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COMPARISON OF LOCAL RELATIVE CHARACTERS AND THE ICHINO–IKEDA CONJECTURE FOR UNITARY GROUPS

RAPHAËL BEUZART-PLESSIS

Université d'Aix-Marseille, I2M-CNRS(UMR 7373), Campus de Luminy, 13288 Marseille Cédex 9, France (rbeuzart@gmail.com)

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Abstract In this paper, we prove a conjecture of Wei Zhang on comparison of certain local relative characters from which we draw some consequences for the Ichino–Ikeda conjecture for unitary groups.

Keywords: relative trace formula; relative characters; Ichino–Ikeda conjecture; automorphic representations; unitary groups

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Contents

1. Introduction

Let E/F be a quadratic extension of number fields. Let V be a $(n+1)$-dimensional Hermitian space over E and let $W \subset V$ be a nondegenerate hyperplane. Set $G = U(W) \times U(V)$ and $H = U(W)$. We view H as a subgroup of G via the natural diagonal embedding. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. Define the H-period of π to be the linear form $\mathcal{P}_H : \pi \to \mathbb{C}$ given by

$$\mathcal{P}_H(\phi) = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)\, dh, \quad \phi \in \pi$$

where dh stands for the Tamagawa Haar measure on $H(\mathbb{A})$ (the integral is absolutely convergent by cuspidality of π). Let $\mathrm{BC}(\pi)$ be the base change of π to $\mathrm{GL}_n(\mathbb{A}_E) \times \mathrm{GL}_{n+1}(\mathbb{A}_E)$ (known to exist thanks to the recent work of Mok [38] and Kaletha, Minguez, Shin and White [28]). We may decompose $\pi = \pi_n \boxtimes \pi_{n+1}$ with π_n, π_{n+1} cuspidal automorphic representations of $U(W)$ and $U(V)$, respectively. We have a similar decomposition $\mathrm{BC}(\pi) = \mathrm{BC}(\pi_n) \boxtimes \mathrm{BC}(\pi_{n+1})$ with $\mathrm{BC}(\pi_n)$, $\mathrm{BC}(\pi_{n+1})$ two automorphic representations of $\mathrm{GL}_{n,E}$ and $\mathrm{GL}_{n+1,E}$, respectively. Let $L(s, \mathrm{BC}(\pi))$ denote the L-function of pair $L(s, \mathrm{BC}(\pi_n) \times \mathrm{BC}(\pi_{n+1}))$ defined by Jacquet, Piatetski-Shapiro and Shalika. If π is tempered everywhere (meaning that for all place v, the local representation π_v is tempered), a famous conjecture of Gan, Gross and Prasad links the nonvanishing of the period $\mathcal{P}_H$ to the nonvanishing of the central value $L(1/2, \mathrm{BC}(\pi))$ (see [16, Conjecture 24.1] for a precise statement). In the influential paper [21], Ichino and Ikeda have proposed a refinement of this conjecture for orthogonal groups in the form of an exact formula relating these two invariants. This conjecture has been suitably extended to unitary groups by N. Harris in his Ph.D. thesis [18]. These formulas are modeled on the celebrated work of Waldspurger [50] on toric periods for GL_2.

In two recent papers [55, 56], W. Zhang has proved both the Gan–Gross–Prasad and the Ichino–Ikeda conjectures for unitary groups under some local assumptions on π. More precisely, Zhang proves the Gan–Gross–Prasad conjecture under some mild local assumptions [55, Theorem 1.1] (mainly that π is supercuspidal at one place of F which splits in E), but he only gets the Ichino–Ikeda conjecture under far more stringent assumptions [56, Theorem 1.2]. This discrepancy is due to some local difficulties that we shall discuss shortly. In [56], Zhang makes a series of conjectures (one for every

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place of F) which, if true, would allow to considerably weaken the assumptions of [56, Theorem 1.2]. The goal of this paper is to prove this conjecture at all non-Archimedean place of F. As a consequence, we derive new cases of the Ichino–Ikeda conjecture.

We now formulate the Ichino–Ikeda conjecture in a form suitable for our purpose. Assume from now on that π is everywhere tempered. Set

$$\mathcal{L}(s,\pi) := \Delta_{n+1}\frac{L(s,\mathrm{BC}(\pi))}{L(s+\frac{1}{2},\pi,Ad)}.$$

Here Δ_{n+1} is the following product of special values of Hecke L-functions

$$\Delta_{n+1} := \prod_{i=1}^{n+1} L(i,\eta_{E/F}^{i}),$$

where $\eta_{E/F}$ denotes the idèle class character associated with the extension E/F and the adjoint L-function of π is defined by

$$L(s,\pi,Ad) := L(s,\mathrm{BC}(\pi_n),As^{(-1)^n})L(s,\mathrm{BC}(\pi_{n+1}),As^{(-1)^{n+1}})$$

(see [16, § 7] for the definition of the Asai L-functions). For every place v of F, we will denote by $\mathcal{L}(s,\pi_v)$ the corresponding quotient of local L-functions. To the period $\mathcal{P}_H$, we associate a global relative character J_π. It is the distribution on the Schwartz space $\mathcal{S}(G(\mathbb{A}))$ of $G(\mathbb{A})$ given by

$$J_\pi(f) = \sum_{\phi\in\mathcal{B}_\pi} \mathcal{P}_H(\pi(f)\phi)\overline{\mathcal{P}_H(\phi)}$$

for every $f \in \mathcal{S}(G(\mathbb{A}))$ where $\mathcal{B}_\pi$ is a (suitable) orthonormal basis of π for the Petersson inner product

$$(\phi,\phi')_{Pet} = \int_{G(F)\backslash G(\mathbb{A})} \phi(g)\overline{\phi(g)}\,dg$$

(again normalized with respect to the Tamagawa Haar measure on $G(\mathbb{A})$). We also define local relative characters as follows. Fix factorizations into local Haar measures $dg = \prod_v dg_v$ and $dh = \prod_v dh_v$ of the Tamagawa measures on $G(\mathbb{A})$ and $H(\mathbb{A})$, respectively. For every place v of F, we define a local relative character $J_{\pi_v} : \mathcal{S}(G(F_v)) \to \mathbb{C}$ (where $\mathcal{S}(G(F_v))$ denotes the Schwartz space of $G(F_v)$) by

$$J_{\pi_v}(f_v) = \int_{H(F_v)} \mathrm{Trace}(\pi_v(h)\pi_v(f_v))\,dh_v, \quad f_v \in \mathcal{S}(G(F_v)).$$

The temperedness of π_v implies that this integral is absolutely convergent. For almost all places v of F, for f_v is the characteristic function of $G(\mathcal{O}_v)$, we have

$$J_{\pi_v}(f_v) = \mathcal{L}(\tfrac{1}{2},\pi_v)\mathrm{vol}(H(\mathcal{O}_v))\mathrm{vol}(G(\mathcal{O}_v)).$$

Therefore, we define a normalized relative character $J^{\natural}_{\pi_v}$ by

$$J^{\natural}_{\pi_v}(f_v) = \frac{J_{\pi_v}(f_v)}{\mathcal{L}(\frac{1}{2},\pi_v)}, \quad f_v \in \mathcal{S}(G(F_v)).$$

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Let S_π be the centralizer of the so-called 'L-parameter of π'. Although strictly speaking such an L-parameter is not well defined (because, among other things, of the lack of a suitable definition for the Langlands group of the number field F), we can use the base-change representation $\mathrm{BC}(\pi)$ as a substitute for it and the group S_π can be defined explicitly using an isobaric decomposition of $\mathrm{BC}(\pi)$. We refer the reader to [28, § 1.3.4] for details on this. The group S_π is always an elementary 2-abelian group (that is a product of copies of $\mathbb{Z}/2\mathbb{Z}$), and, moreover, if $\mathrm{BC}(\pi)$ is cuspidal, we have $S_\pi \simeq (\mathbf{Z}/2\mathbf{Z})^2$. We can now state the Ichino–Ikeda conjecture as follows.

Conjecture 1.0.1 (Ichino–Ikeda). *Assume that π is everywhere tempered. Then, for all factorizable test function $f = \prod_v f_v \in \mathcal{S}(G(\mathbb{A}))$, we have*

$$J_\pi(f) = |S_\pi|^{-1} \mathcal{L}(\tfrac{1}{2}, \pi) \prod_v J^\natural_{\pi_v}(f_v).$$

We remark here that the Ichino–Ikeda conjecture is not usually stated in this way but rather in a form involving directly the (square of the absolute value of the) period $\mathcal{P}_H$ and some local periods (see [21, Conjecture 1.5] and [18, Conjecture 1.2]); see, however, [56, Lemma 1.7] for the equivalence between the two formulations.

The main tool used by Zhang to attack Conjecture 1.0.1 is a comparison of certain (simple) relative trace formulas that have been proposed by Jacquet and Rallis [26]. To carry this comparison, we need a fundamental lemma and the existence of smooth matching. The fundamental lemma for the case at hand has been proved by Yun [54] in a positive characteristic and extended by Gordon to characteristic zero in the appendix to [54]. The existence of smooth matching at non-Archimedean places is one of the main achievements of Zhang in [55]. It has been recently extended in a weak form by Xue [53] to Archimedean places. The comparison between the two trace formulas has been done by Zhang in [55]. The output is an identity relating the relative character J_π (under some mild local assumptions on π) to certain periods on the base change of π. More precisely, there is a certain relative character $I_{\mathrm{BC}(\pi)}$ attached to these periods and we get an equality between $J_\pi(f)$ and $I_{\mathrm{BC}(\pi)}(f')$ up to an explicit factor for *nice* matching functions f and f' (see [56, Theorem 4.3] and Theorem 3.5.1). Thanks to the work of Jacquet, Piatetski-Shapiro and Shalika on Rankin–Selberg convolutions, we know an explicit factorization for $I_{\mathrm{BC}(\pi)}$ in terms of local (normalized) relative characters $I^\natural_{\mathrm{BC}(\pi_v)}$ (see [56, Proposition 3.6]). As a consequence, we also get an explicit factorization of J_π. However, this factorization is still in terms of the local relative characters $I^\natural_{\mathrm{BC}(\pi_v)}$ which are living on (products of) general linear groups. In order to get the Ichino–Ikeda conjecture, we need to compare them with our original local relative characters $J^\natural_{\pi_v}$. It is precisely the content of the following conjecture of Zhang (see [56, Conjecture 4.4] and Conjecture 3.5.5 for precise statements).

Conjecture 1.0.2 (Zhang). *Let v be a place of F. Then for all matching functions $f_v \in \mathcal{S}(G(F_v))$ and $f'_v \in \mathcal{S}(G'(F_v))$, we have*

$$I^\natural_{\mathrm{BC}(\pi)_v}(f'_v) = C(\pi_v) J^\natural_{\pi_v}(f_v),$$

where $C(\pi_v)$ is some explicit constant.

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Together with the above-mentioned comparison of relative trace formulas, this conjecture implies the Ichino–Ikeda conjecture under mild local assumptions (see [56, Proposition 4.5]). Zhang was able to verify his conjecture in certain particular cases. More precisely, in [56], the above conjecture is proved for split places or when the representation π_v is unramified (and the residual characteristic is sufficiently large) or supercuspidal (see Theorem 4.6 of *loc. cit.*). This explains the very strong conditions that are imposed on π in [56, Theorem 1.2]. The main purpose of this paper is to prove Conjecture 1.0.2 at every non-Archimedean place. Our main result thus reads as follows (see Theorem 3.5.7).

Theorem 1.0.3. *For every non-Archimedean place v of F, Conjecture 1.0.2 holds at v.*

As a consequence of this theorem, we obtain the following result toward Conjecture 1.0.1 (see Theorem 3.5.8).

Theorem 1.0.4. *Let π be the cuspidal automorphic representation of $G(\mathbb{A})$ which is everywhere tempered. Assume that all the Archimedean places of F split in E and that there exists a non-Archimedean place v_0 of F such that $\mathrm{BC}(\pi_{v_0})$ is supercuspidal. Then Conjecture 1.0.1 holds for π.*

The main new ingredient in the proof of Theorem 1.0.3 is a group analog of the local relative trace formula for Lie algebras developed by Zhang in [55, § 4.1]. Actually, this local trace formula can be derived directly from results contained in [55] and [6] so that the proof of it is rather brief (see § 4.3). We then deduce Theorem 1.0.3 from a combination of this local trace formula with certain results of Zhang on truncated local expansion of relative characters (see [56, § 8] and § 4.1).

We now briefly describe the content of each section. In § 2, we set up the notation, fix the measures and recall a number of results in particular concerning global and local base change for unitary groups and the local Gan–Gross–Prasad conjecture that will be needed in the sequel. In § 3, we mainly recall the work of Zhang on comparison of global relative trace formulas, and we state precisely Conjecture 1.0.2 as well as the main results (Theorems 3.5.7 and 3.5.8). Section 4 is devoted to the proofs of Theorems 1.0.3 and 1.0.4. Finally, we have included an appendix to prove that the simple Jacquet–Rallis trace formulas are still absolutely convergent for test functions which are not necessarily compactly supported (but nevertheless rapidly decreasing). For this, we define certain norms on the automorphic quotient $[G] := G(F)\backslash G(\mathbb{A})$ and establish their basic properties. This material is certainly classical, but the author has not been able to find a convenient reference; hence, we provide complete proofs. It has, however, interesting consequences, e.g. for H a closed algebraic subgroup of G, we give a simple criterion for the convergence of the H-period of any cusp form on $[G]$: if the center of G does not intersect H, it suffices that the variety $H\backslash G$ is quasi-affine (see Proposition A.1.1(ix)).

2. Preliminaries

2.1. General notations and conventions

In this paper, E/F will always be a quadratic extension of number fields or of local fields of characteristic zero. We denote by $\mathrm{Tr}_{E/F}$ and N the trace and norm of this extension and by $x \mapsto \overline{x}$ the nontrivial F-automorphism of E. Moreover, we fix a nonzero element $\tau \in E$ such that $\mathrm{Tr}_{E/F}(\tau) = 0$. The notation $\mathrm{R}_{E/F}$ stands for the Weil restriction of scalars from E to F. For every finite dimensional Hermitian space V over E, we denote by $U(V)$ the corresponding unitary group and we write $\mathfrak{u}(V)$ for its Lie algebra. We also let $\mathrm{disc}(V) \in F^\times / \mathrm{N}(E^\times)$ be the *discriminant* of V (that is, the determinant of the matrix representing the underlying Hermitian form in any basis of V). The standard maximal unipotent subgroup of GL_n is denoted by N_n. For every connected reductive group G over F, we write Z_G for the center of G. For all $n \geqslant 1$, we define a variety S_n over F by

$$\mathrm{S}_n := \{s \in \mathrm{R}_{E/F}\,\mathrm{GL}_n;\, s\overline{s} = 1\}$$

and its 'Lie algebra' $\mathfrak{s}_n$ by

$$\mathfrak{s}_n := \{X \in \mathrm{R}_{E/F}\, M_n;\, X + \overline{X} = 0\}.$$

We have a surjective map $\nu : \mathrm{R}_{E/F}\,\mathrm{GL}_n / \mathrm{GL}_n \to \mathrm{S}_n$ given by $\nu(g) = g\overline{g}^{-1}$ which, by Hilbert 90, is surjective at the level of k-points for any field k. We denote by $\mathfrak{c}$ the Cayley map $\mathfrak{c} : X \mapsto (X+1)(X-1)^{-1}$ which realizes a birational isomorphism between $\mathfrak{s}_n$ and S_n and also between $\mathfrak{u}(V)$ and $U(V)$ for all finite dimensional Hermitian space V over E.

Assume that the fields E and F are local. We denote by $|.|_F$ the normalized absolute value on F (and similarly for E) and by $\eta_{E/F}$ the quadratic character of $F^\times$ associated with the extension E/F. We will also fix an extension η' of $\eta_{E/F}$ to $E^\times$ and a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. We set $\psi_E(z) = \psi(\frac{1}{2}\mathrm{Tr}_{E/F}(z))$ for all $z \in E$. Let G be a reductive connected group over F. By a representation of $G(F)$, we mean a smooth representation if F is p-adic and an admissible smooth Fréchet representation of moderate growth if F is Archimedean (see [5, 10] and [51, § 11]). We denote by $\mathrm{Irr}(G)$, $\mathrm{Irr}_{\mathrm{unit}}(G)$ and $\mathrm{Temp}(G)$ the set of isomorphism classes of irreducible, irreducible unitary and irreducible tempered representations of $G(F)$, respectively. We endow these sets with the Fell topology (see [49]). For any parabolic subgroup $P = MU$ of G (U denoting the unipotent radical of P and M a Levi factor) and for any irreducible representation σ of $M(F)$, we denote by $i_P^G(\sigma)$ the normalized parabolic induction of σ. The notation $\Psi_{\mathrm{unit}}(G)$ stands for the group of unitary unramified characters of $G(F)$. The space of Schwartz functions $\mathcal{S}(G(F))$ consists of locally constant compactly supported functions if F is p-adic and functions rapidly decreasing together with all their derivatives if F is Archimedean (see [6, § 1.4]). If F is p-adic and Ω is a finite union of Bernstein components of $G(F)$ [4], we denote by $\mathcal{S}(G(F))_\Omega$ the corresponding summand of $\mathcal{S}(G(F))$ (for the action by left translation). Finally, if π is an irreducible generic representation of $\mathrm{GL}_n(E)$, we denote by $\mathcal{W}(\pi, \psi_E)$ the Whittaker model of π with respect to ψ_E. It is a space of smooth functions $W : G(F) \to \mathbb{C}$ satisfying the relation

$$W(ug) = \psi_E\left(\sum_{i=1}^{n-1} u_{i,i+1}\right) W(g)$$

for all $u \in N_n(E)$ and such that π is isomorphic to $\mathcal{W}(\pi, \psi_E)$ equipped with the $G(F)$-action by right translation.

In the number field case, we denote by $\mathbb{A}$ and $\mathbb{A}_E$ the rings of adèles of F and E, respectively, and by $\eta_{E/F}$ the idèle class character associated with the extension E/F. We fix an extension η' of $\eta_{E/F}$ to $\mathbb{A}_E^\times$. For every place v of F, we denote by F_v the corresponding completion, $\mathcal{O}_v \subset F_v$ the ring of integers (if v is non-Archimedean) and we set $E_v = E \otimes_F F_v$, $\mathcal{O}_{E,v} = \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_v$ where $\mathcal{O}_F$, $\mathcal{O}_E$ are the rings of integers in F and E, respectively. If S is a finite set of places of F, we define $F_S = \prod_{v \in S} F_v$. If Σ is a (usually infinite) set of places of F, we write $\mathbb{A}_\Sigma$ for the restricted product of the F_v for $v \in \Sigma$. We also fix a nontrivial additive character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ and we set $\psi_E(z) = \psi(\frac{1}{2}\operatorname{Tr}_{E/F}(z))$ for all $z \in \mathbb{A}_E$. For every place v of F, we denote by ψ_v, $\psi_{E,v}$ and η'_v the local components at v of ψ, ψ_E and η', respectively. Let G be a connected reductive group over F. We set $[G] = G(F)\backslash G(\mathbb{A})$ and for every place v of F, we denote by G_v the base change of G to F_v. The Schwartz space $\mathcal{S}(G(\mathbb{A}))$ of $G(\mathbb{A})$ is, by definition, the restricted tensor product of the local Schwartz spaces $\mathcal{S}(G(F_v))$. We denote by $\mathcal{U}(\mathfrak{g}_\infty)$ the enveloping algebra of the complexification of the Lie algebra $\mathfrak{g}_\infty$ of $\prod_{v|\infty} G(F_v)$ and by $C_G \in \mathcal{U}(\mathfrak{g}_\infty)$ the Casimir element. If a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ has been fixed, we also denote by $C_K \in \mathcal{U}(\mathfrak{g}_\infty)$ the Casimir element of $K_\infty := \prod_{v|\infty} K_v$. Finally, if $\eta : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ is an idèle class character and $g \in \mathrm{GL}_n(\mathbb{A})$, we usually abbreviate $\eta(\det g)$ by $\eta(g)$.

2.2. Analytic families of distributions

Assume that F is a local field. Let G be a connected reductive group over F and let $\pi \mapsto L_\pi$ be a family of (continuous if F is Archimedean) linear forms on $\mathcal{S}(G(F))$ indexed by the set $\mathrm{Temp}(G(F))$ of all irreducible tempered representations of $G(F)$. Assume that the following condition is satisfied:

> For every parabolic subgroup $P = MU$ of G and for every irreducible square-integrable representation σ of $M(F)$, there is at most one irreducible subrepresentation π of $i_P^G(\sigma)$ such that $L_\pi \neq 0$.

This condition is, for example, automatically satisfied if $G = \mathrm{GL}_n$ (as in this case, the representation $i_P^G(\sigma)$ is always irreducible). If this condition is satisfied, we may extend the family of distributions $\pi \mapsto L_\pi$ to any induced representation $i_P^G(\sigma)$ as above by setting $L_{i_P^G(\sigma)} = L_\pi$ if π is the unique irreducible subrepresentation of $i_P^G(\sigma)$ such that $L_\pi \neq 0$ and $L_{i_P^G(\sigma)} = 0$ if no such subrepresentation exists. We then say that this family is *analytic* if for all $f \in \mathcal{S}(G(F))$, all parabolic subgroup $P = MU$ and all square-integrable representation σ of $M(F)$, the function

$$\chi \in \Psi_{\text{unit}}(M) \mapsto L_{i_P^G(\sigma\otimes\chi)}(f)$$

is analytic (recall that $\Psi_{\text{unit}}(M)$ being a compact real torus has a natural structure of analytic variety).

2.3. Base change for unitary groups

Let E/F be a quadratic extension of local fields of characteristic zero (either Archimedean or p-adic). Let V be a n-dimensional Hermitian space over E. Recall that the set

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of Langlands parameters for $U(V)$ is in one-to-one correspondence with the set of $(-1)^{n+1}$-conjugate dual continuous semisimple representations φ of the Langlands group $\mathcal{L}_E$ of E (see [16, § 3] for a definition of ϵ-conjugate dual representations). In what follows, by a Langlands parameter for $U(V)$, we shall mean a representation φ of this sort. By the recent results of Mok [38] and Kaletha–Minguez–Shin–White [28] on the local Langlands correspondence for unitary groups together with the work of Langlands [30] for real groups, we know that there exists a canonical decomposition

$$\mathrm{Irr}(U(V)) = \bigsqcup_{\varphi} \Pi^{U(V)}(\varphi)$$

indexed by the set of all Langlands parameters for $U(V)$. The sets $\Pi^{U(V)}(\varphi)$ are finite (some of them may be empty) and called L-packets. By the Langlands classification, the above decomposition boils down to an analog decomposition of the tempered dual

$$\mathrm{Temp}(U(V)) = \bigsqcup_{\varphi} \Pi^{U(V)}(\varphi)$$

where the union is over the set of *tempered* Langlands parameters for $U(V)$, i.e. the parameters φ whose image is bounded. This last decomposition admits a characterization in terms of endoscopic relations [38, Theorem 3.2.1], [28, Theorem 1.6.1] and of the (known) Langlands correspondence for $\mathrm{GL}_d(E)$ [19, 20, 45]. By this Langlands correspondence, every parameter φ of $U(V)$ determines an irreducible representation $\pi(\varphi)$ of $\mathrm{GL}_n(E)$. If π is in the L-packet corresponding to φ, we will write $\mathrm{BC}(\pi) := \pi(\varphi)$. If π is tempered, then so is $\mathrm{BC}(\pi)$ and conversely. However, it might happen that π is supercuspidal or square-integrable, but $\mathrm{BC}(\pi)$ is not. Aubert, Moussaoui and Solleveld [2] have recently proposed a very general conjecture on how to detect supercuspidal representations in L-packets. Moreover, Moussaoui [39] has been able to verify this conjecture for orthogonal and symplectic groups. Most probably, his work will soon cover unitary groups too. We will need the following particular case of the Aubert–Moussaoui–Solleveld conjecture for which, however, we can give a direct proof.

