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SILVER ANTICHAINS

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Abstract. In this paper we investigate the structure of uncountable maximal antichains of Silver forcing and show that they have to be at least of size $\mathfrak{d}$, where $\mathfrak{d}$ is the dominating number. Part of this work can be used to show that the additivity of the Silver forcing ideal has size at least the unbounding number $\mathfrak{b}$. It follows that every reasonable amoeba Silver forcing adds a dominating real.

§1. Introduction and basic definitions. Silver forcing consists of the set of all Silver trees together with the inclusion ordering. Recall that a *Silver tree* is a perfect tree $T \subseteq 2^{<\omega}$, such that for all nodes $\sigma, \sigma' \in T$ of the same length and $i \in 2$ we have: $\sigma^\frown i \in T \Leftrightarrow \sigma'^\frown i \in T$. So on each level either all nodes go left, all nodes go right, or all nodes split at that level. The set of all Silver trees is denoted as Si.
We will identify a Silver tree with a function $f : dom(f) \to 2$ with $dom(f) \subseteq \omega$ and infinite codomain. Such f represents the Silver tree consisting of all nodes $\sigma \in 2^{<\omega}$ with $\sigma(n) = f(n)$ for $n \in dom(f)$. The respective ordering is reverse inclusion. Sometimes it is convenient to identify such f with $f' \in (2 \cup \{*\})^\omega$ with $f'^{-1}[\{*\}]$ being the codomain of f.

For $f, g \in Si$ Silver functions we write $f \parallel g$, if f and g are compatible, which is equivalent to $f \cup g \in Si$. If f and g are incompatible, we write $f \perp g$. Note that there are two reasons for $f, g \in Si$ to be incompatible: There exists $n \in dom(f) \cap dom(g)$ with $f(n) \neq g(n)$, or $(dom(f))^c \cap (dom(g))^c$ is finite.
For $f \in Si$ and $x \in (2 \cup \{*\})^\omega$ we write $f | x$ if f does not *contradict* x, that is, $\forall n \in dom(f) \cap dom(x)\ f(n) = x(n)$. If the opposite is true, we write $f \nmid x$.
An antichain $A \subseteq Si$ is a set of pairwise incompatible Silver conditions. It is the interplay of the two reasons for incompatibility just mentioned that makes it hard in general to understand the structure of maximal Silver antichains.
For any antichain A of Silver conditions we denote by
$A_{fin} := \{f \in A : |dom(f)| < \omega\}$ the set of finite conditions and by
$A_{inf} := \{f \in A : |dom(f)| = \omega\}$ the set of infinite conditions of the antichain.

In some situations we shall have an infinite subset $X \subseteq \omega$ and we need to consider Silver conditions relative to X, that is, $f \in Si$ with $X \setminus dom(f)$ infinite. The set of all

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these is denoted by $Si(X)$. For $f, g \in Si(X)$ we write $f \parallel_X g$ if f, g are compatible with respect to $Si(X)$ or $f \perp_X g$ otherwise.
For a given set X and a tree $T \subseteq X^{<\omega}$ by $[T]$ we denote the set of all infinite branches of T. Let $T(n) := \{\sigma \in T : |\sigma| = n\}$ and
$T \restriction n := \{\sigma \in T : |\sigma| \leq n\}$. For any $\sigma \in T$ let $T \restriction \sigma := \{\tau \in T : \tau \subseteq \sigma \vee \tau \supseteq \sigma\}$ be the subtree of T consisting of all nodes that are initial segments or extensions of σ. For a given tree T let $sp(T) := \{\sigma \in T : \exists i, j\ (i \neq j \wedge \sigma^\frown i, \sigma^\frown j \in T)\}$ be the set of all splitting nodes of T.
Furthermore, given a subset $Y \subseteq X^{<\omega}$, by $dwcl(Y)$ we denote the downward closure of Y, which is the tree $\{\sigma \in X^{<\omega} : \exists \tau \in Y\ \sigma \subseteq \tau\}$.
If we have some uncountable $Y \subseteq X^{\omega}$, then
$cond(Y) := \{\sigma \in X^{<\omega} : |\{y \in Y : \sigma \subseteq y\}| \geq \aleph_1\}$ is the condensation tree of Y. It is easy to see that then $Y \setminus [cond(Y)]$ is countable and $cond(Y)$ is a perfect tree.
For a given forcing P consisting of trees $T \subseteq 2^{<\omega}$, that are ordered by inclusion, its associated forcing ideal is defined as
$I(P) := \{X \subseteq 2^{\omega} : \forall p \in P\ \exists q \in P\ (q \leq p \wedge [q] \cap X = \emptyset)\}$.
It is easily seen that the sets of the form $X(A) = 2^{\omega} \setminus \bigcup\{[p] : p \in A\}$, where A is a maximal antichain of P, form a basis for $I(P)$. For all standard tree forcings P, $I(P)$ is a forcing ideal in the sense that a real is P generic iff it avoids all sets $X(A)$ for a maximal antichain A in the ground model. Recall that $add(I(P))$, $cov(I(P))$ is the minimal number of sets in $I(P)$ whose union is not in $I(P)$, is all of the underlying space, respectively. By $\mathcal{M}$ we denote the ideal of meager subsets of the reals.

