetilde{\times} S^1$ and gluing them back using some diffeomorphisms. In other words, every closed connected non-orientable 3–manifold is obtained from $S^2 \widetilde{\times} S^1$ by performing surgery along a link $L = \cup L_i$, where L_i is the core of τ_i.*

Proof. Consider Heegaard splittings of M and $S^2 \widetilde{\times} S^1$ of the same genus, $M = H_1 \cup_g H_2$ and $S^2 \widetilde{\times} S^1 = H'_1 \cup_{g'} H'_2$. Now, given a diffeomorphism $h : H'_1 \to H_1$, by the previous lemma, we can extend the map $F = g \circ h \circ g'^{-1} : \partial H'_2 \to \partial H_2$ to $\widetilde{F} : H'_2 \setminus$

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$\{\mathring{\tau}_1, \mathring{\tau}_2, \ldots, \mathring{\tau}_n\} \to H_2 \setminus \{\mathring{\tau}'_1, \mathring{\tau}'_2, \ldots, \mathring{\tau}'_n\}$. Thus, $\widetilde{F}$ together with h will give a diffeomorphism $\widetilde{h} : S^2 \widetilde{\times} S^1 \setminus \{\mathring{\tau}_1, \mathring{\tau}_2, \ldots, \mathring{\tau}_n\} \to M \setminus \{\mathring{\tau}'_1, \mathring{\tau}'_2, \ldots, \mathring{\tau}'_n\}$. Now, the theorem follows easily. □

Note that from the proof of the Lemma 6, it is clear that $\widetilde{h}$ maps meridian of τ_i to some longitude of τ'_i. Also, one can observe that there are two types of components of the surgery link L in $S^2 \widetilde{\times} S^1$ from which we obtained M. Some components of L are unknots, i.e., they bound discs in $S^2 \widetilde{\times} S^1$, and some components are knots which bound Möbius bands in $S^2 \widetilde{\times} S^1$. We call such a surgery link L as a *special surgery link*. For each unknot component of the surgery link, we consider the framing given by a disc bounded by that component and for a component which bounds a Möbius band, we consider the framing given by the Möbius band. From this, we can see that the manifold M can be obtained from the link described in the Lemma 6 by performing ± 1 surgery.

Our aim is to prove the existence of an open book for a closed non-orientable 3–manifold. For a closed orientable 3–manifold, there are various proofs for the existence of an open book, see [**4**]. One such proof given in [**11**] is based on the following Lickorish theorem in [**7**]:

THEOREM 9. *Every closed oriented* 3*–manifold can be obtained by* ± 1 *surgery on a link* L_M *of unknots in* S^3*. Moreover, there is an unknot* K *such that each component of* L_M *wraps once, in monotone sense, around* K*.*

We will use the same idea to prove the existence of an open book decomposition for a non-orientable 3–manifold. To do this, first we will find a knot J in $S^2 \widetilde{\times} S^1$ with the following properties:

LEMMA 10. *Consider the special surgery link* $L = \cup L_i$ *in* $S^2 \widetilde{\times} S^1$ *which yields a closed non-orientable* 3*–manifold* M *as described in the Lemma 6. Then, there exists a knot* J *in* $S^2 \widetilde{\times} S^1$ *which satisfies the following:*

(1) J *bounds a Möbius band in* $S^2 \widetilde{\times} S^1$*.*
(2) Every component L_i *in* L *wraps once to* J *in monotone sense.*

Proof. Consider the genus g Heegaard splitting of $S^2 \widetilde{\times} S^1$, $S^2 \widetilde{\times} S^1 = H_g \cup H_g$ and consider a complex of $3g$ mutually orthogonal annuli—$\alpha_i \times I$, $\beta_i \times I$ $\gamma_i \times I$, $p_j \times I$ and $f'_j(p_j) \times I$, where $I = [0, 1]$ and $p'_j = \gamma_{g-1}$ see Figures 2 and 3—in the genus g handlebody H_g, as shown in Figure 4. A brief examination of the proof of Lemma 6 and Theorem 8 shows that the special surgery link L in H_g can be chosen such that the components L_i of L are disjoint boundary parallel circles on these annuli. Consider a curve J as shown in Figure 4 which passes through the non-orientable handle exactly twice and each L_i wraps J once.

First, we consider the case of the genus one Heegaard splitting of $S^2 \widetilde{\times} S^1$, $S^2 \widetilde{\times} S^1 = H_1 \cup_{id} H_1$, where H_1 is a solid Klein bottle, i.e., a 3–ball with a non-orientable 1–handle attached. In this case, we can choose the special surgery link whose components lie on the complex of annuli, as shown in Figure 5. As J passes through the non-orientable handle twice, it bounds a Möbius band in $S^2 \widetilde{\times} S^1$. Moreover, J is a binding of an open book decomposition of $S^2 \widetilde{\times} S^1$ with pages Möbius band and monodromy the identity. Now, we can isotope the special surgery link such that each component of the special surgery link wraps J once in monotone sense, i.e., each component intersects transversally exactly once to each Möbius band page of the open book.