Lemma 2.3.1. *Assume that F is p-adic. Let $\pi \in \mathrm{Irr}(U(V))$ and assume that $\mathrm{BC}(\pi)$ is supercuspidal. Then so is π.*

Proof. We will use the following characterization of supercuspidal representations:

(2.3.1) π is supercuspidal if and only if the Harish-Chandra character Θ_π of π is compactly supported modulo conjugation.

The necessity follows from a well-known result of Deligne [14]. This sufficiency follows, for example, from Clozel's formula for the character [12, Proposition 1].

Let φ be the Langlands parameter of π. Then, by our assumption, the L-packet $\Pi^{U(V)}(\varphi)$ is a singleton. Introduce the *twisted* group $\widetilde{\mathrm{GL}_n(E)} = \mathrm{GL}_n(E)\theta_n$, where $\theta_n(g) = {}^t\overline{g}^{-1}$. It is the set of F-points of the nonneutral connected component of the non-connected group $G^+ = R_{E/F}\,\mathrm{GL}_n \rtimes \{1, \theta_n\}$. Since φ is a conjugate-dual representation of $\mathcal{L}_E$, it follows that $\mathrm{BC}(\pi)$ may be extended to a representation $\mathrm{BC}(\pi)^+$ of $G^+(F)$. Denote by

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$\widetilde{\mathrm{BC}(\pi)}$ the restriction of $\mathrm{BC}(\pi)^+$ to $\widetilde{\mathrm{GL}_n(E)}$ and denote by $\Theta_{\widetilde{\mathrm{BC}(\pi)}}$ the Harish-Chandra character of $\widetilde{\mathrm{BC}(\pi)}$ (the Harish-Chandra theory of characters has been extended to twisted groups by Clozel [13]). Since $\mathrm{BC}(\pi)$ is supercuspidal, the character $\Theta_{\widetilde{\mathrm{BC}(\pi)}}$ is compactly supported modulo conjugation (this follows, for example, from the equality up to a factor between $\Theta_{\widetilde{\mathrm{BC}(\pi)}}$ and weighted orbital integrals of coefficients of $\widetilde{\mathrm{BC}(\pi)}$; see [36, théorème 7.2]). By the endoscopic characterization of the local Langlands correspondence for unitary groups, there is a relation between Θ_π and $\Theta_{\widetilde{\mathrm{BC}(\pi)}}$. More precisely, there is a correspondence between (stable) regular conjugacy classes in $U(V)(F)$ and $\widetilde{\mathrm{GL}_n(E)}$ (see [7, § 3.2]; in this particular case, the correspondence takes the form of an injective map $U(V)_{\mathrm{reg}}(F)/stab \hookrightarrow \widetilde{\mathrm{GL}_n(E)}_{\mathrm{reg}}/stab$), and for all regular elements $y \in U(V)(F)$, $\widetilde{x} \in \widetilde{\mathrm{GL}_n(E)}$ that correspond to each other, we have (see [38, Theorem 3.2.1] and [28, Theorem 1.6.1])

$$\Theta_{\widetilde{\mathrm{BC}(\pi)}}(\widetilde{x}) = \Delta(y, \widetilde{x}), \Theta_\pi(y)$$

where $\Delta(y, \widetilde{x})$ is (up to a sign) a certain transfer factor. From this relation, we easily infer that Θ_π is compactly supported modulo conjugation and hence that π is supercuspidal by (2.3.1). □

We now move on to a global setting. Thus, E/F is a quadratic extension of number fields and V is an n-dimensional Hermitian space over E. If v is a place of F which splits in E, then we have isomorphisms $U(V)(F_v) \simeq \mathrm{GL}_n(F_v)$ and $(\mathrm{R}_{E/F}\,\mathrm{GL}_n)(F_v) \simeq \mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_v)$ and we define a base-change map $\mathrm{BC} : \mathrm{Irr}(U(V)_v) \to \mathrm{Irr}((\mathrm{R}_{E/F}\,\mathrm{GL}_n)_v)$ by $\pi \mapsto \pi \boxtimes \pi^\vee$. By [38, Theorem 2.5.2] and [28, Theorem 1.7.1/Corollary 3.3.2], we may associate to any cuspidal automorphic representation π of $U(V)$ an isobaric conjugate-dual automorphic representation $\mathrm{BC}(\pi)$ of $\mathrm{GL}_n(E)$, the base change of π, satisfying the following properties:

(2.3.2) The Asai L-function

$$L(s, \mathrm{BC}(\pi), As^{(-1)^{n+1}})$$

has a pole at $s = 1$, and, moreover, if $\mathrm{BC}(\pi)$ is cuspidal, this pole is simple (see [16, § 7] for the definition of the Asai L-functions);

(2.3.3) Let v be a place of F. Then, if $\mathrm{BC}(\pi)$ is generic or v splits in E, we have $\mathrm{BC}(\pi_v) = \mathrm{BC}(\pi)_v$;

(2.3.4) If $\mathrm{BC}(\pi)$ is generic, then the multiplicity of π in $L^2([U(V)])$ is one (see [38, Theorem 2.5.2/Remark 2.5.3] as well as [28, Theorem 5.0.5, Theorem 1.7.1] and the discussion thereafter).

Let v be a place of F and $\pi \in \mathrm{Irr}(U(V)(F_v))$. Assume first that v is inert in E. By the Langlands classification, there exist

- a parabolic subgroup $P = MN$ of $U(V)_v$ with

$$M \simeq \mathrm{R}_{E_v/F_v}\,\mathrm{GL}_{n_1} \times \cdots \times \mathrm{R}_{E_v/F_v}\,\mathrm{GL}_{n_r} \times U(V'),$$

  where $V' \subset V_v$ is a nondegenerate subspace;

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- tempered representations $\pi_i \in \mathrm{Temp}(\mathrm{GL}_{n_i}(E_v))$, $1 \leqslant i \leqslant r$ and $\pi' \in \mathrm{Temp}(U(V'))$;
- real numbers $\lambda_1 > \cdots > \lambda_r > 0$

such that π is the unique irreducible quotient of

$$i_P^{U(V)_v}\left(|\det|_{E_v}^{\lambda_1}\pi_1 \boxtimes \cdots \boxtimes |\det|_{E_v}^{\lambda_r}\pi_r \boxtimes \pi'\right).$$

The r-uple $(\lambda_1, \ldots, \lambda_r)$ only depends on π and we will set $\mathfrak{c}(\pi) = \lambda_1$ if $r \geqslant 1$, $\mathfrak{c}(\pi) = 0$ if $r = 0$ (i.e. if π is tempered). Assume now that v splits in E. Then, we have an isomorphism $U(V)_v \simeq \mathrm{GL}_{n,F_v}$ and there exists a r-uple $(n_1, \ldots, n_r)$ of positive integers such that $n_1 + \cdots + n_r = n$, tempered representations $\pi_i \in \mathrm{Temp}(\mathrm{GL}_{n_1}(F_v))$ $i = 1, \ldots, r$ and real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ such that π is the unique irreducible quotient of

$$i_P^{\mathrm{GL}_n}\left(|\det|_{F_v}^{\lambda_1}\pi_1 \boxtimes \cdots \boxtimes |\det|_{F_v}^{\lambda_r}\pi_r\right),$$

where P denotes the standard parabolic subgroup of GL_n with Levi $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. In this case, we set $\mathfrak{c}(\pi) = \max(|\lambda_1|, |\lambda_r|)$. This depends only on π and, in particular, not on the choice of the isomorphism $U(V)_v \simeq \mathrm{GL}_{n,F_v}$ (which is only defined up to an automorphism of GL_{n,F_v} since it involves the choice of a place of E above v).

In any case, for $c > 0$, we define $\mathrm{Irr}_{\leqslant c}(U(V)_v)$ to be the set of irreducible representations $\pi \in \mathrm{Irr}(U(V)_v)$ such that $\mathfrak{c}(\pi) \leqslant c$. Combining the above global results of Mok and Kaletha–Minguez–Shin–White with the bounds toward the Ramanujan conjecture for GL_n of Luo–Rudnick–Sarnak [32] suitably extended to ramified places independently by Müller–Speh and Bergeron–Clozel [3, 42], we get the following.

Lemma 2.3.2. *Set $c = \frac{1}{2} - \frac{1}{n^2+1}$. Let π be a cuspidal automorphic representation of $U(V)(\mathbb{A})$ such that $\mathrm{BC}(\pi)$ is generic. Then, for all place v of F, we have*

$$\pi_v \in \mathrm{Irr}_{\leqslant c}(U(V)_v).$$

2.4. The local Gan–Gross–Prasad conjecture for unitary groups

Let E/F be a quadratic extension of local fields of characteristic zero (either Archimedean or p-adic). Let W be an n-dimensional Hermitian space over E and define the Hermitian space V by $V = W \oplus^{\perp} e$, where $(e, e) = 1$. Set $H = U(W)$ and $G = U(W) \times U(V)$. We view H as a subgroup of G via the diagonal embedding. We will say that an irreducible representation π of $G(F)$ is *H-distinguished* if the space $Hom_H(\pi, \mathbb{C})$ of $H(F)$-invariant (continuous in the Archimedean case) linear forms on π is nonzero. By multiplicity one results [1, 48], we always have $\dim Hom_H(\pi, \mathbb{C}) \leqslant 1$. We will denote by $\mathrm{Irr}_H(G)$ and $\mathrm{Temp}_H(G)$ the subsets of H-distinguished representations in $\mathrm{Irr}(G)$ and $\mathrm{Temp}(G)$, respectively. Let φ be a generic Langlands parameter for G. We have the following conjecture of Gan, Gross and Prasad [16, Conjecture 17.1].

Conjecture 2.4.1. *The L-packet $\Pi^G(\varphi)$ contains at most one H-distinguished representation.*

By [6, Theorem 12.4.1] and [17, Proposition 9.3], the following cases of this conjecture are known.

Theorem 2.4.2 (Beuzart-Plessis, Gan–Ichino). (i) *Let φ be a tempered Langlands parameter for G. Then Conjecture 2.4.1 holds for φ.*

(ii) *Assume that F is p-adic. Then Conjecture 2.4.1 holds for any generic Langlands parameter φ of G.*

2.5. Measures

We will use the same normalization of measures as in [56, § 2]. Let us recall these choices. We actually define two sets of Haar measures: the *normalized* and the *unnormalized* ones. We will use the normalized Haar measures apart in § 4 where we will use the unnormalized one. From now on and until § 4, where we will switch to a local setting, we fix a quadratic extension E/F of number fields. We will denote by $\eta_{E/F}$ the idèle class character corresponding to this extension. We will also fix a nonzero character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ and a nonzero element $\tau \in E$ such that $\mathrm{Tr}_{E/F}(\tau) = 0$. We will denote by ψ_E the character of $\mathbb{A}_E$ given by $\psi_E(z) = \psi(\frac{1}{2}\mathrm{Tr}_{E/F}(z))$.

Let v be a place of F. We endow F_v with the self-dual Haar measure for ψ_v. Similarly, we endow E_v with the self-dual Haar measure for $\psi_{E,v}$. On $F_v^\times$, we define a normalized measure

$$d^\times x_v = \zeta_{F_v}(1)\frac{dx_v}{|x_v|_{F_v}}$$

and an unnormalized one

$$d^* x_v = \frac{dx_v}{|x_v|_{F_v}}.$$

More generally, for all $n \geqslant 1$, we equip $\mathrm{GL}_n(F_v)$ with the following normalized Haar measure

$$dg_v = \zeta_{F_v}(1)\frac{\prod_{ij} dg_{v,ij}}{|\det g_v|_{F_v}^n}$$

as well as with the following unnormalized one

$$d^* g_v = \frac{\prod_{ij} dg_{v,ij}}{|\det g_v|_{F_v}^n}$$

and similarly for $\mathrm{GL}_n(E_v)$. Recall that N_n denotes the standard maximal unipotent subgroup of GL_n. We will give $N_n(F_v)$ and $N_n(E_v)$ the Haar measures

$$du_v = \prod_{1 \leqslant i < j \leqslant n} du_{v,ij}.$$

We equip $\mathbb{A}^\times$, $N_n(\mathbb{A})$, $N_n(\mathbb{A}_E)$, $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_n(\mathbb{A}_E)$ with the global Tamagawa Haar measures given by

$$d^\times x = \prod_v d^\times x_v, \quad du = \prod_v du_v, \quad dg = \prod_v dg_v.$$

Recall that $\mathrm{S}_n = \{s \in \mathrm{R}_{E/F}\,\mathrm{GL}_n;\, s\overline{s} = 1\}$ and its *Lie algebra* $\mathfrak{s}_n = \{X \in \mathrm{R}_{E/F}\, M_n;\, X + \overline{X} = 0\}$. Let V be an n-dimensional Hermitian space over E and denote by $\mathfrak{u}(V)$ the Lie algebra of $U(V)$. Choosing a basis of V, we get an embedding $\mathfrak{u}(V) \hookrightarrow \mathrm{R}_{E/F} M_n$. Let us denote by $\langle ., \rangle$ the $\mathrm{GL}_n(E_v)$-invariant bilinear pairing on $M_n(E_v)$ given by

$$\langle X, Y\rangle := \mathrm{Trace}(XY).$$

Note that the restrictions of $\langle ., .\rangle$ to $\mathfrak{s}_n(F_v)$ and $\mathfrak{u}(V)(F_v)$ are F_v-valued and nondegenerate. We define a Haar measure dX on $\mathfrak{u}(V)(F_v)$ such that the Fourier transform

$$\widehat{\varphi}(Y) = \int_{\mathfrak{u}(V)(F_v)} \varphi(X)\psi_v(\langle X, Y\rangle)\, dX$$

and its dual

$$\widecheck{\varphi}(X) = \int_{\mathfrak{u}(V)(F_v)} \varphi(Y)\psi_v(-\langle Y, X\rangle)\, dY$$

are inverse of each other. We define similarly a Haar measure and Fourier transforms $\varphi \mapsto \widehat{\varphi}$, $\varphi \mapsto \widecheck{\varphi}$ on $\mathfrak{s}_n(F_v)$.

The Cayley map $\mathfrak{c} : X \mapsto \mathfrak{c}(X) = (1+X)(1-X)^{-1}$ induces birational isomorphisms from $\mathfrak{s}_n$ to S_n and from $\mathfrak{u}(V)$ to $U(V)$. We define the *unnormalized* Haar measure $d^* g_v$ on $U(V)(F_v)$ to be the unique Haar measure such that the Jacobian of $\mathfrak{c}$ at the origin is 1. The normalized Haar measure on $U(V)(F_v)$ is defined by

$$dg_v = L(1, \eta_{E_v/F_v}) d^* g_v.$$

Similarly, we endow $\mathrm{S}_n(F_v)$ with an unnormalized measure $d^* s_v$ which is the unique $\mathrm{GL}_n(E_v)$-invariant measure for which the Jacobian of the Cayley map $\mathfrak{c}$ at the origin is 1. The corresponding normalized measure is given by

$$ds_v = L(1, \eta_{E_v/F_v}) d^* s_v.$$

Note that $d^* s_v$ (resp. ds_v) can also be identified with the quotient of the unnormalized (resp. normalized) Haar measures on $\mathrm{GL}_n(E_v)$ and $\mathrm{GL}_n(F_v)$ via the isomorphism $\nu : \mathrm{GL}_n(E_v)/\mathrm{GL}_n(F_v) \simeq \mathrm{S}_n(F_v)$, $\nu(g) = g\overline{g}^{-1}$. Finally, we equip $U(V)(\mathbb{A})$ with the global Haar measure given by

$$dg = \prod_v dg_v.$$

Note that this is not the Tamagawa measure since the global normalizing factor $L(1, \eta_{E/F})^{-1}$ is missing. The local normalized Haar measure dg_v can be identified with the quotient of the normalized Haar measures on $E_v^\times$ and $F_v^\times$ via the isomorphism $E_v^\times / F_v^\times \simeq U(1)(F_v)$, $x \mapsto x/\overline{x}$. Hence, as the Tamagawa number of $U(1)$ is 2, we have

$$\mathrm{vol}\left(E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times\right) = \mathrm{vol}([U(1)]) = 2L(1, \eta_{E/F}). \tag{2.5.1}$$

3. Spherical characters, the Ichino–Ikeda conjecture and Zhang's conjecture

In this section, E/F will be a quadratic extension of number fields and we will use normalized Haar measures (see § 2.5). Let W be a Hermitian space of dimension n over E. We will set $V = W \oplus^\perp Ee$ where $(e, e) = 1$, $G = U(W) \times U(V)$ and $H = U(W)$. We view H as a subgroup of G via the diagonal embedding. We will fix a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$. We will say that an irreducible representation $\pi = \otimes'_v \pi_v$ of $G(\mathbb{A})$ is *abstractly H-distinguished* if, for all place v of F, the representation π_v is H_v-distinguished, i.e. if $Hom_{H_v}(\pi_v, \mathbb{C}) \neq 0$. Set $G' = \mathrm{R}_{E/F}(\mathrm{GL}_n \times \mathrm{GL}_{n+1})$. We define two subgroups $H'_1 = \mathrm{R}_{E/F}\,\mathrm{GL}_n$ and $H'_2 = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$ of G' (H'_1 is embedded diagonally). We also define a character η of $H'_2(\mathbb{A})$ by

$$\eta(g_1, g_2) = \eta_{E/F}(g_1)^{n+1}\eta_{E/F}(g_2)^n$$

for all $(g_1, g_2) \in H'_2(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_{n+1}(\mathbb{A})$. We will also fix a maximal compact subgroup $K' = \prod_v K'_v$ of $G'(\mathbb{A})$ such that $K'_v = \mathrm{GL}_n(\mathcal{O}_{E,v}) \times \mathrm{GL}_{n+1}(\mathcal{O}_{E,v})$ for all non-Archimedean place v of F. Finally, if π and Π are cuspidal automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$, respectively, then we endow them with the following Petersson inner products:

$$(\phi_1, \phi_2)_{Pet} := \int_{[G]} \phi_1(g)\overline{\phi_2(g)}\, dg, \quad \phi_1, \phi_2 \in \pi,$$

$$(\phi'_1, \phi'_2)_{Pet} := \int_{[Z_{G'}\backslash G']} \phi'_1(g')\overline{\phi'_2(g')}\, dg', \quad \phi'_1, \phi'_2 \in \Pi.$$

3.1. Global relative characters

For any cuspidal automorphic representation π of $G(\mathbb{A})$, we define the H-period $\mathcal{P}_H : \pi \to \mathbb{C}$ by

$$\mathcal{P}_H(\phi) = \int_{[H]} \phi(h)\, dh, \quad \phi \in \pi.$$

The integral is absolutely convergent (see Proposition A.1.1(ix)). We will say that the cuspidal automorphic representation π is *globally H-distinguished* if the period $\mathcal{P}_H$ is not identically zero on π. We may associate to this period a (global) relative character $J_\pi : \mathcal{S}(G(\mathbb{A})) \to \mathbb{C}$ defined as follows. Let $f \in \mathcal{S}(G(\mathbb{A}))$ and choose a compact-open subgroup $K_f \subset G(\mathbb{A}_f)$ by which f is biinvariant. Let $\mathcal{B}_\pi^{K_f}$ be an orthonormal basis for the Petersson inner product of π^{K_f} whose elements are C_G and C_K eigenvectors. Then we set

$$J_\pi(f) = \sum_{\phi \in \mathcal{B}_\pi^{K_f}} \mathcal{P}_H(\pi(f)\phi)\overline{\mathcal{P}_H(\phi)}.$$

The sum is absolutely convergent and does not depend on the choice of the basis $\mathcal{B}_\pi^{K_f}$ (see Proposition A.1.2).

Let Π be a cuspidal automorphic representation of $G'(\mathbb{A})$ whose central character is trivial on $Z_{H'_2}(\mathbb{A}) = \mathbb{A}^\times \times \mathbb{A}^\times$. We define two periods $\lambda : \Pi \to \mathbb{C}$ and $\beta : \Pi \to \mathbb{C}$ by

$$\lambda(\phi) = \int_{[H'_1]} \phi(h_1)\, dh_1,$$

$$\beta(\phi) = \int_{[Z_{H'_2}\backslash H'_2]} \phi(h_2)\eta(h_2)\, dh_2$$

for all $\phi \in \Pi$. The two above integrals are absolutely convergent (see Proposition A.1.1(ix)). We also define a (global) relative character $I_\Pi : \mathcal{S}(G'(\mathbb{A})) \to \mathbb{C}$ as follows.

Let $f' \in \mathcal{S}(G'(\mathbb{A}))$ and choose a compact-open subgroup $K_{f'} \subset G(\mathbb{A}_f)$ by which f' is biinvariant. Let $\mathcal{B}_\Pi^{K_{f'}}$ be an orthonormal basis for the Petersson inner product of $\Pi^{K_{f'}}$ whose elements are $C_{G'}$ and $C_{K'}$ eigenvectors. Then we set

$$I_\Pi(f') = \sum_{\phi \in \mathcal{B}_\Pi^{K_{f'}}} \lambda(\Pi(f')\phi)\overline{\beta(\phi)}.$$

The sum is absolutely convergent and does not depend on the choice of the basis $\mathcal{B}_\Pi^{K_{f'}}$ (see Proposition A.1.2).

3.2. Local relative characters

Let v be a place of F. Let π_v be a tempered representation of $G(F_v)$. We define a distribution $J_{\pi_v} : \mathcal{S}(G(F_v)) \to \mathbb{C}$ (the local relative character associated with π_v) by

$$J_{\pi_v}(f_v) = \int_{H(F_v)} Trace(\pi_v(h)\pi_v(f_v))\, dh_v, \quad f_v \in \mathcal{C}(G(F_v)).$$

By [6, § 8.2], the above integral is absolutely convergent. Choosing models for G and H over $\mathcal{O}_F$, for almost all v if $f_v = \mathbf{1}_{K_v}$, we have

$$J_{\pi_v}(f_v) = \mathcal{L}(\frac{1}{2}, \pi_v)\mathrm{vol}(H(\mathcal{O}_v))\mathrm{vol}(G(\mathcal{O}_v))$$

(see § 1 for the definition of $\mathcal{L}(s, \pi_v)$). Hence, we define a *normalized* relative character $J^\natural_{\pi_v}$ by

$$J^\natural_{\pi_v} = \frac{1}{\mathcal{L}(\frac{1}{2}, \pi_v)} J_{\pi_v}.$$

By [6, Theorem 8.2.1], we have

(3.2.1) π_v is H_v-distinguished if and only if $J_{\pi_v} \neq 0$.

Moreover, by [6, Corollary 8.6.1], for all parabolic subgroup $P = MU$ of G_v and for all square-integrable representation σ of M, there is at most one irreducible subrepresentation $\pi \subset i_P^G(\sigma)$ such that $J_\pi \neq 0$. Thus, we are in the situation of § 2.2 and the family of distributions $\pi_v \in \mathrm{Temp}(G_v) \mapsto J_{\pi_v}$ is analytic (see [8, Proposition 14.2.1] for the analyticity).