A starting point for this research were two questions about the Silver ideal $I(Si)$ that had been asked by G. Laguzzi [2] and others. He had asked whether in ZFC the inequalities $add(I(Si)) \leq \mathfrak{b}$ and $add(I(Si)) \leq cov(\mathcal{M})$ are provable.
For Sacks forcing $\mathbb{S}$ it follows from results by P.Simon [5] and Judah, Miller, Shelah [1] that $add(I(\mathbb{S})) \leq \mathfrak{b}$ holds. It is implicit in [3], that an amoeba forcing for $\mathbb{S}$ can be constructed that does not add Cohen reals and, moreover, has the Laver property. Iterating this forcing one obtains a model for $cov(\mathcal{M}) < add(I(\mathbb{S}))$. In this paper we prove $add(I(Si)) \leq \mathfrak{b}$ in ZFC. The consistency of $cov(\mathcal{M}) < add(I(Si))$ remains open. We conjecture that it is true.
Even though the results seem to be analogous for $\mathbb{S}$ and Si, the methods of proof are not. For example, the main ingredient in [3] is the Halpern–Läuchli Theorem for Sacks trees, which is false for Silver trees. It is obvious that a good understanding of the structure of maximal antichains of a given tree forcing P is relevant for investigating $I(P)$ and its coefficients. We shall show that in this respect, Si behaves quite differently from $\mathbb{S}$. Note that both $\mathbb{S}$ and Si have countably infinite maximal antichains. Let f_n be the finite Silver condition with $dom(f_n) = n + 1$, $f_n(n) = 1$ and $f_n(i) = 0$ for $i < n$. Given some Silver condition $g \in (2 \cup \{*\})^{\omega}$ let n be minimal with $g(n) = 1 \vee g(n) = *$. Then $g \parallel f_n$ holds. On the other hand let $p_n \in \mathbb{S}$ be the full binary tree with stem f_n. Then $\{p_n : n < \omega\}$ is a maximal antichain in $\mathbb{S}$. Therefore it is natural to define $\mathfrak{a}(\mathbb{S}), \mathfrak{a}(Si)$, the *antichain number* of $\mathbb{S}$, Si, respectively, to be the minimal cardinality of an uncountable maximal antichain of $\mathbb{S}$, Si, respectively.
In [4] it has been shown that consistently $\mathfrak{a}(\mathbb{S}) < \mathfrak{c}$. We conjecture that $\mathfrak{a}(Si) = \mathfrak{c}$. We are able to prove this in two cases. The first case is when we restrict to maximal antichains containing no finite Silver functions. Note that by the above example

a finite Silver function corresponds in $\mathbb{S}$ to a full binary tree above some finite stem. The Roslanowski-Shelah example of an uncountable maximal antichain in $\mathbb{S}$ consists of skew trees, that is, trees having at most one splitnode on each level. Clearly such a tree is nowhere full binary. The second case in which we can show that a given maximal antichain $A \subseteq Si$ has size $\mathfrak{c}$ is when A contains uncountably many members that are pairwise incompatible by the second reason mentioned above, that is, these members are all subfunctions of a single $x \in 2^\omega$ and hence their codomains form an almost disjoint family. This result is the core of this paper and it is applied to prove $\mathfrak{d} \leq \mathfrak{a}(Si)$ in ZFC. In the last section $add(I(Si)) \leq \mathfrak{b}$ is proven. This only uses the result of the first case just mentioned. This implies that every reasonable amoeba forcing for Si adds a dominating real. Recall that an *amoeba* for Si is a forcing adding a Silver tree with the property that each of its branches is Si-generic to the ground model. As far as we know , the cardinal $\mathfrak{a}(\mathbb{S})$ has not been investigated except for [4]. In their model $\mathfrak{a}(\mathbb{S}) = \mathfrak{d}$ holds, and hence $\mathfrak{d} \leq \mathfrak{a}(\mathbb{S})$ is conceivable.