Now, the genus g Heegaard splitting can be considered as stabilization of the genus one Heegaard splitting of $S^2 \widetilde{\times} S^1$. Hence, the first $3g - 3$ annuli and the line segment $\overrightarrow{PQ}$ of J in Figure 4 are contained in a 3–ball B^3. By an isotopy of $S^2 \widetilde{\times} S^1$ supported in B^3, we

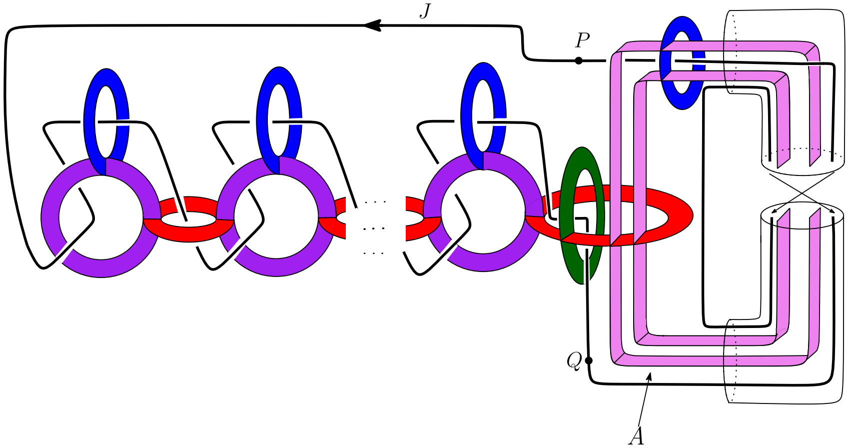

Figure 4. The components of the special surgery link L lie on the annuli, and L is braided about a knot J which bounds a Möbius band in $S^2 \widetilde{\times} S^1$.

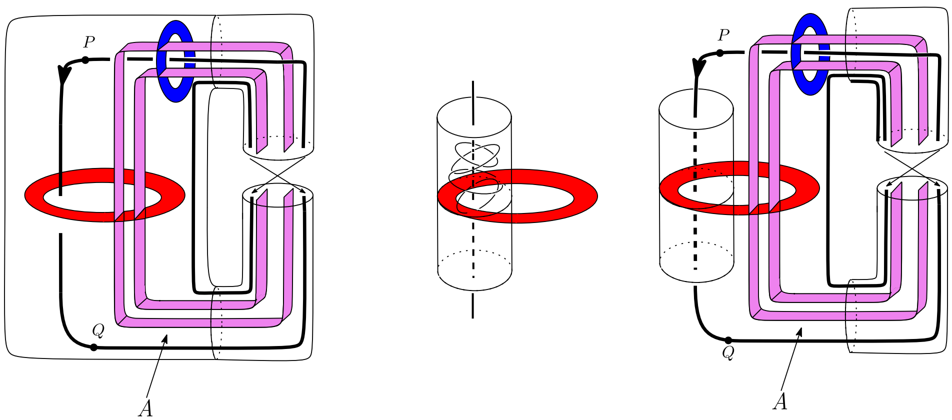

Figure 5. All the components of the special surgery link L which lie on left of the annulus A wrap J once are contained in a vertical cylinder and transverse to every page of the open book decomposition of $S^2 \widetilde{\times} S^1$ with binding J. Also, all annuli in the figure are isotoped such that they are transverse to each page.

can isotope curve J such that J becomes a binding of the open book of $S^2 \widetilde{\times} S^1$ with the pages Möbius band and the components of the special surgery link L which lie on the first $3g-3$ annuli wraps transversally J once in monotone sense, see Figure 5. Now, using the arguments given here and in the genus one case above, we can see that the special surgery link L can be isotoped such that each component of the special surgery link wraps J once in monotone sense. □

Now, we have the following theorem:

THEOREM 11. *Every closed, connected, non-orientable* 3*–manifold M admits an open book decomposition with pages the punctured Möbius band.*

Proof. By the Theorem 8, there exists a special surgery link $L = \cup_i L_i$—described in the Lemma 6—in $S^2 \widetilde{\times} S^1$ which yields the given closed non-orientable 3–manifold M by performing ± 1 surgery along the components of L. Moreover, by the Lemma 10, we can

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choose a knot J which links each component L_i of L exactly once in monotone sense. Consider the open book decomposition of $S^2 \widetilde{\times} S^1$ with pages the Möbius band and the binding J. Each component L_i punctures each Möbius band page of the open book of $S^2 \widetilde{\times} S^1$ exactly once. We remove a tubular neighborhood τ_i of L_i to perform ± 1 surgery to obtain M. Note that the boundary of the puncture on each Möbius band page corresponding to L_i is a meridional curve on $\partial\tau_i$ which becomes a longitude of $\partial\tau_i$ after performing the surgery to obtain M. Thus, we get an open book decomposition of M, where each page of the open book is obtained by gluing to each puncture in the Möbius band corresponding to L_i, an annulus bounded by L_i, and the longitudinal curve on $\partial\tau_i$ corresponding to the puncture. The binding of the open book for M is given by $L \cup J$. The fibration of this open book of M naturally comes from the fibration of the open book of $S^2 \widetilde{\times} S^1$. □

In the next section, we will show how to determine the monodromy associated with the open book decomposition of M described in the Theorem 11.

