ON THE RADICAL OF A GROUP ALGEBRA OVER A COMMUTATIVE RING

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Several people, including Wallace [4] and Passman [3], have studied the Jacobson radical of the group algebra $F[G]$ where F is a field and G is a multiplicative group. In [4], for instance, Wallace proves that if G is an abelian group with Sylow p-subgroup P and if F is a field of characteristic p, then the Jacobson radical of $F[G]$ equals the right ideal generated by the radical of $F[P]$. In this paper we shall study group algebras over arbitrary commutative rings. By a reduction to the case of a semi-simple commutative ring, we obtain Theorem 1 whose corollary contains a generalization of Wallace's theorem. Theorem 2, on the other hand, uses the first theorem to obtain results related to the main theorem of [3].

NOTATION. Throughout this paper, $J(R)$ will denote the Jacobson radical of the ring R, and $0_p(G)$ will denote the (unique) maximal normal p-subgroup of the group G.

LEMMA 1. *Let R be a commutative ring with identity and let G be a locally finite group. Let H be a subgroup of G and suppose $J(\bar{R}[G]) \subseteq J(\bar{R}[H]) \cdot \bar{R}[G]$ where $\bar{R} = R/J(R)$. Then $J(R[G]) \subseteq J(R[H]) \cdot R[G]$.*

Proof. If $\phi : R[G] \to \bar{R}[G]$ is the natural map, then clearly $\ker \phi = J[G]$ where J is the Jacobson radical of R. Since G is a locally finite group, $J[G] \subseteq J(R[G])$ by Proposition 9 of [1], and we have $J(R[G])/J[G] = J(R[G]/J[G]) \cong J(\bar{R}[G])$. It is also true that

$$J(\bar{R}[H]) \cdot \bar{R}[G] \cong (J(R[H]) \cdot R[G])/(J[G] \cdot R[G]) = (J(R[H]) \cdot R[G])/(J[G]).$$

Now let $x = \sum_{i \in I} p_i g_i \in J(R[G])$ with $p_i \in R[H]$ and $\{g_i \mid i \in I\}$ a set of right coset representatives for H in G. Then

$$\phi(x) = \sum_{i \in I} \bar{p}_i g_i \in J(\bar{R}[G]) \subseteq J(\bar{R}[H]) \cdot \bar{R}[H].$$

Thus, $\bar{p}_i \in J(\bar{R}[H]) \cong J(R[H])/J[H]$, and $p_i \in J(R[H])$, as desired.

LEMMA 2. *Let R be a commutative ring with identity and H a locally finite subgroup of a group G. Let $\{I_\nu : \nu \in \Gamma\}$ be a family of ideals of R and m be a positive integer such that $\bigcap_\nu I_\nu = 0$, and for each ν in Γ,*

(i) *$R_\nu = R/I_\nu$ has a composition series of length at most m, and*
(ii) *$J(R_\nu[G]) \subseteq J(R_\nu[H]) \cdot R_\nu[G]$.*

Then $J(R[G]) \subseteq J(R[H]) \cdot R[G]$.

Proof. Let $\{g_j \mid j \in J\}$ be a set of coset representatives for H in G with $g_1 = 1$. Then for any $x \in R[G]$ we can write $x = \sum_j \rho_j g_j$ with $\rho_j \in R[H]$. Let $\pi : R[G] \to R[H]$ be the projection map, i.e., $\pi(x) = \rho_1$. Then $\pi(J(R[G]))$ is an ideal of $R[H]$. We shall show that $\pi(J(R[G]))$

is in fact a nil ideal of $R[H]$. Let $x \in J(R[G])$, and let W be the finite group generated by $\operatorname{supp} \pi(x)$ in H. Let $\phi_\nu : R[G] \to R_\nu[G]$ be the canonical map. Since