Let Π_v be a generic unitary representation of $G'(F_v)$. We may write $\Pi_v = \Pi_{n,v} \boxtimes \Pi_{n+1,v}$ where $\Pi_{n,v}$ and $\Pi_{n+1,v}$ are generic and unitary representations of $\mathrm{GL}_n(E_v)$ and $\mathrm{GL}_{n+1}(E_v)$, respectively. Let $\mathcal{W}(\Pi_{n,v}, \overline{\psi}_E)$ and $\mathcal{W}(\Pi_{n+1,v}, \psi_E)$ be the Whittaker models of $\Pi_{n,v}$ and $\Pi_{n+1,v}$ corresponding to the characters $\overline{\psi}_E$ and ψ_E, respectively. Set $\mathcal{W}(\Pi_v) = \mathcal{W}(\Pi_{n,v}, \overline{\psi}_E) \otimes \mathcal{W}(\Pi_{n+1,v}, \psi_E)$. We define linear forms (the local Flicker–Rallis periods)

$$\beta_{n,v} : \mathcal{W}(\Pi_{n,v}, \overline{\psi}_E) \to \mathbb{C}, \quad \beta_{n+1,v} : \mathcal{W}(\Pi_{n+1,v}, \psi_E) \to \mathbb{C}$$

and scalar products

$$\theta_{n,v} : \mathcal{W}(\Pi_{n,v}, \overline{\psi}_E) \times \mathcal{W}(\Pi_{n,v}, \overline{\psi}_E) \to \mathbb{C}, \quad \theta_{n+1,v} : \mathcal{W}(\Pi_{n+1,v}, \psi_E) \times \mathcal{W}(\Pi_{n+1,v}, \psi_E) \to \mathbb{C}$$

by

$$\beta_{k,v}(W_k) = \int_{N_{k-1}(F_v)\backslash \mathrm{GL}_{k-1}(F_v)} W_k(\epsilon_k(\tau)g_{k-1})\eta_{E_v/F_v}(\det g_{k-1})^{k-1}\, dg_{k-1},$$

$$\theta_{k,v}(W_k, W_k') = \int_{N_{k-1}(E_v)\backslash \mathrm{GL}_{k-1}(E_v)} W_k(g_{k-1})\overline{W_k'(g_{k-1})}\, dg_{k-1}$$

for all $k = n, n+1$, all $W_n, W_n' \in \mathcal{W}(\Pi_{n,v}, \overline{\psi}_E)$ and all $W_{n+1}, W_{n+1}' \in \mathcal{W}(\Pi_{n+1,v}, \psi_E)$, where $\epsilon_k(\tau) = \mathrm{diag}(\tau^{k-1}, \tau^{k-2}, \ldots, \tau, 1)$ (recall that τ is a fixed nonzero element of E such that $\mathrm{Tr}_{E/F}(\tau) = 0$). The above integrals are absolutely convergent (see [27, Propositions 1.3 and 3.16] for the absolute convergence of $\theta_{k,v}$, the proof for $\beta_{k,v}$ is identical). Set $\beta_v = \beta_{n,v} \otimes \beta_{n+1,v}$ and $\theta_v = \theta_{n,v} \otimes \theta_{n+1,v}$. If E_v/F_v, Π_v, $\psi_{E,v}$ are unramified, τ is a unit in E_v and $W_v \in \mathcal{W}(\Pi_v)$ is the unique K_v'-invariant vector such that $W_v(1) = 1$, we have (see [27, Proposition 2.3] and [56, § 3.2])

$$\beta_v(W_v) = \mathrm{vol}(K_v') L(1, \Pi_{n,v}, As^{(-1)^{n-1}}) L(1, \Pi_{n+1,v}, As^{(-1)^n})$$

and

$$\theta_v(W_v) = \mathrm{vol}(K_v') L(1, \Pi_{n,v} \times \Pi_{n,v}^\vee) L(1, \Pi_{n+1,v} \times \Pi_{n+1,v}^\vee).$$

Hence, we define normalized versions $\beta_v^\natural$ and $\theta_v^\natural$ of β_v and θ_v by

$$\beta_v^\natural = \frac{\beta_v}{L(1, \Pi_{n,v}, As^{(-1)^{n-1}}) L(1, \Pi_{n+1,v}, As^{(-1)^n})},$$

$$\theta_v^\natural = \frac{\theta_v}{L(1, \Pi_{n,v} \times \Pi_{n,v}^\vee) L(1, \Pi_{n+1,v} \times \Pi_{n+1,v}^\vee)}.$$

For all $s \in \mathbb{C}$, we also have the local Rankin–Selberg period $\lambda_v(s, .) : \mathcal{W}(\Pi_v) \to \mathbb{C}$ defined by

$$\lambda_v(s, W_n \otimes W_{n+1}) = \int_{N_n(E_v)\backslash \mathrm{GL}_n(E_v)} W_n(g_n) W_{n+1}(g_n) |\det g_n|_{E_v}^s\, dg_n$$

for all $(W_n, W_{n+1}) \in \mathcal{W}(\Pi_{n,v}, \overline{\psi}_E) \times \mathcal{W}(\Pi_{n+1,v}, \psi_E)$, and its normalization $\lambda_v^\natural(s, .)$ is given by

$$\lambda_v^\natural(s, .) = \frac{\lambda_v(s, .)}{L(s + \frac{1}{2}, \Pi_{n,v} \times \Pi_{n+1,v})}.$$

The integral defining $\lambda_v(s, .)$ is absolutely convergent for $Re(s) \gg 0$ and $\lambda_v^\natural(s, .)$ extends to an entire function on $\mathbb{C}$ (see [25] and [24] for the Archimedean case). We will set $\lambda_v^\natural = \lambda_v^\natural(0, .)$. Obviously, $\lambda_v^\natural$ defines a $H_1'(F_v)$-invariant linear form on Π_v. Moreover, by [25] and [24], there exists $W \in \mathcal{W}(\Pi_v)$ such that $\lambda_v^\natural(W) = 1$. Hence, $\lambda_v^\natural$ defines a nonzero element in $Hom_{H_1'}(\Pi_v, \mathbb{C})$. If Π_v is tempered, then $\lambda_v(s, .)$ is absolutely convergent for $Re(s) > -1/2$ and we will set $\lambda_v = \lambda_v(0, .)$.

We are now ready to define the (normalized) local relative character $I_{\Pi_v}^\natural : \mathcal{S}(G'(F_v)) \to \mathbb{C}$ attached to Π_v. Let $f_v' \in \mathcal{S}(G'(F_v))$. If v is non-Archimedean, then choose a compact-open subgroup $K_{f_v'}$ of $G'(F_v)$ by which f_v' is biinvariant and let $\mathcal{B}_{\Pi_v}$ be an orthonormal basis of $\Pi_v^{K_{f_v'}}$ for the scalar product $\theta_v^\natural$. If v is Archimedean, we let $\mathcal{B}_{\Pi_v}$

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be any orthonormal basis of Π_v for the scalar product $\theta_v^\natural$ consisting of $C_{K'_v}$-eigenvectors. Then we set

$$I^\natural_{\Pi_v}(f'_v) = \sum_{W \in \mathcal{B}_{\Pi_v}} \lambda_v^\natural(\Pi_v(f'_v)W)\overline{\beta_v^\natural(W)}.$$

The sum is absolutely convergent and does not depend on the choice of $\mathcal{B}_{\Pi_v}$. If, moreover, Π_v is tempered, then we define an *unnormalized* local relative character $I_{\Pi_v} : \mathcal{S}(G'(F_v)) \to \mathbb{C}$ by using θ_v, β_v and λ_v instead of $\theta_v^\natural$, $\beta_v^\natural$ and $\lambda_v^\natural$. Finally, the proofs of [27, Proposition 1.3] and [25, Theorem 2.7] easily show that the family of distributions $\Pi_v \in \mathrm{Temp}(G'_v) \mapsto I_{\Pi_v}$ is analytic in the sense of § 2.2.

3.3. Orbital integrals

Consider the action of $H \times H$ on G by left and right translations. Then, an element $\delta \in G$ is said to be *regular semisimple* for this action if its orbit is closed and its stabilizer is trivial. Denote by G_{rs} the open subset of regular semisimple elements in G. Let v be a place of F and $\delta \in G_{\mathrm{rs}}(F_v)$ be regular semisimple. We define the (relative) orbital integral associated with δ as the distribution given by

$$O(\delta, f_v) = \int_{H(F_v)\times H(F_v)} f_v(h\delta h')\, dh\, dh', \quad f_v \in \mathcal{S}(G(F_v)).$$

Note that the integrand is compactly supported in the p-adic case. In the Archimedean case, the integral converges by [53, § 2]. There is another way to see these orbital integrals. For all $f_v \in \mathcal{S}(G(F_v))$, we define a function $\widetilde{f_v} \in \mathcal{S}(U(V)(F_v))$ by

$$\widetilde{f_v}(x) = \int_{H(F_v)} f_v(h(1, x))\, dh, \quad x \in U(V)(F_v).$$

This induces a surjective linear map $\mathcal{S}(G(F_v)) \to \mathcal{S}(U(V)(F_v))$ (see [53, Lemma 2.3] in the Archimedean case). Let us say that an element $x \in U(V)$ is regular semisimple if it is so for the action of $U(W)$ by conjugation, i.e. if the $U(W)$-conjugacy class of x is closed and the stabilizer of x in $U(W)$ is trivial. Denote by $U(V)_{\mathrm{rs}}$ the open subset of regular semisimple elements in $U(V)$. For all $x \in U(V)_{\mathrm{rs}}(F_v)$, we define the orbital integral associated with x as the distribution

$$O(x, \varphi) = \int_{U(W)(F_v)} \varphi(hxh^{-1})\, dh, \quad \varphi_v \in \mathcal{S}(U(V)(F_v)).$$

For all $\delta = (\delta_W, \delta_V) \in G_{\mathrm{rs}}$, the element $x = \delta_W^{-1}\delta_V$ is regular semisimple in $U(V)$ and this defines a surjection $G_{\mathrm{rs}} \twoheadrightarrow U(V)_{\mathrm{rs}}$. Moreover, for all $\delta \in G_{\mathrm{rs}}(F_v)$ and all $f \in \mathcal{S}(G(F_v))$, we have the equality

$$O(\delta, f) = O(x, \widetilde{f}),$$

where $x = \delta_W^{-1}\delta_V$.

We can also define orbital integrals on the space $\mathcal{S}(\mathfrak{u}(V)(F_v))$. Call an element $X \in \mathfrak{u}(V)$ regular semisimple if it is so for the adjoint action of $U(W)$. Let us denote by $\mathfrak{u}(V)_{\mathrm{rs}}$ the open subset of regular semisimple elements. Then, for all $X \in \mathfrak{u}(V)_{\mathrm{rs}}(F_v)$, we can define an orbital integral by

$$O(X, \varphi) = \int_{U(W)(F_v)} \varphi(h^{-1}Xh)\, dh, \quad \varphi \in \mathcal{S}(U(V)(F_v))$$

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(again see [53, § 3] for the convergence in the Archimedean case). The Cayley map $\mathfrak{c} : X \mapsto (1+X)(1-X)^{-1}$ realizes a $U(W)$-equivariant isomorphism between the open subsets $\mathfrak{u}(V)^\circ = \{X \in \mathfrak{u}(V);\ \det(1-X) \neq 0\}$ and $U(V)^0 = \{x \in U(V);\ \det(1+x) \neq 0\}$. Assume that v is non-Archimedean and let $\omega \subset \mathfrak{u}(V)^\circ(F_v)$ and $\Omega \subset U(V)^\circ(F_v)$ be open and closed $U(W)(F_v)$-invariant neighborhoods of 0 and 1, respectively, such that the Cayley map restricts to an analytic isomorphism between ω and Ω preserving measures. For all $\varphi \in \mathcal{S}(U(V)(F_v))$, we define a function $\varphi_\natural$ by

$$\varphi_\natural(X) = \begin{cases} \varphi(\mathfrak{c}(X)) & \text{if } X \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $X \in \omega_{\mathrm{rs}} = \omega \cap \mathfrak{u}(V)_{\mathrm{rs}}(F_v)$ and all $\varphi \in \mathcal{S}(U(V)(F_v))$, we have

$$O(\mathfrak{c}(X), \varphi) = O(X, \varphi_\natural).$$

Consider now the action of $H_1' \times H_2'$ on G' by left and right translations. As before, an element $\gamma \in G'$ is said to be *regular semisimple* for this action if its orbit is closed and its stabilizer is trivial. Denote by G'_{rs} the open subset of regular semisimple elements in G'. Let v be a place of F and $\gamma \in G'_{\mathrm{rs}}(F_v)$ be regular semisimple. We define the (relative) orbital integral associated with γ as the distribution given by

$$O(\gamma, f_v') = \int_{H_1'(F_v)\times H_2'(F_v)} f_v'(h_1^{-1}\gamma h_2)\eta(h_2)\, dh_1\, dh_2, \quad f_v' \in \mathcal{S}(G'(F_v)).$$

There is another way to see these orbital integrals. Recall that

$$S_{n+1}(F_v) = \{s \in \mathrm{GL}_{n+1}(E_v);\, s\overline{s} = 1\}$$

and that we have a surjective map $\nu : \mathrm{GL}_{n+1}(E_v) \to S_{n+1}(F_v),\ \nu(g) = g\overline{g}^{-1}$. For all $f_v' \in \mathcal{S}(G'(F_v))$, we define a function $\widetilde{f}_v' \in \mathcal{S}(S_{n+1}(F_v))$ by

$$\widetilde{f}_v'(s) = \int_{H_1'(F_v)} \int_{\mathrm{GL}_{n+1}(F_v)} f_v'(h_1(1, gh_2))\, dh_2\, dh_1, \quad g \in \mathrm{GL}_{n+1}(E_v),\ s = \nu(g)$$

if n is even and

$$\widetilde{f}_v'(s) = \int_{H_1'(F_v)} \int_{\mathrm{GL}_{n+1}(F_v)} f_v'(h_1(1, gh_2))\eta_v'(gh_2)\, dh_2\, dh_1, \quad g \in \mathrm{GL}_{n+1}(E_v),\ s = \nu(g)$$

if n is odd. In any case, this defines a surjective linear map $\mathcal{S}(G'(F_v)) \to \mathcal{S}(S_{n+1}(F_v))$ (see [53, Lemma 2.1] in the Archimedean case). The group GL_n acts on S_{n+1} by conjugation and we shall say that an element $s \in S_{n+1}$ is regular semisimple if it is so for this action, i.e. if the GL_n-conjugacy class of s is closed and the stabilizer of s in GL_n is trivial. We will denote by $S_{n+1,\mathrm{rs}}$ the open subset of regular semisimple elements in S_{n+1}. For all $s \in S_{n+1,\mathrm{rs}}(F_v)$, we define the orbital integral associated with s as the distribution

$$O(s, \varphi') = \int_{\mathrm{GL}_n(F_v)} \varphi'(h^{-1}sh)\eta_{E_v/F_v}(h)\, dh, \quad \varphi' \in \mathcal{S}(S_{n+1}(F_v)).$$

For $\gamma = (\gamma_1, \gamma_2) \in G'_{\mathrm{rs}}$, the element $s = \nu(\gamma_1^{-1}\gamma_2) \in S_{n+1}$ is regular semisimple and this defines a surjection $G'_{\mathrm{rs}} \twoheadrightarrow S_{n+1,\mathrm{rs}}$. Moreover, for all $\gamma \in G'_{\mathrm{rs}}(F_v)$ and all $f' \in \mathcal{S}(G'(F_v))$, we have the equality

$$O(\gamma, f') = \begin{cases} O(s, \widetilde{f}') & \text{if } n \text{ is even,} \\ \eta'_v(\gamma_1^{-1}\gamma_2)O(s, \widetilde{f}') & \text{if } n \text{ is odd.} \end{cases}$$

where $s = \nu(\gamma_1^{-1}\gamma_2)$.

We can also define orbital integrals on the space $\mathcal{S}(\mathfrak{s}_{n+1}(F_v))$. Call an element $X \in \mathfrak{s}_{n+1}$ regular semisimple if it is so for the adjoint action of GL_n. Let us denote by $\mathfrak{s}_{n+1,\mathrm{rs}}$ the open subset of regular semisimple elements. Then, for all $X \in \mathfrak{s}_{n+1,\mathrm{rs}}(F_v)$, we can define an orbital integral by

$$O(X, \varphi') = \int_{\mathrm{GL}_n(F_v)} \varphi'(h^{-1}Xh)\eta_{E_v/F_v}(h)\, dh, \quad \varphi' \in \mathcal{S}(\mathfrak{s}_{n+1}(F_v)).$$

The Cayley map $\mathfrak{c} = \mathfrak{c}_{n+1} : X \mapsto (1+X)(1-X)^{-1}$ realizes a GL_n-equivariant isomorphism between the open subsets $\mathfrak{s}^\circ_{n+1} = \{X \in \mathfrak{s}_{n+1};\ \det(1-X) \neq 0\}$ and $S^\circ_{n+1} = \{s \in S_{n+1};\ \det(1+s) \neq 0\}$. Let $\omega' \subset \mathfrak{s}^\circ_{n+1}(F_v)$ and $\Omega' \subset S^\circ_{n+1}(F_v)$ be open and closed $\mathrm{GL}_n(F_v)$-invariant neighborhoods of 0 and 1, respectively, such that the Cayley map restricts to an analytic isomorphism between ω' and Ω' preserving measures. For all $\varphi' \in \mathcal{S}(S_{n+1}(F_v))$, we define a function $\varphi'_\natural \in \mathcal{S}(\mathfrak{s}_{n+1}(F_v))$ by

$$\varphi'_\natural(X) = \begin{cases} \varphi'(\mathfrak{c}(X)) & \text{if } X \in \omega', \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $X \in \omega'_{\mathrm{rs}} = \omega' \cap \mathfrak{s}_{n+1,\mathrm{rs}}(F_v)$ and all $\varphi' \in \mathcal{S}(S_{n+1}(F_v))$, we have

$$O(\mathfrak{c}(X), \varphi') = O(X, \varphi'_\natural).$$

3.4. Correspondence of orbits and transfer

We now recall the correspondence between orbits following [55, § 2.4]. We will denote by $H'_1\backslash G'/H'_2$, $H\backslash G/H$, S_{n+1}/GL_n and $U(V)/U(W)$ the geometric quotients of G', G, S_{n+1} and $U(V)$ by $H'_1 \times H'_2$, $H \times H$, GL_n and $U(W)$, respectively, (the last two actions being given by conjugation). We will also write $(H'_1\backslash G'/H'_2)_{\mathrm{rs}}$, $(H\backslash G/H)_{\mathrm{rs}}$, $(S_{n+1}/\mathrm{GL}_n)_{\mathrm{rs}}$ and $(U(V)/U(W))_{\mathrm{rs}}$ for the regular semisimple loci in these geometric quotients. These are the image of G'_{rs}, G_{rs}, $S_{n+1,\mathrm{rs}}$ and $U(V)_{\mathrm{rs}}$ by the natural projections. The maps $(\gamma_1, \gamma_2) \in G' \mapsto \nu(\gamma_1^{-1}\gamma_2)$ and $(\delta_W, \delta_V) \in G \mapsto \delta_W^{-1}\delta_V$ induce isomorphisms

$$H'_1\backslash G'/H'_2 \simeq S_{n+1}/\mathrm{GL}_n \quad \text{and} \quad H\backslash G/H \simeq U(V)/U(W)$$

and similarly for the regular semisimple loci. Moreover, there is a natural isomorphism [55, § 3.1]

$$H'_1\backslash G'/H'_2 \simeq H\backslash G/H \tag{3.4.1}$$

which preserves the regular semisimple loci. For all field extension k of F, we have $(H'_1\backslash G'/H'_2)_{\mathrm{rs}}(k) = H'_1(k)\backslash G'_{\mathrm{rs}}(k)/H'_2(k)$ and $H(k)\backslash G_{\mathrm{rs}}(k)/H(k)$ is a subset of $(H\backslash G/H)_{\mathrm{rs}}(k)$. The above isomorphism thus induces injections

$$H(k)\backslash G_{\mathrm{rs}}(k)/H(k) \hookrightarrow H'_1(k)\backslash G'_{\mathrm{rs}}(k)/H'_2(k) \tag{3.4.2}$$

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and

$$U(V)_{\mathrm{rs}}(k)/U(W)(k) \hookrightarrow S_{n+1,\mathrm{rs}}(k)/\mathrm{GL}_n(k). \tag{3.4.3}$$

This last map admits the following explicit description. Choosing a basis of V whose last element is e, we get an embedding $U(V)(k) \hookrightarrow \mathrm{GL}_{n+1}(k \otimes_F E)$. By [57, Lemma 2.3], any regular semisimple element $x \in U(V)_{\mathrm{rs}}(k)$ is $\mathrm{GL}_n(k \otimes_F E)$-conjugated to a regular semisimple element of $S_{n+1}(k)$ which is unique up to $\mathrm{GL}_n(k)$-conjugation. The $\mathrm{GL}_n(k)$-conjugacy class of this element is exactly the image of x by the map (3.4.3).

We have a similar situation at the level of Lie algebras: we have a canonical isomorphism between geometric quotients

$$\mathfrak{s}_{n+1}/\mathrm{GL}_n \simeq \mathfrak{u}(V)/U(W) \tag{3.4.4}$$

which preserves the regular semisimple loci $(\mathfrak{s}_{n+1}/\mathrm{GL}_n)_{\mathrm{rs}} = \mathfrak{s}_{n+1,\mathrm{rs}}/\mathrm{GL}_n$ and $(\mathfrak{u}(V)/U(W))_{\mathrm{rs}} = \mathfrak{u}(V)_{\mathrm{rs}}/U(W)$. For all field extension k of F, this induces an injection

$$\mathfrak{u}(V)_{\mathrm{rs}}(k)/U(W)(k) \hookrightarrow \mathfrak{s}_{n+1,\mathrm{rs}}(k)/\mathrm{GL}_n(k). \tag{3.4.5}$$

We now define, following [56, § 4.1], two families of *transfer factors* $\Omega_v : G'_{\mathrm{rs}}(F_v) \to \mathbb{C}^\times$ and $\omega_v : \mathfrak{s}_{n+1,\mathrm{rs}}(F_v) \to \mathbb{C}^\times$, v a place of F, satisfying the following conditions:

- For all v and all $\gamma \in G'_{\mathrm{rs}}(F_v)$ (resp. all $X \in \mathfrak{s}_{n+1,\mathrm{rs}}(F_v)$), we have $\Omega_v(h_1\gamma h_2) = \eta_v(h_2)\Omega_v(\gamma)$ (resp. $\omega_v(h^{-1}Xh) = \eta_{E_v/F_v}(h)\omega_v(X)$) for all $(h_1, h_2) \in H'_1(F_v) \times H'_2(F_v)$ (resp. for all $h \in \mathrm{GL}_n(F_v)$);
- For all $\gamma \in G'_{\mathrm{rs}}(F)$ (resp. all $X \in \mathfrak{s}_{n+1,\mathrm{rs}}(F)$), we have the product formula $\prod_v \Omega_v(\gamma) = 1$ (resp. $\prod_v \omega_v(X) = 1$) where almost all terms in the product are equal to 1.

Let v be a place of F. For all $s \in S_{n+1,\mathrm{rs}}(F_v)$ and all $X \in \mathfrak{s}_{n+1,\mathrm{rs}}(F_v)$, we set

$$\Omega_v(s) = \eta'_v\left(\det(s)^{-[\frac{n+1}{2}]}\det(e_{n+1}, e_{n+1}s, \ldots, e_{n+1}s^n)\right) \tag{3.4.6}$$

and

$$\omega_v(X) = \eta'_v\left(\det(e_{n+1}, e_{n+1}X, \ldots, e_{n+1}X^n)\right), \tag{3.4.7}$$

where $e_{n+1} = (0, \ldots, 0, 1)$ and η'_v is the local component at v of the character $\eta' : \mathbb{A}_E^\times \to \mathbb{C}^\times$ extending $\eta_{E/F}$ that we fixed at the beginning. Note that (see the proof of [55, Lemma 3.5])

$$\Omega_v(\mathfrak{c}(X)) = \eta_v(2)^{n(n+1)/2}\omega_v(X) \tag{3.4.8}$$

for all $X \in \mathfrak{s}_{n+1,\mathrm{rs}}(F_v)$ sufficiently close to 0. Finally, for all $\gamma = (\gamma_1, \gamma_2) \in G'_{\mathrm{rs}}(F_v)$, we set

$$\Omega_v(\gamma) = \begin{cases} \Omega_v(s) & \text{if } n \text{ is even,} \\ \eta'_v(\gamma_1^{-1}\gamma_2)\Omega_v(s) & \text{if } n \text{ is odd.} \end{cases}$$

where $s = \nu(\gamma_1^{-1}\gamma_2)$. For future reference, we record the following formula:

$$\Omega_v(\gamma)O(\gamma, f') = \Omega_v(s)O(s, \widetilde{f}') \tag{3.4.9}$$

for all $f' \in \mathcal{S}(G'(F_v))$, all $\gamma \in G'_{\mathrm{rs}}(F_v)$ and where we have set $s = \nu(\gamma_1^{-1}\gamma_2)$.