4. Monodromy of the open book. In this section, we will give an algorithm for computing the monodromy of the open book of a non-orientable 3–manifold M with pages the Möbius band minus discs, obtained in the Theorem 11. Note that the special surgery link $L = \cup_i L_i$ described in the Theorem 8 is braided about the knot J in $S^2 \widetilde{\times} S^1$. Therefore, L is the closure of a pure braid $\mathcal{L}$ over a Möbius band, as shown in Figure 12. We will use surgery modification moves described in Subsection 4.2 closely associated with the generators of the pure braid group on Möbius band to compute the monodromy of the associated open book of M.

Here, the idea is to modify the special surgery link in $S^2 \widetilde{\times} S^1$ using the moves to get a new surgery link which is the union of the closure of the trivial braid having 0 surgery coefficients and ± 1 framed knots which lie on the punctured pages—the punctured Möbius band—of the open book of $S^2 \widetilde{\times} S^1$. Then, by projecting the components of the new surgery link which lie on the pages to a punctured Möbius band, we will determine the monodromy of the open book of M as a product of Dehn twists along the projected curves on the punctured Möbius band.

First, we discuss the generators of the pure braid group on the Möbius band.

4.1. Generators for the pure braid group on Möbius band. We begin by the definition of a pure braid group.

DEFINITION 12. A pure n-stranded braid on a surface Σ is a map $F : [0, 1] \to \Sigma^n \times [0, 1]$ defined by $F(t) = (f_1(t), f_2(t), \dots, f_n(t), t)$ such that F satisfies the following:

(a) $f_i(0) = f_i(1) = p_i$, for all i.
(b) For fixed $t \in [0, 1], f_i(t) \neq f_j(t), i \neq j$.

We define i^{th}-*strand* of a pure n-stranded braid F by a map $F_i : [0, 1] \to \Sigma \times [0, 1]$ defined by $F_i(t) = (f_i(t), t)$. The strand $T_i : [0, 1] \to \Sigma \times [0, 1]$ defined by $T_i(t) = (p_i, t)$ is called a *trivial strand* of a braid on Σ and the braid $T(t) = (p_1, p_2, ..., p_n, t)$ is called the trivial braid on Σ.

Note that there is a relation between the fundamental group and the pure braid group on a surface. More precisely, for a given element in the fundamental group of a surface, one can construct a pure n-stranded braid on that surface as follows: Let, $p_1, p_2, \dots, p_n$ be distinct points on a surface Σ. Consider a subset $\mathcal{P}_i = \{p_1, p_2, ..., p_n\} \setminus \{p_i\}$ of Σ. Suppose that $\alpha : [0, 1] \to \Sigma$, where $\alpha(0) = \alpha(1)$ is an element of the fundamental group, $\pi_1(\Sigma \setminus \mathcal{P}_i,\ p_i)$,

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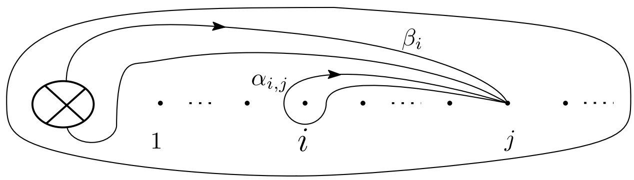

Figure 6. The figure depicts the generators of the fundamental group of Möbius band with punctures, $\pi_1(\mathcal{M} \setminus \mathcal{P}_i,\ p_i)$. Here, we consider a Möbius band as the quotient of a standard annulus by the antipodal identification on exactly one boundary component.

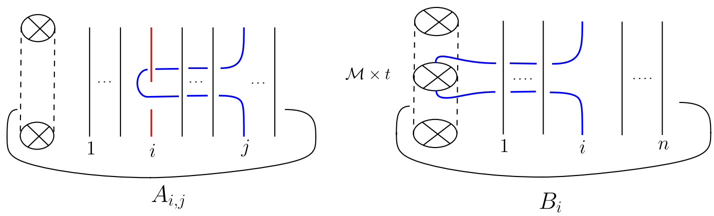

Figure 7. Generators A_{ij} and B_i of the pure braid group on the Möbius band, $\mathcal{P}_n(\mathcal{M})$.

of $\Sigma \setminus \mathcal{P}_i$ based at p_i. Then, we can associate a pure n-stranded braid on Σ, namely, $F_\alpha : [0, 1] \to \Sigma^n \times [0, 1]$ defined by $F_\alpha(t) = (p_1, p_2, \ldots, p_{i-1}, \alpha_i(t), p_{i+1}, \ldots, p_n, t)$ to α. We call such F_α as a lift of α.

