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# NOTE ON MINIMUM DEGREE AND PROPER CONNECTION NUMBER

YUEYU WU, YUNQING ZHANG and YAOJUN CHEN ✉

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**Abstract**

An edge-coloured graph $G$ is called properly connected if any two vertices are connected by a properly coloured path. The proper connection number, $pc(G)$, of a graph $G$, is the smallest number of colours that are needed to colour $G$ such that it is properly connected. Let $\delta(n)$ denote the minimum value such that $pc(G) = 2$ for any 2-connected incomplete graph $G$ of order $n$ with minimum degree at least $\delta(n)$. Brause *et al.* ['Minimum degree conditions for the proper connection number of graphs', *Graphs Combin.* **33** (2017), 833–843] showed that $\delta(n) > n/42$. In this note, we show that $\delta(n) > n/36$.

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## 1. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph. Let $H$ be a subgraph of $G$ and $N_H(v)$ denote the neighbours of $v$ in $H$ and set $N_H[v] = N_H(v) \cup \{v\}$. The *degree* of $v$ is $d(v) = |N(v)|$ where $N(v)$ denotes $N_G(v)$ for simplicity. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of $G$, respectively. If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph induced by $S$. The *edge chromatic number* of $G$ is denoted by $\chi'(G)$.

Motivated by network connectivity and security issues, Borozan *et al.* [1] introduced the concept of proper connection in graphs, which is an extension of classical edge colouring of graphs. An edge-coloured graph $G$ is called *properly connected* if any two vertices $u, v \in V(G)$ are connected by a properly coloured $(u, v)$-path. The *proper connection number* $pc(G)$ is the smallest number of colours needed to colour $G$ such that it is properly connected. Note that $pc(G) \le \chi'(G) \le \Delta(G) + 1$ as a result of Vizing's theorem [4] that $\chi'(G) \le \Delta(G) + 1$ for any simple graph $G$. It is clear that a graph $G$ has $pc(G) = 1$ if and only if $G$ is a complete graph. So we assume that $G$ is an incomplete graph in the rest of this paper. Obviously, $pc(G) \ge 2$. It is a natural question to ask when $pc(G) = 2$. Borozan *et al.* [1] verified that $pc(G) = 2$ for each

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3-connected graph $G$. If $G$ is 2-connected, whether $pc(G) = 2$ remains open. Borozan *et al.* [1] posed the following conjecture.

CONJECTURE 1.1 (Borozan *et al.* [1]). Let $G$ be a 2-connected incomplete graph of order $n$. If $\delta(G) \geq 3$, then $pc(G) = 2$.

Unfortunately, Conjecture 1.1 is not true, as verified independently by Brause *et al.* [2] and Huang *et al.* [3]. Brause *et al.* [2] established a minimum degree condition for a 2-connected graph $G$ having $pc(G) = 2$.

THEOREM 1.2 (Brause *et al.* [2]). *Let $G$ be a 2-connected incomplete graph of order $n$. If $\delta(G) > \max\{2, (n + 8)/20\}$, then $pc(G) = 2$.*

While they could not determine whether the lower bound for the minimum degree is sharp, they constructed a graph $G$ of order $n$ with $\delta(G) = n/42$ such that $pc(G) \geq 3$. Let $\delta(n)$ denote the minimum value such that $pc(G) = 2$ for any 2-connected incomplete graph $G$ of order $n$ with $\delta(G) \geq \delta(n)$. From [2], $\delta(n) > n/42$. An interesting problem is to determine the exact value of $\delta(n)$.

In this paper, we show that $\delta(n)$ is greater than $n/36$, which improves the lower bound of Brause *et al.* [2]. To state our results, we first define a graph as follows: Take 18 vertex disjoint 2-connected bipartite graphs $X_1, \ldots, X_6, Y_1, \ldots, Y_6, Z_1, \ldots, Z_6$, and then add 21 new edges (coloured red) connecting the 18 bipartite graphs as shown in Figure 1. Denote the resulting graph by $H$. Obviously, $H$ is 2-connected.