Using the transfer factors, we can define the notion of matching functions as follows. Let v be a place of F. We say that functions $f' \in \mathcal{S}(G'(F_v))$ and $f \in \mathcal{S}(G(F_v))$ *match* each other or that they are *smooth transfer* of each other if we have the equality

$$O(\delta, f) = \Omega_v(\gamma) O(\gamma, f')$$

for every $\delta \in G_{\mathrm{rs}}(F_v)$ and $\gamma \in G'_{\mathrm{rs}}(F_v)$ whose orbits correspond to each other via the embedding (3.4.2). Similarly, we say that functions $\varphi' \in \mathcal{S}(\mathfrak{s}_{n+1}(F_v))$ and $\varphi \in \mathcal{S}(\mathfrak{u}(V)(F_v))$ *match* each other or that they are *smooth transfer* of each other if we have the equality

$$O(X, \varphi_W) = \omega_v(Y) O(Y, \varphi')$$

for every $X \in \mathfrak{u}(V)_{\mathrm{rs}}(F_v)$ and $Y \in \mathfrak{s}_{n+1,\mathrm{rs}}(F_v)$ whose orbits correspond to each other via the embedding (3.4.5).

If the place v splits in E, then the existence of smooth transfer is easy (see [55, Proposition 2.5]). One of the main achievements of [55] was to prove the existence of smooth transfer for non-Archimedean places. In other words, by [55, Theorem 2.6], we have the following.

Theorem 3.4.1 (Zhang). *Let v be a non-Archimedean place of F.*

(i) *For every function $f' \in \mathcal{S}(G'(F_v))$, there exists a function $f \in \mathcal{S}(G(F_v))$, matching f', and, conversely, for every function $f \in \mathcal{S}(G(F_v))$, there exists a function $f' \in \mathcal{S}(G'(F_v))$ which matches f.*

(ii) *For every function $\varphi' \in \mathcal{S}(\mathfrak{s}_{n+1}(F_v))$, there exists a function $\varphi \in \mathcal{S}(\mathfrak{u}(V)(F_v))$ matching φ', and, conversely, for every function $\varphi \in \mathcal{S}(\mathfrak{u}(V)(F_v))$, there exists a function $\varphi' \in \mathcal{S}(\mathfrak{s}_{n+1}(F_v))$ which matches φ.*

One of the main ingredients in the proof of Zhang is the following result [55, Theorem 4.17]. We refer the reader to [11, Theorem 3.4.2.1] for a precise computation of the constant appearing below.

Theorem 3.4.2 (Zhang). *Let v be a non-Archimedean place of F. If $\varphi \in \mathcal{S}(\mathfrak{u}(V)(F_v))$ and $\varphi' \in \mathcal{S}(\mathfrak{s}_{n+1}(F_v))$ match, then so do*

$$\eta_{E_v/F_v}(-1)^{\frac{n(n+1)}{2}} \eta_{E_v/F_v}(\mathrm{disc}(W))^n \epsilon(\tfrac{1}{2}, \eta_{E_v/F_v}, \psi)^{n(n+1)/2} \widehat{\varphi}$$

and $\widehat{\varphi'}$.

In a recent paper, Xue [53] was able to extend Zhang's result to obtain a weak version of smooth transfer at Archimedean places (which, however, is sufficient for many global applications). In order to state Xue's result in the generality that we need, we have to vary the Hermitian space W. Let us denote momentarily the groups G and H by G^W and H^W. To every Hermitian space W' of rank n over E, we can associate similar groups $G^{W'}$ and $H^{W'}$, and replacing W by W' everywhere in the previous paragraphs, we have a notion of matching between test functions in $\mathcal{S}(G'(F_v))$ and test functions in $\mathcal{S}(G^{W'}(F_v))$, v a place of F. Then, Xue's result reads as follows.

Theorem 3.4.3 (Xue). *Let v be an Archimedean place of F. Then, the space of functions $f' \in \mathcal{S}(G'(F_v))$ admitting a smooth transfer to $\mathcal{S}(G^{W'}(F_v))$ for every Hermitian space W' of rank n over E is dense in $\mathcal{S}(G'(F_v))$. Similarly, the space of functions $f \in \mathcal{S}(G^W(F_v))$ such that there exists a function $f' \in \mathcal{S}(G'(F_v))$ matching f and with the property that for every Hermitian space W' of rank n over E for which $W'_v \not\simeq W_v$ the function f' matches $0 \in \mathcal{S}(G^{W'}(F_v))$, is dense in $\mathcal{S}(G(F_v))$.*

Parallel to the existence of smooth transfer, there is also a fundamental lemma for the case at hand. This fundamental lemma has been proved by Yun in (sufficiently large) positive characteristic [54] and extended to the characteristic zero case by Gordon in the appendix to [54]. It can be stated as follows.

Theorem 3.4.4 (Yun–Gordon). *There exists a constant $c(n)$ depending only on n such that for every place v of F of residual characteristic greater than $c(n)$, the following holds: if W_v admits a self-dual lattice L_v, then the function $f'_v = \mathbf{1}_{K'_v}$ match the function $f_v = \mathbf{1}_{G(\mathcal{O}_v)}$ where we have defined a model of G over $\mathcal{O}_v$ using the self-dual lattice L_v, otherwise the function $f'_v = \mathbf{1}_{K'_v}$ matches the function $f_v = 0$.*

3.5. Transfer of relative characters, Zhang's conjecture and Ichino–Ikeda conjecture

We shall say of a function $f \in \mathcal{S}(G(\mathbb{A}))$ that it is *nice* if it satisfies the following conditions:

- f is factorizable: $f = \prod_v f_v$;
- There exists a non-Archimedean place v_1 of F and a finite union Ω_1 of cuspidal Bernstein components of $G(F_{v_1})$ such that $f_{v_1} \in \mathcal{S}(G(F_{v_1}))_{\Omega_1}$;
- There exists a place $v_2 \neq v_1$ of F such that f_{v_2} is supported in $G_{\mathrm{rs}}(F_{v_2})$.

We define the notion of *nice* function on $G'(\mathbb{A})$ similarly. To state the next theorem, we will need to consider more than one pair of Hermitian spaces (W, V). Recall that we have an orthogonal decomposition $V = W \oplus^{\perp} Ee$, where $(e, e) = 1$. To any (isomorphism class of) n-dimensional Hermitian space W' over E, we associate the pair (W', V') where $V' = W' \oplus^{\perp} Ee$. Using such a pair, we may construct a new pair $(H^{W'}, G^{W'})$ of reductive groups over F where $H^{W'} = U(W')$ and $G^{W'} = U(W') \times U(V')$. Note that if $W' = W$, then $(H^{W'}, G^{W'}) = (H, G)$. The discussions of the previous paragraphs, of course, apply verbatim to $(H^{W'}, G^{W'})$. In particular, we have a notion of matching between functions in $\mathcal{S}(G^{W'}(F_v))$ and $\mathcal{S}(G'(F_v))$, v a place of F and a notion of nice function on $G^{W'}(\mathbb{A})$. We say that a nice function $f' \in \mathcal{S}(G'(\mathbb{A}))$ match a tuple of nice functions $(f^{W'})_{W'}$, $f^{W'} \in \mathcal{S}(G^{W'}(\mathbb{A}))$ where W' runs over all isomorphism classes of n-dimensional Hermitian spaces over E, if for every W' and every place v of F, the functions f'_v and $f_v^{W'}$ match. Comparing two (simple) global relative trace formulas that have been proposed by Jacquet and Rallis [26], Zhang proves the following [55, Proposition 2.10], [56, Theorem 4.3].

Theorem 3.5.1 (Zhang). *Let π be an abstractly H-distinguished cuspidal automorphic representation of $G(\mathbb{A})$ such that $\mathrm{BC}(\pi)$ is cuspidal and for every non-split Archimedean place v, the representation π_v is tempered. Let $f \in \mathcal{S}(G(\mathbb{A}))$ and $f' \in \mathcal{S}(G'(\mathbb{A}))$ be*

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nice functions and assume that there exists a tuple $(f^{W'})_{W'}$, $f^{W'} \in \mathcal{S}(G^{W'}(\mathbb{A}))$ *of nice functions matching* f' *such that* $f^W = f$. *Then, we have*

$$J_\pi(f) = 2^{-2} L(1, \eta_{E/F})^{-2} I_{\mathrm{BC}(\pi)}(f').$$

Remark 3.5.2. The above theorem differs slightly from [55, Proposition 2.10] and [56, Theorem 4.3] essentially because we are using test functions that are not necessarily compactly supported (as they are only rapidly decreasing, i.e. in the Schwartz space, at the Archimedean places). This is necessary if we want to apply this theorem in conjunction with Xue's result (Theorem 3.4.3) as the dense subspace of 'transferable' test functions that he constructs has no reason to be compactly supported. This extension to rapidly decreasing functions is an easy matter using basic estimates for these functions. A convenient way to do this is to introduce some norms on the automorphic quotient $[G] = G(F)\backslash G(\mathbb{A})$. We give definitions and basic properties of these norms in the appendix and, for convenience of the reader, we provide in appendix A a full proof of Theorem 3.5.1.

Thanks to the theory of Rankin–Selberg convolution due to Jacquet, Piatetski-Shapiro and Shalika [25], for every cuspidal automorphic representation π of $G(\mathbb{A})$ whose base change is cuspidal, we know a factorization of the global relative character $I_{\mathrm{BC}(\pi)}$. More precisely, if $f' = \prod_v f'_v \in \mathcal{S}(G'(\mathbb{A}))$ is factorizable, we have [56, Proposition 3.6]

$$I_{\mathrm{BC}(\pi)}(f') = L(1, \eta_{E/F})^2 \frac{L(1/2, \mathrm{BC}(\pi))}{L(1, \pi, Ad)} \prod_v I^\natural_{\mathrm{BC}(\pi)_v}(f'_v). \tag{3.5.1}$$

A direct consequence of this factorization and Theorem 3.5.1 is the following.

Corollary 3.5.3. *Let* π *be a globally* H*-distinguished (i.e. such that* $J_\pi \neq 0$*) cuspidal automorphic representation of* $G(\mathbb{A})$ *satisfying the following two conditions:*

- *For all non-split Archimedean place* v *of* F, *the representation* π_v *is tempered;*
- *There exist two non-Archimedean places* v_1, v_2 *split in* E *such that* π_{v_1} *is supercuspidal and* π_{v_2} *is tempered.*

Then, for every place v_0 *of* F *different from* v_1, v_2 *and with* π_{v_0} *tempered, there exists a constant* $C(\pi_{v_0}) \in \mathbb{C}$ *such that for all pairs* $(f_{v_0}, f'_{v_0}) \in \mathcal{S}(G(F_{v_0})) \times \mathcal{S}(G'(F_{v_0}))$ *of matching functions with the property that* f'_{v_0} *has also a matching test function* $f^{W'}_{v_0} \in \mathcal{S}(G^{W'}(F_{v_0}))$ *for every Hermitian space* W' *of rank* n, *we have*

$$J_{\pi_{v_0}}(f_{v_0}) = C(\pi_{v_0}) I_{\mathrm{BC}(\pi_{v_0})}(f'_{v_0}).$$

Remark 3.5.4. Note that the condition of matching of the function f'_{v_0} is empty if v_0 is non-Archimedean (by Theorem 3.4.1) or splits in E.

Proof. Let $\mathbb{A}^{v_0}$ denote the adèles outside of v_0 and let $f^{v_0} = \prod_{v \neq v_0} f_v \in \mathcal{S}(G(\mathbb{A}^{v_0}))$ be a factorizable test function. By the multiplicity one results of [1] and [48] and (3.2.1), there exists a constant $C \in \mathbb{C}$ such that

$$J_\pi(f^{v_0} \otimes f_{v_0}) = C J_{\pi_{v_0}}(f_{v_0}) \tag{3.5.2}$$

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for every $f_{v_0} \in \mathcal{S}(G(F_{v_0}))$. Since $J_\pi \neq 0$, we may choose the function f^{v_0} so that $C \neq 0$. Up to replacing f_{v_1} by its projection to $\mathcal{S}(G(F_{v_1}))_{\Omega_1}$, where Ω_1 denotes the Bernstein component of π_{v_1}, we may assume that $f_{v_1} \in \mathcal{S}(G(F_{v_1}))_{\Omega_1}$. Also, by [23, Theorem A.2], we can choose f_{v_2} with support in $G_{\mathrm{rs}}(F_{v_2})$, and by Theorem 3.4.3, we can arrange f_v for v non-split Archimedean to admit a transfer $f'_v \in \mathcal{S}(G'(F_v))$ which itself matches $0 \in \mathcal{S}(G^{W'}(F_v))$ for every other Hermitian space W' over E_v of dimension n. As v_1, v_2 split in E, by the explicit transfer of [55, Proposition 2.5], we can find functions $f'_{v_1} \in \mathcal{S}(G'(F_{v_1}))$ and $f'_{v_2} \in \mathcal{S}(G'(F_{v_2}))$ matching f_{v_1}, f_{v_2} with f'_{v_1} supported on one supercuspidal Bernstein component and f'_{v_2} supported in $G'_{\mathrm{rs}}(F_{v_2})$. Choose for every other place $v \neq v_0$ a matching test function $f'_v \in \mathcal{S}(G'(F_v))$ with $f'_v = \mathbf{1}_{K'_v}$ for almost all places v (this is possible by Theorem 3.4.4) and set $(f^{v_0})' = \prod_{v \neq v_0} f'_v$. Then, for every pair of matching functions $(f_{v_0}, f'_{v_0}) \in \mathcal{S}(G(F_{v_0})) \times \mathcal{S}(G'(F_{v_0}))$ satisfying the assumption of the corollary (when v_0 is split Archimedean), the functions $f = f^{v_0} f_{v_0}$, $f' = f'_{v_0}(f^{v_0})'$ are nice and satisfy the assumption of Theorem 3.5.1. The result now follows from 3.5.2 combined with Theorems 3.5.1 and 3.5.1 applied to the pair (f, f') (note that the assumption that π_{v_0} is supercuspidal implies that $\mathrm{BC}(\pi)$ is cuspidal). □

In [56, Conjecture 4.4], Zhang makes the following conjecture.

Conjecture 3.5.5. *Let v be a place of F and let $\pi_v = \pi_{n,v} \boxtimes \pi_{n+1,v}$ be an irreducible tempered unitary H_v-distinguished representation of $G(F_v)$. Then, for all matching functions $f_v \in \mathcal{S}(G(F_v))$ and $f'_v \in \mathcal{S}(G'(F_v))$, we have*

$$I_{\mathrm{BC}(\pi_v)}(f'_v) = \kappa_v(\pi_v) L(1, \eta_{E_v/F_v})^{-1} J_{\pi_v}(f_v)$$

the constant $\kappa_v(\pi_v)$ being given by

$$\kappa_v(\pi_v) = |\tau|_{E_v}^{(d_n+d_{n+1})/2} \left(\frac{\epsilon(1/2, \eta_{E_v/F_v}, \psi_v)}{\eta'_v(2\tau)} \right)^{n(n+1)/2} \eta_{E_v/F_v}(\mathrm{disc}(W))^n \omega_{\mathrm{BC}(\pi_{n,v})}(\tau),$$

where $\omega_{\mathrm{BC}(\pi_{n,v})}$ denotes the central character of $\mathrm{BC}(\pi_{n,v})$ *and $d_n = \binom{n}{3}$, $d_{n+1} = \binom{n+1}{3}$.*

Remark 3.5.6. The above conjecture actually differs slightly from [56, Conjecture 4.4]. Indeed, there is a discrepancy in the definition of the constant $\kappa_v(\pi_v)$. In *loc. cit.*, the exponent of $\eta_v(\mathrm{disc}(W))$ is 1 rather than n and the factor $\eta'_v(-2\tau)$ is replaced by $\eta'_v(\tau)$. This discrepancy seems to originate from the precise computation of the constant up to which 'transfer commutes with Fourier transform' (see [11, Theorem 3.4.2.1] and Theorem 3.4.2) as well as from the compatibility relation between transfer factors on the group and on the Lie algebra (compare [56, Lemma 9.1] to (3.4.8)). Of course, this difference has no impact for global applications since in any case if π is an automorphic representation, then $\prod_v \kappa_v(\pi_v) = 1$.

Obviously, we may deduce from the conjunction of the above conjecture, of Theorems 3.5.1, 3.4.1 and 3.4.3 and of the factorization (3.5.1) some instances of the Ichino–Ikeda conjecture (as stated in § 1). In [56, Theorem 4.6], Zhang was able to verify Conjecture 3.5.5 in certain particular cases. More precisely, he proves the conjecture when any of the following conditions are satisfied:

- The place v splits in E;
- v is non-Archimedean, π_v is unramified and the residue characteristic of v is sufficiently large;
- v is non-Archimedean and π_v is supercuspidal.

The main goal of this paper is to prove Conjecture 3.5.5 for all non-Archimedean places v. Namely, we prove the following.

Theorem 3.5.7. *Conjecture 3.5.5 holds for every non-Archimedean place v of F.*

As in [56], this theorem has consequences for the Ichino–Ikeda conjecture. Namely, we will deduce from it the following.

Theorem 3.5.8. *Assume that all the Archimedean places of F split in E and let π be a cuspidal automorphic representation of $G(\mathbb{A})$ which is everywhere tempered and such that there exists a non-Archimedean place v of F with $\mathrm{BC}(\pi_v)$ is supercuspidal. Then Conjecture 1.0.1 holds for π.*

The proofs of Theorems 3.5.7 and 3.5.8 will be given in §§ 4.4 and 4.6.

3.6. A globalization result

Until the end of this paragraph, we make the following assumption:

$$\text{The Hermitian space } W \text{ is anisotropic.}$$

This implies that $H = U(W)$ is an anisotropic group over F.

Let v_0 be a non-Archimedean place of F which is inert in E and let S be a finite set of non-Archimedean places of F which split in E. Let σ be a unitary supercuspidal representation of $G(F_S)$. Recall that $\mathrm{Temp}_H(G_{v_0})$ denotes the set of (isomorphism classes of) tempered irreducible $H(F_{v_0})$-distinguished representations π_0 of $G(F_{v_0})$ (see § 2.4). Let $\mathrm{Irr}_{v_0,\sigma,H}(G)$ be the set of irreducible representations $\pi_0 \in \mathrm{Irr}(G_{v_0})$ for which there exists a cuspidal automorphic representation π of $G(\mathbb{A})$ which is globally H-distinguished (i.e. such that $J_\pi \neq 0$) such that $\pi_{v_0} \simeq \pi_0$ and $\pi_S \simeq \sigma \otimes \chi$ for some unramified character $\chi \in \Psi_{\mathrm{unit}}(G_S)$. The goal of this section is to prove the following result.

Proposition 3.6.1. *The set $\mathrm{Irr}_{v_0,\sigma,H}(G) \cap \mathrm{Temp}(G_{v_0})$ is dense in $\mathrm{Temp}_H(G_{v_0})$.*

The proof of this proposition follows closely that of [22, Corollary A.8]. We will need a lemma which is the analog of Lemma A.2 of *loc. cit.* Before stating it, we need to introduce some more notations.

Let $P = MN$ be a parabolic subgroup of G_{v_0} and let σ be a square-integrable representation of $M(F_{v_0})$. We will say that the tempered representation $i_P^{G_{v_0}}\sigma$ of $G(F_{v_0})$ is *regular* if for all $w \in W(G_{v_0}, M)$, we have $w\sigma \not\simeq \sigma$, where $W(G_{v_0}, M)$ stands for the Weyl group of M in G_{v_0} that is the normalizer of M in $G(F_{v_0})$ modulo $M(F_{v_0})$. Recall that this implies that the representation $i_M^{G_{v_0}}\sigma$ is irreducible. We will denote by $\mathrm{Temp}_{\mathrm{reg}}(G_{v_0})$ the set of all regular tempered representations of $G(F_{v_0})$. It is an open subset of $\mathrm{Temp}(G_{v_0})$. Recall that in § 2.3, for every $c > 0$, we have defined subsets $\mathrm{Irr}_{\leqslant c}(U(W)_{v_0})$ and $\mathrm{Irr}_{\leqslant c}(U(V)_{v_0})$ of $\mathrm{Irr}(U(W)_{v_0})$ and $\mathrm{Irr}(U(V)_{v_0})$, respectively. In what follows, we set $\mathrm{Irr}_{\leqslant c}(G_{v_0}) = \mathrm{Irr}_{\leqslant c}(U(W)_{v_0}) \boxtimes \mathrm{Irr}_{\leqslant c}(U(V)_{v_0})$ and $\mathrm{Irr}_{\mathrm{unit},\leqslant c}(G_{v_0}) = \mathrm{Irr}_{\mathrm{unit}}(G_{v_0}) \cap \mathrm{Irr}_{\leqslant c}(G_{v_0})$.

Lemma 3.6.2. *Let $0 < c < \frac{1}{2}$. Then, $\mathrm{Temp}_{\mathrm{reg}}(G_{v_0})$ is open in $\mathrm{Irr}_{\mathrm{unit},\leqslant c}(G_{v_0})$ (for the Fell topology).*

Proof. The proof is the same as in [22, Lemma A.2] the key fact being that the real exponents of tempered representations of unitary groups are all half integers. In *loc. cit.*, the authors use the work of Muic on generic square-integrable representations of classical groups [40] to deduce this fact for tempered generic representations. However, as already noted in [22, Remark A.3], the same result holds for all tempered representation, thanks to the work of Mœglin and Mœglin–Tadic on classification of square-integrable representations of classical groups [33, 35]. Note that the *basic assumption* made by Mœglin and Tadic (see [35, § 2] for a precise statement) to prove their classification is now known since it follows from the canonical normalization of intertwining operators for unitary groups proved by Mok [38, Proposition 3.3.1] and Kaletha–Minguez–Shin–White [28, Lemma 2.2.3] together with the classical reducibility criterion of Silberger and Harish-Chandra [46, § 5.4], [47, Lemma 1.2; Lemma 1.3]. For quasi-split unitary groups, a different proof has been given by Mœglin [34] using twisted endoscopy. For a proof of the basic assumption for quasi-split symplectic and orthogonal groups using the normalization of intertwining operators, see [52, Proposition 3.2]. □

Proof of Proposition 3.6.1. By [6, Corollary 8.6.1], the closure of $\mathrm{Temp}_H(G_{v_0})$ is an union of connected components of $\mathrm{Temp}(G_{v_0})$. Hence, $\mathrm{Temp}_{H,\mathrm{reg}}(G_{v_0}) := \mathrm{Temp}_H(G_{v_0}) \cap \mathrm{Temp}_{\mathrm{reg}}(G_{v_0})$ is dense in $\mathrm{Temp}_H(G_{v_0})$. Let $\pi_0 \in \mathrm{Temp}_{H,\mathrm{reg}}(G_{v_0})$. It is sufficient to show that π_0 belongs to the closure of $\mathrm{Irr}_{v_0,\sigma,H}(G) \cap \mathrm{Temp}(G_{v_0})$. Since σ is $H(F_S)$-distinguished (see § 3.2), by [44, Theorems 6.2.1 and 6.4.1], we know that π_0 and σ belong to the support of the Plancherel measures for $L^2(H(F_{v_0})\backslash G(F_{v_0}))$ and $L^2(H(F_S)\backslash G(F_S))$, respectively. From [44, Theorem 16.3.2], it follows that there exists a sequence of globally H-distinguished automorphic representations $(\pi_k)_k$ of $G(\mathbb{A})$ such that $\pi_{k,v_0} \to \pi_0$ and $\pi_{k,S} \to \sigma$ for the Fell topology. Since $\sigma \otimes \Psi_{\mathrm{unit}}(G_S)$ is open in $\mathrm{Irr}_{\mathrm{unit}}(G_S)$, we have $\pi_{k,S} \in \sigma \otimes \Psi_{\mathrm{unit}}(G_S)$ for k sufficiently large. This implies that π_{k,v_0} belongs to $\mathrm{Irr}_{v_0,\sigma,H}(G)$, and, therefore, π_k, $\mathrm{BC}(\pi_k)$ are cuspidal for k sufficiently large.