We define the notion of an isotopy for pure braids on a surface as follows:

DEFINITION 13. Let F, G be two pure n-stranded braids on a surface Σ with $F(0) = G(0)$. We say that F is isotopic to G, which we denote by $F \sim G$ if there exists a map $H : [0, 1] \times [0, 1] \to \Sigma^n \times [0, 1]$ such that

(a) $H(t, 0) = F(t)$ and $H(t, 1) = G(t)$
(b) $H(0, s) = F(0)$
(c) For fixed s, $H(t, s)$ is a pure n-stranded braid on Σ.

DEFINITION 14. A pure n-stranded braid group on a surface Σ is a group of all isotopy classes of pure n-stranded braids on Σ, where the group operation is the usual product of paths. We denote pure n-stranded braid group on a surface Σ by $\mathcal{P}_n(\Sigma)$.

Now, we describe the generators of the pure braid group on the Möbius band $\mathcal{M}$. Consider the curves α_{ij}, $j \neq i$, $1 \leq i < j \leq n$ and β_i, $1 \leq i \leq n$ as shown in Figure 6. Note that α_{ij}'s and β_i's are the generators of the fundamental group $\pi_1(\mathcal{M} \setminus \mathcal{P}_i, p_i)$ of $\mathcal{M} \setminus \mathcal{P}_i$ based at p_i. One can easily see that the lifts, $A_{ij} = F_{\alpha_{ij}}$'s and $B_i = F_{\beta_i}$'s of α_{ij}'s and β_i's are the generators of $\mathcal{P}_n(\mathcal{M})$, see Figure 7. Hence, $\mathcal{P}_n(\mathcal{M}) = \langle \{A_{ij} : 1 \leq i < j \leq n\}, \{B_i : 1 \leq i \leq n\} \rangle$.

4.2. Surgery modification moves. We will consider framed pure braids on the Möbius band, i.e., a surgery coefficient (framing) is assigned to each strand in the braid. We will describe the surgery modification moves defined on the framed pure braid on the Möbius band and use these moves to change the framed pure braid $\mathcal{L}$ to a trivial braid with 0 framing associated with each strand. These moves can be described as twisting along a disc or twisting along a Möbius band as follows:

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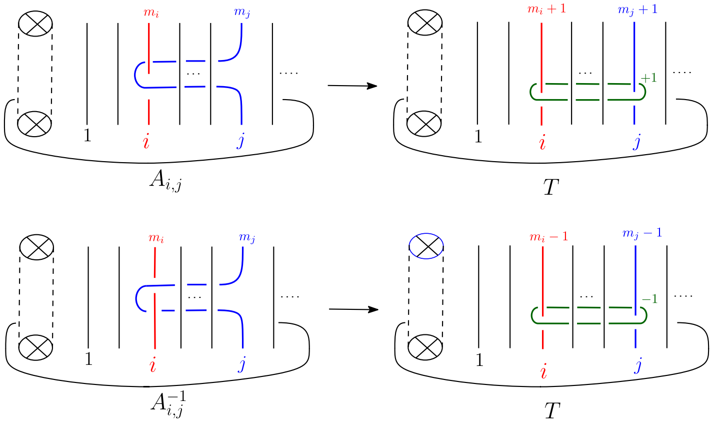

Figure 8. The α–moves with framing changes. The left (right) figure shows the effect of left (right) twist of the disc.

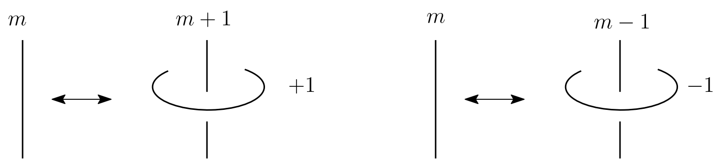

Figure 9. $\mathcal{R}$–twists with framing changes.

α–moves: This move is to change a framed braid $A_{ij}^{\pm 1}$ to a framed trivial braid T with appropriate framing changes by adding a ± 1 framed unknot which winds only i-th and j-th strands of the trivial braid. Here, the framing of only i-th and j-th strands will change by ± 1, while framing of other strands will remain unchanged. See Figure 8.

$\mathcal{R}$–moves: This move is to change the framing n of a j-th strand in a braid to the framing $n \pm 1$ by adding a ± 1 framed unknot which winds only the j-th strand, as shown in Figure 9. The framing of other strands in the braid remains unchanged.

β–moves: This move can be seen as twisting along a curve which bounds a Möbius band. This move is to change the braid generator $B_i^{\pm 1}$ to $B_i^{\mp 1}$ by adding a ± 1 framed knot which bounds a Möbius band such that the Möbius band transversely intersects only the i-th strand and disjoint from other. Here, the framing of each strand remains unchanged. See Figure 10.

