

Pseudorotations of the 2-disc and Reeb flows on the 3-sphere

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To the memory of Anatole Katok

Abstract. We use Lerman’s contact cut construction to find a sufficient condition for Hamiltonian diffeomorphisms of compact surfaces to embed into a closed 3-manifold as Poincaré return maps on a global surface of section for a Reeb flow. In particular, we show that the irrational pseudorotations of the 2-disc constructed by Fayad and Katok embed into the Reeb flow of a dynamically convex contact form on the 3-sphere.

Key words: area-preserving diffeomorphisms of the disc, contact cut, global surface of section, open book decomposition, pseudorotation

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1. Introduction

A *global surface of section* for the flow of a smooth non-singular vector field X on a closed three-dimensional manifold M is an embedded compact surface $\Sigma \subset M$ with the following properties.

- (i) Each component of the boundary $\partial\Sigma$ is a periodic orbit of X .
- (ii) The interior $\text{Int}(\Sigma)$ is transverse to X , and the orbit of X through any point in $M \setminus \partial\Sigma$ intersects $\text{Int}(\Sigma)$ in forward and backward time.

The Poincaré return map $\psi: \text{Int}(\Sigma) \rightarrow \text{Int}(\Sigma)$ sends a point $p \in \text{Int}(\Sigma)$ to the first intersection point in forward time of the flow line of X through p . In general, ψ need not extend smoothly to a diffeomorphism of Σ ; if the return time of the flow goes to

infinity as one approaches $\partial\Sigma$, the rescaled vector field with return time 2π , say, will blow up near $\partial\Sigma$.

Global surfaces of section were introduced by Poincaré in the context of celestial mechanics, allowing him to reduce the search for periodic orbits in the three-body problem to finding periodic points of the return map. The most celebrated instance of this approach is Poincaré's last geometric theorem on area-preserving twist maps of the annulus, as proved by Birkhoff; see [32, §8.2]. Hofer, Wysocki and Zehnder [21] developed holomorphic curve techniques for finding global surfaces of section for Reeb flows, and established the existence of those surfaces for Hamiltonian flows on strictly convex energy hypersurfaces in \mathbb{R}^4 . (In this context, there is an area form on the surface of section preserved by the return map.) This has provided fresh impetus for the study of the three-body problem; see [2, 3, 36] for recent applications of such global symplectic methods to this problem.

In this paper, we study what in some sense is the dual or converse problem. Our goal is to realise certain Hamiltonian diffeomorphisms of compact surfaces with boundary as the return map of a Reeb flow on a closed 3-manifold. Specifically, we are interested in achieving this for the irrational pseudorotations constructed by Fayad and Katok [12] as an application of the 'approximation by conjugation' method introduced by Anosov and Katok [4].

Definition 1.1. An irrational pseudorotation is a diffeomorphism ψ of D^2 with the following properties:

- (i) ψ is area-preserving for the standard area form of D^2 ;
- (ii) ψ has $0 \in D^2$ as a fixed point, and no other periodic points.

Here is our first main result. For the definition of dynamical convexity, see §5.2.

THEOREM 1.2. *Let $\psi: D^2 \rightarrow D^2$ be an irrational pseudorotation as constructed by Fayad and Katok. Then there is a dynamically convex contact form on the 3-sphere S^3 , inducing the standard contact structure, whose Reeb flow has a disc-like surface of section on which the return map equals $\psi|_{\text{Int}(D^2)}$.*

In particular, the Reeb flow has exactly two (simple) periodic orbits: the boundary of the surface of section, and the one corresponding to the fixed point 0 of ψ . By the work of Cristofaro-Gardiner and Hutchings [8], two is the minimum number of periodic Reeb orbits on any closed three-dimensional contact manifold. Also, our construction produces a contact open book in the sense of Giroux [16] (cf. [13, §4.4.2]): the binding is given by the boundary of the surface of section, and the pages are the translates of this surface by the Reeb flow, suitably reparametrized.

The proof of Theorem 1.2 will be given in §4, except for the statement about dynamical convexity, which will be established in §5, where we compute Conley–Zehnder indices and other invariants of the Reeb flows we construct, including the (minimal) periods of the two periodic orbits.

This leads to the following interesting corollary of Theorem 1.2. Recall that for a closed 3-manifold M with contact form α , the (prime) action spectrum of (M, α) is the subset of

\mathbb{R} made up of the (minimal) periods of the periodic Reeb orbits. The action spectrum is said to be of rank one if it is contained in a subset of the form $\{nT : n \in \mathbb{N}\}$ for some $T > 0$. Cristofaro-Gardiner and Mazzucchelli [9] established the following spectral recognition result: if the action spectrum is of rank one, then the contact form α (on a given M) is determined by its prime action spectrum.

Our examples constructed in Theorem 1.2 show that this result fails for higher ranks; cf. [9, Remark 1.6].

COROLLARY 1.3. *A contact form on S^3 with an action spectrum of rank greater than one cannot, in general, be recovered from its prime action spectrum.*

Proof. The Reeb flow on S^3 coming from an irrational pseudorotation has only two periodic orbits, and their minimal periods are rationally independent. Our computations in §5 show that a Reeb flow with the same prime action spectrum exists on a suitable irrational ellipsoid. The two Reeb flows are distinguished by the fact that the former has a dense orbit, while the latter does not. \square

Given an open book on a 3-manifold adapted to a contact structure $\ker \alpha$, the Reeb flow preserves the area form on the *interior* of the pages induced by $d\alpha$. If the Reeb flow is tangent to the binding (that is, the common boundary of the pages), this area form degenerates along the boundary. So it is to be expected that we cannot work with an embedding of a page smooth up to the boundary if we want to realise a return map preserving the standard area form. Indeed, our construction for proving Theorem 1.2 will produce a topological embedding $D^2 \hookrightarrow S^3$ smooth only on the interior of the disc. This embedding differs from a smooth embedding by a radial reparametrization of the disc, and the image is a smooth disc in S^3 . The following definition is to be understood in the same vein.

Definition 1.4. When an area-preserving diffeomorphism $\psi : \Sigma \rightarrow \Sigma$ can be realised, on $\text{Int}(\Sigma)$, as the Poincaré return map on a global surface of section for a Reeb flow on a closed 3-manifold M , we say that ψ *embeds into a Reeb flow on M* .

Remark 1.5. Theorem 1.2 is actually a corollary of the much more general Theorem 4.12 that we formulate in §4.9. We see there that any Hamiltonian diffeomorphism $\psi : \Sigma \rightarrow \Sigma$ embeds into a Reeb flow, subject to a condition on the ∞ -jet at the boundary $\partial\Sigma$ of the Hamiltonian function generating ψ .

For clarity of exposition, we proceed from the particular to the general. That is, we first prove the embeddability of Hamiltonian diffeomorphisms whose generating Hamiltonian is particularly well behaved near $\partial\Sigma$ (Proposition 1.7). We then perform a limit process to demonstrate Theorem 1.2. An inspection of that proof will yield the general result alluded to above.

The condition on the ∞ -jet of the Hamiltonian can be verified directly, so it applies to Hamiltonian functions that do not necessarily arise as a limit of ‘well-behaved’ Hamiltonians, as is the case in the Fayad–Katok examples.

The pseudorotations of Fayad and Katok have precisely three ergodic invariant measures: the Lebesgue measure on the disc, the δ -measure at the fixed point, and the

Lebesgue measure on the boundary. Thus, the Reeb flows we construct are in some sense as exotic as possible. However, even disregarding the two periodic orbits, the Reeb flow will not be minimal, since by the work of Le Calvez and Yoccoz [29] there will always be other non-dense orbits. We refer the reader to [12, §3.1] for further historical comments. Concerning the minimality issue, see also the discussion in [15].

Remark 1.6. In [28], Katok describes a perturbation construction of integrable degenerate Hamiltonian systems into systems generating an ergodic flow on each energy hypersurface with a small number of periodic orbits. By applying his main result [28, Theorem A] to the periodic Reeb flow of the standard contact form on S^3 , one arrives at a Reeb flow which is ergodic and has precisely two periodic orbits.

Another construction of ‘exotic’ Reeb flows is mentioned in [21, p. 200]. In private communication to those authors, M. Herman has constructed hypersurfaces in \mathbb{R}^4 that are C^∞ -close to an irrational ellipsoid and admit precisely two periodic orbits, but have a Reeb flow with a dense orbit.

From the viewpoint of contact homology, dynamically convex contact forms inducing the standard contact structure on S^3 , and whose Reeb flow has precisely two periodic orbits, have been studied by Bourgeois, Cieliebak and Ekholm in [5]. They mention that in the context of their main theorem, there is a disc-like global surface of section on which the return map has a single fixed point and no further periodic points, but they leave open the question of whether pseudorotations are actually realised in this way.

For a recent survey on global surfaces of section for Reeb flows, see [26].

The embedding of the Fayad–Katok pseudorotations into a Reeb flow on a closed manifold may pave the way to using global symplectic methods for studying these pseudorotations. For recent applications of pseudoholomorphic curves methods to the study of pseudorotations, see [6, 7].

The irrational pseudorotations of Fayad and Katok are C^∞ -limits

$$\lim_{v \rightarrow \infty} \varphi_v \circ \mathfrak{R}_{p_v/q_v} \circ \varphi_v^{-1}$$

of conjugates of 2π -rational rotations \mathfrak{R}_{p_v/q_v} , where the conjugating maps φ_v are area-preserving diffeomorphisms of D^2 that are the identity on a small and, for $v \rightarrow \infty$, shrinking neighbourhood of the boundary ∂D^2 . We describe these pseudorotations in more detail later. In order to prove Theorem 1.2, we first establish the analogous statement for area-preserving diffeomorphisms of D^2 that equal a rigid rotation near the boundary. Such a result is essentially contained in [27] or [1, §3]. We present an alternative proof that relies on the notion of contact cuts in the sense of Lerman [30]. This will be the key to embedding area-preserving diffeomorphisms of D^2 with much weaker boundary conditions into Reeb flows on S^3 .

Contact cuts provide the natural language for constructing contact *forms* on manifolds obtained from a manifold with boundary by collapsing the orbits of an S^1 -action on the boundary, allowing one to control the Reeb dynamics on such quotients. Therefore the cut construction is ideally suited for formulating the general condition on a Hamiltonian diffeomorphism to embed into a Reeb flow. For a brief introduction to contact cuts in the context of Reeb dynamics, see [14].

As an instructive first step towards the general result, with this approach one easily sees how one can relax the condition that the diffeomorphism be a rigid rotation near the boundary, as in the following proposition.

PROPOSITION 1.7. *Let ψ be the time 2π map of a Hamiltonian isotopy of D^2 generated by a 2π -periodic Hamiltonian function $H_s: D^2 \rightarrow \mathbb{R}$, $s \in \mathbb{R}/2\pi\mathbb{Z}$. If the Hamiltonian function is autonomous on a collar neighbourhood of ∂D^2 and depends only on the radial coordinate in that neighbourhood, then ψ embeds into a Reeb flow on S^3 .*

This proposition will be given a short proof in §2, after a discussion of contact cuts and their relation to contact open books.

In order to use Proposition 1.7 for proving Theorem 1.2, in particular for the limit process in the Fayad–Katok construction, we need to write the area-preserving diffeomorphisms under consideration in a canonical fashion as the time 2π map of a non-autonomous Hamiltonian function. This is done in §3. The discussion there includes a proof of the following result, which is probably folklore.

THEOREM 1.8. *The space $\text{Diff}_c(D^2, \omega)$ of area-preserving diffeomorphisms of D^2 with compact support in the interior $\text{Int}(D^2)$ has $\{\text{id}_{D^2}\}$ as a strong deformation retract.*

In Proposition 4.13 we see that if $\psi: D^2 \rightarrow D^2$ embeds into a Reeb flow, then so does its conjugate $\varphi^{-1} \circ \psi \circ \varphi$ under any area-preserving diffeomorphism φ of D^2 . Strictly speaking, the embeddability property has to be formulated for a pair (H_s, λ) , where λ is a primitive of the area form on D^2 . In §4.11 we see that, at least up to C^2 -differentiability, the choice of primitive is irrelevant.

2. Contact open books as contact cuts

In this section we prove Proposition 1.7. We begin by describing the cut construction and how it can be used to construct open book decompositions. We then define a contact form on the solid torus $S^1 \times D^2$ whose Reeb flow gives the solid torus the structure of a mapping torus of (D^2, ψ) , where ψ is the given Hamiltonian diffeomorphism. The desired contact form on S^3 is then produced by a contact cut.

2.1. Open books via the cut construction. An open book decomposition of a 3-manifold M consists of a link $B \subset M$, called the *binding*, and a smooth, locally trivial fibration $p: M \setminus B \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$. It is assumed that p is well behaved near the binding. By this we mean that one can find a tubular neighbourhood $B \times D^2$ of B in M on which the map p is given by the angular coordinate in the D^2 -factor. The closures Σ_s of the fibres $p^{-1}(s)$, $s \in S^1$, are called the *pages*. The binding is the common boundary of the pages.

Every closed, orientable 3-manifold admits an open book decomposition; see [35].