THEOREM 1.3. *Let $H$ be the graph as shown Figure 1. Then $pc(H) \geq 3$.*

For any $d \geq 3$, if we take the 18 bipartite graphs as complete bipartite graph $K_{d,d}$ in the graph $H$, then $|H| = 36d$ and $\delta(H) = d$. Thus, by Theorem 1.3, we have the following result.

COROLLARY 1.4. $\delta(n) > n/36$.

## 2. Proof of Theorem 1.3

Since $H \neq K_n$, we have $pc(H) \geq 2$. Suppose to the contrary that $pc(H) = 2$. Let $c$ be a 2-edge-colouring of $H$ such that it is properly connected. Assume without loss of generality that $c(v_1v_2) = 1$. Set

$$X = \bigcup_{i=1}^{6} V(X_i), \quad Y = \bigcup_{i=1}^{6} V(Y_i), \quad Z = \bigcup_{i=1}^{6} V(Z_i);$$

$$\mathbb{X} = \bigcup_{i=2}^{5} V(X_i), \quad \mathbb{Y} = \bigcup_{i=2}^{5} V(Y_i), \quad \mathbb{Z} = \bigcup_{i=2}^{5} V(Z_i).$$

For any $x \in \mathbb{X}$, $y \in \mathbb{Y}$, $z \in \mathbb{Z}$, define

$$\mathcal{P}_x = \{P \mid P \text{ is a properly coloured } (x, u)\text{-path}, u \in Y \cup Z\},$$
$$\mathcal{Q}_y = \{P \mid P \text{ is a properly coloured } (y, u)\text{-path}, u \in X \cup Z\},$$
$$\mathcal{R}_z = \{P \mid P \text{ is a properly coloured } (z, u)\text{-path}, u \in X \cup Y\}.$$

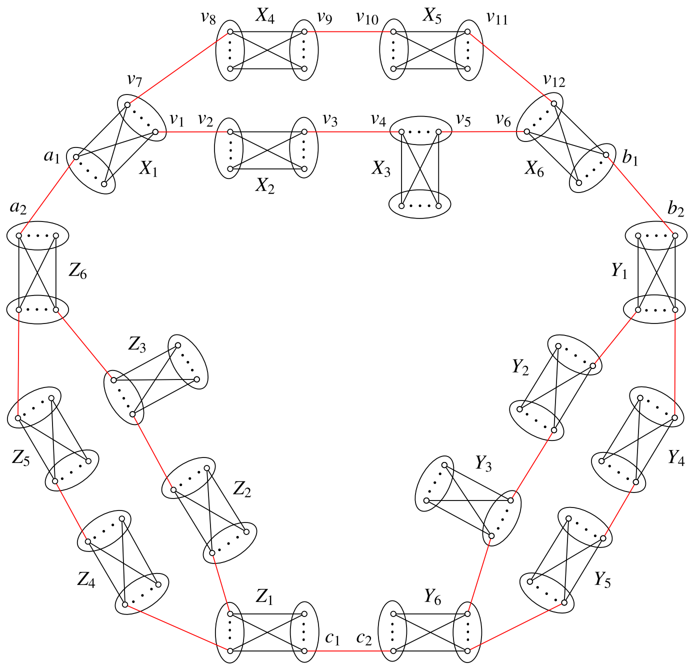


FIGURE 1. A graph $H$ with $pc(H) \geq 3$. Colour available online.

CLAIM 2.1. There is a vertex $x \in \mathbb{X}$ such that $|(\bigcup_{P \in \mathcal{P}_x} E(P)) \cap \{a_1a_2, b_1b_2\}| = 1$.

PROOF OF CLAIM 2.1. Suppose that $|(\bigcup_{P \in \mathcal{P}_x} E(P)) \cap \{a_1a_2, b_1b_2\}| = 2$ for any $x \in \mathbb{X}$. Let

$$H_1 = H[V(X_2) \cup \{v_1, v_4\}], \quad H_2 = H[V(X_3) \cup \{v_3, v_6\}],$$
$$H_3 = H[V(X_4) \cup \{v_7, v_{10}\}], \quad H_4 = H[V(X_5) \cup \{v_9, v_{12}\}],$$

and $\mathbb{H}_i = H[V(H) \backslash V(X_{i+1})]$ for $1 \leq i \leq 4$. We first show that

$$c(v_3v_4) = c(v_1v_2), \tag{2.1}$$
$$c(v_5v_6) \neq c(v_3v_4), \tag{2.2}$$
$$c(v_7v_8) = c(v_9v_{10}), \tag{2.3}$$
$$c(v_9v_{10}) = c(v_{11}v_{12}), \tag{2.4}$$

hold. If (2.$i$) does not hold for some $i$ with $1 \leq i \leq 4$, then there exists no properly coloured path connecting two vertices of degree one in $H_i$. Thus, we have established the following fact.