§2. Antichain number of Silver forcing.

We start with a well known fact.

LEMMA 2.1. *Let $\{X_\xi : \xi < \gamma < \mathfrak{c}\}$ be a family of sets $X_\xi \in [\omega]^\omega$. Then there exists a $X \in [\omega]^\omega$ with $|\omega \setminus X| = \omega$, such that $|X \cap X_\xi| = \omega$ for all $\xi < \gamma$.*

PROOF. Let $\{X_\xi : \xi < \gamma\}$ be as above. Let $T \subseteq \omega^{<\omega}$ be a perfect tree such that each natural number occurs exactly once and thus uniquely determines a node that has this number as its last value.

Pick some $x \in [T]$ that is not definable from $T \cup \{T\} \cup \{X_\xi : \xi < \gamma\}$. What is meant by this phrase is that we choose $\lambda \in OR$ sufficiently large (such that the relevant parameters belong to H_λ), and then we pick some $x \in (H_\lambda \cap [T]) \setminus N$ for some $N \prec H_\lambda$ with $T \cup \{T\} \cup \cdots \subseteq N$ and $|N| < \mathfrak{c}$. (Note, that from now on we will use this phrase to simplify the notation).

This implies that $\forall \xi < \gamma \; X_\xi \not\subseteq^* ran(x)$. Hence, if we define $X := \omega \setminus ran(x)$ we have $\forall \xi < \gamma \; |X \cap X_\xi| = \omega$ and $|\omega \setminus X| = \omega$. ⊣

Next we use this fact to handle the case of maximal antichains that solely consist of conditions with infinite domain.

THEOREM 2.2. *Let $A \subseteq Si$ be a set of Silver conditions with $|A| < \mathfrak{c}$ and $\forall f \in A \; |dom f| = \omega$. Let $\langle e_n : n \in \omega\rangle$ be an enumeration of $[\omega]^{<\omega}$ and let $A_n := \{f \in A : f^{-1}[\{1\}] = e_n\}$. If every A_n is an antichain, there exists $h \in Si$ such that $h \nmid f$ for every $f \in A$.*

PROOF. Let $B_n := \{(dom f)^c : f \in A_n\}$. Obviously every B_n is an a.d. family. Now, because we have countably many a.d. families, we can easily choose some $b \in [\omega]^\omega$ with the property that for all $n \in \omega$ there is at most one $c \in B_n$ with $|c \cap b| = \omega$, and if c is like that then actually $b \subseteq^* c$.
For all $f \in A$ with $|f^{-1}[\{1\}]| < \omega$ we have $f \in A_n$ for some $n \in \omega$.
Hence, by choice of b we have either $b \subseteq^* (dom f)^c$ from which we can conclude $|dom f \cap (\omega \setminus b)| = \omega$,
or $|b \cap (dom f)^c| < \omega$, hence $b \subseteq^* dom f$.

This enables us to define a partition of A as follows: Letting
$C_0 := \{f \in A : |f^{-1}[\{1\}]| < \omega \wedge b \subseteq^* dom(f)\}$,
$C_1 := \{f \in A : |f^{-1}[\{1\}]| < \omega \wedge b \subseteq^* (dom(f))^c\}$,

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$D_0 := \{f \in A : |f^{-1}[\{1\}]| = \omega \wedge |b \cap f^{-1}[\{1\}]| = \omega\}$,
$D_1 := \{f \in A : |f^{-1}[\{1\}]| = \omega \wedge |b \cap f^{-1}[\{1\}]| < \omega\}$,
we have $A = C_0 \dot\cup C_1 \dot\cup D_0 \dot\cup D_1$ and C_1 is countable.

By Lemma 2.1 we can pick some infinite $b_0 \subseteq b$ with $|b_0 \cap f^{-1}[\{1\}]| = \omega$ for all $f \in D_0$ as well as $|b \setminus b_0| = \omega$.
For $b_1 := b \setminus b_0$ we have $\forall f \in C_0\ b_1 \subseteq^* dom(f)$.