The vector field ∂_s on S^1 lifts to a vector field on $M \setminus B$ that coincides near B with the angular vector field on the D^2 -factor of $B \times D^2$. The time 2π flow of this vector field defines a diffeomorphism ψ of $\Sigma := \Sigma_0$ to itself, equal to the identity near the boundary $\partial\Sigma = B$. This diffeomorphism is called the *monodromy* of the open book.

Conversely, an open book can be built starting from a compact surface Σ with boundary, and a diffeomorphism ψ of Σ that equals the identity near the boundary. This construction is well known; see [13, §4.4.2]. Here we interpret it as a cut construction in the sense of Lerman [30] (cf. [31, Remark 5.6] and [11, §2.2.3]).

This construction starts with the mapping torus

$$V := \Sigma \times [0, 2\pi] / (x, 2\pi) \sim (\psi(x), 0)$$

of (Σ, ψ) . The boundary of V is $\partial V = \partial \Sigma \times S^1$. Write θ , by a slight abuse of notation, for the S^1 -coordinate on the components of the boundary $\partial \Sigma$, and s for the S^1 -coordinate on V given by the projection onto the second factor.

Consider the S^1 -action on the boundary ∂V of the mapping torus generated by the vector field $\partial_s - h\partial_\theta$, where h is an integer. If ∂V has several components $\partial_i V$, $i = 1, \dots, k$, one may choose an integer h_i for each component. Let $M := V / \sim$ be the quotient space obtained by identifying points on ∂V that lie on the same S^1 -orbit. The idea of Lerman’s cut construction is to identify this seemingly singular quotient space with the quotient of a larger manifold under a free S^1 -action. In the present setting, the details will be given in the following proposition and its proof; for the general construction, see [30].

PROPOSITION 2.1. *The space $M = V / \sim$ is a smooth closed 3-manifold. It carries the structure of an open book with binding $B := (\partial V / \sim) \cong \partial \Sigma$ and projection map*

$$p: M \setminus B = \text{Int}(V) \longrightarrow S^1$$

given by the s -coordinate. The monodromy of the open book equals the composition of ψ with an h_i -fold right-handed Dehn twist along a curve parallel to the boundary circle $\partial_i V$, $i = 1, \dots, k$.

Remark 2.2. On a first reading, the details of the following proof may be skipped. The relation between contact cuts and open books is of general interest from the topological viewpoint. For the proof of Theorem 1.2, the key point is the coordinate description (4) near the binding, which will be used for the analysis of smoothness issues, starting in §4.3.

Proof of Proposition 2.1. Since we have to consider the components of ∂V separately, we may as well pretend that ∂V is connected. Write $(-\varepsilon, 0] \times \partial \Sigma$ for a collar neighbourhood of $\partial \Sigma$ in Σ on which ψ acts as the identity. Then

$$V_\varepsilon := (-\varepsilon, 0] \times \partial \Sigma \times S^1$$

is a collar neighbourhood of ∂V in V . We think of V_ε as a subset of the open bicollar

$$N := (-\varepsilon, \varepsilon) \times \partial \Sigma \times S^1.$$

Lift the S^1 -action on $\partial V = \partial \Sigma \times S^1$ in the obvious way to an S^1 -action on N . Then the function $N \rightarrow (-\varepsilon, \varepsilon)$ assigning to each point $(\tau, \theta, s) \in N$ its bicollar parameter τ is smooth and S^1 -invariant, and its 0-level set ∂V is regular. The function

$$\begin{aligned} \mu: \quad N \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (\tau, \theta, s; z) &\longmapsto \tau + |z|^2 \end{aligned} \tag{1}$$

is invariant under the anti-diagonal S^1 -action

$$e^{i\varphi}(\tau, \theta, s; z) := (\tau, \theta - h\varphi, s + \varphi; e^{-i\varphi}z), \tag{2}$$

and $\mu^{-1}(0)$ is a regular level set on which the S^1 -action is free. It follows that $\mu^{-1}(0)/S^1$ is a smooth manifold.

Observe that $\mu^{-1}(0) = P \times \partial V$, where P is the paraboloid

$$P := \{(\tau, z) \in (-\varepsilon, \varepsilon) \times \mathbb{C} : \tau = -|z|^2\}.$$

The S^1 -action on the P -factor is free away from the apex $(0, 0)$, which is a fixed point of the action. It follows that taking the quotient of $\mu^{-1}(0)$ under the S^1 -action is the same as forming the quotient space V_ε / \sim . Thus, $M = V / \sim$ is a smooth manifold. The homeomorphism

$$(V_\varepsilon / \sim) \longrightarrow \mu^{-1}(0)/S^1$$

induced by

$$\begin{aligned} V_\varepsilon &\longrightarrow \mu^{-1}(0) \\ (\tau, \theta, s) &\longmapsto (\tau, \theta, s; \sqrt{-\tau}) \end{aligned} \tag{3}$$

defines the smooth manifold structure of M near $B = \partial V / \sim$.

The manifold $\mu^{-1}(0)/S^1$ is diffeomorphic to $\partial\Sigma \times \text{Int}(D^2_{\sqrt{\varepsilon}})$, which can be seen as follows. Consider the differentiable map

$$\begin{aligned} \mu^{-1}(0) &\longrightarrow \partial\Sigma \times \text{Int}(D^2_{\sqrt{\varepsilon}}) \\ (\tau, \theta, s; z) &\longmapsto (\theta + hs, e^{is}z). \end{aligned}$$

Notice that on the left-hand side, τ is determined by $\tau = -|z|^2$. Points on the same orbit of the S^1 -action (2) have the same image, so the map descends to

$$\mu^{-1}(0)/S^1 \longrightarrow \partial\Sigma \times \text{Int}(D^2_{\sqrt{\varepsilon}}).$$

This induced map is a diffeomorphism with inverse map

$$\begin{aligned} \partial\Sigma \times \text{Int}(D^2_{\sqrt{\varepsilon}}) &\longrightarrow \mu^{-1}(0)/S^1 \\ (b, \rho e^{i\vartheta}) &\longmapsto [(-\rho^2, b - h\vartheta, \vartheta; \rho)]. \end{aligned} \tag{4}$$

This map is well defined even for $\rho = 0$, since the points $(0, b - h\vartheta, \vartheta; 0)$ precisely make up the S^1 -orbit through the point $(0, b, 0; 0)$ as ϑ varies over S^1 .

This diffeomorphism identifies $B = \partial V / \sim$ with $\partial\Sigma \times \{0\}$. The S^1 -valued function

$$(\tau, \theta, s; z) \longmapsto s + \arg z$$

on $\mu^{-1}(0) \setminus \{z = 0\}$ is S^1 -invariant, and under the identification

$$\text{Int}(V_\varepsilon) \cong (\mu^{-1}(0) \setminus \{z = 0\})/S^1$$

coming from (3), this function coincides with s , that is, the fibration \mathfrak{p} defining the open book. On the other hand, under the identification

$$\partial\Sigma \times (\text{Int}(D^2_{\sqrt{\varepsilon}}) \setminus \{0\}) \cong (\mu^{-1}(0) \setminus \{z = 0\})/S^1$$

coming from (4), that function coincides with ϑ , that is, the angular coordinate in the disc factor.

It remains to determine the monodromy. On the mapping torus V , the monodromy ψ is the return map on $\Sigma \times \{0\}$ given by the flow $[(x, 0)] \mapsto [(x, t)]$ at time 2π . On the collar V_ε , this flow is given by

$$(\tau, \theta, s) \mapsto (\tau, \theta, s + t),$$

and the return map is the identity. On the other hand, on the neighbourhood $\partial\Sigma \times \text{Int}(D^2_{\sqrt{\varepsilon}})$ of the binding, the monodromy should also be the identity near $\rho = 0$, realised as the time 2π map of the flow

$$(b, \rho e^{i\vartheta}) \mapsto (b, \rho e^{i(\vartheta+t)})$$

in an angular direction along the disc factor. Under the identification of

$$\partial\Sigma \times (\text{Int}(D^2_{\sqrt{\varepsilon}}) \setminus \{0\})$$

with $\text{Int}(V_\varepsilon)$, this flow becomes (near $\tau = 0$)

$$(\tau, \theta, s) \mapsto (\tau, \theta - ht, s + t).$$

This implies that the monodromy on V_ε has to be of the form

$$(\tau, \theta, s) \mapsto (\tau, \theta + \chi(\tau)t, s + t),$$

where χ interpolates smoothly between 0 near $\tau = -\varepsilon$ and $-h$ near $\tau = 0$. This amounts to an h -fold right-handed Dehn twist along a θ -circle, that is, a boundary parallel curve. □

2.2. Hamiltonian disc maps and contact forms. The mapping torus of any orientation-preserving diffeomorphism ψ of the closed unit disc D^2 is a copy of the solid torus $S^1 \times D^2$. Our aim in this section is to construct contact forms on $S^1 \times D^2$, starting from a diffeomorphism ψ that arises as the time 2π map of a non-autonomous Hamiltonian. This construction is standard; see [1]. Much of our discussion generalizes to Hamiltonian diffeomorphisms of arbitrary compact, oriented surfaces with boundary. We restrict attention to the 2-disc, since this is the case that will interest us later when we construct Reeb flows on S^3 , and it allows us to work with global coordinates.

Write (r, θ) for polar coordinates on the closed unit 2-disc D^2 . As area form on D^2 we take $\omega := 2r \, dr \wedge d\theta$, with primitive 1-form $\lambda := r^2 \, d\theta$. Let $H_s, s \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, be a 2π -periodic Hamiltonian function on D^2 . Throughout the present section, the following assumption, which is part of the hypotheses in Proposition 1.7, will be understood.

ASSUMPTION 2.3. *There is a neighbourhood of the boundary ∂D^2 in D^2 on which H_s depends only on the radial coordinate r , not on θ or the ‘time’ parameter s .*

The Hamiltonian vector field X_s is defined by

$$\omega(X_s, \cdot) = dH_s.$$

This is the sign convention of [1, 32], and it is the one which is convenient in the present context. By our assumption on H_s , the vector field X_s will be a multiple of the angular

vector field ∂_θ near the boundary of $S^1 \times D^2$. Without changing the Hamiltonian vector field, we may assume that H_s is as large as we like, and that

$$H_s|_{\partial D^2} =: h \in \mathbb{N}, \tag{5}$$

by adding a positive constant to the Hamiltonian function.

LEMMA 2.4. *For H_s sufficiently large, the 1-form*

$$\alpha := H_s ds + \lambda$$

is a positive contact form on $S^1 \times D^2$. Specifically, the condition for α to be a positive contact form is given by

$$H_s + \lambda(X_s) > 0. \tag{6}$$

Proof. We compute

$$\begin{aligned} \alpha \wedge d\alpha &= (H_s ds + \lambda) \wedge (dH_s \wedge ds + \omega) \\ &= ds \wedge (H_s \omega + \lambda \wedge dH_s). \end{aligned}$$

By adding a large constant to the Hamiltonian function, we can make the first summand in parentheses large without changing the second summand.

A word on notation is in order. When we write dH_s , we mean the differential of the function $H_s : D^2 \rightarrow \mathbb{R}$ for a fixed value of the parameter s , that is, there is no summand $(\partial H_s / \partial s) ds$.

With this understood, we have the identity

$$\lambda \wedge dH_s = \lambda(X_s) \cdot \omega,$$

which can be verified by taking the interior product with X_s on both sides. (At points where $X_s = 0$, the 2-forms on either side vanish.) It follows that the contact condition for α is equivalent to (6). □

Remark 2.5. Since λ equals the interior product of ω with $r\partial_r/2$, we have $\lambda(X_s) = -dH_s(r\partial_r/2)$, so the contact condition (6) can equivalently be written as

$$r \frac{\partial H_s}{\partial r} < 2H_s. \tag{6'}$$

LEMMA 2.6. *When the contact condition (6) is satisfied, the vector field*

$$R := \partial_s + X_s$$

equals, up to positive scale, the Reeb vector field of α .

Proof. We have

$$i_R d\alpha = i_R(dH_s \wedge ds + \omega) = -dH_s + dH_s = 0$$

and

$$\alpha(R) = H_s + \lambda(X_s),$$

so the contact condition (6) is the same as $\alpha(R) > 0$. □

LEMMA 2.7. *On a collar neighbourhood of $\partial(S^1 \times D^2)$ in $S^1 \times D^2$ where H_s depends only on r and $\partial H_s/\partial s = 0$, the contact form α is invariant under the S^1 -action generated by the vector field $Y := \partial_s - h\partial_\theta$.*

Proof. The Lie derivative of α with respect to Y is, by the Cartan formula,

$$\begin{aligned} L_Y\alpha &= d(\alpha(Y)) + i_Y d\alpha \\ &= d(H_s - hr^2) + i_Y(dH_s \wedge ds + \omega). \end{aligned}$$

Beware that in the first summand we also get a term $(\partial H_s/\partial s) ds$, but this term vanishes on a collar neighbourhood of the boundary. In that neighbourhood, where H_s depends only on r , we have $i_Y(dH_s \wedge ds) = -dH_s$. Then all terms in the expression for $L_Y\alpha$ cancel in pairs. □

An S^1 -action that preserves the contact form, not just the contact structure, is called a *strict contact S^1 -action*.