FACT 2.2. Any properly coloured path connecting two vertices of $V(\mathbb{H}_i)$ must lie in $\mathbb{H}_i$, where $1 \leq i \leq 4$.

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Since $\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3$ and $\mathbb{H}_4$ can be seen as symmetric in a way that takes each bipartite graph of $X_2, \ldots, X_5$, $Y_1, \ldots, Y_6$, $Z_1, \ldots, Z_6$ as a whole in $\mathbb{H}_i$ for $1 \le i \le 4$, it suffices to show that the equality (2.1) holds.

Suppose that (2.1) does not hold. Since $|(\bigcup_{P\in\mathcal{P}_x} E(P)) \cap \{a_1a_2, b_1b_2\}| = 2$ for any $x \in \mathbb{X}$, there is a vertex $u_1 \in Y \cup Z$ such that there is a properly coloured $(v_{11}, u_1)$-path $Q_1$ passing $b_1b_2$. Since $v_5v_6$ is a cut edge of $\mathbb{H}_1$, $\{v_{11}v_{12}, b_1b_2\}$ is an edge cut of $\mathbb{H}_1$ and, by Fact 2.2, $Q_1$ traverses $H[V(X_6) \cup \{v_{11}, b_2\}]$ to reach $u_1$. Since $H[V(X_6) \cup \{v_{11}, b_2\}]$ is bipartite and $b_2, v_{11}$ belong to different parts of its bipartition, $|E(v_{11}Q_1b_2)|$ is odd and hence $c(v_{11}v_{12}) = c(b_1b_2)$. Since $v_5v_6$ is a cut edge of $\mathbb{H}_1$, $H[V(X_6) \cup \{v_5, v_{11}, b_2\}]$ is bipartite and $c(v_{11}v_{12}) = c(b_1b_2)$, there exists no properly coloured $(v_5, v_{11})$-path in $H[V(X_6) \cup \{v_5, v_{11}\}]$ if $c(v_5v_6) = c(v_{11}v_{12})$, and hence there exists no properly coloured $(v_5, v_{11})$-path in $\mathbb{H}_1$. Likewise, there exists no properly coloured $(v_5, b_2)$-path in $\mathbb{H}_1$ if $c(v_5v_6) \neq c(v_{11}v_{12})$. From these remarks and Fact 2.2 and since $c$ is a 2-edge-colouring of $H$ and $\{v_{11}v_{12}, b_1b_2\}$ and $\{a_1a_2, b_1b_2\}$ are edge cuts of $\mathbb{H}_1$, it follows that $|(\bigcup_{P\in\mathcal{P}_{v_5}} E(P)) \cap \{a_1a_2, b_1b_2\}| \le 1$, a contradiction.

Therefore, (2.1)–(2.4) hold. Since $c(v_1v_2) = 1$, we have $c(v_3v_4) = 1$, $c(v_5v_6) = 2$ and $c(v_7v_8) = c(v_9v_{10}) = c(v_{11}v_{12})$. This establishes the following fact.

Fact 2.3. There is no properly coloured $(v_8, v_2)$-path in $H[V(X_1) \cup \{v_2, v_8\}]$ if $c(v_7v_8) = 1$ and no properly coloured $(v_5, v_{11})$-path in $H[V(X_6) \cup \{v_5, v_{11}\}]$ if $c(v_7v_8) = 2$.