Next, let $\langle f_n : n \in \omega\rangle$ enumerate the elements of C_1 and construct a perfect tree $T \subseteq (\omega \setminus b)^{<\omega}$ with $\forall \sigma \in T \setminus \{\emptyset\}\ (\sigma(|\sigma| - 1) \in (\omega \setminus b) \cap f^{-1}_{|\sigma|}[\{0\}])$ and $\forall \sigma, \sigma' \in T \setminus \{\emptyset\}\ (\sigma \neq \sigma' \rightarrow (\sigma(|\sigma| - 1) \neq \sigma'(|\sigma'| - 1)))$, such that each natural number occurs in the tree at most once.
By our assumption $|A| < \mathfrak{c}$ we then pick some $b_2 \in [T]$ that is not definable by $T \cup \{T\} \cup D_1 \cup \{D_1\}$.

Assume by contradiction, that there exists $f \in D_1$, such that $f^{-1}[\{1\}] \subseteq^* b_2$, hence $f^{-1}[\{1\}] \setminus N \subseteq b_2$ for some $N \in \omega$.
Then by construction of the tree b_2 would be determined by
$b_2 = \bigcup\{\sigma \in T : \sigma(|\sigma| - 1) \in f^{-1}[\{1\}] \setminus N\}$, which is clearly not possible.

So in conclusion we get: $\forall f \in C_1\ f^{-1}[\{0\}] \cap b_2 \neq \emptyset$ by construction of our tree, as well as $\forall f \in D_1\ (|(\omega \setminus b) \setminus b_2 \cap f^{-1}[\{1\}]| = \omega)$ as stated above.
Again by Lemma 2.1 we can split up $(\omega \setminus b) \setminus b_2$ into two disjoint infinite sets b_3 and b_4 with $\forall f \in D_1\ |f^{-1}[\{1\}] \cap b_3| = \omega$.

If we define $h \in Si$ by:

$$h(n) := \begin{cases} 0 & \text{if } n \in b_0 \cup b_3, \\ 1 & \text{if } n \in b_1 \cup b_2, \\ * & \text{otherwise.} \end{cases}$$

we can easily check that $h \nmid A$. ⊣

In particular, we get the following corollary:

Corollary 2.3. *If $A \subseteq Si$ is an antichain of Silver conditions with $|A| < \mathfrak{c}$ and $\forall f \in A\ |dom f| = \omega$, then A is not maximal.*

As a consequence of Theorem 2.2 we can show that the additivity of the Silver ideal is at most $\mathfrak{b}$ (see Theorem 3.1 below).

The core of this paper is the following result which analyzes maximal Silver antichains that contain uncountably many elements which are pairwise incompatible by the second reason (see §1).

Theorem 2.4. *Let $A \subseteq Si$ be an antichain with $\omega_1 \leq |A| < \mathfrak{c}$. If for some real $x \in 2^\omega$ the set $\{f \in A : f|x\}$ is uncountable, then A is not maximal.*

Proof. We assume without loss of generality that
$\{f \in A : \forall n \in dom f\ f(n) = 0\}$ is uncountable. Let $\langle e_n : n \in \omega\rangle$ enumerate the set $\{f^{-1}[\{1\}] : f \in A_{inf} \wedge |f^{-1}[\{1\}]| < \omega\} \subseteq \omega^{<\omega}$ with $e_0 = \emptyset$ and as in the previous proof define
$A_n := \{f \in A_{inf} : f^{-1}[\{1\}] = e_n\}$ and $B_n := \{(dom f)^c : f \in A_n\}$. Clearly each B_n is an almost disjoint family and B_0 is uncountable.
Also for each $n \in \omega$ define $x_n \in 2^\omega$ by:

$$x_n(k) := \begin{cases} 1 & \text{if } k \in e_n, \\ 0 & \text{otherwise.} \end{cases}$$

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If for some $n \in \omega$ we have $b \in B_n$ and if it is clear which B_n we are referring to, we will often write f_b for a Silver condition that has b as its codomain with $f_b^{-1}[\{1\}] = e_n$.
We will also identify each $b \in B_n$ with its strictly increasing enumeration.