2.3. *Contact cuts.* Recall from [13, §7.7] that for a strict contact S^1 -action on a contact manifold (N, α) generated by a vector field Y , the *momentum map* $\mu_N: N \rightarrow \mathbb{R}$ is defined as $\mu_N = \alpha(Y)$. From the identity

$$d\mu_N = d(\alpha(Y)) = L_Y\alpha - i_Y d\alpha = -i_Y d\alpha \tag{7}$$

it follows that the vector field Y is tangent to the level sets of μ_N . We also see that the level set $\mu_N^{-1}(0)$ is regular if and only if Y is nowhere zero along this level. In that case, the S^1 -action is locally free on the 0-level. If the action is free, α induces a contact form on the quotient $\mu_N^{-1}(0)/S^1$. This process is known as *contact reduction*. By (7), the Reeb vector field of α is likewise tangent to the level sets of μ , and it descends to the Reeb vector field of the contact form on the reduced manifold.

The *contact cut*, introduced by Lerman [30], produces a contact form on the manifold obtained from the bounded manifold $\mu_N^{-1}([0, \infty))$ by collapsing the S^1 -orbits on the boundary $\mu_N^{-1}(0)$. Again, it is assumed that the S^1 -action is free on $\mu_N^{-1}(0)$. This contact cut is constructed as follows. Consider the contact manifold

$$(N \times \mathbb{C}, \alpha + x dy - y dx),$$

with circle action generated by $Y - (x\partial_y - y\partial_x)$. The momentum map of this S^1 -action is

$$\mu(p, z) = \mu_N(p) - |z|^2, \quad (p, z) \in N \times \mathbb{C}. \tag{8}$$

Then the reduced contact manifold $\mu^{-1}(0)/S^1$ is the desired cut.

Write $\pi: \mu^{-1}(0) \rightarrow \mu^{-1}(0)/S^1$ for the projection onto the orbit space. The contact form $\bar{\alpha}$ on the quotient is characterized by

$$\pi^*\bar{\alpha} = (\alpha + x dy - y dx)|_{T(\mu^{-1}(0))}.$$

It follows that the composition of maps

$$\begin{array}{ccc} \{p \in N : \mu_N(p) > 0\} & \longrightarrow & \mu^{-1}(0) \\ p & \longmapsto & (p, \sqrt{\mu_N(p)}) \end{array} \xrightarrow{\pi} \begin{array}{ccc} \mu^{-1}(0)/S^1 & & \\ [(p, \sqrt{\mu_N(p)})] & & \end{array} \tag{9}$$

is an equidimensional strict contact embedding.

Likewise, the embedding

$$\begin{array}{ccc} \mu_N^{-1}(0) & \longrightarrow & \mu^{-1}(0) \\ p & \longmapsto & (p, 0) \end{array}$$

induces a codimension-2 strict contact embedding of reduced manifolds,

$$\mu_N^{-1}(0)/S^1 \longrightarrow \mu^{-1}(0)/S^1.$$

2.4. *Disc maps and contact cuts.* We now combine the themes of the two preceding sections. Start with the solid torus $V = S^1 \times D^2$ with contact form $\alpha = H_s ds + \lambda$, subject to the contact condition (6). (If you prefer, you may work with a slight thickening N of the bounded manifold V , but this is not essential.) As before, we choose a Hamiltonian function H_s that satisfies Assumption 2.3 and Condition (5).

Then the vector field $Y := \partial_s - h\partial_\theta$ generates a strict contact S^1 -action near the boundary ∂V . Along this boundary, the momentum map $\mu_V = \alpha(Y) = H_s - hr^2$ takes the value zero.

LEMMA 2.8. *Subject to the contact condition (6'), the boundary ∂V is a regular component of the 0-level set of the momentum map μ_V .*

Proof. The contact condition gives

$$\left. \frac{\partial H_s}{\partial r} \right|_{\{s\} \times \partial D^2} < 2h, \tag{10}$$

which implies $d\mu_V(\partial_r) < 0$ along ∂V . □

Remark 2.9. The contact condition implies $\mu_V > 0$ on the interior of V near ∂V . So the definition of the function μ in (1) accords with the one in (8) up to a global minus sign.

LEMMA 2.10. *The manifold $(S^1 \times D^2)/\sim$ obtained by collapsing the orbits of $Y = \partial_s - h\partial_\theta$ along the boundary $\partial(S^1 \times D^2)$ is the 3-sphere S^3 .*

Proof. The map

$$\begin{array}{ccc} S^1 \times D^2 & \longrightarrow & S^3 \subset \mathbb{C}^2 \\ (s; r, \theta) & \longmapsto & (\sqrt{1-r^2} e^{is}, r e^{i(\theta+hs)}) \end{array}$$

is an explicit description of the quotient map. Observe that the quotient map is not differentiable in $r = 1$. Thus, strictly speaking, we have shown only that the quotient is homeomorphic to S^3 . Thanks to the existence and uniqueness of differential structures on topological 3-manifolds, this is not something to worry about. See also the following remark.

Alternatively, one may appeal to Proposition 2.1. This says that $(S^1 \times D^2)/\sim$ is an open book with page D^2 and monodromy an h -fold right-handed Dehn twist along a curve parallel to ∂D^2 . This monodromy is isotopic to the identity, so the manifold is S^3 (cf. [13, §4.4.2]). □

Remark 2.11. The quotient map in the proof is obtained by parametrizing the closed northern hemisphere S^2_+ of $S^2 = S^3 \cap (\mathbb{R} \times \mathbb{C})$ as the graph of the map $z \mapsto \sqrt{1 - |z|^2}$ on the closed unit disc in the equatorial plane $\{0\} \times \mathbb{C}$, and then rotating the graph under the S^1 -action $e^{is}(z_1, z_2) = (e^{is}z_1, e^{ihs}z_2)$.

If instead we parametrize S^2_+ by the stereographic projection of the equatorial unit disc from the south pole, we obtain the smooth quotient map

$$(s; r, \theta) \mapsto \left(\frac{1 - r^2}{1 + r^2} e^{is}, \frac{2r}{1 + r^2} e^{i(\theta + hs)} \right).$$

There are other quotient maps one could consider, and in what follows we choose one that is adapted to the contact form in question. The options correspond to different choices of the collar parameter in the cut construction. We elaborate on this issue in §4.3.

Now consider the contact form $\alpha = H_s ds + \lambda$ on $S^1 \times D^2$, subject to the contact condition (6). The contact cut construction yields a contact form $\bar{\alpha}$ on $S^3 = (S^1 \times D^2)/\sim$.

LEMMA 2.12. *The contact structure $\ker \bar{\alpha}$ on S^3 is diffeomorphic to the standard tight contact structure.*

Proof. By the contact condition (10), on a collar neighbourhood V_ε of ∂V in $V = S^1 \times D^2$, the function $H_s - hr^2$ is strictly monotonically decreasing in r , and by (5) it takes the value zero on the boundary. In particular, the function is positive on $V_\varepsilon \setminus \partial V$. Consider the map

$$\begin{aligned} V_\varepsilon &\longrightarrow \mathbb{C}^2 \\ (s; r, \theta) &\longmapsto (\sqrt{H_s - hr^2} e^{is}, r e^{i(\theta + hs)}). \end{aligned}$$

This, too, is a model for the quotient map $V \rightarrow V/\sim$ near ∂V . Again, the map is not smooth, but its image is a piece of a smooth star-shaped hypersurface in \mathbb{C}^2 . (We expand on this point in Remark 2.13.)

The pull-back of the standard Liouville 1-form $\lambda_{\mathbb{R}^4} = r_1^2 d\theta_1 + r_2^2 d\theta_2$ under this map equals $\alpha|_{V_\varepsilon}$. So the restriction of $\lambda_{\mathbb{R}^4}$ to the hypersurface describes the contact form on the contact cut $S^3 = V/\sim$ near the circle $\partial V/\sim$. Notice that the quotient map identifies this circle with the unit circle in $\{0\} \times \mathbb{C}$, regardless of the choice of H_s .

For the contact form $(1 + (h - 1)r^2) ds + r^2 d\theta$, the quotient map is the one in the proof of Lemma 2.10, with image $S^3 \subset \mathbb{R}^4$. The contact condition (6') is convex in H_s . Thus, the convex linear interpolation between the given H_s and $1 + (h - 1)r^2$ (and the corresponding interpolation of star-shaped hypersurfaces) induces a smooth homotopy of contact forms on S^3 . The result then follows from Gray stability [13, Theorem 2.2.2]. □

Remark 2.13. As promised, here is the argument for why the image of V_ε under the quotient map is a smooth star-shaped hypersurface in \mathbb{C}^2 . Smoothness is only an issue

near $r = 1$. There, by the inverse function theorem, r is a smooth function of $H_s - hr^2$. This means that the points

$$(\sqrt{H_s - hr^2} e^{is}, r)$$

on the hypersurface, corresponding to $\theta = hs$, form a smooth surface of revolution, with $\sqrt{H_s - hr^2}$ playing the part of the radius, and r a function of that radius squared.

The three-dimensional hypersurface is then obtained by rotating this surface under the S^1 -action $e^{i\varphi}(z_1, z_2) = (z_1, e^{i\varphi}z_2)$; in other words, we think of θ as $\theta = hs + \varphi$. Since $|z_2| \in (1 - \varepsilon, 1]$ is bounded away from zero on the surface, this rotation produces a smooth hypersurface.

To see that the hypersurface is star-shaped with respect to the origin in \mathbb{C}^2 , it is enough to observe that the value of $|z_2|$ of image points increases with increasing r (and s, θ fixed), while that of $|z_1|$ decreases by the contact condition (10). Alternatively, one can reach the same conclusion by observing that $\lambda_{\mathbb{R}^4}$ pulls back to a contact form on the hypersurface. Since the radial vector field $(r_1\partial_{r_1} + r_2\partial_{r_2})/2$ is a Liouville vector field for $\omega_{\mathbb{R}^4} = d\lambda_{\mathbb{R}^4}$, the hypersurface must be transverse to the radial vector field.

2.5. Proof of Proposition 1.7. The Reeb vector field of $\alpha + x dy - y dx$, which is simply the pull-back of the Reeb vector field R_α from $V = S^1 \times D^2$ to $N \times \mathbb{C}$, descends to the Reeb vector field of the induced contact form on the 3-sphere $(S^1 \times D^2)/\sim$. On $\text{Int}(V)$, this coincides with the old Reeb vector field R_α by the strict contact embedding (9), and hence up to positive scale with $R = \partial_s + X_s$ by Lemma 2.6. So the inclusion $\{0\} \times D^2 \subset S^1 \times D^2$ descends to the desired embedding $D^2 \hookrightarrow S^3$, smooth on $\text{Int}(D^2)$.

On ∂V we have $R = \partial_s + a\partial_\theta$ for some $a \in \mathbb{R}$. The contact condition (6), which we have seen to be equivalent to $\alpha(R) > 0$, translates along the boundary into $h + a > 0$. When we take the quotient of ∂V with respect to the S^1 -action generated by $Y = \partial_s - h\partial_\theta$, the vector field R descends to $(h + a)\partial_\theta$ on $\partial V/S^1 = \partial D^2$. The time 2π flow of this vector field coincides with that of R , regarded as a map of $\{0\} \times \partial D^2$ to itself.

This completes the proof of Proposition 1.7.

2.6. Contact structures supported by open books. Let M be a closed, oriented 3-manifold with an open book decomposition $\mathfrak{p}: M \setminus B \rightarrow S^1$. The standard orientation of S^1 defines a co-orientation of the fibres $\mathfrak{p}^{-1}(s)$; with the orientation of M this determines the positive orientation of the pages. The binding B is endowed with the orientation as boundary of the pages.

A contact structure $\xi = \ker \alpha$ on M is said to be *supported* by the open book if the following compatibility conditions are satisfied.

- (o) The 3-form $\alpha \wedge d\alpha$ is a positive volume form on M .
- (i) The 2-form $d\alpha$ induces a positive area form on each fibre of \mathfrak{p} .
- (ii) The 1-form α is positive on each component of the link B .

As shown by Giroux [16], every contact structure on a closed, oriented 3-manifold is supported by an open book.

The contact form on S^3 constructed in the proof of Proposition 1.7 is supported by an open book with disc-like pages. Condition (i) is guaranteed by the transversality of R

to the disc factor in $\text{Int}(S^1 \times D^2)$. The orientation condition in (ii) is satisfied thanks to $h + a > 0$.

3. *Area-preserving diffeomorphisms of the disc*

In this section we aim to describe how to make a sufficiently canonical choice of Hamiltonian function generating any given area-preserving diffeomorphism of D^2 compactly supported in the interior. This, as mentioned earlier, is essential for giving us the necessary control over the limit process in the Fayad–Katok construction.

As before, we fix the area form $\omega = 2r \, dr \wedge d\theta$ on D^2 with primitive 1-form $\lambda = r^2 \, d\theta$. Write $\text{Diff}_c(D^2, \omega)$ for the group of area-preserving diffeomorphisms of D^2 with compact support in $\text{Int}(D^2)$. Similarly, $\text{Diff}_c(D^2)$ denotes the group of all diffeomorphisms with the same condition on their support. The space of area forms on D^2 of total area 2π , and which coincide with ω near ∂D^2 , will be denoted by $\Omega_c(D^2)$.