By symmetry, assume that $c(v_7v_8) = 1$. Since $|(\bigcup_{P\in\mathcal{P}_x} E(P)) \cap \{a_1a_2, b_1b_2\}| = 2$ for any $x \in \mathbb{X}$, there exists a vertex $u_2 \in Y \cup Z$ such that there is a properly coloured $(v_8, u_2)$-path $Q_2$ passing $b_1b_2$. Since $\{a_1a_2, b_1b_2\}$ is an edge cut of $H$ and by Fact 2.3, $Q_2$ passes through $H[\bigcup_{i=4}^{6} V(X_i) \cup \{b_2\}]$ to reach $u_2$ and hence $c(b_1b_2) = 1$. Therefore, $c(v_5v_6) \neq c(b_1b_2)$ and it follows that there exists no properly coloured $(v_5, b_2)$-path in $H[V(X_6) \cup \{v_5, b_2\}]$ as $H[V(X_6) \cup \{v_5, b_2\}]$ is bipartite. Together with Fact 2.3, this shows that there exists no properly coloured $(v_5, b_2)$-path in $H[X \cup \{b_2\}]$ and hence there exists no properly coloured $(v_5, u)$-path in $H$ passing $b_1b_2$ for any $u \in Y \cup Z$ since $\{a_1a_2, b_1b_2\}$ is an edge cut of $H$. Thus, $((\bigcup_{P\in\mathcal{P}_{v_5}} E(P)) \cap \{a_1a_2, b_1b_2\}) \subseteq \{a_1a_2\}$, that is, $|(\bigcup_{P\in\mathcal{P}_{v_5}} E(P)) \cap \{a_1a_2, b_1b_2\}| \le 1$, a contradiction. □

By Claim 2.1, there is a vertex $x \in \mathbb{X}$ such that $|(\bigcup_{P\in\mathcal{P}_x} E(P)) \cap \{a_1a_2, b_1b_2\}| = 1$. By the symmetry of $\mathbb{Y}, \mathbb{Z}$ together with $\mathbb{X}$ and Claim 2.1, there are two vertices $y \in \mathbb{Y}$ and $z \in \mathbb{Z}$ with

$$\left|\left(\bigcup_{P\in\mathcal{Q}_y} E(P)\right) \cap \{c_1c_2, b_1b_2\}\right| = 1 \quad \text{and} \quad \left|\left(\bigcup_{P\in\mathcal{R}_z} E(P)\right) \cap \{a_1a_2, c_1c_2\}\right| = 1.$$

Assume without loss of generality that $(\bigcup_{P\in\mathcal{P}_x} E(P)) \cap \{a_1a_2, b_1b_2\} = \{a_1a_2\}$. Since $H$ is properly connected under $c$ and $\{a_1a_2, b_1b_2\}$ is an edge cut of $H$, any properly coloured $(x, u)$-path passes $a_1a_2$ for any $u \in Y \cup Z$, and so it passes $c_1c_2$ to reach $u$ for any $u \in Y$. Thus,

$$\left(\bigcup_{P\in\mathcal{R}_z} E(P)\right) \cap \{a_1a_2, c_1c_2\} = \{a_1a_2\} \quad \text{and} \quad \left(\bigcup_{P\in\mathcal{Q}_y} E(P)\right) \cap \{c_1c_2, b_1b_2\} = \{c_1c_2\}.$$

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Since $H$ is properly connected under $c$, $\{a_1a_2, c_1c_2\}$ is an edge cut of $H$ and $(\bigcup_{P\in\mathcal{R}_z} E(P)) \cap \{a_1a_2, c_1c_2\} = \{a_1a_2\}$. Any properly coloured $(z, u)$-path passes $a_1a_2$ for any $u \in Y$ and it follows that it passes through $b_1b_2$ to reach $u$. Consequently, $b_1b_2 \in (\bigcup_{P\in Q_y} E(P)) \cap \{c_1c_2, b_1b_2\}$, a contradiction.

This completes the proof.

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YUEYU WU, Department of Mathematics,
Nanjing University, Nanjing 210093, PR China
e-mail: yueyuw@smail.nju.edu.cn

YUNQING ZHANG, Department of Mathematics,
Nanjing University, Nanjing 210093, PR China
e-mail: yunqingzh@nju.edu.cn

YAOJUN CHEN, Department of Mathematics,
Nanjing University, Nanjing 210093, PR China
e-mail: yaojunc@nju.edu.cn