Let $T \subseteq \omega^{<\omega}$ be the tree on ω consisting of all $\sigma \in \omega^{<\omega}$ that are initial segments of uncountably many members of B_0.
Thus, $T := cond(\{\sigma \in \omega^{<\omega} : \exists b \in B_0\ \sigma \subseteq b\})$.
As B_0 is an uncountable a.d family, T is perfect.

Let us first sketch the basic idea of the proof in the special situation that $B_n = \emptyset$ for all $n \geq 1$. It is clear that for no $f \in A_{fin}$ can there be $\sigma \in T$ with $f^{-1}[\{1\}] \subseteq ran(\sigma)$. Hence every Silver condition that is constantly 0 outside some fixed branch of T is incompatible with every $f \in A_{fin}$.
We distinguish two cases:

Case 1:

$$\exists \sigma \in T\ \exists m \in \omega\ \exists b_0 \neq b_1 \in B_0 \cap [T \restriction \sigma]\ \forall f \in A_{fin}\ (f^{-1}[\{1\}] \subseteq ran(b_0) \cup ran(b_1) \\ \to f^{-1}[\{1\}] \cap m \neq \emptyset).$$

In this case choose b_0, b_1, and m as above and define a function $h \in (2 \cup \{*\})^\omega$ by:

$$h_0(n) := \begin{cases} 0 & \text{if } n \notin ran(b_0) \cup ran(b_1), \\ 0 & \text{if } n \in (ran(b_0) \cup ran(b_1)) \cap m, \\ * & \text{otherwise.} \end{cases}$$

Then obviously $h_0 \perp A_{fin}$ and because B_0 is an a.d. family we have for all $f, g \in A_{inf}$ that $|dom(f) \cap (b_0 \cup b_1)| = \omega$ and $f \perp_{(b_0 \cup b_1) \setminus m} g$. Hence, by the latter fact and by Theorem 2.2 we can pick $h_1 : (b_0 \cup b_1) \setminus m \to 2 \cup \{*\}$ with $h_1 \perp_{(b_0 \cup b_1) \setminus m} A_{inf}$. We can conclude that for $h := h_0 \cup h_1$ we have $h \perp A$.

Case 2: If the first case does not hold true, we can construct recursively $\langle \sigma_s : s \in 2^{<\omega} \rangle$ and $\langle f_s : s \in 2^{<\omega} \rangle$ with $\sigma_s \in T$ and $f_s \in A_{fin}$ for all $s \in 2^{<\omega}$ in the following way:
For a given $\sigma_s \in T$ we pick branches $b_0, b_1 \in [T]$ with $b_0 \neq b_1$ and $b_0, b_1 \supseteq \sigma_s$. Let n be the length of the common initial segment. Because we are not in Case 1 we can find $f_s \in A_{fin}$ with $f_s^{-1}[\{1\}] \subseteq ran(b_0) \cup ran(b_1)$ and $f_s^{-1}[\{1\}] \cap n = \emptyset$. Let $\sigma_{s^\frown 0}, \sigma_{s^\frown 1} \in T$ be sufficiently long initial segments of b_0 and b_1 with $f_s^{-1}[\{1\}] \subseteq ran(\sigma_{s^\frown 0}) \cup ran(\sigma_{s^\frown 1})$.
After we have got all σ_s, f_s let $T_0 := dwcl(\{\sigma_s : s \in 2^{<\omega}\})$. Choose a branch $x \in [T_0]$ that is not definable from $T \cup \{T\} \cup A \cup \{A\}$. Note that because x is a branch in T we have $f^{-1}\{1\} \cap (\omega \setminus x) \neq \emptyset$ for all $f \in A_{fin}$.
Thus, if we define $h_0 \in (2 \cup \{*\})^\omega$ by

$$h_0(n) := \begin{cases} 0 & \text{if } n \notin x, \\ * & \text{if } n \in x, \end{cases}$$

we have $h_0 \perp A_{fin}$.
Now, assume by contradiction that we have $b \in B_0$ with $b \supseteq^* ran(x)$. Then we know by choice of x that $\{y \in [T_0] : b \supseteq^* ran(y)\}$ is uncountable (otherwise x would be definable by b). Then there exists $N \in \omega$ with $\{y \in [T_0] : b \cup N \supseteq ran(y)\}$