We also require an isotopy as shown in Figure 11 where we first isotope a braid P to a braid Q and then Q to a braid R. We call this isotopy from P to R an $\mathcal{I}$–move.

$\mathcal{I}$–moves:

Consider the braid $P = B_i^{-1} H_{j_1} H_{j_2} \dots H_{j_s} B_i$, where $H_{j_k} = B_{j_k}^{\pm 1}$, $j_k > i$, as shown in Figure 11. The $\mathcal{I}$–move isotopes the braid P to the braid $R = H'_{j_1} H'_{j_2} \dots H'_{j_s}$ as shown in Figure 11, where

$$H'_{j_k} = \begin{cases} A_{ij_k} B_{j_k} & \text{if } H_{j_k} = B_{j_k} \\ B_{j_k}^{-1} A_{ij_k}^{-1} & \text{if } H_{j_k} = B_{j_k}^{-1}. \end{cases}$$

Note that the framing of the strands remains unchanged under an $\mathcal{I}$–move.

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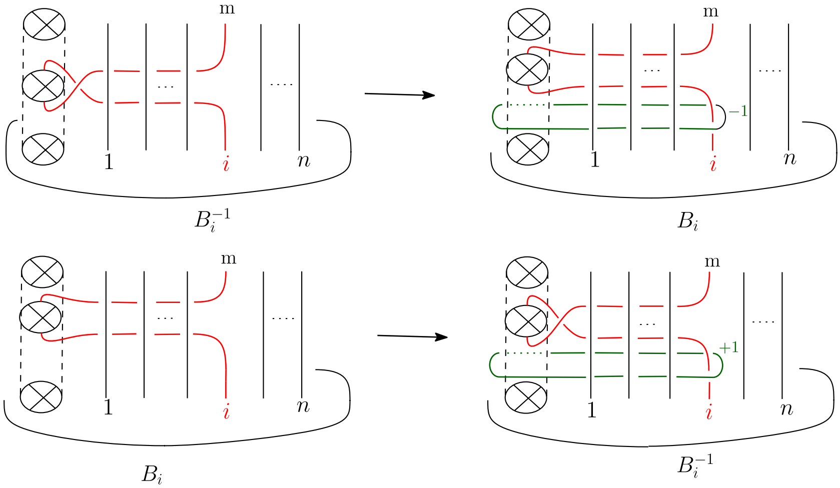

Figure 10. β–twists: changing B_i^{-1} to B_i and vice versa.

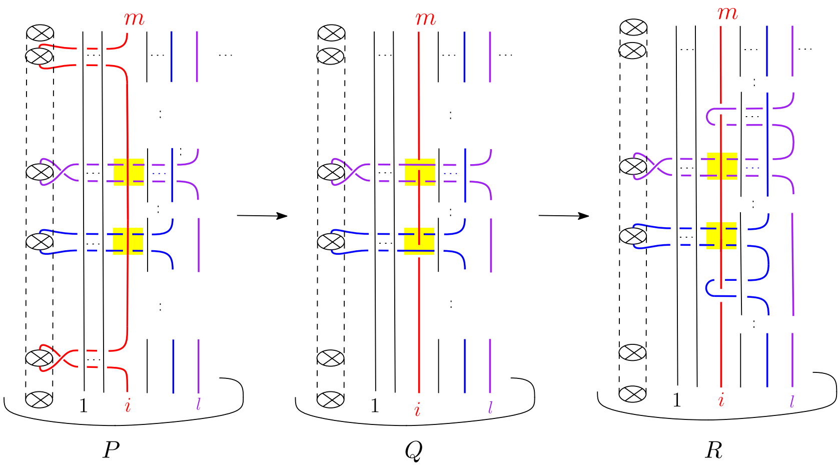

Figure 11. The isotopy which gives an $\mathcal{I}$–move

Any braid H of the form $H = B_i^{-1} H_{j_1} H_{j_2} \ldots H_{j_s} B_i$, where $H_{j_k} = B_{j_k}^{\pm 1}$, $j_k > i$, we call a $P^{(i)}$-type braid. If H is a $P^{(i)}$-type braid, then we denote its image under an $\mathcal{I}$–move by $\mathcal{I}H$ which is a braid of the form $H'_{j_1} H'_{j_2} \ldots H'_{j_s}$, where H'_{j_k} is as described above.

4.3. Monodromy computation. Recall that M is obtained from $S^2 \widetilde{\times} S^1$ by performing surgery along a ± 1 framed special link $L = \cup_i L_i$. Note that L is the closure of a framed pure braid $\mathcal{L}$ on a Möbius band $\mathcal{M}$, where $\mathcal{M}$ is a page of the open book of $S^2 \widetilde{\times} S^1$ with the binding J, the boundary of the Möbius band page. This gives an open book for M with pages a punctured Möbius band and the binding $L \cup J$.