By Moser stability [32, Theorem 3.2.4], we have a Serre fibration

$$\begin{array}{ccc} \text{Diff}_c(D^2, \omega) & \longrightarrow & \text{Diff}_c(D^2) \\ & & \downarrow f \\ & & \Omega_c(D^2) \\ & & \uparrow f^* \omega \end{array}$$

cf. [18, Lemma 1.1]. The base $\Omega_c(D^2)$ of this fibration is contractible: convex linear interpolation between any given area form and the base point ω defines a strong deformation retraction to $\{\omega\}$. The total space $\text{Diff}_c(D^2)$ is likewise contractible. As proved by Munkres [34] and, independently, Smale [37], it admits a strong deformation retraction to $\{\text{id}_{D^2}\}$.

From the Serre fibration property it then follows that the fibre $\text{Diff}_c(D^2, \omega)$, too, is contractible. In fact, as claimed in Theorem 1.8, the fibre also has $\{\text{id}_{D^2}\}$ as a strong deformation retract.

Proof of Theorem 1.8. Write

$$E_s: \text{Diff}_c(D^2) \longrightarrow \text{Diff}_c(D^2), \quad s \in [0, 1],$$

for the strong deformation retraction of $\text{Diff}_c(D^2)$ to $\{\text{id}_{D^2}\}$, that is:

- E_0 is the identity map on the space $\text{Diff}_c(D^2)$;
- E_1 maps the whole space to $\{\text{id}_{D^2}\}$;
- $E_s(\text{id}_{D^2}) = \text{id}_{D^2}$ for all $s \in [0, 1]$.

Similarly, let

$$B_t: \Omega_c(D^2) \longrightarrow \Omega_c(D^2), \quad t \in [0, 1],$$

be the strong deformation retraction of $\Omega_c(D^2)$ to $\{\omega\}$. The retractions E_s, B_t may be taken smooth in the respective deformation parameter s, t .

Given $\psi \in \text{Diff}_c(D^2, \omega)$, the contraction E_s defines a path $s \mapsto E_s(\psi)$ in the larger space $\text{Diff}_c(D^2)$. This maps to a loop $(p \circ E_s(\psi))_{s \in [0,1]}$ in $\Omega_c(D^2)$ based at ω . The deformation retraction B_t of $\Omega_c(D^2)$ then defines a homotopy $\text{rel } \{0, 1\}$ from the constant loop at ω to that loop $(p \circ E_s(\psi))_{s \in [0,1]}$, where we take the retraction in backwards time:

$$(s, t) \longmapsto \omega_{s,t} := B_{1-t} \circ p \circ E_s(\psi).$$

Notice that $\omega_{s,1} = p \circ E_s(\psi) = E_s(\psi)^*\omega$. Also, the $\omega_{s,t}$ coincide with ω in some neighbourhood of the boundary ∂D^2 .

When one applies the Moser stability argument to the homotopy $t \mapsto \omega_{s,t}$ (for each fixed s), one needs to choose a family of 1-forms $\sigma_{s,t}$, compactly supported in $\text{Int}(D^2)$, with

$$d\sigma_{s,t} = \frac{d}{dt}\omega_{s,t}.$$

Since

$$\int_{D^2} \frac{d}{dt}\omega_{s,t} = \frac{d}{dt} \int_{D^2} \omega_{s,t} = 0,$$

such forms exist by the Poincaré lemma for compactly supported cohomology. In Lemma 3.1 below we make this explicit in order to see that the $\sigma_{s,t}$ can be chosen canonically and smoothly dependent on s and t .

Define the vector field $X_{s,t}$ on D^2 by

$$\sigma_{s,t} + i_{X_{s,t}}\omega_{s,t} = 0.$$

This is compactly supported in $\text{Int}(D^2)$, so its flow $\psi_{s,t}$ (for each fixed s) is defined for all times $t \in [0, 1]$. By the usual Moser argument (see [32, p. 108]), this flow satisfies $\psi_{s,t}^*\omega_{s,t} = \omega$. Notice that $\omega_{0,t} = \omega = \omega_{1,t}$ for all $t \in [0, 1]$. This entails $\psi_{0,t} = \text{id}_{D^2} = \psi_{1,t}$.

The map

$$\begin{aligned} F_s : \text{Diff}_c(D^2, \omega) &\longrightarrow \text{Diff}_c(D^2, \omega) \\ \psi &\longmapsto E_s(\psi) \circ \psi_{s,1} \end{aligned}$$

for $s \in [0, 1]$ then defines the desired strong deformation retraction of $\text{Diff}_c(D^2, \omega)$, since $\psi_{s,1}^* E_s(\psi)^*\omega = \psi_{s,1}^*\omega_{s,1} = \omega$. □

It remains to discuss the canonical choice of the 1-forms $\sigma_{s,t}$. Here is the relevant version of the Poincaré lemma for compactly supported forms. It shows that the $\sigma_{s,t}$ depend only on an *a priori* choice of a bump function. For simplicity of notation, we work on the unit square I^2 , with $I := [0, 1]$, instead of the unit disc.

LEMMA 3.1. *Choose a bump function $y \mapsto \chi(y)$ on I , compactly supported in $\text{Int}(I)$, with $\int_I \chi(y) dy = 1$. Let $\eta = g(x, y) dx \wedge dy$ be a 2-form on I^2 with g compactly supported in $\text{Int}(I^2)$ and $\int_{I^2} \eta = 0$. Set*

$$\begin{aligned} a(x) &:= \int_0^1 g(x, y) dy, \\ b(x) &:= \int_0^x a(s) ds, \\ u(x, y) &:= -g(x, y) + a(x)\chi(y), \\ v(x, y) &:= \int_0^y u(x, t) dt. \end{aligned}$$

Then the 1-form

$$\beta := v(x, y) dx + b(x)\chi(y) dy$$

is compactly supported in $\text{Int}(I^2)$ and satisfies $d\beta = \eta$.

Proof. The fact that the functions b and v are compactly supported in I and I^2 , respectively, follows from $\int_I a(x) dx = 0$ and $\int_I u(x, y) dy = 0$. The computation showing that β is a primitive of η is straightforward. \square

We now want to show how the strong deformation retraction of Theorem 1.8 translates into a canonical choice of Hamiltonian function generating a given $\psi \in \text{Diff}_c(D^2, \omega)$. Up to some sign changes and with a little care concerning the boundary behaviour, this is exactly the argument in [32, Proposition 9.3.1]. We shall assume that the strong deformation retraction F_s has been chosen as a *technical* homotopy, that is, F_s is the identity map on $\text{Diff}_c(D^2, \omega)$ for s near 0, and $F_s \equiv \text{id}_{D^2}$ for s near 1.

Given $\psi \in \text{Diff}_c(D^2, \omega)$, we define the path $s \mapsto \psi_s := F_{1-s}(\psi)$ in $\text{Diff}_c(D^2, \omega)$ from id_{D^2} to ψ . Define the vector field X_s on D^2 by

$$\frac{d}{ds} \psi_s = X_s \circ \psi_s.$$

This vector field is compactly supported in $\text{Int}(D^2)$, and $X_s \equiv 0$ for s near 0 or 1. There is a unique function $G_s: D^2 \rightarrow \mathbb{R}$ that is compactly supported in $\text{Int}(D^2)$ and satisfies

$$\psi_s^* \lambda - \lambda = dG_s.$$

The function

$$H_s := -\lambda(X_s) + \left(\frac{d}{ds} G_s\right) \circ \psi_s^{-1}$$

is compactly supported in $\text{Int}(D^2)$, and it is identically zero for s near 0 or 1, so it may be regarded as a 1-periodic function in s . One then computes that $dH_s = i_{X_s} \omega$, so ψ_s is the Hamiltonian isotopy generated by H_s .

4. Pseudorotations

We now want to prove Theorem 1.2 by performing a limit process in the argument for proving Proposition 1.7. To this end, we need to describe pseudorotations as Hamiltonian diffeomorphisms.

4.1. *Hamiltonian description of pseudorotations.* Write \mathfrak{R}_a for the rotation of D^2 through an angle $2\pi a$. As mentioned in the introduction, the irrational pseudorotations constructed by Fayad and Katok [12] are C^∞ -limits

$$\lim_{\nu \rightarrow \infty} \varphi_\nu \circ \mathfrak{R}_{p_\nu/q_\nu} \circ \varphi_\nu^{-1},$$

where $(p_\nu/q_\nu)_{\nu \in \mathbb{N}}$ is a sequence of rational numbers, which we take to be positive, converging sufficiently fast to a (Liouvillean) irrational number, and the φ_ν are area-preserving diffeomorphisms of D^2 . Each φ_ν is the identity on a neighbourhood of ∂D^2 . For $\nu \rightarrow \infty$,

that neighbourhood shrinks to ∂D^2 . The most relevant statements can be found in Theorem 3.3 and Lemma 3.5 of [12].

By the preceding section, where we now take our Hamiltonian isotopies to be parametrized on the interval $[0, 2\pi]$, we can write the area-preserving diffeomorphism

$$\varphi_\nu \circ \mathfrak{R}_{p_\nu/q_\nu} \circ \varphi_\nu^{-1} \circ \mathfrak{R}_{p_\nu/q_\nu}^{-1} \in \text{Diff}_c(D^2, \omega)$$

in a canonical fashion as the time 2π map of a Hamiltonian isotopy Ψ_s^ν generated by a 2π -periodic Hamiltonian function K_s^ν with compact support in $\text{Int}(D^2)$. The rotation $\mathfrak{R}_{p_\nu/q_\nu}$ is the time 2π map of the Hamiltonian isotopy generated by the function

$$R^\nu := h + \frac{p_\nu}{q_\nu} - \frac{p_\nu}{q_\nu} r^2, \tag{11}$$

where h is chosen as a large natural number. By the well-known formula for composing Hamiltonian diffeomorphisms (see [32, Exercise 3.1.14]), the diffeomorphism

$$\psi_\nu := \varphi_\nu \circ \mathfrak{R}_{p_\nu/q_\nu} \circ \varphi_\nu^{-1} = \Psi_{2\pi}^\nu \circ \mathfrak{R}_{p_\nu/q_\nu}$$

is the time 2π map of the Hamiltonian isotopy generated by

$$H_s^\nu := K_s^\nu + R^\nu \circ (\Psi_s^\nu)^{-1}. \tag{12}$$

Observe that this Hamiltonian satisfies Assumption 2.3 and the boundary condition (5), since $H_s^\nu = R^\nu$ near $r = 1$.

By the discussion in the preceding section, the fact that the pseudorotation $\psi := \lim_{\nu \rightarrow \infty} \psi_\nu$ is a C^∞ -limit implies that we have a C^∞ -limit $H_s^\infty := \lim_{\nu \rightarrow \infty} H_s^\nu$, and ψ equals the time 2π map of the Hamiltonian isotopy generated by this limit Hamiltonian H_s^∞ . The smoothness of H_s^∞ follows from the fact that the pointwise convergence $f_\nu \rightarrow f$ of a sequence $(f_\nu)_{\nu \in \mathbb{N}}$ of C^1 -functions and the uniform convergence of its partial derivatives $\partial f_\nu / \partial x_i \rightarrow f_i$ implies, by the fundamental theorem of calculus, that the limit function f is a C^1 -function with $\partial f / \partial x_i = f_i$.

4.2. *The cut construction for circle actions on the boundary.* Our aim will be to show that the contact cut construction in §2 can be performed for the contact form $\alpha_\infty := H_s^\infty ds + \lambda$ on $V = S^1 \times D^2$. Notice that H_s^∞ still satisfies the boundary condition (5), but it violates Assumption 2.3, in general.

This means that the S^1 -action on $\partial(S^1 \times D^2)$ defined by $Y = \partial_s - h\partial_\theta$ may not extend to a strict contact S^1 -action on a collar neighbourhood of the boundary. However, since the cut construction only affects the boundary, one can sometimes perform a cut even when the S^1 -action does not extend. As we shall see, this is the case here.

Remark 4.1. Topologically, one can always extend an S^1 -action on the boundary to one on a collar neighbourhood, and hence perform a cut. In the symplectic setting, one can appeal to an equivariant coisotropic embedding theorem and conclude likewise; see [30, Proposition 2.7].

In the contact setting, Giroux’s neighbourhood theorem for surfaces in contact 3-manifolds (see [13, Theorem 2.5.22], or its higher-dimensional analogue [10, Proposition 6.4]), gives an extension of the S^1 -action to one preserving only the contact structure. By

averaging the contact form, one may assume it to be S^1 -invariant, but this would of course alter the Reeb dynamics.

Since we are interested in preserving the Reeb dynamics on $\text{Int}(V)$, we shall explicitly analyse the 1-form on the quotient V/\sim induced by the contact form α_∞ near the binding $B := (\partial(S^1 \times D^2))/S^1 \cong \partial D^2$ and discuss its extendability to the binding.

4.3. *The neighbourhood of the binding.* The diffeomorphism (4) from §2.1, for $\Sigma = D^2$, gives us an embedding Φ of a pointed neighbourhood of the binding $B \cong \partial D^2$ into the interior of the solid torus $V = S^1 \times D^2$. This embedding is given by setting $\tau = -\rho^2$, so it depends on a choice of collar parameter $\tau = \tau(r, s, \theta)$. This function should be chosen to be invertible with respect to r , that is, we require that r can be written as a smooth function $r = g(\tau, s, \theta)$.