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uncountable. Let T' the perfect condensation tree of the latter set. We can pick some σ_s that is a member of T' of length greater or equal to N, such that $\sigma_{s^\frown 0}$ and $\sigma_{s^\frown 1}$ are members of T' as well. For branches b_0, b_1 of T' with $ran(b_0) \supseteq ran(\sigma_{s^\frown 0})$ and $ran(b_1) \supseteq ran(\sigma_{s^\frown 1})$ we have by definition of T_0 that $(ran(b_0) \cup ran(b_1)) \setminus N \supseteq f_s^{-1}[\{1\}]$. Hence $b \supseteq f_s^{-1}[\{1\}]$. This implies $f_s \parallel f_b$, which is a contradiction.
Therefore, $\{f \restriction x : f \in A_{inf}\}$ is an antichain solely consisting of infinite conditions relative to x.
By Theorem 2.2 we can choose $h_1 \in Si(x)$ with $h_1 \perp_x A_{inf}$. For $h := h_0 \cup h_1$ we get $h \perp A$.

For the general case we can repeatedly apply this basic idea in a quite delicate recursion (to take care of all B_n) together with some new ideas, which in particular exploit the freedom we had to choose the two branches b_0, b_1 in Case 2.
We construct a subtree $T_0 \subseteq T$ by recursion in the following way:

At the beginning of the recursion:
Choose a splitting node $\tau \in sp(T)$ and $\overline{\sigma_0}, \overline{\sigma_1} \in T$, $\overline{\sigma_0}, \overline{\sigma_1} \supseteq \tau$ with $\overline{\sigma_0}(|\tau|) \neq \overline{\sigma_1}(|\tau|)$.
For $\overline{\sigma_0}$ fix $\overline{b} \in B_0 \cap [T \restriction \overline{\sigma_0}]$, which we call the "reference branch". First, we check whether there exist $f \in A_{fin}$ and $\sigma \in T$ with $\sigma \supseteq \overline{\sigma_1}$ and $f^{-1}[\{1\}] \subseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau|$.
If we find such σ, f, we define $\sigma_1 := \sigma$, $f_{0,\emptyset} := f$, $c_0 := \{\omega\} =: c_1$ and $\sigma_0 := \overline{b} \restriction k$ with $k \in \omega$ sufficiently large that $f^{-1}[\{1\}] \subseteq ran(\sigma_0) \cup ran(\sigma_1)$.
In addition set $\overline{T}^{(0)} := T \restriction \sigma_0$ and $\overline{T}^{(1)} := T \restriction \sigma_1$
If we are not able to find any such σ, f we try to construct a sequence $\langle \mu^{(m)} \mid m \in \omega\rangle$ recursively with
$\overline{\sigma_1} \subseteq \mu^{(0)} \subseteq \mu^{(1)} \subseteq \cdots$
and obtain by $y := \bigcup_{m \in \omega} \mu^{(m)} \in \omega^\omega$ an "opponent" to $\overline{b}$ as follows:
$m = 0$:

Case 1: For all $a \in B_1$ we have that $a \not\supseteq^* \overline{b}$:
In this case simply define $\mu^{(0)} := \overline{\sigma_1}$.
Case 2: There exists $a \in B_1$ with $a \supseteq^* \overline{b}$:
Case 2.1: $\forall \sigma \in T\ (\sigma \supseteq \overline{\sigma_1} \to\ f_a^{-1}[\{1\}] \not\subseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau|)$:
In this case define $\mu^{(0)} := \overline{\sigma_1}$.
Case 2.2: $\exists \sigma \in T\ (\sigma \supseteq \overline{\sigma_1} \wedge f_a^{-1}[\{1\}] \subseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau|)$:
Then let $\overline{\mu}^{(0)} \in T$ be a witness for this statement.
Consider the following two subcases:
Case 2.2.1: The set $X := \{b \in B_0 : b \in [T \restriction \overline{\mu}^{(0)}] \wedge |b \cap a| = \omega\}$ is uncountable:
In this case the construction of the $\mu^{(m)}$ stops.
We define $f_{0,\emptyset} :\equiv f_a$, $c_{\langle 0\rangle} := \{\omega\}$, $c_{\langle 1\rangle} := \{a\}$ and $\sigma_1 := \overline{\mu}^{(0)}$, as well as $\sigma_0 := \overline{b} \restriction k$ with $k \in \omega$ sufficiently large, such that $f_a^{-1}[\{1\}] \subseteq ran(\sigma_0) \cup ran(\sigma_1)$.
We also set $\overline{T}^{(0)} := T \restriction \sigma_0$ and $\overline{T}^{(1)} := cond(X)$
and go on with the construction of the tree T_0 (as described later).
Case 2.2.2: The X of the former case is countable:
Then choose some $\mu^{(0)} \in T$, $\mu^{(0)} \supseteq \overline{\mu}^{(0)}$, such that $|\overline{b}^c \cap ran(\mu^{(0)})| \geq 1$ and $|a^c \cap ran(\mu^{(0)})| \geq 1$. This is clearly possible since uncountably many $b \in B_0$ have $\overline{\mu}^{(0)}$ as an initial segment and B_0 is an a.d. family.