We decompose our pure braid $\mathcal{L}$ in terms of the generators of $\mathcal{P}_n(\mathcal{M})$, see Figure 12. Suppose that $\mathcal{L} = g_1 g_2 \ldots g_m$, where g_i's are from the generating set of $\mathcal{P}_n(\mathcal{M})$.

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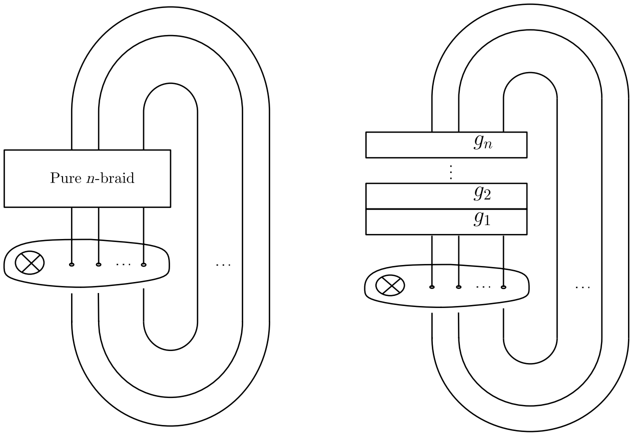

Figure 12. Decomposition of a closed braid on Möbius band.

Note that each component of our special link L has a trivial normal bundle, the number of times B_i occurs minus the number of times B_i^{-1} occurs in the product $g_1 g_2 \ldots g_m$ is 0 or ± 2.

Now, we modify the special surgery link L to a surgery link L' such that the manifold obtained by surgery on L' is diffeomorphic to M. If two surgery links L and L' give the same manifold, we call them equivalent surgery links.

We prove the following :

THEOREM 15. *There exists a sequence of links $N_0, N_1, N_2, \ldots, N_n$ equivalent to the special link L in $S^2 \widetilde{\times} S^1$ such that each N_i is the union of the closure of a framed pure braid $\mathcal{N}_i$ and ± 1 framed knots lying in different punctured Möbius band pages of the open book of $S^2 \widetilde{\times} S^1$. Each braid $\mathcal{N}_i$ is a product of the braid generators $B_j^{\pm 1}$'s, where $j > i$ and the framed pure braid $\mathcal{N}_{i+1}$ is obtained from the framed pure braid $\mathcal{N}_i$ by performing the β–moves, $\mathcal{I}$–moves, and α–moves.*

Moreover, by performing $\mathcal{R}$–moves, N_n can be chosen to be the union of the closure of the trivial braid with 0 surgery coefficients and ± 1 framed knots lying on different pages.

Proof. First, decompose the pure braid $\mathcal{L}$ in terms of the generators of $\mathcal{P}_n(\mathcal{M})$ in $\mathcal{M} \times [0, 1]$. Suppose that $\mathcal{L} = g_1 g_2 \ldots g_m$, where g_k's are from the generating set of $\mathcal{P}_n(\mathcal{M})$. Here, g_k is a braid in $\mathcal{M} \times [\frac{k-1}{m}, \frac{k}{m}]$ for $i = 1, 2, \ldots, m$.

Now, we construct the sequence of equivalent surgery links in an inductive way.

We obtain N_0 as follows:

If $g_k = A_{ij}^{\pm 1}$, then we use an α–move as shown in Figure 8 in $\mathcal{M} \times [\frac{k-1}{m}, \frac{k}{m}]$ to replace g_k by a trivial braid T with a ± 1 framed unknot in $\mathcal{M} \times t$, where $t \in [\frac{k-1}{m}, \frac{k}{m}]$. We make appropriate framing changes to i-th and j-th strands of the modified braid $\mathcal{L}$. These α-moves can be performed step by step from bottom to top of $\mathcal{M} \times [0, 1]$ by appropriately modifying the framings of the strands in each step. After performing α– moves for all such g_k's, we get a new braid $\mathcal{N}_0$ which is a product of the braid generators $B_j^{\pm 1}$ for $j > 0$. Hence, we obtain the link N_0 as the closure of the braid $\mathcal{N}_0$ together with ± 1 framed unknots a_{ij}'s lying on the distinct punctured pages of the open book of $S^2 \widetilde{\times} S^1$.

To obtain the link N_1 from the link N_0, we proceed as follows: We perform β–moves on the braid $\mathcal{N}_0$ at appropriate places to obtain a braid $\mathcal{N}_0'$ which is a product of $P^{(1)}$-type

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braids. Recall by a $P^{(i)}$-type braid, we mean a braid of the form $B_i^{-1}H_{j_1}H_{j_2}\dots H_{j_s}B_i$, where $H_{j_k}=B_{j_k}^{\pm 1},\ j_k>i$.