Then the embedding Φ takes the form

$$\begin{aligned} \Phi: \quad B \times (\text{Int}(D^2_{\sqrt{\varepsilon}}) \setminus \{0\}) &\longrightarrow \text{Int}(S^1 \times D^2) \\ (b; \rho, \vartheta) &\longmapsto \begin{cases} s = \vartheta, \\ r = g(-\rho^2, \vartheta, b - h\vartheta), \\ \theta = b - h\vartheta. \end{cases} \end{aligned} \tag{13}$$

One obvious choice for the collar parameter is $\tau = r - 1$. Alternatively, one can choose a collar parameter adapted to the contact form $\alpha = H_s ds + \lambda$. Here, the natural collar parameter to use is the one coming from the momentum map

$$\mu_V = \alpha(Y) = H_s - hr^2.$$

The collar parameter τ would simply be the negative of that.

Thus, when we consider the sequence of contact forms $\alpha_v := H_s^v ds + \lambda$ with limit α_∞ on $S^1 \times D^2$, we could opt to work with a fixed collar parameter, or with one that changes with each element in the sequence. We briefly describe the advantages of either choice.

4.3.1. *Collar parameter depending on α .* We first consider the collar parameter

$$\tau = \tau(r, s, \theta) = hr^2 - H_s(r, \theta)$$

adapted to the contact form $\alpha = H_s ds + \lambda$. We have

$$\left. \frac{\partial \tau}{\partial r} \right|_{r=1} = 2h - \left. \frac{\partial H_s}{\partial r} \right|_{r=1} > 0$$

by the contact condition (6'). This means that near $r = 1$, we can write r as a smooth function $r = g(\tau, s, \theta)$. Then

$$\Phi^* \alpha = (g(-\rho^2, \vartheta, b - h\vartheta))^2 (db - h d\vartheta) + (H_s \circ \Phi) d\vartheta.$$

We have

$$(H_s - hr^2) \circ \Phi(b, \rho, \vartheta) = -\tau(g(-\rho^2, \vartheta, b - h\vartheta), \vartheta, b - h\vartheta) = \rho^2,$$

and hence

$$\Phi^*\alpha = (g(-\rho^2, \vartheta, b - h\vartheta))^2 db + \rho^2 d\vartheta.$$

The second summand obviously extends smoothly over the binding $\{\rho = 0\}$, so the only question is whether the function

$$(b; \rho, \vartheta) \mapsto (g(-\rho^2, \vartheta, b - h\vartheta))^2$$

extends smoothly. When it does, the extended 1-form is easily seen to be a contact form.

Notice that in the case where the Hamiltonian function satisfies Assumption 2.3, in which case the S^1 -action on the boundary of the solid torus extends to a collar neighbourhood, τ is a function of r only near $r = 1$, and hence g is a function of ρ only. So in this case the contact form extends, which is of course not surprising, since this is what the cut construction tells us.

4.3.2. *Collar parameter independent of α .* When we take $\tau = r - 1$ as collar parameter, the function g is simply given by $r = 1 + \tau$, so Φ takes the form

$$\Phi: (b; \rho, \vartheta) \mapsto \begin{cases} s = \vartheta; \\ r = 1 - \rho^2, \\ \theta = b - h\vartheta. \end{cases} \tag{14}$$

It follows that

$$\Phi^*\alpha = (1 - \rho^2)^2 db + (H_s \circ \Phi - h(1 - \rho^2)^2) d\vartheta. \tag{15}$$

Now the extension problem is located in the second summand, and the dependence on H_s is more transparent than with the choice made in §4.3.1, where this dependence is hidden in the function g .

4.4. *Ellipsoids.* By way of example, we now describe ellipsoids in this language. This will be relevant in §5, where we compute the dynamical invariants of the Reeb flows on S^3 coming from pseudorotations and compare them with the Reeb flows on ellipsoids. This information is used in the proof of Corollary 1.3.

Consider the Hamiltonian function $H(re^{i\theta}) = a_2r^2 + a_0$ with $a_0, a_2 \in \mathbb{R}, a_0 > 0$, and $a_0 + a_2 = h \in \mathbb{N}$. This satisfies the contact condition (6'). The function H defines the Hamiltonian vector field $X = -a_2\partial_\theta$, and the Reeb vector field of the contact form $\alpha = H ds + \lambda$ is $R_\alpha = (\partial_s + X)/a_0$. We compute

$$\Phi^*\alpha = (1 - \rho^2)^2 db + a_0(2 - \rho^2)\rho^2 d\vartheta;$$

this formula also defines the extension of $\Phi^*\alpha$ as a contact form $\hat{\alpha}$ over $\rho = 0$. The Reeb vector field of $\hat{\alpha}$ is $R_{\hat{\alpha}} = \partial_b + \partial_\vartheta/a_0$. In Cartesian coordinates $u + iv = \rho e^{i\vartheta}$ we have $\partial_\vartheta = u\partial_v - v\partial_u$, so along the binding B the Reeb vector field equals ∂_b . Notice that if we fix a_0 and allow a_2 to vary (by integers), the dynamics around the periodic Reeb orbit corresponding to the fixed point $0 \in D^2$ change, while those around the periodic orbit B do not.

This example gives the intrinsic description of the Reeb flow on ellipsoids in \mathbb{R}^4 . Apart from the two periodic orbits just mentioned, we have a foliation by 2-tori, which in turn

are linearly foliated by Reeb orbits. Depending on a_0 being rational or not, the Reeb orbits on these tori are periodic or dense.

Indeed, we can adapt the quotient map in the proof of Lemma 2.10 to this example. Consider the ellipsoid

$$E_{a_0} := \left\{ (z_1, z_2) \in \mathbb{C}^2 : \frac{|z_1|^2}{a_0} + |z_2|^2 = 1 \right\}.$$

The quotient map

$$\begin{aligned} \Psi : S^1 \times D^2 &\longrightarrow E_{a_0} \\ (s; r, \theta) &\longmapsto (\sqrt{a_0(1-r^2)} e^{is}, r e^{i(\theta+hs)}) \end{aligned}$$

pulls back the standard contact form $r_1^2 d\theta_1 + r_2^2 d\theta_2$ on E_{a_0} to

$$\Psi^*(r_1^2 d\theta_1 + r_2^2 d\theta_2) = (a_0 + a_2 r^2) ds + r^2 d\theta = \alpha,$$

and $T\Psi(R_\alpha) = \partial_{\theta_1}/a_0 + \partial_{\theta_2}$.

4.5. *The extension problem.* We now return to the irrational pseudorotations of Fayad and Katok. Thus, from now on, the Hamiltonian functions H_s^ν , $\nu \in \mathbb{N}$, and their C^∞ -limit H_s^∞ are those corresponding to an irrational pseudorotation, as found in §4.1.

We choose to work with a fixed collar parameter as in §4.3.2. Then the question of whether $\Phi^*\alpha_\infty$ extends as a smooth 1-form to the binding B reduces to the following statement.

PROPOSITION 4.2. *The function $f : B \times \text{Int}(D_{\sqrt{\varepsilon}}^2) \rightarrow \mathbb{R}$ defined by*

$$f(b, \rho e^{i\vartheta}) := \begin{cases} (H_s^\infty \circ \Phi - h(1 - \rho^2)^2)/\rho^2 & \text{for } \rho \neq 0, \\ 2(h + a) & \text{for } \rho = 0, \end{cases} \tag{16}$$

where $a = \lim_{\nu \rightarrow \infty} p_\nu/q_\nu$, is smooth.

The 1-form $\Phi^*\alpha_\infty$ then extends smoothly over $\rho = 0$ as

$$\hat{\alpha} := (1 - \rho^2)^2 db + f\rho^2 d\vartheta.$$

The functions f^ν , defined as in Proposition 4.2, with H_s^∞ replaced by H_s^ν and a by p_ν/q_ν , are easily shown to be smooth; see Lemma 4.5 below.

LEMMA 4.3. *When $h \in \mathbb{N}$ in the above construction is chosen such that $h + a > 0$, the extended 1-form $\hat{\alpha}$ is a contact form, and $B \times \{0\}$ is a (positively oriented) Reeb orbit.*

Proof. The contact condition needs to be verified along $B \times \{0\}$. We have

$$d\hat{\alpha} = -4(1 - \rho^2)\rho d\rho \wedge db + 2f\rho d\rho \wedge d\vartheta + df \wedge \rho^2 d\vartheta,$$

and hence

$$\hat{\alpha} \wedge d\hat{\alpha}|_{\rho=0} = 2f db \wedge \rho d\rho \wedge d\vartheta > 0,$$

provided that $f|_{B \times \{0\}} > 0$. Moreover, we have $\hat{\alpha}(\partial_b)|_{\rho=0} = 1$ and $i_{\partial_b} d\hat{\alpha}|_{\rho=0} = 0$, so ∂_b is the Reeb vector field of $\hat{\alpha}$ along $B \times \{0\}$. □

Remark 4.4. The condition $h + p_\nu/q_\nu > 0$ is precisely the contact condition (6') for the 1-form $R^\nu ds + \lambda$, where R^ν is the standard quadratic Hamiltonian in (11). So the condition $h + a$ in the lemma is simply saying that the strict inequality should also hold in the limit $\nu \rightarrow \infty$.

Thus, in order to demonstrate Theorem 1.2, it only remains to prove Proposition 4.2. The further statements in Theorem 1.2, apart from the dynamical convexity, then follow as in the proof of Proposition 1.7 in §2.

The embedding Φ in (14) extends to a smooth map

$$\begin{aligned} \tilde{\Phi}: B \times [0, \sqrt{\varepsilon}) \times S^1 &\longrightarrow S^1 \times D^2 \\ (b, \rho, \vartheta) &\longmapsto (\vartheta, (1 - \rho^2)e^{i(b-h\vartheta)}). \end{aligned}$$

The function

$$\begin{aligned} \tilde{f}: B \times [0, \sqrt{\varepsilon}) \times S^1 &\longrightarrow \mathbb{R} \\ (b, \rho, \vartheta) &\longmapsto f(b, \rho e^{i\vartheta}) \end{aligned}$$

lifting f from (16) can then be written as

$$\tilde{f}(b, \rho, \vartheta) = \begin{cases} (H_s^\infty \circ \tilde{\Phi} - h(1 - \rho^2)^2)/\rho^2 & \text{for } \rho \neq 0, \\ 2(h + a) & \text{for } \rho = 0. \end{cases} \tag{17}$$

Similarly, we have functions \tilde{f}^ν , $\nu \in \mathbb{N}$, when we replace H_s^∞ by H_s^ν and a by p_ν/q_ν in the definition of \tilde{f} .

LEMMA 4.5. *The functions \tilde{f}, \tilde{f}^ν on $B \times [0, \sqrt{\varepsilon}) \times S^1$ are smooth.*

Proof. By equations (11) and (12), we have

$$H_s^\nu = R_s^\nu = h + \frac{p_\nu}{q_\nu} - \frac{p_\nu}{q_\nu} r^2 \quad \text{near } \partial D^2.$$

It follows that

$$H_s^\nu \circ \tilde{\Phi} - h(1 - \rho^2)^2 = \left(h + \frac{p_\nu}{q_\nu} \right) \cdot (2\rho^2 - \rho^4)$$

for ρ near and including 0. This shows that the \tilde{f}^ν are smooth, and so are the f^ν .

Since H_s^∞ is the C^∞ -limit of the H_s^ν , the function

$$H_s^\infty \circ \tilde{\Phi} - h(1 - \rho^2)^2$$

vanishes to second order in ρ at $\rho = 0$, and its second partial derivative with respect to ρ at $\rho = 0$ equals $4(h + a)$. By a well-known lemma of Morse [33, p. 349] (cf. [38, Lemma 1.2.3]), this means that \tilde{f} is smooth. □

In [17], a function $u : D_\delta^2 \rightarrow \mathbb{R}$ having the property that the lifted function $\tilde{u} : [0, \delta] \times S^1 \rightarrow \mathbb{R}$ is smooth is called *weakly smooth*.

4.6. *C^1 -functions in polar coordinates.* We now discuss the general question of under which conditions a C^1 -function \tilde{u} on $[0, \delta] \times S^1$ descends to a C^1 -function u on D_δ^2 when

$(\rho, \vartheta) \in [0, \delta] \times S^1$ are interpreted as polar coordinates. We write the partial derivatives of \tilde{u} as \tilde{u}_ρ and \tilde{u}_ϑ , respectively.

The considerations in this and the two following subsections are elementary, and we do not make any claim to originality. For instance, the sufficient condition for C^∞ -differentiability formulated in Lemma 4.11 below can be found in similar form in [19, Lemma 2]. We begin with a necessary and sufficient condition for C^1 -differentiability.