Set $c = \frac{1}{2} - \frac{1}{(n+1)^2+1}$. By Lemma 2.3.2, π_{k,v_0} belongs to $\mathrm{Irr}_{\mathrm{unit},\leqslant c}(G_{v_0})$ for k sufficiently large. Hence, by Lemma 3.6.2, $\pi_{k,v_0} \in \mathrm{Temp}_{H,\mathrm{reg}}(G_{v_0})$ for k sufficiently large, and this ends the proof of the proposition. □

4. Proof of Zhang's conjecture

In this section, we will prove Theorems 3.5.7 and 3.5.8. As Theorem 3.5.7 has already been proved by Zhang at every split place v, we only need to prove it at every non-Archimedean place v of F which is inert in E. Fix such a place v. We will now drop all the index v: $E/F = E_v/F_v, G = G_v$, $H = H_v$, $G' = G'_v$, $H'_1 = H'_{1,v}$, $H'_2 = H'_{2,v}$, $\psi = \psi_v$, $\psi_E = \psi_{E,v}$ and so on. Also, to ease notation, we will just write $\mathfrak{s} = \mathfrak{s}_{n+1}$. Finally, we will now use

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unnormalized Haar measures (see § 2.5). In particular, Theorem 3.5.7 now takes the following form (see [56, Lemma 4.7]).

Theorem 4.0.1. *Let $\pi = \pi_n \boxtimes \pi_{n+1}$ be a H-distinguished irreducible tempered representation of $G(F)$. Then, for all matching functions $f \in \mathcal{S}(G(F))$ and $f' \in \mathcal{S}(G'(F))$, we have*

$$I_{\mathrm{BC}(\pi)}(f') = \kappa(\pi) J_\pi(f)$$

where

$$\kappa(\pi) = |\tau|_E^{(d_n+d_{n+1})/2} \left(\frac{\epsilon(1/2, \eta_{E/F}, \psi)}{\eta'(2\tau)} \right)^{n(n+1)/2} \eta_{E_v/F_v}(\mathrm{disc}(W))^n \omega_{\mathrm{BC}(\pi_n)}(\tau).$$

4.1. A result of Zhang on truncated local expansion of the relative character I_Π

In this section, we recall a result of Zhang [56] on the existence of truncated local expansion for the relative characters I_Π. This result is the main ingredient in the proof by Zhang of some particular cases of Conjecture 3.5.5. It will also play a crucial role in the proof of Theorem 3.5.7.

Let us set

$$\xi_- = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ \tau & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \tau & 0 \end{pmatrix} \in \mathfrak{s}(F).$$

It is a regular nilpotent element for the $\mathrm{GL}_n(F)$-action by conjugation [56, Lemma 6.1]. Zhang has defined a *regularized* orbital integral μ_{ξ_-} over the orbit of ξ_- (see [56, Definition 6.10]). It is a $\mathrm{GL}_n(F)$-invariant linear form $\varphi \in \mathcal{S}(\mathfrak{s}(F)) \mapsto \mu_{\xi_-}(\varphi)$ which coincide with the usual orbital integral when the support of φ intersect the orbit of ξ_- in a compact set.

For all $X = \begin{pmatrix} A & u \\ v & w \end{pmatrix} \in \mathfrak{s}_{n+1}(F)$, we define

$$\Delta_-(X) := \det\left(v, vA, \dots, vA^{n-1}\right).$$

Note that (see (3.4.7))

$$\omega(X) = \eta(-1)^n \eta'(\Delta_-(X)) \tag{4.1.1}$$

for all $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ and

$$\Delta_-(\xi_-) = (-1)^{n(n-1)/2} \tau^{n(n+1)/2}. \tag{4.1.2}$$

Let $r > m' > m > 0$ be positive integers. In [56, Definition 8.1], Zhang defines a notion of (m, m', r)-admissible test functions on $G'(F)$. They span a finite dimensional subspace of $\mathcal{S}(G'(F))$. In what follows, when we say that (m, m', r) is *sufficiently large*, we shall

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mean that m is sufficiently large, that m' is sufficiently large depending on m and that r is sufficiently large depending on (m, m'). Recall that in § 3.3, we have, using a Cayley map, associated with any function $f' \in \mathcal{S}(G'(F))$ a function $f'_\natural$ on $\mathfrak{s}(F)$. Also, in § 2.5, we have defined a certain Fourier transform $\varphi \mapsto \widehat{\varphi}$ on $\mathcal{S}(\mathfrak{s}(F))$. We extract from [56] the two following results (see [56, Lemma 8.8, Theorem 8.5] and their proofs).

Proposition 4.1.1 (Zhang). *Let U and $\mathcal{Z}$ be relatively compact neighborhood of 1 and 0 in $G'(F)$ and $(\mathfrak{s}/\mathrm{GL}_n)(F)$, respectively. Then, if (m, m', r) is sufficiently large, for every (m, m', r)-admissible function f', we have $Supp(f') \subseteq U$ and the function $X \in \mathcal{Z}_{rs} \mapsto \eta'(\Delta_-(X))O(X, \widehat{f'_\natural})$ is constant and equal to $\eta'(\Delta_-(\xi_-))\mu_{\xi_-}(\widehat{f'_\natural})$. Moreover, we can find a (m, m', r)-admissible function f' such that $\mu_{\xi_-}(\widehat{f'_\natural}) \neq 0$.*

Theorem 4.1.2 (Zhang). *Let $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ be an irreducible tempered representation of $G'(F)$. Then, if (m, m', r) is sufficiently large (depending on Π), we have the equality*

$$I_\Pi(f') = |\tau|_E^{(d_n+d_{n+1})/2}\omega_{\Pi_n}(\tau)\mu_{\xi_-}(\widehat{f'_\natural})$$

for all (m, m', r)-admissible function f', where $d_n = \binom{n}{3}$ and ω_{Π_n} denotes the central character of Π_n.

A direct consequence of Proposition 4.1.1 and Theorem 4.1.2 is the following.

Corollary 4.1.3. *Let $C \subseteq \mathrm{Temp}(G')$ be a compact subset and let U and $\mathcal{Z}$ be relatively compact neighborhood of 1 and 0 in $G'(F)$ and $(\mathfrak{s}/\mathrm{GL}_n)(F)$, respectively. Then, there exists a test function $f' \in \mathcal{S}(G'(F))$ satisfying the following conditions:*

(i) *$Supp(f') \subseteq U$ and the function $X \in \mathcal{Z}_{rs} \mapsto \eta'(\Delta_-(X))O(X, \widehat{f'_\natural})$ is constant and equal to $\eta'(\Delta_-(\xi_-))\mu_{\xi_-}(\widehat{f'_\natural})$*

(ii) *$\mu_{\xi_-}(\widehat{f'_\natural}) \neq 0$;*

(iii) *For all $\Pi \in C$, we have the equality*

$$I_\Pi(f') = |\tau|_E^{(d_n+d_{n+1})/2}\omega_{\Pi_n}(\tau)\mu_{\xi_-}(\widehat{f'_\natural}).$$

Proof. For all $r > m' > m > 0$, let us denote by $C[m, m', r]$ the set of $\Pi \in C$ such that the equality

$$I_\Pi(f') = |\tau|_E^{(d_n+d_{n+1})/2}\omega_{\Pi_n}(\tau)\mu_{\xi_-}(\widehat{f'_\natural})$$

holds for all (m, m', r)-admissible function f'. Note that $C[m, m', r]$ is a closed subset of C. Obviously, by Proposition 4.1.1, it suffices to show that if (m, m', r) is sufficiently large, then $C[m, m', r] = C$ and for that, we may assume C to be connected. By Theorem 4.1.2, we have

$$\bigcup_{m_0>0}\bigcap_{m\geqslant m_0}\bigcup_{m'_0>m}\bigcap_{m'\geqslant m'_0}\bigcup_{r_0>m'}\bigcap_{r\geqslant r_0} C[m, m', r] = C.$$

Now, by Baire category theorem, this implies that for (m, m', r) sufficiently large, the set $C[m, m', r]$ is not meager, i.e. it has non-empty interior (since it is closed). By connectedness of C and analyticity of $\Pi \mapsto I_\Pi$, this implies $C[m, m', r] = C$, and this ends the proof. □

4.2. Weak comparison of local relative characters

Proposition 4.2.1. *For all $\pi \in \mathrm{Temp}_H(G)$, there exists a nonzero constant $C(\pi) \in \mathbb{C}$ such that for all matching functions $f \in \mathcal{S}(G(F))$ and $f' \in \mathcal{S}(G'(F))$, we have*

$$J_\pi(f) = C(\pi) I_{\mathrm{BC}(\pi)}(f').$$

Moreover, the function $\pi \in \mathrm{Temp}_H(G) \mapsto C(\pi)$ is analytic.

Proof. Assume that we have proved the existence of a constant $C(\pi)$ as in the proposition for a dense set of π in $\mathrm{Temp}_H(G)$. We claim that the proposition can be deduced from this. Indeed, for all $\pi \in \mathrm{Temp}_H(G)$, we can define a constant $C(\pi)$ as follows: choose any set $(f, f') \in \mathcal{S}(G(F)) \times \mathcal{S}(G'(F))$ of matching functions such that $I_{\mathrm{BC}(\pi)}(f') \neq 0$ (the existence of such a pair follows from Theorems 3.4.1 and 4.1.2) and set $C(\pi) = J_\pi(f) I_{\mathrm{BC}(\pi)}(f')^{-1}$. Of course, this constant may *a priori* depend on the choice of f and f', but it follows from the analyticity of $\pi \mapsto J_\pi$ and $\Pi \mapsto I_\Pi$ and our assumption that, in fact, it is independent of such a choice. Still by analyticity of the relative characters, the equality of the proposition is true for all $\pi \in \mathrm{Temp}_H(G)$ and all pair of matching functions (f, f'), and the function $\pi \in \mathrm{Temp}_H(G) \mapsto C(\pi)$ is analytic. Moreover, it is nowhere zero since for all $\pi \in \mathrm{Temp}_H(G)$, there exists $f \in \mathcal{S}(G(F))$ such that $J_\pi(f) \neq 0$ and there exists a $f' \in \mathcal{S}(G'(F))$ matching f (by Theorem 3.4.1).

We now prove the existence of a dense subset of π satisfying the proposition. To this end, we will use Proposition 3.6.1. We first need to globalize the situation at hand. Let

- $\mathbb{E}/\mathbb{F}$ be a quadratic extension of number fields such that all Archimedean places of $\mathbb{F}$ are non-split in $\mathbb{E}$ and v_0 be a place of $\mathbb{F}$ such that $\mathbb{E}_{v_0}/\mathbb{F}_{v_0} \simeq E/F$;
- $\mathbb{W}$ a n-dimensional Hermitian space over $\mathbb{E}$ such that
  - for all Archimedean place v of $\mathbb{F}$, the group $U(\mathbb{W})_v$ is anisotropic (in particular, $\mathbb{W}$ is anisotropic);
  - $\mathbb{W}_{v_0} \simeq W$.

We will set $\mathbb{V} = \mathbb{W} \oplus^\perp \mathbb{E}e$ where $(e, e) = 1$ (so that $\mathbb{V}_{v_0} \simeq V$), $\mathbb{H} = U(\mathbb{W})$ and $\mathbb{G} = U(\mathbb{V})$. Let v_1, v_2 be two non-Archimedean places of $\mathbb{F}$ which split in $\mathbb{E}$ and let σ_1, σ_2 be a supercuspidal representations of $\mathbb{G}(\mathbb{F}_{v_1})$, $\mathbb{G}(\mathbb{F}_{v_2})$ respectively. Applying Proposition 3.6.1 to $S = \{v_1, v_2\}$, we deduce the existence of a dense subset $D \subset \mathrm{Temp}_H(G)$ such that for all $\pi \in D$, there exists a globally $\mathbb{H}$-distinguished cuspidal automorphic representation Π of $\mathbb{G}(\mathbb{A})$ such that $\Pi_{v_0} \simeq \pi$ and Π_{v_i} is supercuspidal for $i = 1, 2$. Applying Corollary 3.5.3 to such representations Π, which we remark are necessarily tempered at all Archimedean places since $U(\mathbb{W})$ is anisotropic there, we deduce that for every $\pi \in D$, there exists a constant $C(\pi)$ as in the proposition. □

4.3. A local trace formula

Let $f_1, f_2 \in \mathcal{S}(G(F))$. Then

(4.3.1) The integral

$$J(f_1, f_2) = \int_{H(F)} \int_{H(F)} \int_{G(F)} f_1(h_1 g h_2) f_2(g) dg dh_1 dh_2$$

is absolutely convergent.

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This follows from [55, Lemma A.4].

By [6, Proposition 8.2.1(v)], we have

$$J(f_1, f_2) = \int_{\mathrm{Temp}_H(G)} J_\pi(f_1) J_{\pi^\vee}(f_2) d\mu_G(\pi), \tag{4.3.2}$$

where $d\mu_G(\pi)$ denotes the Harish-Chandra–Plancherel measure of $G(F)$. We also have (see § 3.3 for the definition of $\widetilde{f}_i$)

$$J(f_1, f_2) = \int_{U(W)(F)} \int_{U(V)(F)} \widetilde{f}_1(h^{-1}xh)\widetilde{f}_2(x)\, dx\, dh.$$

Let us fix open and closed $U(W)(F_v)$-invariant neighborhoods $\omega \subset \mathfrak{u}(V)(F_v)$ and $\Omega \subset U(V)(F_v)$ of 0 and 1 as in § 3.3. Assume that $\widetilde{f}_2$ is supported in Ω. Then, we have (see § 3.3 for the definition of $f_{i,\natural}$)

$$J(f_1, f_2) = \int_{U(W)(F)} \int_{\mathfrak{u}(V)(F)} f_{1,\natural}(h^{-1}Xh) f_{2,\natural}(X)\, dX\, dh.$$

By Fourier transform, we also have

$$J(f_1, f_2) = \int_{U(W)(F)} \int_{\mathfrak{u}(V)(F)} \check{f}_{1,\natural}(h^{-1}Xh) \widehat{f}_{2,\natural}(X)\, dX\, dh.$$

By [55, Corollary 4.5], this expression is absolutely convergent so that we can switch the two integrals and we finally get

$$J(f_1, f_2) = \int_{\mathfrak{u}(V)(F)} \check{f}_{1,\natural}(X) O(X, \widehat{f}_{2,\natural})\, dX. \tag{4.3.3}$$

Summing up, from (4.3.2) and (4.3.3), we deduce that

$$\int_{\mathrm{Temp}_H(G)} J_\pi(f_1) J_{\pi^\vee}(f_2)\, d\mu_G(\pi) = \int_{\mathfrak{u}(V)(F)} \check{f}_{1,\natural}(X) O(X, \widehat{f}_{2,\natural})\, dX \tag{4.3.4}$$

for all functions $f_1, f_2 \in \mathcal{S}(G(F))$ with $Supp(\widetilde{f}_2) \subseteq \Omega$.

We will also need the following formula from [6, Proposition 8.2.1(iv)]:

$$\widetilde{f}(1) = \int_{\mathrm{Temp}_H(G)} J_\pi(f)\, d\mu_G(\pi) \tag{4.3.5}$$

for all $f \in \mathcal{S}(G(F))$.

4.4. Proof of Theorem 4.0.1

We keep the notations of the previous paragraph. Let $f \in \mathcal{S}(G(F))$. Denote by $C \subseteq \mathrm{Temp}_H(G)$ the support of the function $\pi \mapsto J_\pi(f)$. It is a compact set and so is its dual $C^\vee$. Let us denote by $\mathcal{Y}$ the image of Ω in $(H\backslash G/H)(F) = (U(V)/U(W))(F)$ and by $\mathcal{Z}$ the image of the support of $\check{f}_\natural$ in $(\mathfrak{u}(V)/U(W))(F)$. We will denote by the same letters the corresponding subsets in $(H'_1\backslash G'/H'_2)(F)$ and $(\mathfrak{s}/\mathrm{GL}_n)(F)$, respectively (see (3.4.1) and (3.4.4)). By Corollary 4.1.3, there exists a function $f' \in \mathcal{S}(G'(F))$ such that

- f' is supported in the inverse image of $\mathcal{Y}$ in $G'(F)$;

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- The function $Y \in \mathcal{Z}_{\mathrm{rs}} \mapsto \eta'(\Delta_-(Y))O(Y, \widehat{f'_\natural})$ is constant and equal $\eta'(\Delta_-(\xi_-))\mu_{\xi_-}(\widehat{f'_\natural})$;
- $\mu_{\xi_-}(\widehat{f'_\natural}) \neq 0$;
- For all $\Pi \in \mathrm{BC}(C^\vee)$, we have

$$I_\Pi(f') = |\tau|_E^{(d_n+d_{n+1})/2}\omega_{\Pi_n}(\tau)\mu_{\xi_-}(\widehat{f'_\natural}).$$

Let $f_2 \in \mathcal{S}(G(F))$ be a function matching f' (whose existence is guaranteed by Theorem 3.4.1). Up to multiplying f_2 by the characteristic function of $\mathcal{Y}$, we may assume that $\widetilde{f_2}$ is supported in Ω. By (3.4.8), the functions $\eta(2)^{n(n+1)/2}f'_\natural$ and $f_{2,\natural}$ match. Hence, by Theorem 3.4.2, so do $\eta(2)^{n(n+1)/2}\widehat{f'_\natural}$ and $\eta_{E/F}(-1)^{n(n+1)/2}\eta_{E/F}(\mathrm{disc}(W))^n\epsilon(\frac{1}{2}, \eta_{E/F}, \psi)^{n(n+1)/2}\widehat{f_{2,\natural}}$. Thus, by (4.1.1) and (4.1.2), for all $X \in \Omega_{\mathrm{rs}}$, denoting by $Y \in \mathcal{Z}_{\mathrm{rs}}$ the corresponding element, we have

$$\begin{aligned}
&\eta_{E/F}(-1)^{n(n+1)/2}\eta_{E/F}(\mathrm{disc}(W))^n\epsilon(\frac{1}{2}, \eta_{E/F}, \psi)^{n(n+1)/2}O(X, \widehat{f_{2,\natural}})\\
&= \eta(2)^{n(n+1)/2}\eta(-1)^n\eta'(\Delta_-(Y))O(Y, \widehat{f'_\natural})\\
&= \eta(2)^{n(n+1)/2}\eta(-1)^n\eta'(\Delta_-(\xi_-))\mu_{\xi_-}(\widehat{f'_\natural})\\
&= \eta'(-2\tau)^{n(n+1)/2}\mu_{\xi_-}(\widehat{f'_\natural}).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
(4.4.1)\quad &\eta_{E/F}(\mathrm{disc}(W))^n\left(\frac{\epsilon(\frac{1}{2}, \eta_{E/F}, \psi)}{\eta'(2\tau)}\right)^{n(n+1)/2}\int_{\mathfrak{u}(V)(F)} \widetilde{f}_\natural(X)O(X, \widehat{f_{2,\natural}})\,dX\\
&= \mu_{\xi_-}(\widehat{f'_\natural})\int_{\mathfrak{u}(V)(F)} \widetilde{f}_\natural(X)\,dX\\
&= \mu_{\xi_-}(\widehat{f'_\natural})f_\natural(0) = \mu_{\xi_-}(\widehat{f'_\natural})\widetilde{f}(1)\\
&= \mu_{\xi_-}(\widehat{f'_\natural})\int_{\mathrm{Temp}_H(G)} J_\pi(f)\,d\mu_G(\pi)
\end{aligned}$$

where the last equality follows from (4.3.5)

On the other hand, by Proposition 4.2.1, for all $\pi \in C$, we have

$$J_{\pi^\vee}(f_2) = C(\pi^\vee)I_{\mathrm{BC}(\pi^\vee)}(f') = C(\pi^\vee)|\tau|_E^{(d_n+d_{n+1})/2}\omega_{\mathrm{BC}(\pi^\vee)_n}(\tau)\mu_{\xi_-}(\widehat{f'_\natural}).$$

It follows that

$$\begin{aligned}
(4.4.2)\quad &\int_{\mathrm{Temp}_H(G)} J_\pi(f)J_{\pi^\vee}(f_2)\,d\mu_G(\pi)\\
&= \mu_{\xi_-}(\widehat{f'_\natural})|\tau|_E^{(d_n+d_{n+1})/2}\int_{\mathrm{Temp}_H(G)} J_\pi(f)C(\pi^\vee)\omega_{\mathrm{BC}(\pi^\vee)_n}(\tau)\,d\mu_G(\pi).
\end{aligned}$$

Since $\mu_{\xi_-}(\widehat{f'_\natural}) \neq 0$, we deduce from (4.4.1), (4.4.2) and (4.3.4) that

$$(4.4.3)\qquad \int_{\mathrm{Temp}_H(G)} J_\pi(f)\left(\kappa(\pi^\vee)C(\pi^\vee) - 1\right)\,d\mu_G(\pi) = 0$$

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for all $f \in \mathcal{S}(G(F))$ and where

$$\kappa(\pi) = |\tau|_E^{(d_n+d_{n+1})/2} \left(\frac{\epsilon(\frac{1}{2}, \eta_{E/F}, \psi)}{\eta'(2\tau)} \right)^{n(n+1)/2} \eta_{E/F}(\mathrm{disc}(W))^n \omega_{\mathrm{BC}(\pi)_n}(\tau).$$

It only remains to separate each spectral contribution in 4.4.3. For this, we apply a standard argument using the Bernstein center and inspired from [44, Proposition 6.1.1]. Thus, let $\mathcal{Z}(G)$ denote the Bernstein center of G [4]. We may see $\mathcal{Z}(G)$ as a unital subalgebra of the space of continuous functions on $\mathrm{Temp}(G)$ which, moreover, acts on $\mathcal{S}(G(F))$ with the property that $J_\pi(z \star f) = z(\pi) J_\pi(f)$ for all $z \in \mathcal{Z}(G)$, all $f \in \mathcal{S}(G(F))$ and all $\pi \in \mathrm{Temp}(G)$. Thus, by (4.4.3), we get

$$\int_{\mathrm{Temp}_H(G)} z(\pi) J_\pi(f) \left(\kappa(\pi^\vee) C(\pi^\vee) - 1 \right) d\mu_G(\pi) = 0 \tag{4.4.4}$$

for all $f \in \mathcal{S}(G(F))$ and all $z \in \mathcal{Z}(G)$. For all $\pi \in \mathrm{Temp}(G)$, let us denote by χ_π the 'infinitesimal character' of π, that is, the algebra homomorphism $\chi_\pi : \mathcal{Z}(G) \to \mathbb{C}$ given by $\chi_\pi(z) := z(\pi)$ for all $z \in \mathcal{Z}(G)$. Set $Y := \mathrm{Specmax}\, \mathcal{Z}(G)$. Then, the map $\mathrm{Temp}(G) \to Y$, $\pi \mapsto \chi_\pi$ is continuous and proper. Let $Y_{temp} \subseteq Y$ be the image of this map and μ_Y be the push-forward of the Plancherel measure μ_G to Y_{temp}. Then, by the disintegration of measures, there exists a measurable mapping $\chi \mapsto \mu_\chi$ from Y_{temp} to the space of measures on $\mathrm{Temp}(G)$ such that

$$\int_{\mathrm{Temp}(G)} \varphi(\pi)\, d\mu_G(\pi) = \int_{Y_{temp}} \int_{\mathrm{Temp}(G)} \varphi(\pi)\, d\mu_\chi(\pi)\, d\mu_Y(\chi) \tag{4.4.5}$$

for all continuous compactly supported function $\varphi : \mathrm{Temp}(G) \to \mathbb{C}$ and such that for all $\chi \in Y_{temp}$, μ_χ is supported on $\mathrm{Temp}_\chi(G) := \{\pi \in \mathrm{Temp}(G) \mid \chi_\pi = \chi\}$. By (4.4.4), we get

$$\int_{Y_{temp}} z(\chi) \int_{\mathrm{Temp}_{H,\chi}(G)} J_\pi(f) \left(\kappa(\pi^\vee) C(\pi^\vee) - 1 \right) d\mu_\chi(\pi)\, d\mu_Y(\chi) = 0 \tag{4.4.6}$$

for all $f \in \mathcal{S}(G(F))$ and all $z \in \mathcal{Z}(G)$ where we have set $\mathrm{Temp}_{H,\chi}(G) := \mathrm{Temp}_H(G) \cap \mathrm{Temp}_\chi(G)$. Since the restriction of $\mathcal{Z}(G)$ to Y_{temp} is self-adjoint (i.e. for all $z \in \mathcal{Z}(G)$, there exists $z^* \in \mathcal{Z}(G)$ such that $z^*(\chi) = \overline{z(\chi)}$ for all $\chi \in Y_{temp}$), separates points and for all $f \in \mathcal{S}(G(F))$, the function $\pi \in \mathrm{Temp}(G) \mapsto J_\pi(f)$ is compactly supported, by (4.4.6) and the Stone–Weierstrass theorem for μ_Y-almost all $\chi \in Y_{temp}$, we get

$$\int_{\mathrm{Temp}_{H,\chi}(G)} J_\pi(f) \left(\kappa(\pi^\vee) C(\pi^\vee) - 1 \right) d\mu_\chi(\pi) = 0$$

for all $f \in \mathcal{S}(G(F))$. Since $\mathrm{Temp}_\chi(G)$ is finite, we have

$$\int_{\mathrm{Temp}_{H,\chi}(G)} J_\pi(f) \left(\kappa(\pi^\vee) C(\pi^\vee) - 1 \right) d\mu_\chi(\pi) = \sum_{\pi \in \mathrm{Temp}_{H,\chi}(G)} J_\pi(f) \left(\kappa(\pi^\vee) C(\pi^\vee) - 1 \right) \mu_\chi(\pi)$$

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for μ_Y-almost all $\chi \in Y_{temp}$ and all $f \in \mathcal{S}(G(F))$. Finally, as the relative characters J_π for $\pi \in \mathrm{Temp}_{H,\chi}(G)$ are linearly independent, we get that

$$\left(\kappa(\pi^\vee)C(\pi^\vee)-1\right)\mu_\chi(\pi)=0$$

for μ_Y-almost all $\chi \in Y_{temp}$ and all $\pi \in \mathrm{Temp}_{H,\chi}(G)$ which by (4.4.5) means that $\kappa(\pi)C(\pi)=1$ for μ_G-almost all $\pi \in \mathrm{Temp}_H(G)$. Since $\pi \in \mathrm{Temp}_H(G) \mapsto \kappa(\pi)C(\pi)$ is analytic and the support of μ_G is precisely $\mathrm{Temp}(G)$, it follows that $\kappa(\pi)C(\pi)=1$ for all $\pi \in \mathrm{Temp}_H(G)$ which is what we wanted. □

4.5. A first corollary

In this paragraph, we prove the following corollary to Theorem 4.0.1. It will be needed for the proof of Theorem 3.5.8.