Now, we perform $\mathcal{I}$–moves on each $P^{(1)}$-type braids in $\mathcal{N}_0'$ to isotope the braid $\mathcal{N}_0'$ to a braid $\mathcal{I}\mathcal{N}_0'$ such that $\mathcal{I}\mathcal{N}_0'$ has trivial 1-st strand. Then perform α–moves to eliminate the $A_{1j}^{\pm 1}$'s, $j>1$ from the braid $\mathcal{I}\mathcal{N}_0'$ to obtain the braid $\mathcal{N}_1$. Note that the braid $\mathcal{N}_1$ is a product of $B_j^{\pm 1}, j>1$. Let the link N_1 be the union of the closure of the braid $\mathcal{N}_1$ together with ± 1 framed knots which we added while performing β and α–moves to obtain the braid $\mathcal{N}_1$ from the braid $\mathcal{L}$ associated with the special link L.

Hence, inductively, we can obtain the equivalent surgery link N_{i+1} from the link N_i as described in the above paragraph by performing $\beta, \mathcal{I}$, and α–moves on the braid $\mathcal{N}_i$. Thus, finally by applying the $\mathcal{R}$–moves on the braid $\mathcal{N}_n$, we get the equivalent surgery link N_n as described in the statement of the theorem. □

Let us recall the following lemma to determine the monodromy:

LEMMA 16. *Let (Σ, ϕ) be an open book decomposition for a closed 3–manifold M.*

(a) Consider a knot $K=\{p\}\times S^1\subset \mathcal{MT}(\Sigma,\phi)$, where $p\in\Sigma$ such that $\phi|_{D_p}=Id$, for some disc neighborhood D_p of p in Σ. Let $\tilde{M}$ be the manifold obtained by 0–surgery—preferred longitude for $\partial D_p\times S^1$ is $\{p_0\}\times S^1$, $p_0\in\partial D_p$—along a knot K then $\tilde{M}$ have open book decomposition with the page $\tilde{\Sigma}=\Sigma\setminus \mathring{D}_p$ and the monodromy $\tilde{\phi}=\phi|_{\tilde{\Sigma}}$.

(b) Consider a knot $K\subset\Sigma\times\{t\}\subset\mathcal{MT}(\Sigma,\phi)$. Let $\tilde{M}$ be the manifold obtained by ± 1–surgery—preferred framing obtained by $\Sigma\times\{t\}$—along the knot K then $\tilde{M}$ have open book decomposition with the page $\tilde{\Sigma}=\Sigma$ and the monodromy $\tilde{\phi}=\phi\circ d_K^{\mp 1}$, where d_K^{-1}/d_K^{+1} is left-/right-handed Dehn twist along K.

We have obtained a surgery link N_n equivalent to L in the Theorem 15. The link N_n is the union of the closure of the trivial braid with 0 surgery coefficients and ± 1 framed knots lying on different punctured pages of $S^2\widetilde{\times}S^1$. We label these ± 1 framed knots $c_1, c_2, \dots, c_r$ according to their position in $\mathcal{M}\times[0,1]$ from bottom to top. We project these curves to a fixed page, the punctured Möbius band. Then, the monodromy ϕ of the open book of M with pages the punctured Möbius band is the product $\phi=\phi_1\phi_2\dots\phi_r$, where $\phi_k=d_{c_k}^{\pm 1}$ if c_k has the framing ∓ 1, which follows from the Lemma 16. Here, $d_c^{\pm 1}$ is a right/left Dehn twist along the curve c.

5. Illustration of monodromy calculation. In this section, we will illustrate the algorithm which we describe in the proof of the Theorem 15 with an example.

Consider a closed non-orientable 3–manifold M obtained by special surgery link L with surgery coefficients, as shown in Figure 13. We change the braid associated with our link step by step, as shown in Figures 13 and 14.

Finally, we got a link N_3 as the union of the closure of trivial braid with 0–surgery coefficients and some knots—ordered from bottom to top of $\mathcal{M}\times[0,1]$—with surgery coefficient ± 1.

Now, by projecting these knots in different pages to a page we get the monodromy generating curves as shown in Figure 15. Hence, we get the monodromy

$$\phi=d_{c_1}^{+1}d_{c_2}^{-1}d_{c_3}^{-1}d_{c_4}^{-1}d_{c_5}^{-1}d_{c_6}^{+1}d_{c_7}^{+1}d_{c_8}^{-1}d_{c_9}^{-1}d_{c_{10}}^{+1}d_{c_{11}}^{+1}d_{c_{12}}^{+1}.$$