LEMMA 4.6. *Let $\tilde{u}: [0, \delta] \times S^1 \rightarrow \mathbb{R}$ be a C^1 -function with $\tilde{u}(0, \vartheta)$ independent of ϑ . Define*

$$\begin{aligned} u: D_\delta^2 &\longrightarrow \mathbb{R} \\ \rho e^{i\vartheta} &\longmapsto \tilde{u}(\rho, \vartheta). \end{aligned}$$

Then u is a C^1 -function if and only if

$$\tilde{u}_\rho(0, 0) = -\tilde{u}_\rho(0, \pi), \quad \tilde{u}_\rho(0, \pi/2) = -\tilde{u}_\rho(0, 3\pi/2),$$

as well as

$$\lim_{\rho \rightarrow 0} \left(\cos \vartheta \tilde{u}_\rho - \frac{\sin \vartheta}{\rho} \tilde{u}_\vartheta \right) = \tilde{u}_\rho(0, 0)$$

and

$$\lim_{\rho \rightarrow 0} \left(\sin \vartheta \tilde{u}_\rho + \frac{\cos \vartheta}{\rho} \tilde{u}_\vartheta \right) = \tilde{u}_\rho(0, \pi/2).$$

Here the limits $\lim_{\rho \rightarrow 0}$ are to be read as $\lim_{m \rightarrow \infty}$ for any sequence (ρ_m, ϑ_m) with $\rho_m \rightarrow 0$; the sequence $(\vartheta_m)_{m \in \mathbb{N}}$ need not converge.

Proof. In Cartesian coordinates $z = x + iy$ on D_δ^2 we have $\rho = \sqrt{x^2 + y^2}$ and $\vartheta = \arctan(y/x)$. It follows that, for $z \neq 0$,

$$u_x = \cos \vartheta \tilde{u}_\rho - \frac{\sin \vartheta}{\rho} \tilde{u}_\vartheta$$

and

$$u_y = \sin \vartheta \tilde{u}_\rho + \frac{\cos \vartheta}{\rho} \tilde{u}_\vartheta.$$

In $z = 0$, we have

$$u_x(0) = \lim_{t \rightarrow 0} \frac{u(t) - u(0)}{t},$$

which, depending on the sign of $t \in \mathbb{R} \setminus \{0\}$, gives the limit

$$\lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{\tilde{u}(t, 0) - \tilde{u}(0, 0)}{t} = \tilde{u}_\rho(0, 0)$$

or

$$\lim_{\substack{t \rightarrow 0 \\ t < 0}} \frac{\tilde{u}(-t, \pi) - \tilde{u}(0, \pi)}{t} = -\tilde{u}_\rho(0, \pi).$$

For the partial derivative u_y , the computations are analogous. The lemma follows. □

When we verify the conditions of Lemma 4.6 in the application to proving Proposition 4.2, we compute the limit $\lim_{\rho \rightarrow 0}$ as a double limit $\lim_{m,n \rightarrow \infty}$ for a sequence (ρ_m, ϑ_n) with $\rho_m \rightarrow 0$ and ϑ_n arbitrary. There we shall need the following elementary lemma.

LEMMA 4.7. *Let $(a_{mn})_{m,n \in \mathbb{N}}$ be a double sequence of real numbers. Suppose the following conditions are satisfied.*

- (i) *For $m \rightarrow \infty$, each of the sequences $(a_{mn})_{m \in \mathbb{N}}$ converges to some real number a_n , uniformly in n .*
- (ii) *The limit $\lim_{n \rightarrow \infty} a_n =: a$ exists.*

Then the limit $\lim_{m,n \rightarrow \infty} a_{mn}$ exists and equals a .

Proof. Uniform convergence in n of the sequences $(a_{mn})_{m \in \mathbb{N}}$ means that for any $\varepsilon > 0$ there is an $M(\varepsilon) \in \mathbb{N}$ such that

$$|a_{mn} - a_n| < \varepsilon \quad \text{for all } m \geq M(\varepsilon) \text{ and } n \in \mathbb{N}.$$

Convergence of $(a_n)_{n \in \mathbb{N}}$ means that there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon \quad \text{for all } n \geq N(\varepsilon).$$

Hence, for $m, n \geq \max\{M(\varepsilon), N(\varepsilon)\}$ we have

$$|a_{mn} - a| \leq |a_{mn} - a_n| + |a_n - a| < 2\varepsilon,$$

which proves the lemma. □

4.7. *Proof of Proposition 4.2—the first derivative.* We now apply Lemma 4.6 to the function \tilde{f} in (17) corresponding to the function f in (16). We suppress the b -coordinate, which is irrelevant to the argument.

LEMMA 4.8. *The function \tilde{f} in (17) satisfies*

$$\tilde{f}_\rho|_{\rho=0} = 0 \quad \text{and} \quad \tilde{f}_{\rho\rho}|_{\rho=0} = -2(h + a). \tag{18}$$

All other higher or mixed derivatives of \tilde{f} vanish at $\rho = 0$.

Proof. From the proof of Lemma 4.5 we have

$$\tilde{f}^v = \left(h + \frac{p_v}{q_v} \right) \cdot (2 - \rho^2)$$

near $\rho = 0$. Since \tilde{f} is the C^∞ -limit of the \tilde{f}^v , the lemma follows. □

Let $(\rho_n, \vartheta_n)_{n \in \mathbb{N}}$ be a sequence in $(0, \delta] \times S^1$ with $\rho_n \rightarrow 0$ for $n \rightarrow \infty$. We need to verify that

$$\tilde{f}_\rho(\rho_n, \vartheta_n) \longrightarrow 0 \tag{19}$$

and

$$\frac{1}{\rho_n} \tilde{f}_{\vartheta}(\rho_n, \vartheta_n) \longrightarrow 0. \tag{20}$$

For the limit in (19), set

$$a_{mn} := \tilde{f}_\rho(\rho_m, \vartheta_n) = \rho_m \cdot \frac{\tilde{f}_\rho(\rho_m, \vartheta_n)}{\rho_m}.$$

The limit $\lim_{m \rightarrow \infty} a_{mn}$ is uniform in n (and equals 0) thanks to the following lemma. With Lemma 4.7 we then conclude $\lim_{m,n \rightarrow \infty} a_{mn} = 0$.

LEMMA 4.9. *For $\rho \rightarrow 0$, the difference quotient $\tilde{f}_\rho(\rho, \vartheta)/\rho$ converges to the derivative $\tilde{f}_{\rho\rho}(0, \vartheta) = -2(h + a)$ uniformly in ϑ .*

Proof. We make the following estimates with the mean value theorem:

$$\begin{aligned} |\tilde{f}_\rho(\rho_0, \vartheta_1) - \tilde{f}_\rho(\rho_0, \vartheta_0)| &\leq \max_{\vartheta} |\tilde{f}_{\rho\vartheta}(\rho_0, \vartheta)| \cdot |\vartheta_1 - \vartheta_0| \\ &\leq \max_{\rho, \vartheta} |\tilde{f}_{\rho\vartheta\rho}(\rho, \vartheta)| \cdot \rho_0 \cdot |\vartheta_1 - \vartheta_0|. \end{aligned}$$

Here $|\vartheta_1 - \vartheta_0|$ denotes the length of a circular arc between ϑ_0 and ϑ_1 ; the maximum is taken over $\vartheta \in S^1$ and $\rho \in [0, \rho_0]$.

We then estimate

$$\begin{aligned} \left| \frac{\tilde{f}_\rho(\rho_0, \vartheta_1)}{\rho_0} + 2(h + a) \right| &\leq \left| \frac{\tilde{f}_\rho(\rho_0, \vartheta_1)}{\rho_0} - \frac{\tilde{f}_\rho(\rho_0, \vartheta_0)}{\rho_0} \right| + \left| \frac{\tilde{f}_\rho(\rho_0, \vartheta_0)}{\rho_0} + 2(h + a) \right| \\ &\leq \max_{\rho, \vartheta} |\tilde{f}_{\rho\vartheta\rho}(\rho, \vartheta)| \cdot |\vartheta_1 - \vartheta_0| + \left| \frac{\tilde{f}_\rho(\rho_0, \vartheta_0)}{\rho_0} + 2(h + a) \right|, \end{aligned}$$

which, together with the compactness of S^1 , gives the desired uniformity in ϑ . □

For the limit in (20), one can apply completely analogous arguments to the double sequence $\tilde{f}_\vartheta(\rho_m, \vartheta_n)/\rho_m$.

This shows that the function f in Proposition 4.2 is continuously differentiable.

4.8. *Proof of Proposition 4.2—higher derivatives.* In principle, higher derivatives can be dealt with by iterating Lemma 4.6. In order to establish that f is C^2 , we write out explicitly the second derivative f_{xx} . For f_{xy} and f_{yy} the considerations are analogous.

In $z \neq 0$ we have

$$\begin{aligned} f_{xx} &= \tilde{f}_{\rho\rho} \cos^2 \vartheta - \tilde{f}_{\rho\vartheta} \frac{2 \sin \vartheta \cos \vartheta}{\rho} + \tilde{f}_\rho \frac{\sin^2 \vartheta}{\rho} \\ &\quad + \tilde{f}_{\vartheta\vartheta} \frac{\sin^2 \vartheta}{\rho^2} + \tilde{f}_\vartheta \frac{2 \sin \vartheta \cos \vartheta}{\rho^2}. \end{aligned} \tag{21}$$

In $z = 0$, we find

$$f_{xx}(0) = \tilde{f}_{\rho\rho}(0, 0) = \tilde{f}_{\rho\rho}(0, \pi).$$

Recall the properties of \tilde{f} stated in Lemma 4.8. The derivative $f_{xx}(0)$ exists thanks to $\tilde{f}_{\rho\rho}(0, 0)$ and $\tilde{f}_{\rho\rho}(0, \pi)$ both being equal to $-2(h + a)$. For the continuity of f_{xx} in $z = 0$, we consider the summands on the right-hand side of (21) in turn. We evaluate these summands at a point (ρ_m, ϑ_n) and consider the limit $m \rightarrow \infty$, assuming that $\rho_m \rightarrow 0$ in this limit.

(i) The term $\tilde{f}_{\rho\rho} \cos^2 \vartheta_n$ converges to

$$\tilde{f}_{\rho\rho}|_{\rho=0} \cos^2 \vartheta_n = -2(h + a) \cos^2 \vartheta_n.$$

(ii) The term $-2\tilde{f}_{\rho\vartheta} \sin \vartheta_n \cos \vartheta_n/\rho$ converges to zero, since $\tilde{f}_{\rho\vartheta}|_{\rho=0} = 0$.

(iii) The term $\tilde{f}_{\rho} \sin^2 \vartheta_n/\rho$ converges to

$$\tilde{f}_{\rho}|_{\rho=0} \sin^2 \vartheta_n = -2(h + a) \sin^2 \vartheta_n.$$

(iv) The term $\tilde{f}_{\vartheta\vartheta} \sin^2 \vartheta_n/\rho^2$ is seen to converge to zero by applying l'Hôpital's rule twice, since $\tilde{f}_{\vartheta\vartheta\rho\rho}|_{\rho=0} = 0$.

(v) The limit of the term $2\tilde{f}_{\vartheta} \sin \vartheta_n \cos \vartheta_n/\rho^2$ equals zero by the same argument as in (iv).

The uniformity of these limits can be seen by the same reasoning as above. Thus, we have

$$\lim_{m,n \rightarrow \infty} f_{xx}(\rho_m, \vartheta_n) = -2(h + a) = f_{xx}(0).$$

This argument, applied analogously to f_{xy} and f_{yy} , shows that the function f in Proposition 4.2 is C^2 .

In order to establish that the function f in question is C^∞ near $z = 0$, we need a more systematic approach. We shall describe one such approach that is general enough to apply to the Fayad–Katok examples.

LEMMA 4.10. *Let $\tilde{u}: [0, \delta] \times S^1 \rightarrow \mathbb{R}$ be a smooth function with $\tilde{u}(0, \vartheta)$ independent of ϑ , and let $u(x + iy) = u(\rho e^{i\vartheta}) = \tilde{u}(\rho, \vartheta)$ be the induced function on D_δ^2 . For $k \in \mathbb{N}$ and $j \in \{0, 1, \dots, k\}$, the partial derivatives*

$$\frac{\partial^k u}{\partial x^j \partial y^{k-j}}$$

in $z \neq 0$ are sums of terms

$$\frac{\partial^\ell \tilde{u}}{\partial \rho^i \partial \vartheta^{\ell-i}} \cdot \frac{a_{ji}^{k\ell}(\vartheta)}{\rho^{k-i}},$$

where $1 \leq \ell \leq k$, $0 \leq i \leq \ell$, and $a_{ji}^{k\ell}$ is a polynomial in $\sin \vartheta$ and $\cos \vartheta$.

Proof. For $k = 1$ this is confirmed by the formulae for u_x and u_y in the proof of Lemma 4.6. Then, we argue by induction over k , using the fact that $\partial\rho/\partial x = \cos \vartheta$ and $\partial\vartheta/\partial x = -\sin \vartheta/\rho$, and similar expressions for the derivatives with respect to y . □

LEMMA 4.11. *Let $\tilde{u}: [0, \delta] \times S^1 \rightarrow \mathbb{R}$ be a smooth function satisfying*

$$\frac{\partial^\ell \tilde{u}}{\partial \rho^i \partial \vartheta^{\ell-i}}(0, \vartheta) = 0$$

for all $\ell \in \mathbb{N}_0$, $0 \leq i \leq \ell$ and $\vartheta \in S^1$. Then the function $u: D_\delta^2 \rightarrow \mathbb{R}$, defined as in Lemma 4.10, is smooth and vanishes to infinite order at $z = 0$.

Proof. Again we argue by induction on the order of derivatives. The limit conditions in Lemma 4.6 are satisfied by l'Hôpital's rule, so the function u is C^1 .