Corollary 4.5.1. *Let $f \in \mathcal{S}(G(F))$ and $f' \in \mathcal{S}(G'(F))$. Then f and f' match if and only if we have*

$$I_{\mathrm{BC}(\pi)}(f')=\kappa(\pi)J_\pi(f)$$

for all $\pi \in \mathrm{Temp}_H(G)$ and where, as before, we have set

$$\kappa(\pi)=|\tau|_E^{(d_n+d_{n+1})/2}\left(\frac{\epsilon(\frac{1}{2},\eta_{E/F},\psi)}{\eta'(2\tau)}\right)^{n(n+1)/2}\eta_{E/F}(\mathrm{disc}(W))^n\omega_{\mathrm{BC}(\pi)_n}(\tau).$$

Proof. The necessity follows from Theorem 4.0.1. Let us prove the sufficiency. Thus, we assume that

$$I_{\mathrm{BC}(\pi)}(f')=\kappa(\pi)J_\pi(f)$$

for all $\pi \in \mathrm{Temp}_H(G)$ and we want to prove that f and f' match. Let $f_2 \in \mathcal{S}(G(F))$ be a function which matches f' (such a function exists by Theorem 3.4.1). Then, by Theorem 4.0.1 and the assumption, for all $\pi \in \mathrm{Temp}_H(G)$, we have $J_\pi(f)=J_\pi(f_2)$. Thus, by (4.3.2), for all $f_1 \in \mathcal{S}(G(F))$, we have

$$J(f_1,f)=J(f_1,f_2). \tag{4.5.1}$$

Let $x_0 \in U(V)_{\mathrm{rs}}(F)$ and choose f_1 so that $\widetilde{f_1}$ is supported in a small neighborhood of x_0 in $U(V)_{\mathrm{rs}}(F)$. Then a formal manipulation, which is justified since everything is absolutely convergent here, yields

$$J(f_1,f)=\int_{U(V)(F)} f_1(x)O(x,f)\,dx \tag{4.5.2}$$

and

$$J(f_1,f_2)=\int_{U(V)(F)} f_1(x)O(x,f_2)\,dx. \tag{4.5.3}$$

Since the functions $x \in U(V)_{\mathrm{rs}}(F) \mapsto O(x,f)$ and $x \in U(V)_{\mathrm{rs}}(F) \mapsto O(x,f_2)$ are locally constant [55, Proposition 3.13], we may choose f_1 such that $\int_{U(V)(F)} f_1(x)O(x,f)\,dx = O(x_0,f)$ and $\int_{U(V)(F)} f_1(x)O(x,f_2)\,dx=O(x_0,f_2)$. For such a choice, it follows from (4.5.1), (4.5.2) and (4.5.3) that $O(x_0,f)=O(x_0,f_2)$. As x_0 was arbitrary, we see that f and f_2 have the same regular semisimple orbital integrals, and, hence, f and f' match. □

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4.6. Proof of Theorem 3.5.8

We may assume that π is abstractly $H(\mathbb{A})$-distinguished (hence, for all v, π_v is H_v-distinguished) as otherwise both sides of Conjecture 1.0.1 are identically zero. By the multiplicity one theorems of [1, 48], there exists a constant C such that

$$J_\pi(f) = C \prod_v J^\natural_{\pi_v}(f_v)$$

for every factorizable test function $f = \prod_v f_v \in \mathcal{S}(G(\mathbb{A}))$, and we only need to show that $C = 4^{-1}\mathcal{L}(\pi, 1/2)$. For this, it is sufficient to prove the existence of $f \in \mathcal{S}(G(\mathbb{A}))$ with

$$J_\pi(f) = 4^{-1}\mathcal{L}(\pi, \tfrac{1}{2}) \prod_v J^\natural_{\pi_v}(f_v)$$

and $J^\natural_{\pi_v}(f_v) \neq 0$ for all places v. Let v_1 be a (non-Archimedean) place of F such that $\mathrm{BC}(\pi_{v_1})$ is supercuspidal (such a place exists by assumption). This implies, in particular, that $\mathrm{BC}(\pi)$ is cuspidal and π_{v_1} supercuspidal (by Lemma 2.3.1). By Theorems 3.5.1, 3.5.7 and identity (3.5.1), it suffices to show that there exists a nice function $f' \in \mathcal{S}(G'(\mathbb{A}))$ matching a tuple of nice functions $(f^{W'})_{W'}$, $f^{W'} \in \mathcal{S}(G^{W'}(\mathbb{A}))$ such that $I_{\mathrm{BC}(\pi_v)}(f'_v) \neq 0$ for all v. Let Ω_1 be the Bernstein component of $\mathrm{BC}(\pi_{v_1})$ in $G'(F_{v_1})$. Then, we can find a function $f'^{\circ}_{v_1} \in \mathcal{S}(G'(F_{v_1}))_{\Omega_1}$ such that $I_{\mathrm{BC}(\pi_{v_1})}(f'^{\circ}_{v_1}) \neq 0$. Let $f' = \prod_v f'_v$ be a factorizable test function in $\mathcal{S}(G'(\mathbb{A}))$ such that $f'_{v_1} = f'^{\circ}_{v_1}$ and $I_{\mathrm{BC}(\pi_v)}(f'_v) \neq 0$ for all other places v. By [23, Theorem A.2], we can assume that for some split non-Archimedean place $v_2 \neq v_1$, we have $\mathrm{Supp}(f'_{v_2}) \subset G'_{\mathrm{rs}}(F_{v_2})$. Then, by construction, the function f' is nice. Moreover, by the assumption on Archimedean places, Theorems 3.4.1 and 3.4.4, we can find a tuple of functions $(f^{W'})_{W'}$, $f^{W'} \in \mathcal{S}(G^{W'}(\mathbb{A}))$, matching f'. Of course, the functions $f^{W'}$ have no reason of being nice. However, by Lemma 2.3.1, for every W', there exists a finite union $\Omega_1^{W'}$ of cuspidal Bernstein components of $G^{W'}(F_{v_1})$ such that $\Omega_1^{W'}$ contains all irreducible representations of $G^{W'}(F_{v_1})$ whose base change belongs to Ω_1 and, by Corollary 4.5.1, up to replacing $f^{W'}_{v_1}$ by its projection $f^{W'}_{v_1,\Omega_1^{W'}}$ onto $\mathcal{S}(G^{W'}(F_{v_1}))_{\Omega_1^{W'}}$, we may assume that $f^{W'}_{v_1} = f^{W'}_{v_1,\Omega_1^{W'}}$. Also, by the explicit transfer of [55, Proposition 2.5], we can choose the tuple $(f^{W'})_{W'}$ such that $f^{W'}_{v_2}$ is supported in $G^{W'}_{\mathrm{rs}}(F_{v_2})$ for every W'. Then, for every W', the function $f^{W'}$ is nice and we are done. □

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Appendix A. Comparison of relative trace formulas

The goal of this appendix is to provide a proof of Theorem 3.5.1. Inspired by [29, § 18], we start by introducing a convenient notion of norms on the adelic points of a variety over F.

A.1. Norms on adelic varieties

We will use the following convenient, although not very precise, notations. If f_1, f_2 are positive valued functions on a set X, we will write

$$f_1(x) \ll f_2(x), \quad \text{for all } x \in X$$

to mean that there exists a constant $C > 0$ such that $f_1(x) \leqslant Cf_2(x)$ for all $x \in X$. We will also write

$$f_1(x) \prec f_2(x), \quad \text{for all } x \in X$$

or just $f_1 \prec f_2$ if there exist constants $C, d > 0$ such that $f_1(x) \leqslant Cf_2(x)^d$ for all $x \in X$. Finally, we will write

$$f_1(x) \sim f_2(x), \quad \text{for all } x \in X$$

or simply $f_1 \sim f_2$ if $f_1 \prec f_2$ and $f_2 \prec f_1$.

Let X be a set. By an abstract norm on X, we will just mean a function $\|.\| : X \to [1, +\infty[$. Let $\|.\|_1$ and $\|.\|_2$ be two abstract norms on X. We will say that $\|.\|_1$ dominates $\|.\|_2$ if $\|x\|_2 \prec \|x\|_1$ for all $x \in X$ and we will say that $\|.\|_1$ and $\|.\|_2$ are equivalent if $\|.\|_1$ dominates $\|.\|_2$ and $\|.\|_2$ dominates $\|.\|_1$, i.e. if $\|.\|_1 \sim \|.\|_2$. Let $f : X \to Y$ be a map between two sets and let $\|.\|_Y$ be an abstract norm on Y. Then, we define an abstract norm $f^*\|.\|_Y$ on X by

$$f^*\|x\|_Y := \|f(x)\|_Y$$

for all $x \in X$.

Let F be a number field and $\mathbb{A}$ its ring of adeles, and for every place v of F, we will denote by F_v the corresponding completion. For every finite extension F' of F, we will write $\mathbb{A}_{F'} = \mathbb{A} \otimes_F F'$ for the adele ring of F'. We fix algebraic closures $\overline{F}$ of F and $\overline{F}_v$ of F_v. For every place v of F, we will denote by $|.|_v$ the normalized absolute value on F_v. This absolute value extends uniquely to an absolute value on $\overline{F}_v$ that we will also denote by $|.|_v$. We define

$$\mathbb{A}_{\overline{F}} = \overline{F} \otimes_F \mathbb{A} = \varinjlim_{F'} \mathbb{A}_{F'},$$

where the limit is taken over all finite subextension of $\overline{F}/F$. Let X be an algebraic variety over F (i.e. a reduced separated scheme of finite type over F). Since X is of finite type, we have $X(\mathbb{A}_{\overline{F}}) = \varinjlim_{F'} X(\mathbb{A}_{F'})$. We are going to define certain (equivalence classes of) abstract norms on $X(\mathbb{A}_{\overline{F}})$ and $X(\overline{F}_v)$, v a place of F. The definition of these abstract norms is mainly inspired by [29, § 18]. First, assume that X is affine and choose a set $\{P_1, \ldots, P_k\}$ of generators for the F-algebra $F[X]$. For every place v of F, we define an abstract norm $\|.\|_{X_v}$ on $X(\overline{F}_v)$ by

$$\|x\|_{X_v} := \max(1, |P_1(x)|_v, \ldots, |P_k(x)|_v)$$

for all $x \in X(\overline{F}_v)$. Choosing a different generating set $\{Q_1, \ldots, Q_\ell\}$ would yield another family of abstract norms $(\|.\|'_{X_v})_v$ with the following properties:

- For all place v, $\|.\|'_{X_v} \sim \|.\|_{X_v}$;
- There exists $d > 0$ such that for almost all place v, we have

$$\|.\|_{X_v}^{1/d} \leqslant \|.\|'_{X_v} \leqslant \|.\|_{X_v}^d.$$

In particular, for all v, the equivalence class of the abstract norm $\|.\|_{X_v}$ does not depend on the particular generating set chosen, and by a *norm* on $X(\overline{F}_v)$, we will mean any abstract norm in this equivalence class. Note that the norms $(\|.\|_{X_v})_v$ constructed above are Galois invariant in the sense that $\|^\sigma x\|_{X_v} = \|x\|_{X_v}$ for all $x \in X(\overline{F}_v)$ and all $\sigma \in Gal(\overline{F}_v/F_v)$. This allows us to extend the norm $\|.\|_{X_v}$ to $X(K)$ for any finite extension K of F_v: choosing any embedding $\iota : K \hookrightarrow \overline{F}_v$, we set

$$\|x\|_{X_v} := \|\iota(x)\|_{X_v}$$

for any $x \in X(K)$.

We now define an abstract norm $\|.\|_X$ on $X(\mathbb{A}_{\overline{F}})$ as follows. Let $x \in X(\mathbb{A}_{\overline{F}})$ and choose a finite extension F'/F such that $x \in X(\mathbb{A}_{F'})$. Then, we may write x as a product $\prod_w x_w$, $x_w \in X(F'_w)$, indexed by the set of places of F' and we set

$$\|x\|_X := \prod_v \left(\prod_{w|v} \|x_w\|_{X_v}^{[F'_w:F_v]} \right)^{1/[F':F]}$$

where the first product is over the set of places v of F and the second product is over the set of places w of F' above v. Note that this definition does not depend on the choice of the finite extension F'/F such that $x \in X(\mathbb{A}_{F'})$. Moreover, choosing a different generating set would give an equivalent abstract norm. By a *norm* on $X(\mathbb{A}_{\overline{F}})$, we will mean any abstract norm in this equivalence class. We will assume from now on that for any affine variety X over F, we have fixed norms $\|.\|_X$ on $X(\mathbb{A}_{\overline{F}})$ and norms $\|.\|_{X_v}$ on $X(\overline{F}_v)$, for all place v of F, as above (i.e. by choosing a finite generating set of $F[X]$). In the particular case $X = \mathbb{A}^1$ (the affine line), we will even take

$$\|x\|_{\mathbb{A}^1_v} = \max(1, |x|_v)$$

for all place v of F and for all $x \in X(\overline{F}_v) = \overline{F}_v$. Note that by the product formula, we then have

$$\|x\|_{\mathbb{A}^1} = \|x^{-1}\|_{\mathbb{A}^1} \tag{A.1.1}$$

for every $x \in \overline{F}^\times$.

We continue to assume that X is affine. Let $\mathcal{U} = (U_i)_{i\in I}$ be a finite covering of X by affine open subsets. We can define another abstract norm $\|.\|_{X_v,\mathcal{U}}$ on $X(\overline{F}_v)$ by

$$\|x\|_{X_v,\mathcal{U}} := \min\{\|x\|_{U_{i,v}}; i \in I \text{ such that } x \in U_i(\overline{F}_v)\}, \quad x \in X(\overline{F}_v).$$

Then we have [29, Proposition 18.1(6)]

- For all place v, $\|.\|_{X_v,\mathcal{U}} \sim \|.\|_{X_v}$;
- There exists $d > 0$ such that for almost all place v, we have

$$\|.\|_{X_v}^{1/d} \leqslant \|.\|_{X_v,\mathcal{U}} \leqslant \|.\|_{X_v}^d;$$

- For all place v, $\|.\|_{X_v,\mathcal{U}}$ is Galois invariant.

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We can also define an abstract norm $\|.\|_{X,\mathcal{U}}$ on $X(\mathbb{A}_{\overline{F}})$ by sending $x \in X(\mathbb{A}_{F'})$, F'/F a finite extension, to

$$\|x\|_{X,\mathcal{U}} := \prod_v \left(\prod_{w|v} \|x_w\|_{X_v,\mathcal{U}}^{[F'_w:F_v]} \right)^{1/[F':F]}.$$

Then $\|.\|_{X,\mathcal{U}}$ is a norm on $X(\mathbb{A}_{\overline{F}})$ (i.e. $\|.\|_{X,\mathcal{U}} \sim \|.\|_X$). This allows us to extend the definition of the abstract norms $\|.\|_X$ and $\|.\|_{X_v}$ to any algebraic variety X over F as follows. Let X be such a variety and choose a finite covering $\mathcal{U} = (U_i)_{i\in I}$ of X by affine open subsets. Then the definitions of the abstract norms $\|.\|_{X,\mathcal{U}}$ and $\|.\|_{X_v,\mathcal{U}}$ as above still make sense and we will set $\|.\|_X := \|.\|_{X,\mathcal{U}}$, $\|.\|_{X_v} := \|.\|_{X_v,\mathcal{U}}$. Choosing a different covering $\mathcal{V}$ of X would give abstract norms $(\|.\|'_{X_v})_v$ and $\|.\|'_X$ satisfying the following:

- For all v, $\|.\|'_{X_v} \sim \|.\|_{X_v}$ and there exists $d > 0$ such that for almost all v, we have

$$\|.\|_{X_v}^{1/d} \leqslant \|.\|'_{X_v} \leqslant \|.\|_{X_v}^d;$$

- $\|.\|'_X \sim \|.\|_X$.

In particular, the equivalence class of $\|.\|_X$ (resp. of $\|.\|_{X_v}$ for v a place of F) does not depend on the particular choice of $\mathcal{U}$, and by a *norm* on $X(\mathbb{A}_{\overline{F}})$ (resp. on $X(\overline{F}_v)$), we will mean any abstract norm in this equivalence class. From now on, we assume that every algebraic variety over F has been equipped with a family of norms as above (i.e. by choosing a finite covering $\mathcal{U}$ by affine open subsets). If X is affine, we also assume that these norms have been defined using the trivial covering $\mathcal{U} = \{X\}$ so that they coincide with the ones we already fixed. If G is an affine algebraic group over F, we also define a norm $\|.\|_{[G]}$ on $[G] = G(F)\backslash G(\mathbb{A})$ by

$$\|x\|_{[G]} := \inf_{\gamma\in G(F)} \|\gamma x\|_G$$

for all $x \in [G]$.

Proposition A.1.1. *Let X and Y be algebraic varieties over F and let G be an affine algebraic group over F.*

(i) *The function $x \mapsto \|x\|_X$ is locally bounded on $X(\mathbb{A})$.*

(ii) *Let $f : X \to Y$ be a morphism of algebraic varieties. Then $f^*\|.\|_Y \prec \|.\|_X$. In particular, we have $\|gg'\|_G \prec \|g\|_G\|g'\|_G$ and $\|g^{-1}\|_G \sim \|g\|_G$ for all $g, g' \in G(\mathbb{A}_{\overline{F}})$. If, moreover, f is a finite morphism (in particular, if it is a closed embedding), then $f^*\|.\|_Y \sim \|.\|_X$.*

(iii) *Let $f \in F[X]$ and let $X_f = D(f)$ be the principal open subset of X defined by the nonvanishing of f. Then, we have*

$$\|x\|_{X_f} \sim \|x\|_X \|f(x)^{-1}\|_{\mathbb{A}^1}$$

for all $x \in X_f(\mathbb{A}_{\overline{F}})$.

(iv) *Let $U \subset X$ be an open subset and assume that X is quasi-affine. Then, we have*

$$\|x\|_U \sim \|x\|_X$$

for all $x \in U(\overline{F})$. More generally, if $p : X \to Y$ is a regular map and Y is quasi-affine, then for all open subset $V \subset Y$, we have

$$\|x\|_{p^{-1}(V)} \sim \|x\|_X$$

for all $x \in p^{-1}(V)(\mathbb{A}_{\overline{F}})$ such that $p(x) \in V(\overline{F})$.

(v) *If X is quasi-affine, then there exists $d > 0$ such that*

$$\sum_{x \in X(F)} \|x\|_X^{-d}$$

converges.

(vi) *Let $d_r g$ be a right Haar measure on $G(\mathbb{A})$. Then there exists $d > 0$ such that the two integrals*

$$\int_{G(\mathbb{A})} \|g\|_G^{-d}\, d_r g, \quad \int_{[G]} \|x\|_{[G]}^{-d}\, dx$$

converge.

(vii) *Assume that X carries a G-action and that we have a regular map $p : X \to Y$ making X into a G-torsor over Y. Fix a right Haar measure $d_r g$ on $G(\mathbb{A})$. Then for all $d > 0$, there exists $d' > 0$ such that*

$$\int_{G(\mathbb{A})} \|gx\|_X^{-d'}\, d_r g \ll \|p(x)\|_Y^{-d}$$

for all $x \in X(\mathbb{A})$.

(viii) *Assume that G is connected and reductive, and let $\mathfrak{S} \subset G(\mathbb{A})$ be a Siegel domain (see [37, § I.2.1]). Then, we have*

$$\|g\|_G \sim \|g\|_{[G]}$$

for all $g \in \mathfrak{S}$.

(ix) *Let $H < G$ be a closed subgroup such that G/H is quasi-affine (this is the case if, for example, H is reductive or if there is no nontrivial morphism $H \to \mathbb{G}_m$). Then, we have*

$$\|x\|_{[H]} \sim \|x\|_{[G]}$$

for all $x \in [H]$. In particular, by (vi), there exists $d > 0$ such that the integral

$$\int_{[H]} \|x\|_{[G]}^{-d}\, dx$$

converges.

Proof. (i) This follows from the fact that for all v, the function $x \in X(F_v) \mapsto \|x\|_{X_v}$ is locally bounded and the fact that for almost all v, we have $\|x_v\|_{X_v} = 1$ for all $x_v \in X(\mathcal{O}_v)$.

(ii) It suffices to prove the following:

- For all place v, we have $f^*\|.\|_{Y_v} \prec \|.\|_{X_v}$ and if f is finite, $\|.\|_{X_v} \prec f^*\|.\|_{Y_v}$;

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- There exists $d > 0$ such that for almost all place v, we have $f^*\|.\|_{Y_v} \leqslant \|.\|_{X_v}^d$ and if f is finite, $\|.\|_{X_v}^{1/d} \leqslant f^*\|.\|_{Y_v}$.

Assume that the norms $(\|.\|_{X_v})_v$ have been defined using the finite affine open covering $\mathcal{U} = (U_i)_{i\in I}$ of X and that the norms $(\|.\|_{Y_v})_v$ have been defined using the finite affine open covering $\mathcal{V} = (V_j)_{j\in J}$ of Y. Up to refining $\mathcal{U}$, we may assume that for all $j \in J$, there exists a subset $I(j) \subset I$ such that $f^{-1}(V_j) = \bigcup_{i\in I(j)} U_i$. If, moreover, f is finite, then for all $j \in J$, the open subset $f^{-1}(V_j)$ is affine so that we may assume that $\mathcal{U} = (f^{-1}(V_j))_{j\in J}$. This allows us to reduce to the case where both X and Y are affine in which case the statement can be proved much the same way as [29, Proposition 18.1(1)].