In this case we go on with the construction of the $\mu^{(m)}$.

$m \rightsquigarrow m+1$:

For $m \in \omega$ let $\mu^{(m)}$ be constructed:

Case 1: For all $a \in B_{m+2}$ we have $a \not\supseteq^* \overline{b}$:

In this case choose $\mu^{(m+1)} \in T,\ \mu^{(m+1)} \supseteq \mu^{(m)}$, such that $|ran(\mu^{(m+1)}) \cap \overline{b}^c| \geq m+2$ and such that for all $k \leq m$, where we had Case 2.2.2 witnessed by $\tilde{a} \in B_{k+1}$, we have: $|ran(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m+2-k$.

Case 2: There exists $a \in B_{m+2}$ with $a \supseteq^* \overline{b}$:

Case 2.1: $\forall \sigma \in T\ (\sigma \supseteq \mu^{(m)} \to f_a^{-1}[\{1\}] \nsubseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau|)$:

In this case also choose $\mu^{(m+1)} \in T,\ \mu^{(m+1)} \supseteq \mu^{(m)}$, such that $|ran(\mu^{(m+1)}) \cap \overline{b}^c| \geq m+2$ and such that for all $k \leq m$, where we had Case 2.2.2 witnessed by $\tilde{a} \in B_{k+1}$, we have: $|ran(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m+2-k$.

Case 2.2: $\exists \sigma \in T\ (\sigma \supseteq \mu^{(m)} \wedge f_a^{-1}[\{1\}] \subseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau|)$:

Then let $\overline{\mu}^{(m+1)}$ be a witness.

Again we have two subcases:

Case 2.2.1: The set $X := \{b \in B_0 :\ b \in [T \restriction \overline{\mu}^{(m+1)}] \wedge |b \cap a| = \omega\}$ is uncountable:

Then the construction of the $\mu^{(m)}$ stops.

Define $f_{0,\emptyset} :\equiv f_a,\ c_{\langle 0 \rangle} := \{\omega\},\ c_{\langle 1 \rangle} := \{a\}$

and $\sigma_1 := \overline{\mu}^{(m+1)}$ and $\sigma_0 := \overline{b} \restriction k$ with $k \in \omega$ sufficiently large, such that $f_a^{-1}[\{1\}] \subseteq ran(\sigma_0) \cup ran(\sigma_1)$.

Also define: $\overline{T}^{(0)} := T \restriction \sigma_0$ and $\overline{T}^{(1)} := cond(X)$

Case 2.2.2: The set X defined above is countable:

Then choose $\mu^{(m+1)} \in T,\ \mu^{(m+1)} \supseteq \overline{\mu}^{(m+1)}$, such that:

$|ran(\mu^{(m+1)}) \cap a^c| \geq 1$, $|ran(\mu^{(m+1)}) \cap \overline{b}^c| \geq m+2$ and for all $k \leq m$, where we had Case 2.2.2 witnessed by $\tilde{a} \in B_{k+1}$, we have $|ran(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m+2-k$.