By a repeated application of l'Hôpital's rule, we also see that the terms from the preceding lemma satisfy

$$\lim_{\rho \rightarrow 0} \frac{\partial^\ell \tilde{u}}{\partial \rho^i \partial \vartheta^{\ell-i}} \cdot \frac{a_{ji}^{k\ell}(\vartheta)}{\rho^{k-i}} = 0.$$

These limits are uniform in ϑ by arguments as in the proof of Lemma 4.9. It follows that

$$\lim_{z \rightarrow 0} \frac{\partial^k u}{\partial x^j \partial y^{k-j}}(z) = 0.$$

For the inductive step, we need to show that u is of class C^{k+1} , presuming that we have already established it to be of class C^k , with vanishing partial derivatives at $z = 0$. Thus, let v be a k^{th} partial derivative of u , and \tilde{v} its lift to $[0, \delta] \times S^1$. Then, as in Lemma 4.6,

$$v_x(0) = \lim_{t \rightarrow 0} \frac{v(t)}{t} = \begin{cases} \tilde{v}_\rho(0, 0) = 0 & \text{for } t > 0, \\ -\tilde{v}_\rho(0, \pi) = 0 & \text{for } t < 0, \end{cases}$$

so the derivative $v_x(0)$ exists. The continuity of $v_x(z)$ at $z = 0$ follows from the limit behaviour of the derivatives described above. For the derivative v_y the argument is analogous. □

By applying this lemma to the functions

$$u = f - (h + a)(2 - x^2 - y^2) \quad \text{and} \quad \tilde{u} = \tilde{f} - (h + a)(2 - \rho^2),$$

we see that u and hence f is smooth. This proves Proposition 4.2.

This concludes the proof of Theorem 1.2, except for the dynamical convexity, which will be established in §5.2.2.

4.9. *A more general sufficient criterion.* We continue to write Φ for the embedding (14).

THEOREM 4.12. *Let $\psi : D^2 \rightarrow D^2$ be an area-preserving diffeomorphism generated by a 2π -periodic Hamiltonian function H_s with $H_s|_{\partial(S^1 \times D^2)} \equiv h \in \mathbb{N}$. Consider the function*

$$f(b, \rho e^{i\vartheta}) := (H_s \circ \Phi - h(1 - \rho^2)^2) / \rho^2$$

on $B \times (\text{Int}(D^2_{\sqrt{\varepsilon}}) \setminus \{0\})$. If f extends continuously over $B \times \{0\}$, and if the lifted function \tilde{f} on $B \times [0, \sqrt{\varepsilon}] \times S^1$ is smooth and has the ∞ -jet, along $B \times \{0\} \times S^1$, of the lift of a smooth function, then ψ embeds into a Reeb flow on S^3 .

Proof. By the discussion in §4.5, a sufficient condition for ψ to embed into a Reeb flow on S^3 is that the function f extends smoothly as a positive function over $B \times \{0\}$. By the analysis in the preceding section, this in turn is equivalent to the conditions on f stated in the theorem. □

The condition of the theorem is satisfied, as one ought to expect, when H_s satisfies Assumption 2.3. More generally, it suffices to assume, for instance, that the ∞ -jet of H_s along $\partial(S^1 \times D^2)$ is that of a function depending only on r .

4.10. *Conjugation invariance.* In our proof of Theorem 1.2 and the more general statement in the preceding section, we have relied on an explicit coordinate description of S^3 as a contact cut of $S^1 \times D^2$. In this section we discuss the conjugation invariance of the construction, which amounts to saying that the specific coordinates are irrelevant.

There is one version of conjugation invariance that is completely tautological. Let

$$\varphi: D^2 \xrightarrow{\cong} \{0\} \times D^2 \subset S^1 \times D^2$$

be any embedding of D^2 with image $\{0\} \times D^2$, and assume that

$$\psi: \{0\} \times D^2 \rightarrow \{0\} \times D^2$$

is the return map of a given Reeb flow. Then, with respect to the embedding

$$D^2 \rightarrow S^3 = (S^1 \times D^2) / \sim$$

given by φ , the return map is $\varphi^{-1} \circ \psi \circ \varphi$, which preserves the area form $\varphi^* \omega$.

More restrictively, we may fix the disc $\{0\} \times D^2 \subset S^1 \times D^2$ and ask whether it is possible to find a new contact form on $S^1 \times D^2$ whose return map on $\{0\} \times D^2$ is the conjugate of the previous one. The following proposition gives the (still quite tautological) answer.

PROPOSITION 4.13. *Let $\psi: D^2 \rightarrow D^2$ be an area-preserving diffeomorphism generated by a 2π -periodic Hamiltonian function H_s with $H_s|_{\partial(S^1 \times D^2)} \equiv h \in \mathbb{N}$. Assume that ψ embeds into a Reeb flow on S^3 by the cut construction described in §4, starting from the contact form $\alpha = H_s ds + \lambda$. Let $\varphi: D^2 \rightarrow D^2$ be a further area-preserving diffeomorphism. Then the conjugate $\varphi^{-1} \circ \psi \circ \varphi$ likewise embeds into a Reeb flow.*

Proof. The diffeomorphism $\varphi^{-1} \circ \psi \circ \varphi$ is generated by the Hamiltonian function $H_s \circ \varphi$; see [32, Exercise 3.1.14]. Regard φ as a diffeomorphism of $S^1 \times D^2$. Then

$$\varphi^* \alpha = (H_s \circ \varphi) ds + \varphi^* \lambda$$

and

$$d(\varphi^* \alpha) = d(H_s \circ \varphi) \wedge ds + \omega,$$

since φ is area-preserving. With $R = \partial_s + X_s$ as before, where X_s is the Hamiltonian vector field X_{H_s} of H_s , it follows that $\varphi^* R = \partial_s + X_{H_s \circ \varphi}$. Incidentally, this provides a quick solution to the cited exercise.

We take the cut of $S^1 \times D^2$ with respect to the S^1 -action by $\varphi^* Y$ on the boundary, and we replace the embedding Φ of a pointed neighbourhood of the binding by the composition $\varphi^{-1} \circ \Phi$. The pull-back of $\varphi^* \alpha$ under this embedding equals $\Phi^* \alpha$, which extends by assumption. \square

4.11. *The choice of primitive.* In some sense, Proposition 4.13 is not entirely satisfactory, since it really talks about the embeddability not of ψ but of the pair (ψ, λ) . It says that the

embeddability of (ψ, λ) is equivalent to that of

$$(\varphi^{-1} \circ \psi \circ \varphi, \varphi^*\lambda),$$

and this just amounts to a global change of coordinates.

We want to show that, at least up to C^2 -differentiability of the Hamiltonian function H_s , the question of whether ψ embeds is independent of the choice of primitive for the area form ω , where it is understood that the choice of allowable primitives is among those compatible with the cut construction.

A general primitive of ω is of the form $\lambda + dF$, with $F: D^2 \rightarrow \mathbb{R}$ a smooth function. We assume that, possibly after adding a large integer to H_s , the 1-form

$$\alpha_F := H_s ds + \lambda + dF$$

is a contact form, that is, it satisfies (6) with λ replaced by $\lambda + dF$. We want the new contact form α_F to be invariant under the S^1 -action on $\partial(S^1 \times D^2)$ generated by $Y = \partial_s - h\partial_\theta$. Near $r = 1$ we have

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta,$$

so the invariance requirement (cf. Lemma 2.7) becomes

$$0 = L_Y dF|_{r=1} = d(dF(\partial_\theta))|_{r=1}.$$

This means that

$$\frac{\partial^2 F}{\partial \theta^2} \Big|_{r=1} = 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial r \partial \theta} \Big|_{r=1} = 0. \tag{22}$$

The first condition forces $(\partial F/\partial \theta)|_{r=1} = 0$. Notice that $\alpha_F(Y)|_{r=1} = 0$, so α_F descends to a 1-form on the quotient $S^3 = (S^1 \times D^2)/\sim$.

With the embedding Φ from (14), we have

$$\Phi^* dF = -\frac{\partial F}{\partial r}(1 - \rho^2, b - h\vartheta) \rho d\rho + \frac{\partial F}{\partial \theta}(1 - \rho^2, b - h\vartheta) \cdot (db - h d\vartheta).$$

In $\Phi^*\alpha_F$ there are no other terms in $d\rho$, so if we want $\Phi^*\alpha_F$ to extend as a contact form over $\rho = 0$, the least we need to require is that

$$\tilde{f}(b, \rho, \vartheta) := \frac{\partial F}{\partial r}(1 - \rho^2, b - h\vartheta), \quad (b, \rho, \vartheta) \in B \times [0, \delta] \times S^1, \tag{23}$$

is the lift of a smooth function $f: B \times D_\delta^2 \rightarrow \mathbb{R}$. As before, we suppress the b -coordinate.

With this requirement understood, we have the following proposition, which says that the remaining terms in $\Phi^* dF$ extend as a C^2 -form over $\rho = 0$. So we do not, up to C^2 , gain more flexibility in the conditions on H_s by adding dF to the primitive λ .

PROPOSITION 4.14. *The function*

$$(\rho, \vartheta) \mapsto \frac{\partial F}{\partial \theta}(1 - \rho^2, b - h\vartheta)$$

on $[0, \delta] \times S^1$ (for any fixed $b \in B$) equals ρ^2 times the lift of a C^2 -function $D_\delta^2 \rightarrow \mathbb{R}$.

Proof. By (22) and the lemma of Morse [38, Lemma 1.2.3] used earlier, we can write

$$\frac{\partial F}{\partial \theta}(r, \theta) = (r - 1)^2 G(r, \theta) \tag{24}$$

with a smooth function G . Hence,

$$\frac{\partial F}{\partial \theta}(1 - \rho^2, b - h\vartheta) = \rho^4 G(1 - \rho^2, b - h\vartheta).$$

We therefore need to show that

$$\tilde{g}(\rho, \vartheta) := \rho^2 G(1 - \rho^2, b - h\vartheta), \quad (\rho, \vartheta) \in [0, \delta] \times S^1,$$

is the lift of a C^2 -function $g: D_\delta^2 \rightarrow \mathbb{R}$.

The derivative $\partial \tilde{f} / \partial \vartheta$, where \tilde{f} is the function defined in (23), is the lift of the smooth function $\partial f / \partial \vartheta$. On the other hand, we can write this derivative upstairs as

$$\frac{\partial \tilde{f}}{\partial \vartheta}(\rho, \vartheta) = -h \frac{\partial^2 F}{\partial \theta \partial r}(1 - \rho^2, b - h\vartheta).$$

In other words, the function

$$\tilde{k}(\rho, \vartheta) := \frac{\partial^2 F}{\partial \theta \partial r}(1 - \rho^2, b - h\vartheta)$$

is the lift of a smooth function $k: D_\delta^2 \rightarrow \mathbb{R}$. But, by (24),

$$\begin{aligned} \tilde{k}(\rho, \vartheta) &= \frac{\partial^2 F}{\partial r \partial \theta}(1 - \rho^2, b - h\vartheta) \\ &= -2\rho^2 G(1 - \rho^2, b - h\vartheta) + \rho^4 \frac{\partial G}{\partial r}(1 - \rho^2, b - h\vartheta) \\ &= -2\tilde{g}(\rho, \vartheta) + \rho^4 \frac{\partial G}{\partial r}(1 - \rho^2, b - h\vartheta). \end{aligned}$$

The second summand in this last expression is the lift of a C^2 -function $D_\delta^2 \rightarrow \mathbb{R}$, since the factor ρ^4 guarantees that the C^1 -condition in Lemma 4.6 and the C^2 -condition, as analysed for f at the beginning of §4.8, are satisfied. It follows that \tilde{g} is the lift of a C^2 -function, as we wanted to show. □

5. Dynamical invariants

In this section we compute some invariants of the Reeb flows on S^3 constructed via the cut construction, *viz.*, the Conley–Zehnder indices of the periodic Reeb orbits, and the self-linking number of the binding orbit.

Throughout this section we assume, as before, that we are dealing with a contact form on $S^3 = (S^1 \times D^2) / \sim$ coming from a contact form $\alpha = H_s ds + \lambda$ on $S^1 \times D^2$, where H_s satisfies the boundary condition (5), and the quotient is taken with respect to the S^1 -action on $\partial(S^1 \times D^2)$ defined by the flow of $Y = \partial_s - h\partial_\theta$. Moreover, it is of course assumed that H_s has been chosen such that $\Phi^*\alpha$ in (15) extends as a contact form over $\rho = 0$; for instance, one may assume the sufficient condition described in §4.9.

In addition, we impose the condition $H_s > 0$. Notice that adding a large natural number to the Hamiltonian does not change the vector field $R = \partial_s + X_s$, so this merely leads to

a reparametrization of the Reeb orbits. This assumption on H_s simplifies the discussion of framings.

5.1. *Framings.* In this section we describe trivializations of the contact plane fields over $\text{Int}(S^1 \times D^2)$ and near the binding orbit $B = (\partial(S^1 \times D^2))/S^1$. The comparison of these two framings will allow us to compute the dynamical invariants.