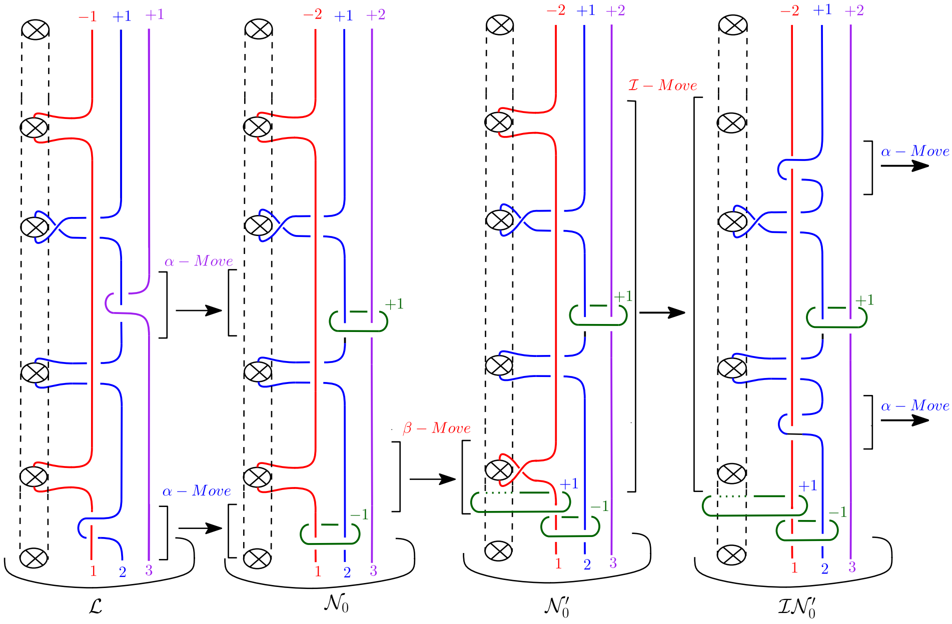

Figure 13. Surgery modification moves on the given special surgery braid $\mathcal{L}$ to get the braid $\mathcal{I}\mathcal{N}'_0$.

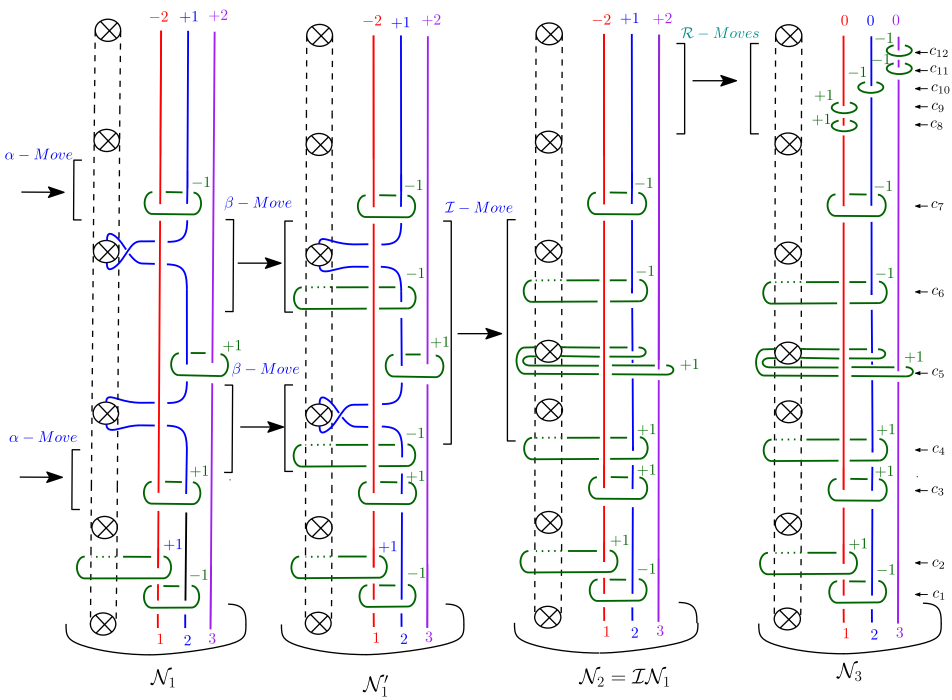

Figure 14. Surgery modification moves on the braid $\mathcal{I}\mathcal{N}'_0$ to get the braid $\mathcal{N}_3$.

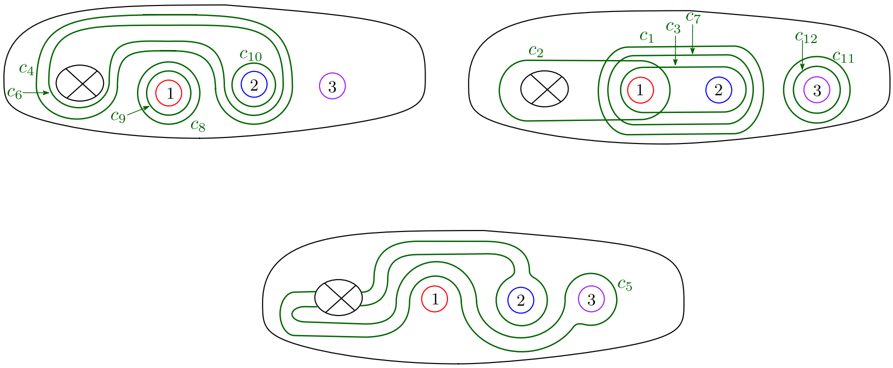

Figure 15. The monodromy is the product of Dehn twists along the projected knots as shown in the figure.

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