$$\phi_\nu(x) \in J(R_\nu[G]) \subseteq J(R_\nu[H]) \cdot R_\nu[G],$$

we have $\phi_\nu(\pi(x)) \in J(R_\nu[H]) \cdot R_\nu[G] \cap R_\nu[W] \subseteq J(R_\nu[W])$. But $|W| = n < \infty$ and the R_ν have composition length $\leqq m$, so that $J(R_\nu[W])$ is nilpotent. Hence, $\pi(x)^{mn} \in \ker \phi_\nu$ for all ν in Γ. This proves that $\pi(x)^{mn} \in \cap I_\nu G = 0$ so $\pi(J(R[G]))$ is a nil ideal in $R[H]$. This puts ρ_1 in $J(R[H])$. Now suppose $\rho_j, j \neq 1$, occurs in the sum $\sum \rho_j g_j = x$. Since $x g_j^{-1}$ is another element of $J(R[G])$, we can repeat the above argument with x replaced by $x g_j^{-1}$ to get

$$\pi(x g_j^{-1}) = \rho_j \in J(R[H]).$$

Thus, $J(R[G]) \subseteq J(R[H]) \cdot R[G]$, which was to be proved.

THEOREM 1. *Let R be a commutative ring with identity of prime characteristic p and H be a subgroup of a locally finite group G such that $J(F[G]) \subseteq J(F[H]) \cdot F[G]$ for all fields F of characteristic p. Then $J(R[G]) \subseteq J(R[H]) \cdot R[G]$ and, provided H is normal in G,*

$$J(R[G]) = J(R[H]) \cdot R[G].$$

Proof. Let $\bar{R} = R/J(R)$, and let $\{I_\nu \mid \nu \in \Gamma\}$ be the collection of maximal ideals of $\bar{R}$. Then conditions (i) and (ii) of Lemma 2 are easily seen to be satisfied, and

$$J(\bar{R}[G]) \subseteq J(\bar{R}[H]) \cdot \bar{R}[G].$$

The result now follows from Lemma 1 and Lemma 4.1 of [**5**].

COROLLARY. *Let F be a field of characteristic $p > 0$ and let G be a locally finite group. Suppose H is a normal subgroup of G such that G/H has no elements of order p. Then $J(R[G]) = J(R[H]) \cdot R[G]$ for any commutative F-algebra R.*

Proof. Since G/H is a locally finite p'-group, $J(K[G]) \subseteq J(K[H]) \cdot K[G]$ for any field K of characteristic p by Theorem 16.6 of [**2**].

In [**3**] Passman proposes a candidate for the Jacobson radical of $F[G]$ when F is a field of characteristic $p > 0$ and G is a locally finite group. Define $\int(G)$ to be the subgroup of G generated by all finite, locally subnormal subgroups of G and let $\int^p(G) = [\int(G)]^p$ be the subgroup of $\int(G)$ generated by its p-elements. Then $J(F[\int^p(G)]) \cdot F[G]$ is Passman's candidate and as evidence, Passman proves that this ideal is a semiprime ideal whenever $0_p(G) = 1$. Also, if H is a normal subgroup of G and $J(F[G]) = J(F[H]) \cdot F[G]$, then Passman proves $H \supseteq \int^p(G)$. Since there are no known examples of locally finite groups G and normal subgroups H with $J(F[G]) \neq J(F[H]) \cdot F[G]$ and $H \supseteq \int^p(G)$, it makes sense to study subgroups H containing $\int^p(G)$. Therefore, we offer:

THEOREM 2. *Let K be a field of characteristic $p > 0$ and let G be a locally finite group such that $0_p(G) = 1$ and $J(K[G]) \subseteq J(K[\int^p(G)]) \cdot K[G]$. Then $J(R[G]) = J(R[H]) \cdot R[G]$ for any commutative K-algebra R and normal subgroup H of G with $\int^p(G) \subseteq H$.*

Proof. By Theorem 1 we need only prove that if F is any field of characteristic p, then $J(F[G]) \subseteq J(F[H]) \cdot F[G]$. For this, it suffices to show that $J(F[G]) \subseteq J(F[\int^p(G)]) \cdot F[G]$. This is so by virtue of Woods' theorem (Lemma 4.1 of [5]) and the fact that $\int^p(G)$ is a normal subgroup of G.