(iii) Assume that the family of norms $(\|.\|_{X_v})_v$ has been defined using the finite affine open covering $\mathcal{U} = (U_i)_{i\in I}$ of X. Set $U_{i,f} = U_i \cap X_f$ for all $i \in I$. Obviously, we may assume that the family of norms $(\|.\|_{X_{f,v}})_v$ has been defined using the affine open covering $\mathcal{U}_f = (U_{i,f})_{i\in I}$ of X_f and that

$$\|x\|_{U_{i,f,v}} = \max\left(\|x\|_{U_{i,v}}, |f(x)|_v^{-1}\right)$$

for all place v of F and all $x \in U_{i,f}(\overline{F}_v)$. Then we have

$$\sqrt{\|x\|_{U_{i,v}} \max(1, |f(x)|_v^{-1})} \leqslant \|x\|_{U_{i,f,v}} \leqslant \|x\|_{U_{i,v}} \max(1, |f(x)|_v^{-1})$$

for all place v of F and all $x \in U_{i,f}(\overline{F}_v)$. It follows that

$$\sqrt{\|x\|_{X_v} \max(1, |f(x)|_v^{-1})} \leqslant \|x\|_{X_{f,v}} \leqslant \|x\|_{X_v} \max(1, |f(x)|_v^{-1})$$

for all place v of F and all $x \in X_f(\overline{F}_v)$. Taking the product, we get

$$\sqrt{\|x\|_X \|f(x)^{-1}\|_{\mathbb{A}^1}} \leqslant \|x\|_{X_f} \leqslant \|x\|_X \|f(x)^{-1}\|_{\mathbb{A}^1}$$

for all $x \in X_f(\mathbb{A}_{\overline{F}})$.

(iv) We prove the second claim which is more general than the first. Let $p : X \to Y$ be a regular map, $V \subset Y$ an open subset and assume that Y is quasi-affine. It already follows from (ii) that we have

$$\|x\|_X \prec \|x\|_{p^{-1}(V)}$$

for all $x \in X(\mathbb{A}_{\overline{F}})$. Hence, it suffices to prove the reverse inequality for all $x \in p^{-1}(V)(\mathbb{A}_{\overline{F}})$ such that $p(x) \in V(\overline{F})$. As Y is quasi-affine, up to replacing V by a finite affine open cover, we may assume that $V = Y_f$ for some $f \in F[Y]$. Still denoting by f its image in $F[X]$, we then have $p^{-1}(V) = X_f$. Then by (ii), (iii) and (A.1.1), we have

$$\|x\|_{X_f} \sim \|x\|_X \|f(x)^{-1}\|_{\mathbb{A}^1} = \|x\|_X \|f(x)\|_{\mathbb{A}^1} \prec \|x\|_X$$

for all $x \in X_f(\mathbb{A}_{\overline{F}})$ such that $f(x) \in \overline{F}^\times$. This implies the desired inequality.

(v) As there exists an open embedding of X into an affine variety, by (iv), we immediately reduce to the case where X itself is affine. Then, we can find a closed embedding $\iota : X \hookrightarrow \mathbb{A}^n$ for some integer $n > 0$, and by (ii), we are reduced to prove the statement for $X = \mathbb{A}^n$ and then eventually for $X = \mathbb{A}^1$ in which case the statement is easily checked.

(vi) Note that

$$\int_{[G]} \|x\|_{[G]}^{-d}\, dx \leqslant \int_{[G]} \sum_{\gamma \in G(F)} \|\gamma x\|_G^{-d}\, dx = \int_{G(\mathbb{A})} \|g\|_G^{-d}\, d_r g$$

for all $d > 0$. Hence, it suffices to show that for d sufficiently large, the last integral above is convergent. Assume that H is a closed distinguished subgroup of G isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_a$. We first show that if the statement is true for both H and G/H, then it is true for G. For this, we write

$$\int_{G(\mathbb{A})} \|g\|_G^{-d}\, d_r g = \int_{(G/H)(\mathbb{A})} \int_{H(\mathbb{A})} \|\dot{g}h\|_G^{-d}\, d_r h d_r \dot{g}$$

for all $d > 0$ and where $d_r h$, $d_r \dot{g}$ are suitable right Haar measures on $H(\mathbb{A})$ and $(G/H)(\mathbb{A})$, respectively (note that $(G/H)(\mathbb{A}) = G(\mathbb{A})/H(\mathbb{A})$). Let $d_0, d_1 > 0$. Setting $d = d_0 + d_1$, we get

$$\int_{H(\mathbb{A})} \|gh\|_G^{-d}\, d_r h \leqslant \left(\inf_{h \in H(\mathbb{A})} \|gh\|_G\right)^{-d_0} \int_{H(\mathbb{A})} \|gh\|_G^{-d_1}\, d_r h$$

for all $g \in G(\mathbb{A})$. By (ii), there exists $c > 0$ such that $\|h\|_H \ll \|gh\|_G^c \|g\|_G^c$ for all $(h, g) \in H(\mathbb{A}) \times G(\mathbb{A})$. Hence,

$$\int_{H(\mathbb{A})} \|gh\|_G^{-d}\, d_r h \ll \left(\inf_{h \in H(\mathbb{A})} \|gh\|_G\right)^{-d_0} \|g\|_G^{d_1} \int_{H(\mathbb{A})} \|h\|_H^{-d_1/c}\, d_r h$$

for all $g \in G(\mathbb{A})$. As the left-hand side above is, as a function of g, invariant by the right translation by $H(\mathbb{A})$, we also get

$$\int_{H(\mathbb{A})} \|gh\|_G^{-d}\, d_r h \ll \left(\inf_{h \in H(\mathbb{A})} \|gh\|_G\right)^{d_1 - d_0} \int_{H(\mathbb{A})} \|h\|_H^{-d_1/c}\, d_r h$$

for all $g \in G(\mathbb{A})$. By assumption for d_1 sufficiently large, the last integral above is convergent. Thus, it only remains to show that for $d' > 0$ sufficiently large, the integral

$$\int_{(G/H)(\mathbb{A})} \left(\inf_{h \in H(\mathbb{A})} \|\dot{g}h\|_G\right)^{-d'} d_r \dot{g}$$

converges. By (ii), we have $\|\dot{g}\|_{G/H} \prec \inf_{h \in H(\mathbb{A})} \|\dot{g}h\|_G$ for all $\dot{g} \in G(\mathbb{A})/H(\mathbb{A})$. Consequently, the convergence of the last integral above for d' sufficiently large follows from the assumption on G/H.

Let P_0 be a minimal parabolic subgroup of G over F. Then, by the Iwasawa decomposition, there exists a compact subgroup $K \subset G(\mathbb{A})$ such that $G(\mathbb{A}) =$

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$P_0(\mathbb{A})K$. As K is compact, by (i), the norm $\|.\|_G$ is bounded on K. Moreover, we have

$$\int_{G(\mathbb{A})} \|g\|_G^{-d}\, d_r g = \int_{P_0(\mathbb{A})} \int_K \|kp_0\|_G^{-d}\, dk\, d_r p_0$$

for suitable (right) Haar measures $d_r p_0$ and dk on $P_0(\mathbb{A})$ and K, respectively. By (i) and (ii), it follows that we may assume $G = P_0$. Let $P_0 = M_0 N_0$ be a Levi decomposition. Then the Haar measure $d_r p_0$ decomposes as $d_r p_0 = dn_0 dm_0$ according to the decomposition $P_0(\mathbb{A}) = N_0(\mathbb{A}) M_0(\mathbb{A})$. Moreover, we have $\|n_0 m_0\|_{P_0} \sim \|m_0\|_{M_0} \|n_0\|_{N_0}$ for all $(m_0, n_0) \in M_0(\mathbb{A}) \times N_0(\mathbb{A})$. This allows us to reduce to the case where $G = M_0$ or $G = N_0$. If $G = N_0$, then it admits a composition series whose successive quotients are isomorphic to $\mathbb{G}_a$ and we are reduced to the case $G = \mathbb{G}_a$ where the statement can be checked directly. Assume now that $G = M_0$ and denote by A_0 the maximal split torus in the center of G. Then A_0 is isomorphic to a product of $\mathbb{G}_m$ and M_0/A_0 is anisotropic. Thus, we only need to treat the cases $G = \mathbb{G}_m$ and G anisotropic. Once again, if $G = \mathbb{G}_m$, the statement can be checked directly. Now if G is anisotropic, we write

$$\int_{G(\mathbb{A})} \|g\|_G^{-d}\, dg = \int_{G(F)\backslash G(\mathbb{A})} \sum_{\gamma \in G(F)} \|\gamma g\|_G^{-d}\, dg. \tag{A.1.2}$$

By (i), (ii) and (v), if d is sufficiently large, the function

$$g \in G(\mathbb{A}) \mapsto \sum_{\gamma \in G(F)} \|\gamma g\|_G^{-d}$$

is locally bounded. Moreover, by [9], the quotient $G(F)\backslash G(\mathbb{A})$ is compact. The result then follows from (A.1.2).

(vii) Let $d > 0$. As p is a G-torsor and Y is separated, the action of G on X is free, i.e. the regular map

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

is a closed embedding. By (ii), it follows that there exists $c > 0$ such that $\|g\|_G \ll \|gx\|_X^c \|x\|_X^c$ for all $(g, x) \in G(\mathbb{A}) \times X(\mathbb{A})$. Let $d_0, d_1 > 0$. Using the same trick as in the first part of the proof of (vi), we show that for $d' = d_0 + d_1$, we have

$$\int_{G(\mathbb{A})} \|gx\|_X^{-d'}\, d_r g \ll \left(\inf_{g \in G(\mathbb{A})} \|gx\|_X \right)^{d_1 - d_0} \int_{G(\mathbb{A})} \|g\|_G^{-d_1/c}\, d_r g$$

for all $x \in X(\mathbb{A})$. By (vi), the last integral above is convergent for d_1 sufficiently large. Moreover, by (ii), we have $\|p(x)\|_Y \prec \inf_{g \in G(\mathbb{A})} \|gx\|_X$ for all $x \in X(\mathbb{A})$. Thus, the statement follows by choosing d_0 sufficiently large (depending on d_1).

(viii) Let T_0 be a maximal split torus in $\mathrm{R}_{F/\mathbb{Q}}\, G$. Then, up to conjugating $\mathfrak{S}$ by an element of $G(F)$, there exists a compact subset $\Omega \subset G(\mathbb{A})$ such that

$$\mathfrak{S} \subseteq T_0(\mathbb{R})\Omega.$$

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Hence, by (i) and (ii), it is sufficient to show that

$$\|a\|_G \sim \|a\|_{[G]}$$

for all $a \in T_0(\mathbb{R})$. The inequality $\|a\|_{[G]} \leqslant \|a\|_G$ is obvious so that we only need to show that $\|a\|_G \prec \|a\|_{[G]}$ for all $a \in T_0(\mathbb{R})$. Let $\chi_1, \dots, \chi_n$ be a basis of $X^*(T_0)$. Then we have

$$\|a\|_G \sim \max\left(|\chi_1(a)|, |\chi_1(a)|^{-1}, \dots, |\chi_n(a)|, |\chi_n(a)|^{-1}\right)$$

for all $a \in T_0(\mathbb{R})$. Thus, it suffices to show that for all character $\chi \in X^*(T_0)$, we have $|\chi(a)| \prec \|a\|_{[G]}$ for all $a \in T_0(\mathbb{R})$. Let χ be such a character and let V be a rational representation of $\mathrm{R}_{F/\mathbb{Q}}\, G$ containing a nonzero vector v_0 such that $a.v_0 = \chi(a)v_0$ for all $a \in T_0$. Let $v_1, \dots, v_r$ be a basis of V. Set $V_{\mathbb{A}} = V \otimes_{\mathbb{Q}} \mathbb{A}$ and define a nonnegative function $|.|_V$ on $V_{\mathbb{A}}$ by

$$|\lambda_1 v_1 + \dots + \lambda_r v_r|_V = \prod_v \max(|\lambda_{1,v}|_v, \dots, |\lambda_{r,v}|_v)$$

for all $\lambda_1, \dots, \lambda_r \in \mathbb{A}$. Note that there exist nonzero vectors $v \in V_{\mathbb{A}}$ such that $|v|_V = 0$ but that, however, if $v \in V_F = V \otimes_{\mathbb{Q}} F$ is nonzero, then $|v|_V \geqslant 1$. We have $|v|_V \prec \|v\|_{V_F}$ for all $v \in V_{\mathbb{A}}$, where V_F is considered as an algebraic variety over F. Note that G acts on V_F via the natural embedding $G \hookrightarrow (\mathrm{R}_{F/\mathbb{Q}}\, G)_F$. Hence, by (ii), we have

$$|\chi(a)|^d \leqslant |\chi(a)|^d |\gamma v_0|_V = |\gamma a v_0|_V \prec \|\gamma a v_0\|_{V_F} \prec \|\gamma a\|_G$$

for all $a \in T_0(\mathbb{R})$, $\gamma \in G(F)$ and where we have set $d = [F : \mathbb{Q}]$. Taking the infimum over γ yields the desired inequality.

(ix) By (ii), the inequality $\|x\|_{[G]} \prec \|x\|_{[H]}$ is obvious so that we only need to show that $\|x\|_{[H]} \prec \|x\|_{[G]}$ for all $x \in [H]$. We will need the following fact (which is where the assumption G/H quasi-affine is crucial):

(A.1.3) There exists a (set-theoretic) section $s : (H\backslash G)(\overline{F}) \to G(\overline{F})$ such that $\|s(x)\|_G \prec \|x\|_{H\backslash G}$ for all $x \in (H\backslash G)(\overline{F})$.

Proof of (A.1.3): Let $p : G \to H\backslash G$ be the natural surjection. Since $H\backslash G$ is quasi-affine, by (iv), it suffices to find an open covering $(U_i)_{i\in I}$ of $H\backslash G$ and sections $s_i : U_i(\overline{F}) \to p^{-1}(U_i)(\overline{F})$ such that $\|s_i(x)\|_{p^{-1}(U_i)} \prec \|x\|_{U_i}$ for all $i \in I$ and all $x \in U_i(\overline{F})$. It is even sufficient to construct one non-empty open subset $U \subseteq H\backslash G$ and a section $s_U : U(\overline{F}) \to p^{-1}(U)(\overline{F})$ such that $\|s_U(x)\|_{p^{-1}(U)} \prec \|x\|_U$ for all $x \in U(\overline{F})$. Indeed, if such a pair (U, s_U) exists, we can find a finite number of translates $U_i := U\gamma_i$, $\gamma_i \in G(\overline{F})$, $i \in I$, covering $H\backslash G$ and then the sections $s_i : U_i(\overline{F}) \to p^{-1}(U_i)(\overline{F})$ given by $s_i(x) := s(x\gamma_i^{-1})\gamma_i$, for all $i \in I$ and $x \in U_i(\overline{F})$, satisfy the desired condition. As $p : G \to H\backslash G$ is a torsor for the étale topology, we can find a non-empty open subset $U \subseteq H\backslash G$ and a finite étale map $U' \to U$ such that $U' \times_U G$ is the trivial G-torsor over U'. In particular, there exists a regular section $s_{U'} : U' \to U' \times_U G$. Let $s_0 : U(\overline{F}) \to U'(\overline{F})$ be any set-theoretic section. Then, by (ii) and since $U' \to U$ is finite, the section $s_U := pr_2 \circ s_{U'} \circ s_0 : U(\overline{F}) \to$

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$p^{-1}(U)(\overline{F})$, where pr_2 denotes the projection $U' \times_U G \to G$, satisfies the desired condition.

Let $s : (G/H)(\overline{F}) \to G(\overline{F})$ be a section as in (A.1.3). We have $\|\gamma\|_{G/H} \prec \|\gamma h\|_G$, for all $(\gamma, h) \in G(F) \times H(\mathbb{A})$ (by (ii)) and thus

$$\inf_{\gamma' \in H(\overline{F})} \|\gamma' h\|_H \leqslant \|s(\gamma)^{-1}\gamma h\|_H \prec \|s(\gamma)\|_G \|\gamma h\|_G \prec \|\gamma\|_{G/H} \|\gamma h\|_G \prec \|\gamma h\|_G$$

for all $(\gamma, h) \in G(F) \times H(\mathbb{A})$. Taking the infimum over γ, it follows that

$$\inf_{\gamma' \in H(\overline{F})} \|\gamma' h\|_H \prec \inf_{\gamma \in G(F)} \|\gamma h\|_G = \|h\|_{[G]}$$

for all $h \in H(\mathbb{A})$. Hence, it suffices to show

$$\|h\|_{[H]} \prec \inf_{\gamma \in H(\overline{F})} \|\gamma h\|_H \tag{A.1.4}$$

for all $h \in H(\mathbb{A})$. Denote by N_H the unipotent radical of H and let L_H be a Levi component of H (so that $H = L_H \ltimes N_H$). As $[N_H]$ is compact, we can easily infer from (i) and (ii) that

$$\|\ell n\|_{[H]} \sim \|\ell\|_{[L_H]} \quad \text{and} \quad \inf_{\gamma \in H(\overline{F})} \|\gamma \ell n\|_H \sim \inf_{\gamma_L \in L_H(\overline{F})} \|\gamma_L \ell\|_{L_H}$$

for all $\ell \in L_H(\mathbb{A})$ and all $n \in N_H(\mathbb{A})$. We are thus reduced to prove (A.1.4) in the case where H is reductive. Denote by H^0 the connected component of the identity in H. Since $H(F)/H^0(F)$, $H(\overline{F})/H^0(\overline{F})$ are finite and $H(\mathbb{A})/H^0(\mathbb{A})$ is compact, we may assume that $H = H^0$. Let T_0 be a maximal split torus of $\mathrm{R}_{F/\mathbb{Q}} H$ and let $\chi \in X^*(T_0)$. By (viii), it is sufficient to show that

$$|\chi(a)| \prec \|\gamma a\|_H \tag{A.1.5}$$

for all $a \in T_0(\mathbb{R})$ and all $\gamma \in H(\overline{F})$. Let V be a rational representation of $\mathrm{R}_{F/\mathbb{Q}} H$ containing a nonzero vector v_0 such that $a.v_0 = \chi(a)v_0$ for all $a \in T_0$. Fix a basis $v_1, \dots, v_n$ of V and let $|.|_V$ be the nonnegative function on $V_{\mathbb{A}_{\overline{F}}} = V \otimes_{\mathbb{Q}} \mathbb{A}_{\overline{F}}$ defined by

$$|\lambda_1 v_1 + \dots + \lambda_r v_r|_V = \prod_v \left(\prod_{w|v} \max(|\lambda_{1,w}|_v, \dots, |\lambda_{r,w}|_v)^{[F'_w:F_v]} \right)^{1/[F':F]}$$

for all $\lambda_1, \dots, \lambda_r \in \mathbb{A}_{F'}$, F'/F a finite extension. Note that $|v|_V \geqslant 1$ for all nonzero vector $v \in V_{\overline{F}} = V \otimes_{\mathbb{Q}} \overline{F}$ and $|v|_V \prec \|v\|_{V_F}$ for all $v \in V_{\mathbb{A}_{\overline{F}}}$. It follows that

$$|\chi(a)|^d \leqslant |\chi(a)|^d |\gamma v_0|_V = |\gamma a v_0|_V \prec \|\gamma a v_0\|_{V_F} \prec \|\gamma a\|_H$$

for all $(a, \gamma) \in T_0(\mathbb{R}) \times H(\overline{F})$. Taking the infimum over γ, we get (A.1.5), and this ends the proof of (ix). □

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Let G be a connected reductive group over F. Fix a maximal compact subgroup K_∞ of $G(\mathbb{A}_\infty)$ and a Haar measure dg on $G(\mathbb{A})$. We will denote by $\mathcal{U}(\mathfrak{g}_\infty)$ the universal enveloping algebra of (the complexification of) the Lie algebra of $G(\mathbb{A}_\infty)$. For simplicity, we will assume that the split center of G is trivial. Denote by $\mathcal{A}([G])$ the space of automorphic functions on $[G]$ by which we mean functions $\phi : [G] \to \mathbb{C}$ satisfying the following conditions:

- ϕ is smooth: there exists a compact-open subgroup K of $G(\mathbb{A}_f)$ such that ϕ is right K-invariant and for all $g_f \in G(\mathbb{A}_f)$, the function $g_\infty \in G(\mathbb{A}_\infty) \mapsto \phi(g_\infty g_f)$ is C^∞;
- ϕ is uniformly of moderate growth: there exists $d > 0$ such that for all $u \in \mathcal{U}(\mathfrak{g}_\infty)$, we have $|(R(u)\phi)(g)| \ll \|g\|_{[G]}^d$ for all $g \in G(\mathbb{A})$.

Note that we do not impose any condition of K_∞-finiteness or $\mathfrak{z}_\infty$-finiteness (where $\mathfrak{z}_\infty$ denotes the center of $\mathcal{U}(\mathfrak{g}_\infty)$). The space $\mathcal{A}([G])$ is naturally equipped with an LF topology (see [6, Appendix A] for basic facts about LF vector spaces). As usual, we define $\mathcal{A}_{\mathrm{cusp}}([G])$ to be the subspace of cuspidal functions in the following sense: $\phi \in \mathcal{A}([G])$ is cuspidal if for all proper parabolic subgroup $P = MN$ of G, we have

$$\int_{[N]} \phi(ng)\, dn = 0$$

for all $g \in G(\mathbb{A})$. The space $\mathcal{A}_{\mathrm{cusp}}([G])$ is a closed subspace of $\mathcal{A}([G])$ from which it inherits an LF topology and, moreover, every cuspidal function $\phi \in \mathcal{A}_{\mathrm{cusp}}([G])$ is of rapid decay in the following sense: for all $u \in \mathcal{U}(\mathfrak{g}_\infty)$ and for all $d > 0$, we have

$$|(R(u)\phi)(g)| \ll \|g\|_{[G]}^{-d}$$

for all $g \in [G]$ (see [37, Corollary I.2.12]). By the open mapping theorem, for all compact-open subgroup K of $G(\mathbb{A}_f)$, the topology on $\mathcal{A}_{\mathrm{cusp}}([G])^K$ is also induced by the family of seminorms

$$\|\phi\|_{d,u} = \sup_{g \in [G]} |(R(u)\phi)(g)| \|g\|_{[G]}^d, \quad d > 0, u \in \mathcal{U}(\mathfrak{g}_\infty).$$

There is another natural family of seminorms inducing the given topology on $\mathcal{A}_{\mathrm{cusp}}([G])^K$. Let $C_G \in \mathcal{U}(\mathfrak{g}_\infty)$ and $C_K \in \mathcal{U}(\mathfrak{k}_\infty)$ denote the Casimir elements of $G(\mathbb{A}_\infty)$ and K_∞, respectively, and set $\Delta = C_G^2 + C_K^2$. Then the family of Sobolev seminorms

$$\|\phi\|_k = \|R(1+\Delta)^k \phi\|_{L^2([G])}, \quad k \geqslant 0, \phi \in \mathcal{A}_{\mathrm{cusp}}([G])$$

where $\|.\|_{L^2([G])}$ denotes the L^2-norm on $L^2([G])$, induce on $\mathcal{A}_{\mathrm{cusp}}([G])^K$ its LF topology (this follows essentially from strong approximation together with the Sobolev lemma). We will denote $L^2_{\mathrm{cusp}}([G])$ the completion of $\mathcal{A}_{\mathrm{cusp}}([G])$ in $L^2([G])$. It is a unitary representation of $G(\mathbb{A})$ which decomposes discretely.