Over $\text{Int}(S^1 \times D^2)$, the contact structure $\ker \alpha$ is trivialized by the oriented frame

$$\begin{cases} \mathbf{e}_1 = H_s \partial_x + y \partial_s, \\ \mathbf{e}_2 = H_s \partial_y - x \partial_s. \end{cases}$$

Write $\Phi^* \alpha$ from (15) as

$$\Phi^* \alpha = (1 - \rho^2)^2 db + f(b, \rho e^{i\vartheta}) \cdot (u dv - v du), \tag{25}$$

where $u + iv = \rho e^{i\vartheta}$. By assumption, f extends smoothly to $\rho = 0$. We then see that the contact structure $\ker(\Phi^* \alpha)$ is trivialized by the oriented frame

$$\begin{cases} \mathbf{e}'_1 = (1 - \rho^2)^2 \partial_u + f v \partial_b, \\ \mathbf{e}'_2 = (1 - \rho^2)^2 \partial_v - f u \partial_b. \end{cases}$$

Away from $r = 0$ we have

$$\begin{cases} \partial_x = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta, \\ \partial_y = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta. \end{cases}$$

There are analogous expressions for ∂_u, ∂_v , with (r, θ) replaced by (ρ, ϑ) .

The differential $T\Phi$ of Φ in (14) is given by

$$\begin{cases} T\Phi(\partial_b) = \partial_\theta, \\ T\Phi(\partial_\rho) = -2\sqrt{1-r} \partial_r, \\ T\Phi(\partial_\vartheta) = \partial_s - h \partial_\theta. \end{cases}$$

It follows that

$$\begin{aligned} T\Phi(\mathbf{e}'_1) &= -r^2 \left(2\sqrt{1-r} \cos s \partial_r + \frac{\sin s}{\sqrt{1-r}} (\partial_s - h \partial_\theta) \right) \\ &\quad + f(\theta + hs, \sqrt{1-r} e^{is}) \sqrt{1-r} \sin s \partial_\theta \end{aligned}$$

and

$$\begin{aligned} T\Phi(\mathbf{e}'_2) &= -r^2 \left(2\sqrt{1-r} \sin s \partial_r - \frac{\cos s}{\sqrt{1-r}} (\partial_s - h \partial_\theta) \right) \\ &\quad - f(\theta + hs, \sqrt{1-r} e^{is}) \sqrt{1-r} \cos s \partial_\theta. \end{aligned}$$

One particular case of interest will be when $C := S^1 \times \{0\} \subset S^1 \times D^2$ is a periodic Reeb orbit. Observe that the annulus

$$\{(s, r e^{-ihs}) : s \in S^1, 0 \leq r \leq 1\} \subset S^1 \times D^2$$

descends to a disc Δ in $S^3 = (S^1 \times D^2)/\sim$ with boundary $\partial\Delta = S^1 \times \{0\}$. Along the Reeb orbit C , the surface framing given by Δ defines a trivialization of $\ker \alpha|_{S^1 \times \{0\}}$. In a neighbourhood of C , the contact planes project isomorphically onto the tangent planes to the D^2 -factor. With respect to this projection, the oriented frame of the contact structure defined by Δ is then given by $(\partial_r, \partial_\theta)$ in a pointed neighbourhood of $S^1 \times \{0\}$.

5.2. *Conley–Zehnder indices.* We can now compute the Conley–Zehnder indices μ_{CZ} of the periodic Reeb orbits in some examples. Recall that a contact form on the 3-sphere is called *dynamically convex* if every periodic Reeb orbit has index $\mu_{CZ} \geq 3$ [21, Definition 1.2].

5.2.1. *Irrational ellipsoids.* We begin with the ellipsoids from §4.4, that is, we consider the Hamiltonian function $H(re^{i\theta}) = a_2r^2 + a_0$. The condition $a_0 > 0$ is equivalent to the contact condition (6'). The condition $a_0 + a_2 = h \in \mathbb{N}$ guarantees that $H > 0$. We assume $a_0 \in \mathbb{R}^+ \setminus \mathbb{Q}$. Then there are precisely two periodic Reeb orbits: the binding orbit B , and the central orbit $C = S^1 \times \{0\} \subset S^1 \times D^2$.

The meridian $\{0\} \times \partial D^2$ of $S^1 \times D^2$ may be taken as a representative of the binding orbit B , so B bounds the disc $\{0\} \times D^2$ in $S^1 \times D^2$. The vector field $T\Phi(\mathbf{e}'_1)$, for r near but different from 1, and for $s = 0$, takes the form

$$T\Phi(\mathbf{e}'_1) = -2r^2\sqrt{1-r} \partial_r.$$

This makes one positive twist with respect to the frame $(\mathbf{e}_1, \mathbf{e}_2)$ as we go once along the meridian.

Near B the Reeb vector field equals $\partial_b + \partial_\theta/a_0$, so as we go once along B , the Reeb flow defines a rotation through an angle $2\pi/a_0$ with respect to the frame $(\mathbf{e}'_1, \mathbf{e}'_2)$. Thus, with respect to the frame $(\mathbf{e}_1, \mathbf{e}_2)$, we have a rotation through an angle $2\pi(1 + (1/a_0))$.

By the definition of the Conley–Zehnder index μ_{CZ} (see [22, §8.1] or [23, §2.2]), we have $\mu_{CZ}(B) = 2n + 1$, where $n \in \mathbb{N}$ is the natural number determined by $1 + (1/a_0) \in (n, n + 1)$.

Near C the Reeb vector field equals $(\partial_s - a_2\partial_\theta)/a_0$. The normalization with return time 2π is $\partial_s - a_2\partial_\theta$. Thus, with respect to the frame $(\mathbf{e}_1, \mathbf{e}_2)$, we make $-a_2$ twists as we go once along the central orbit. The frame defined by the disc Δ makes $-h$ twists relative to $(\mathbf{e}_1, \mathbf{e}_2)$. It follows that the Reeb flow rotates through an angle $2\pi(h - a_2) = 2\pi a_0$ with respect to the surface framing. Finally, the frame of $\ker \alpha$ that extends over Δ is the one defined by $(\mathbf{e}'_1, \mathbf{e}'_2)$ near the centre of the disc. Up to positive factors, the projection of $T\Phi(\mathbf{e}'_1)$ onto the tangent planes to the D^2 -factor is of the form

$$-\cos s \partial_r + \sin s \partial_\theta.$$

As we go once along C , this makes one negative twist with respect to the frame $(\partial_r, \partial_\theta)$. It follows that the Reeb flow makes $a_0 + 1$ twists relative to the frame $(\mathbf{e}'_1, \mathbf{e}'_2)$ (or its image under $T\Phi$). This gives $\mu_{CZ}(C) = 2m + 1$, where $m \in \mathbb{N}$ is determined by $a_0 + 1 \in (m, m + 1)$.

We see that, no matter what choice we make for $a_0 \in \mathbb{R}^+ \setminus \mathbb{Q}$, one of the periodic orbits B, C has $\mu_{CZ} = 3$; the other has $\mu_{CZ} = 2n + 1$ with $n \geq 2$. For an earlier proof of this well-known result, see [20, Lemma 1.6].

5.2.2. *Irrational pseudorotations.* As we have seen in the proof of Lemma 4.5, the Hamiltonian function describing an irrational pseudorotation arises as the C^∞ -limit of Hamiltonians H_s^v which near ∂D^2 are given by

$$H_s^v(re^{i\theta}) = h + \frac{p_v}{q_v} - \frac{p_v}{q_v}r^2.$$

In fact, the conjugating diffeomorphisms φ_v in the Fayad–Katok construction equal the identity map also near $0 \in D^2$, so there we have the same description of H_s^v . These Hamiltonians give rise to functions f^v in the description (25) of $\Phi^*\alpha$ of the form

$$f^v(b, \rho e^{i\vartheta}) = \left(h + \frac{p_v}{q_v} \right) \cdot (2 - \rho^2).$$

It follows that the ∞ -jet of the limit Hamiltonian H_s^∞ along the central orbit C equals that of

$$H(re^{i\theta}) = h + a - ar^2,$$

and the ∞ -jet of the extended contact form along B equals that of

$$(1 - \rho^2)^2 db + (h + a)(2 - \rho^2)\rho^2 d\vartheta.$$

This is precisely the situation of the irrational ellipsoids with $a_0 = h + a$ and $a_2 = -a$. Recall from Lemma 4.3 that $h + a > 0$. Summarizing our arguments, we have the following result, which completes the proof of Theorem 1.2.

PROPOSITION 5.1. *The irrational pseudorotations of Fayad and Katok embed into a Reeb flow on S^3 whose periodic orbits B, C have Conley–Zehnder indices*

$$\mu_{CZ}(B) = 2n + 1 \quad \text{where } n \in \mathbb{N} \text{ is determined by } 1 + \frac{1}{h + a} \in (n, n + 1),$$

and

$$\mu_{CZ}(C) = 2m + 1 \quad \text{where } m \in \mathbb{N} \text{ is determined by } h + a + 1 \in (m, m + 1).$$

In particular, the contact form defining this Reeb flow is dynamically convex. □

Similar computations can be performed for general Hamiltonian functions H_s on D^2 that give rise to a contact form on S^3 . The considerations above suggest that the the Conley–Zehnder indices of periodic Reeb orbits corresponding to periodic points of the diffeomorphism ψ defined by H_s can be determined from the local behaviour of H_s near the periodic point in question.

Remark 5.2. For Reeb flows on the 3-sphere with two periodic orbits forming a Hopf link, Hryniewicz, Momin and Salomão [24, Theorem 1.2] describe a non-resonance condition that forces the existence of infinitely many periodic orbits. This condition is formulated in terms of the so-called transverse rotation number of the two given periodic orbits. Our argument leading to Proposition 5.1 shows that in the situation of that proposition the transverse rotation numbers are given by $\rho_0 = 1 + 1/(h + a)$ and $\rho_1 = 1 + h + a$, respectively. The numbers $\theta_i = \rho_i - 1$ defined in [24, Theorem 1.2] then become $\theta_0 = 1/(h + a)$ and $\theta_1 = h + a$. This means that the vectors $(\theta_0, 1)$ and $(1, \theta_1)$ in

\mathbb{R}^2 are proportional to each other, which is precisely the resonance condition that would have to be violated to guarantee infinitely many periodic Reeb orbits.

5.3. The self-linking number. A periodic Reeb orbit in a contact 3-manifold (M, α) constitutes a transverse knot K for the contact structure $\xi = \ker \alpha$. When K is homologically trivial in M , it bounds a Seifert surface Σ , over which the 2-plane field ξ is trivial. Choose a non-vanishing section Z of $\xi|_{\Sigma}$, and push K in the direction of $Z|_K$ to obtain a parallel copy K' of K . The self-linking number $\text{sl}(K, \Sigma)$ is then defined as the linking number of K and K' , that is, the oriented intersection number of K' and Σ ; see [13, Definition 3.5.28]. When the Euler class of ξ vanishes, the self-linking number is independent of the choice of Seifert surface; see [13, Proposition 3.5.30]. In that case, we write $\text{sl}(K)$ for the self-linking number.

Going back to the contact forms on S^3 found via a cut construction on $S^1 \times D^2$, the self-linking number $\text{sl}(B)$ of the binding orbit is defined. As Seifert surface we take the meridional disc $\Delta_0 := \{0\} \times D^2 \subset S^1 \times D^2$ as before, and the trivialization of $\ker \alpha|_{\{0\} \times D^2}$ given by \mathbf{e}_1 . Strictly speaking, we cannot push $B = \{0\} \times \partial D^2$ in the direction of \mathbf{e}_1 , but one can make sense of this as one passes to the quotient $S^3 = (S^1 \times D^2)/\sim$.

At the two points of B where $y = 0$, we have $\mathbf{e}_1 = H_s \partial_x = h \partial_x$, that is, \mathbf{e}_1 is a positive multiple of ∂_x . This means that at the point $(x, y) = (-1, 0)$, we push B into Δ_0 ; at $(x, y) = (1, 0)$, into the opposite disc in the open book $(S^1 \times D^2)/\sim$, that is, the one descending from $\{\pi\} \times D^2$. At all other points of B , the vector field \mathbf{e}_1 has a non-trivial ∂_s -component, which in the open book amounts to an angular component about the binding B .

Thus, the parallel knot B' intersects the meridional disc Δ_0 at a single point on the negative x -axis. In order to determine the sign of this intersection, we may consider the situation near this intersection point in $S^1 \times D^2$ rather than in the quotient S^3 . For $y > 0$, B' lies above the $\{s = 0\}$ -plane; for $y < 0$, it lies below. It follows that the intersection point of B' and Δ_0 is a negative one, that is, $\text{sl}(B) = -1$.

This accords with [25, Theorem 1.5]. That theorem establishes the conditions $\text{sl}(P) = -1$ and $\mu_{\text{CZ}}(P) \geq 3$ as necessary for a (simply covered) periodic Reeb orbit P to bound a disc-like global surface of section. The general assumption there is that P is an unknotted periodic Reeb orbit in S^3 for a contact form defining the standard contact structure.

Remark 5.3. Much of our discussion carries over to contact structures on lens spaces, provided we start with a Hamiltonian function H_s on $S^1 \times D^2$ invariant under rotations of the D^2 -factor about an angle $2\pi/p$. See [23] for a dynamical characterization of universally tight contact structures on lens spaces.

Also, one may replace $S^1 \times D^2$ by $S^1 \times \Sigma$, where Σ is any compact surface with boundary. As diffeomorphisms $\psi: \Sigma \rightarrow \Sigma$ we may take any Hamiltonian diffeomorphism whose generating Hamiltonian H_s satisfies criteria as in §4.9.

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