First, let F be a field contained in K. Then since G is locally finite we have $J(F[G]) \subseteq J(K[G])$; for if $x \in J(F[G])$, $x(K[G])$ is easily seen to be a nil ideal. Therefore $J(F[G]) \subseteq J(K[\int^p(G)]) \cdot K[G] \cap F[G] \subseteq J(F[\int^p(G)]) \cdot F[G]$ by Proposition 9 of [1].

Secondly, let F be a field extension of K. If F has finite degree n over K then by Theorem 16.10 of [2] and Woods' theorem we have

$$(J(F[G]))^n \subseteq F \otimes_K J(K[G]) \subseteq$$
$$F \otimes_K J\left(K\left[\int^p(G)\right]\right) \cdot K[G] \subseteq J\left(F\left[\int^p(G)\right]\right) \cdot F[G] \subseteq J(F[G]).$$

Since $J(F[\int^p(G)]) \cdot F[G]$ is a semiprime ideal, $F[G]/(J(F[\int^p(G)]) \cdot F[G])$ has no nilpotent ideals. Therefore, $J(F[G]) \subseteq J(F[\int^p(G)]) \cdot F[G]$. Next suppose F is a purely transcendental extension of K. By Theorems 16.11 and 17.10 of [2], $J(F[G]) = F \otimes_K (J(F[G]) \cap K[G]) \subseteq F \otimes_K J(K[G])$. By hypothesis, $J(K[G]) \subseteq J(K[\int^p(G)]) \cdot K[G] \subseteq J(F[\int^p(G)]) \cdot F[G]$.

Now let F be an arbitrary field extension of K. If $x \in J(F[G])$, then there exists a field L with $K \subseteq L \subseteq F$ such that L is finitely generated over K and $x \in L[G]$. Again using Theorem 16.11 of [2] we have $x \in J(F[G] \cap L[G] \subseteq J(L[G])$. But L is a finite extension of a purely transcendental extension of K, so that the above cases yield

$$x \in J(L[G]) \subseteq J\left(L\left[\int^p(G)\right]\right) \cdot L[G] \subseteq F \otimes_L J\left(L\left[\int^p(G)\right]\right) \cdot L[G] \subseteq J\left(F\left[\int^p(G)\right]\right) \cdot F[G].$$

Finally, if F is an arbitrary field of characteristic p, then let L be a field of characteristic p containing F and K. Since L is an extension of K, we know by the previous step that $J(L[G]) \subseteq J(L[\int^p(G)]) \cdot L[G]$ and by the first step, since $F \subseteq L$, that

$$J(F[G]) \subseteq J(F[\int^p(G)]) \cdot F[G].$$

This completes the proof of the theorem.

COROLLARY. *Let R be a commutative ring with identity and G be a locally finite group such that for all primes p, $0_p(G) = 1$ and $J(K[G]) \subseteq J(K[\int^p(G)]) \cdot K[G]$ for all prime fields of characteristic p. Then $J(R[G]) = J(R[H]) \cdot R[G]$ for any normal subgroup H of G with $\int(G) \subseteq H$.*

Proof. First note that $\int(G)$ contains $\int^p(G)$ for any prime p. Therefore, by the theorem $J(F[G]) \subseteq J(F[H]) \cdot F[G]$ for any field of characteristic $p > 0$. If F is a field of characteristic zero then $F[G]$ is semi-simple by Lemma 18.7 of [2], so trivially $J(F[G]) \subseteq J(F[H]) \cdot F[G]$. Now take $\{I_\nu \mid \nu \in \Gamma\}$ to be the collection of all the maximal ideals of R. Then all the hypotheses of Lemma 2 are satisfied and $J(\bar{R}[G]) \subseteq J(\bar{R}[H]) \cdot \bar{R}[G]$. Now apply Lemma 1 and Lemma 4.1 of [5].

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