Let now $f \in \mathcal{S}(G(\mathbb{A}))$ be a Schwartz function on $G(\mathbb{A})$. We denote as usual by

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y), \quad x, y \in [G]$$

the automorphic kernel of f. Note that the sum is absolutely convergent by Proposition A.1.1. Let $\pi \subset \mathcal{A}_{\mathrm{cusp}}([G])$ be a cuspidal automorphic representation and let $\mathcal{B}_\pi$ be an

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orthonormal basis of (the completion of) π for the L^2 scalar product. We define

$$K_{f,\pi}(x,y) = \sum_{\phi \in \mathcal{B}_\pi} (R(f)\phi)(x)\overline{\phi(y)}, \quad x, y \in [G].$$

Then $K_{f,\pi}$ is the orthogonal projection of K_f, seen as a function in x, onto π or, what amounts to the same, the orthogonal projection of K_f, seen as a function of y, onto $\overline{\pi}$. Finally, letting $\mathcal{B} \subset \mathcal{A}_{\mathrm{cusp}}([G])$ be an orthonormal basis of $L^2_{\mathrm{cusp}}([G])$, we set

$$K_{f,\mathrm{cusp}}(x,y) = \sum_{\phi \in \mathcal{B}} (R(f)\phi)(x)\overline{\phi(y)}, \quad x, y \in [G].$$

Note that

$$K_{f,\mathrm{cusp}} = \sum_{\pi} K_{f,\pi},$$

where the sum is over a complete family of orthogonal cuspidal automorphic representations $\pi \subset \mathcal{A}_{\mathrm{cusp}}([G])$ (all of them if there is multiplicity one).

Proposition A.1.2. *Let $H_1, H_2 \subset G$ be closed algebraic subgroups such that the quotients G/H_1 and G/H_2 are quasi-affine. Then the integral*

$$\int_{[H_1]} \int_{[H_2]} \sum_{\pi} |K_{f,\pi}(h_1,h_2)|\, dh_1\, dh_2,$$

the sum running over a complete family of orthogonal cuspidal automorphic representations, converges. We even have the stronger following result: let K_0 be a compact-open subgroup of $G(\mathbb{A}_f)$ such that f is right K_0-invariant and let $\mathcal{B} \subset \mathcal{A}_{\mathrm{cusp}}([G])^{K_0}$ be an orthonormal basis of $L^2_{\mathrm{cusp}}([G])^{K_0}$ consisting of functions which are C_K and C_G eigenvectors, then the integral

$$\int_{[H_1]} \int_{[H_2]} \sum_{\phi \in \mathcal{B}} |(R(f)\phi)(h_1)||\phi(h_2)|\, dh_1\, dh_2$$

converges.

Proof. The second statement is obviously stronger than the first since for every cuspidal automorphic representation π, we can find an orthonormal basis of π^{K_0} consisting of C_K- and C_G-eigenvectors. Let $\mathcal{B}$ be an orthonormal basis of $L^2_{\mathrm{cusp}}([G])^{K_0}$ as in the proposition. By Proposition A.1.1(ix), it suffices to prove that for all $d > 0$, we have

$$\sum_{\phi \in \mathcal{B}} |(R(f)\phi)(x)||\phi(y)| \ll \|x\|_{[G]}^{-d} \|y\|_{[G]}^{-d} \tag{A.1.6}$$

for all $x, y \in [G]$. Let $d > 0$. Since the family of norms $(\|.\|_k)_k$ generates the topology on $\mathcal{A}_{\mathrm{cusp}}([G])^{K_0}$, there exists $k > 0$ such that

$$|\phi(x)| \ll \|\phi\|_k \|x\|_{[G]}^{-d}$$

for all $\phi \in \mathcal{A}_{\mathrm{cusp}}([G])^{K_0}$ and all $x \in [G]$.

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For all $\phi \in \mathcal{B}$, let us denote by $\lambda_K(\phi), \lambda_G(\phi) \in \mathbb{R}$ the eigenvalues of C_K and C_G acting on ϕ. Let N be a positive integer. For all $\phi \in \mathcal{B}$, we have

$$R(f)\phi = (1+\lambda_G(\phi)^2+\lambda_K(\phi)^2)^{-N} R(f^{(N)})\phi,$$

where $f^{(N)} = (1+\Delta)^N f$. Hence, we have

$$\begin{aligned}\sum_{\phi\in\mathcal{B}} |R(f)\phi(x)||\phi(y)| &= \sum_{\phi\in\mathcal{B}} (1+\lambda_G(\phi)^2+\lambda_K(\phi)^2)^{-N} |R(f^{(N)})\phi(x)||\phi(y)| \\ &\ll \|x\|_{[G]}^{-d}\|y\|_{[G]}^{-d} \sum_{\phi\in\mathcal{B}} (1+\lambda_G(\phi)^2+\lambda_K(\phi)^2)^{-N} \|R(f^{(N)})\phi\|_k \|\phi\|_k \\ &= \|x\|_{[G]}^{-d}\|y\|_{[G]}^{-d} \sum_{\phi\in\mathcal{B}} (1+\lambda_G(\phi)^2+\lambda_K(\phi)^2)^{k-N} \|R(f^{(N+k)})\phi\|_{L^2} \|\phi\|_{L^2} \\ &\leqslant \|x\|_{[G]}^{-d}\|y\|_{[G]}^{-d} \|f^{(N+k)}\|_{L^1} \sum_{\phi\in\mathcal{B}} (1+\lambda_G(\phi)^2+\lambda_K(\phi)^2)^{k-N}\end{aligned}$$

for all $x, y \in [G]$ and where $\|.\|_{L^1}$ denotes the L^1-norm on $L^1([G])$. By [41], for $N \gg 1$, the last sum above converges. This proves (A.1.6) and ends the proof of the proposition. □

Remark A.1.3. • Fix $f^\infty \in \mathcal{S}(G(\mathbb{A}^\infty))$. Then, the proof of the proposition actually shows that for every $d > 0$, there exists a continuous seminorm ν_d on $\mathcal{S}(G(F_\infty))$ so that

(A.1.7) $$\left|K_{f_\infty \otimes f^\infty, \mathrm{cusp}}(x, y)\right| \leqslant \nu_d(f_\infty)\|x\|_{[G]}^{-d}\|y\|_{[G]}^{-d}$$

for all $f_\infty \in \mathcal{S}(G(F_\infty))$ and all $x, y \in [G]$.

- We can prove the first part of the proposition directly by using the Selberg *trick*. Indeed, it suffices to show that the series

$$\sum_\pi K_{f,\pi}$$

converges absolutely in $\mathcal{A}_{\mathrm{cusp}}([G\times G])$ or, what amounts to the same, that it converges absolutely in $\mathcal{A}([G\times G])$. To prove this, we only need to show that the sum

$$\sum_\pi |K_{f,\pi}(x, y)|$$

converges absolutely for all $x, y \in [G]$ and is bounded uniformly in x, y. By a theorem of Dixmier–Malliavin [15], we may write f as a finite sum of convolutions $f_{1,i} \star f_{2,i}$, $f_{1,i}, f_{2,i} \in \mathcal{S}(G(\mathbb{A}))$, $i = 1, \dots, k$. By the Cauchy–Schwarz inequality, we have

$$|K_{f,\pi}(x, y)| \leqslant \sum_{i=1}^k K_{h_{2,i},\pi}(x, x)^{1/2} K_{h_{1,i},\pi}(y, y)^{1/2}$$

for all π, all $x, y \in [G]$ and where we have set $h_{j,i} = f_{j,i}^* \star f_{j,i}$, where, by definition, $f_{j,i}^*(g) = \overline{f_{j,i}(g^{-1})}$. Thus, by another application of Cauchy–Schwarz, we get

$$\sum_\pi |K_{f,\pi}(x, y)| \leqslant \sum_{i=1}^k K_{h_{2,i},\mathrm{cusp}}(x, x)^{1/2} K_{h_{1,i},\mathrm{cusp}}(y, y)^{1/2}$$

for all $x, y \in [G]$ and the right-hand side is uniformly bounded (even of rapid decay).

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A.2. Relative trace formulas

We now return to the situation considered in § 3. In particular, we adopt the notation and normalization of measures of this section (thus, our global Haar measures are Tamagawa measures). For every test function $f \in \mathcal{S}(G(\mathbb{A}))$, $f' \in \mathcal{S}(G'(\mathbb{A}))$ and regular semisimple elements $\delta \in G_{\mathrm{rs}}(F)$, $\gamma \in G'_{\mathrm{rs}}(F)$, we denote by

$$O(\delta, f) := \int_{H(\mathbb{A})\times H(\mathbb{A})} f(h_1\delta h_2)\, dh_1\, dh_2,$$
$$O(\gamma, f') := \int_{H'_1(\mathbb{A})\times H'_2(\mathbb{A})} f'(h_1\gamma h_2)\eta(h_2)\, dh_2\, dh_1$$

the corresponding global orbital integrals. We also define, whenever convergent, the following expressions:

$$J(f) = \int_{[H]}\int_{[H]} K_f(h_1, h_2)\, dh_1\, dh_2, \quad f \in \mathcal{S}(G(\mathbb{A})),$$
$$I(f') = \int_{[H'_1]}\int_{[H'_2]} K_{f'}(h_1, h_2)\eta(h_2)\, dh_2\, dh_1, \quad f' \in \mathcal{S}(G'(\mathbb{A})).$$

Proposition A.2.1. (i) *Assume that $f \in \mathcal{S}(G(\mathbb{A}))$ is a nice function (see § 3.5). Then the expressions defining $J(f)$ and $O(\delta, f)$, $\delta \in G_{\mathrm{rs}}(F)$, are absolutely convergent and we have the equalities*

$$\sum_{\delta \in H(F)\backslash G_{\mathrm{rs}}(F)/H(F)} O(\delta, f) = J(f) = \sum_{\pi} J_\pi(f)$$

where the left sum is absolutely convergent and the right sum is over the set of cuspidal automorphic representations π of $G(\mathbb{A})$.

(ii) *Assume that $f' \in \mathcal{S}(G'(\mathbb{A}))$ is a nice function (see § 3.5). Then the expressions defining $I(f')$ and $O(\gamma, f')$, $\gamma \in \mathcal{B}'(F)$, are absolutely convergent and we have the equalities*

$$\sum_{\gamma \in H'_1(F)\backslash G'_{\mathrm{rs}}(F)/H'_2(F)} O(\gamma, f') = I(f') = \sum_{\Pi} 2^{-2} L(1, \eta_{E/F})^{-2} I_\Pi(f')$$

where the left sum is absolutely convergent and the right sum is over the set of cuspidal automorphic representations Π of $G'(\mathbb{A})$ whose central character is trivial on $Z_{H'_2}(\mathbb{A})$.

Proof. We only prove (ii) the proof of (i) being similar.

Set $\widetilde{G}' = G'/Z_{H'_2}$ and define $\widetilde{f}' \in \mathcal{S}(\widetilde{G}'(\mathbb{A}))$ by

$$\widetilde{f}'(\widetilde{g}) = \int_{Z'_{H_2}(\mathbb{A})} f'(z\widetilde{g})\, dz.$$

Then we have, at least formally,

$$I(f') = \int_{[H'_1]}\int_{[H'_2/Z_{H'_2}]} K_{\widetilde{f}'}(h_1, h_2)\eta(h_2)\, dh_2\, dh_1.$$

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As f' is a nice function, we have $K_{\widetilde{f'}} = K_{\widetilde{f'},\mathrm{cusp}}$. Thus, by Proposition A.1.2, it follows that the expression defining $I(f')$ is absolutely convergent and that

$$I(f') = \sum_{\Pi} \int_{[H_1']} \int_{[H_2'/Z_{H_2'}]} K_{\widetilde{f'},\Pi}(h_1, h_2)\eta(h_2)\, dh_2\, dh_1$$

where the sum is over the set of all cuspidal automorphic representations Π of $G'(\mathbb{A})$ with a central character trivial on $Z_{H_2'}(\mathbb{A})$. We would like to identify the term indexed by Π above with the global relative character $I_\Pi(f')$. However, we do not have equality on the nose because the scalar products used to define $I_\Pi(f')$ and $K_{\widetilde{f'},\Pi}$ are not the same. More precisely, $I_\Pi(f')$ is defined using the Petersson scalar product $(.,.)_{Pet}$ of § 3, whereas in the definition of $K_{\widetilde{f'},\Pi}$, we have used the scalar product

$$\begin{aligned}(\phi, \phi')_{L^2([\widetilde{G}'])} &= \int_{[\widetilde{G}']} \phi(\widetilde{g})\overline{\phi'(\widetilde{g})}d\widetilde{g} \\ &= \mathrm{vol}\left(Z_{G'}(F)Z_{H_2'}(\mathbb{A})\backslash Z_{G'}(\mathbb{A})\right)(\phi, \phi')_{Pet}.\end{aligned}$$

Thus, we get

$$\int_{[H_1']} \int_{[H_2'/Z_{H_2'}]} K_{\widetilde{f'},\Pi}(h_1, h_2)\eta(h_2)\, dh_2\, dh_1 = \mathrm{vol}\left(Z_{G'}(F)Z_{H_2'}(\mathbb{A})\backslash Z_{G'}(\mathbb{A})\right)^{-1} I_\Pi(f').$$

By (2.5.1), we have

$$\mathrm{vol}\left(Z_{G'}(F)Z_{H_2'}(\mathbb{A})\backslash Z_{G'}(\mathbb{A})\right) = \mathrm{vol}\left(E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times\right)^2 = 2^2 L(1, \eta_{E/F})^2$$

and the second equality of (ii) follows.

Since the test function f' is nice, we have

$$K_{f'}(h_1, h_2) = \sum_{\gamma \in G'_{\mathrm{rs}}(F)} f'(h_1 \gamma h_2^{-1}), \quad h_1 \in [H_1'], h_2 \in [H_2'].$$

From this, the first equality follows from formal manipulations. To justify these manipulations, we need to establish that the following expression is convergent:

(A.2.1)
$$\begin{aligned}&\int_{[H_1']} \int_{[H_2']} \sum_{\gamma \in G'_{\mathrm{rs}}(F)} |f'(h_1 \gamma h_2^{-1})|\, dh_2\, dh_1 \\ &= \sum_{\gamma \in H_1'(F)\backslash G'_{\mathrm{rs}}(F)/H_2'(F)} \int_{H_1'(\mathbb{A}) \times H_2'(\mathbb{A})} |f'(h_1 \gamma h_2^{-1})|\, dh_2\, dh_1.\end{aligned}$$

Let $\mathcal{B}_{\mathrm{rs}}$ be the GIT quotient $H_1' \backslash G'_{\mathrm{rs}}/H_2'$. It follows from Luna's étale slice theorem [31] that G'_{rs} is a $H_1' \times H_2'$-torsor over $\mathcal{B}_{\mathrm{rs}}$. Moreover, the first Galois cohomology set of the group $H_1' \times H_2'$ being trivial (by Hilbert 90), we have $\mathcal{B}_{\mathrm{rs}}(F) = H_1'(F)\backslash G'_{\mathrm{rs}}(F)/H_2'(F)$. Thus, by Proposition A.1.1 (vii) and since the function f' is Schwartz, for every $d > 0$, the expression

$$\int_{H_1'(\mathbb{A}) \times H_2'(\mathbb{A})} |f'(h_1 \gamma h_2^{-1})|\, dh_2\, dh_1$$

is essentially bounded by $\|\gamma\|_{\mathcal{B}_{\mathrm{rs}}}^{-d}$. By Proposition A.1.1 (v) for d sufficiently large, the sum

$$\sum_{\gamma \in H_1'(F)\backslash G_{\mathrm{rs}}'(F)/H_2'(F)} \|\gamma\|_{\mathcal{B}_{\mathrm{rs}}}^{-d} = \sum_{\gamma \in \mathcal{B}_{\mathrm{rs}}(F)} \|\gamma\|_{\mathcal{B}_{\mathrm{rs}}}^{-d}$$

converges. This shows that A.2.1 is convergent and ends the proof of the proposition. $\square$

A.3. Proof of Theorem 3.5.1

Let $f \in \mathcal{S}(G(\mathbb{A}))$ and $f' \in \mathcal{S}(G'(\mathbb{A}))$ be nice functions and assume that there exists a tuple $(f^{W'})_{W'}$, $f^{W'} \in \mathcal{S}(G^{W'}(\mathbb{A}))$, of nice functions matching f' such that $f^W = f$. By Theorem 3.4.4, we may assume that $f^{W'} = 0$ for almost all W'. By the trace formulas of Proposition A.2.1 applied to the functions f' and $f^{W'}$, we have

$$\sum_{W'} \sum_{\delta \in H^{W'}(F)\backslash G_{\mathrm{rs}}^{W'}(F)/H^{W'}(F)} O(\delta, f^{W'}) = \sum_{W'} \sum_{\pi_{W'}} J_{\pi_{W'}}(f^{W'}), \tag{A.3.1}$$

where the inner right sum is over the set of cuspidal automorphic representations of $G^{W'}(\mathbb{A})$, and

$$\sum_{\gamma \in H_1'(F)\backslash G_{\mathrm{rs}}'(F)/H_2'(F)} O(\gamma, f') = 2^{-2} L(1, \eta_{E/F})^{-2} \sum_{\Pi} I_{\Pi}(f') \tag{A.3.2}$$

where the right sum is over the set of cuspidal automorphic representations Π of $G'(\mathbb{A})$ whose central character is trivial on $Z_{H_2'}(\mathbb{A})$. Moreover, the correspondence 3.4.2 induces a bijection (see [57, § 2])

$$\bigsqcup_{W'} H^{W'}(F)\backslash G_{\mathrm{rs}}^{W'}(F)/H^{W'}(F) \simeq H_1'(F)\backslash G_{\mathrm{rs}}'(F)/H_2'(F)$$

and the condition that f' matches $f^{W'}$ for every W' implies that

$$O(\delta, f^{W'}) = O(\gamma, f')$$

for every elements $\delta \in G_{\mathrm{rs}}^{W'}(F)$, $\gamma \in G_{\mathrm{rs}}'(F)$, whose orbits correspond to each other via the above bijection. Therefore, the left-hand sides of A.3.1 and A.3.2 are equal, and we obtain

$$\sum_{W'} \sum_{\pi_{W'}} J_{\pi_{W'}}(f^{W'}) = 2^{-2} L(1, \eta_{E/F})^{-2} \sum_{\Pi} I_{\Pi}(f'). \tag{A.3.3}$$

Fix a maximal compact subgroup $K^{W'} = \prod_v K_v^{W'}$ of $G^{W'}(\mathbb{A})$ for every W' and let Σ be the infinite set of places v of F which split in E and where π, f and f' are unramified. By Theorem 3.4.4, we may assume that for every W' and every $v \in \Sigma$, the function $f_v^{W'}$ is unramified (i.e. it equals $\mathrm{vol}(K_v^{W'})^{-1}\mathbf{1}_{K_v^{W'}}$). Then, in the equality (A.3.3), only the $\pi_{W'}$ and the Π which are unramified at all places in Σ contribute. Define the Hecke algebra $\mathcal{H}_{G,\Sigma} = C_c(G(\mathbb{A}_\Sigma)//K_\Sigma)$ of compactly supported and K_Σ-biinvariant functions on $G(\mathbb{A}_\Sigma)$. This is the restricted tensor product over $v \in \Sigma$ of the local Hecke algebras $\mathcal{H}_{G,v} = C_c(G(F_v)//K_v)$. We define similarly the Hecke algebra $\mathcal{H}_{G',\Sigma} = C_c(G'(\mathbb{A}_\Sigma)//K'_\Sigma)$

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and the local Hecke algebras $\mathcal{H}_{G',v} = C_c(G'(F_v)//K'_v)$. Note that for every n-dimensional Hermitian space W' over E, we have an isomorphism $G(\mathbb{A}_\Sigma) \simeq G^{W'}(\mathbb{A}_\Sigma)$ canonical up to conjugation which induces a canonical isomorphism $\mathcal{H}_{G,\Sigma} \simeq C_c(G^{W'}(\mathbb{A}_\Sigma)//K_\Sigma^{W'})$. There is a base-change homomorphism $\mathcal{H}_{G',\Sigma} \to \mathcal{H}_{G,\Sigma}$, $h \mapsto h^{bc}$ and for every $v \in \Sigma$, every W' and every $h_v \in \mathcal{H}_{G',v}$, $h_v \star f'_v$ and $h_v^{bc} \star f_v^{W'}$ match each other (see [55, Proposition 2.5]). For an irreducible unitary representation Π of $G'(\mathbb{A})$ which is unramified at all places in Σ, let us denote by $h \mapsto \widehat{h}(\Pi)$ the corresponding character of the Hecke algebra $\mathcal{H}_{G',\Sigma}$. Then, for every W', every cuspidal automorphic representation $\pi_{W'}$ which is unramified at all places in Σ and every $h \in \mathcal{H}_{G',\Sigma}$ the element $h^{bc} \in \mathcal{H}_{G,\Sigma}$ acts on $\pi_{W'}^{K_\Sigma^{W'}}$ by $\widehat{h}(\mathrm{BC}(\pi_{W'}))$. Let $h \in \mathcal{H}_{G',\Sigma}$. Since the functions $h \star f'$ and $(h^{bc} \star f^{W'})_{W'}$ are nice and match each other, we can apply equality (A.3.3) to these functions to get

$$\sum_{W'} \sum_{\pi_{W'}} \widehat{h}(\mathrm{BC}(\pi_{W'})) J_{\pi_{W'}}(f^{W'}) = \sum_{\Pi} 2^{-2} L(1, \eta_{E/F})^{-2} \widehat{h}(\Pi) I_\Pi(f'). \tag{A.3.4}$$

Let $\mathrm{Irr}_{\mathrm{unit},\Sigma}(G'(\mathbb{A}))$ be the set of all irreducible unitary representations of $G'(\mathbb{A})$ which are unramified at all places in Σ. The functions $\Pi \in \mathrm{Irr}_{\mathrm{unit},\Sigma}(G'(\mathbb{A})) \mapsto \widehat{h}(\Pi)$, $h \in \mathcal{H}_{G',\Sigma}$, are bounded nowhere vanishing identically and we have $\widehat{h^*} = \overline{\widehat{h}}$ where $h^*(g) = \overline{h(g^{-1})}$. Hence, by the Stone–Weierstrass theorem, from (A.3.4), we deduce

$$\sum_{W'} \sum_{\pi_{W'}} J_{\pi_{W'}}(f^{W'}) = \sum_{\Pi} 2^{-2} L(1, \eta_{E/F})^{-2} I_\Pi(f') \tag{A.3.5}$$

where this time $\pi_{W'}$ and Π run over the sets of cuspidal automorphic representations of $G^{W'}(\mathbb{A})$ and $G'(\mathbb{A})$ such that $\mathrm{BC}(\pi_{W',v}) = \Pi_v = \mathrm{BC}(\pi_v)$ for all $v \in \Sigma$. Recall the following *automorphic-Chebotarev-density* theorem due to Ramakrishnan [43].

Theorem A.3.1 (Ramakrishnan). *Let Π_1, Π_2 be two isobaric automorphic representations of $\mathrm{GL}_d(\mathbb{A}_E)$ such that $\Pi_{1,v} \simeq \Pi_{2,v}$ for almost all places v of F that are split in E. Then, $\Pi_1 = \Pi_2$.*

As $\mathrm{BC}(\pi_{W'})$ is always isobaric, it follows from this theorem that the right-hand side of (A.3.5) reduces to $2^{-2} L(1, \eta_{E/F})^{-2} I_{\mathrm{BC}(\pi)}(f')$ and that if $\pi_{W'}$ contributes to the left-hand side, then $\mathrm{BC}(\pi_{W'}) = \mathrm{BC}(\pi)$. In particular, $\pi_{W'}$ and π belong to the same (global) Vogan L-packet. By the local Gan–Gross–Prasad conjecture (see § 2.4), and since by assumption π is tempered at all Archimedean places, we know that there is at most one abstractly $H^{W'}$-distinguished representation in this L-packet. By assumption, π is such a representation. Hence, the left-hand side of (A.3.5) reduces to $J_\pi(f)$, and this ends the proof of Theorem 3.5.1. □

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