Let us first consider the situation, where the construction of the $\mu^{(m)}$ does not stop (which means that Case 2.2.1 does not occur):

Define by $y := \bigcup_{m \in \omega} \mu^{(m)}$ the opponent to $\overline{b}$. By construction the following propositions are true:

i) For all $f \in A_{fin}$ we have:
$f^{-1}[\{1\}] \cap |\tau| \neq \emptyset$ or $f^{-1}[\{1\}] \cap (\omega \setminus (y \cap \overline{b})) \neq \emptyset$.

ii) For $f = f_d$ with $d \in B_{k+1}$ for some $k \in \omega$ we have:
$|\overline{b} \cap dom(f_d)| = \omega$ (Case 1 or Case 2 with $a \supseteq^* \overline{b},\ a \neq d$)
or $f_d^{-1}[\{1\}] \cap |\tau| \neq \emptyset$ or $f_d^{-1}[\{1\}] \cap (\omega \setminus \overline{b} \cup y) \neq \emptyset$ (Case 2.1 with $d \supseteq^* \overline{b}$)
or $|y \cap dom(f_d)| = \omega$ (Case 2.2.2 with $d \supseteq^* \overline{b}$).

iii) For $f = f_{\overline{b}}$ we have $|y \cap dom(f)| = \omega$.

iv) For $f = f_d$ with $d \in B_0 \setminus \{\overline{b}\}$ we have $|dom(f) \cap \overline{b}| = \omega$.

v) For $f \in A$ with $|f^{(-1)}[\{1\}]| = \omega$ the following holds: $f^{-1}[\{1\}] \cap (\omega \setminus (y \cup \overline{b})) \neq \emptyset$ or $|dom(f) \cap (y \cup \overline{b})| = \omega$.

Thus, if we define $h \in Si$ by $h \restriction (\omega \setminus (\overline{b} \cup y)) \cup |\tau| :\equiv 0$, each condition of the antichain is either incompatible to h or is an infinite Silver condition relativized to $(\overline{b} \cup y) \setminus |\tau|$. This means, we can use Corollary 2.3 to obtain $h \in Si$ with $h \perp A$.

$\Rightarrow A$ is not maximal.

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If on the other hand we did not start to construct an opponent or the construction of the opponent stops (Case 2.2.1), we go on with the construction of T_0 in the following way:

For $i \in 2$ we have either:

Case a: There exists $a \in B_1$ with $X := \{b \in B_0 \cap [\overline{T}^{(i)}] : |b \cap \bigcup c_i| = \omega \wedge |b \cap a| = \omega\}$ uncountable:
In this case define $z_{\langle i \rangle} := a$ and $T^{(\langle i \rangle)} := cond(X)$ $(\subseteq \overline{T}^{(i)})$.
Or:
Case b:
$\forall a \in B_1$ $\{b \in B_0 \cap [\overline{T}^{(i)}] : |b \cap \bigcup c_i| = \omega \wedge |b \cap a| = \omega\}$ is countable:
In this case define $z_{\langle i \rangle} := *$ and $T^{(\langle i \rangle)} := \overline{T}^{(i)}$.

Remark: Because $Z := \{b \in B_0 : |b \cap \bigcup c_i| = \omega\}$ is uncountable and $\overline{T}^{(i)} = cond(Z)$ it follows in case b, that for all $a \in B_1$ the set
$\{\sigma \in T^{(\langle i \rangle)} : \sigma(|\sigma| - 1) \in a^c\}$ is dense in $T^{(\langle i \rangle)}$.
In fact more is true:
For each subtree $T' \subseteq T^{(\langle i \rangle)}$ with the property
$\forall \sigma \in T'$ $([T' \restriction \sigma] \cap Z$ is uncountable$)$
we have $\forall a \in B_1$ $\{\sigma \in T' : \sigma(|\sigma| - 1) \in a^c\}$ is dense in T'.
(Note that in case b all subtrees $T^{(s)}$ with $s \in 2^{<\omega}$ and $s(0) = i$ in the following construction actually will have the above property.)

This finishes the first step in the construction of T_0.

Suppose that for some $n \in \omega$ and for all $s \in 2^n$ all σ_s together with the c_t and z_t for all $t \in 2^{\leq n} \setminus \emptyset$ have been constructed
such that for each $s \in 2^n$ we have gotten $T^{(s)} = cond(X)$ with uncountable

$$X = \{b \in B_0 : b \supseteq \sigma_s \wedge \forall 1 \leq k \leq |s| \forall d \in c_{s \restriction k} \ (|b \cap d| = \omega) \\ \wedge \forall 1 \leq k \leq |s| \ ((z_{s \restriction k} \neq *) \rightarrow |b \cap z_{s \restriction k}| = \omega)\}.$$

Now fix some $s \in 2^n$.

Choose a splitting node $\tau_s \in sp(T^{(s)})$, such that
$ran(\tau_s \restriction [|\sigma_s|, |\tau_s|)) \cap d \neq \emptyset$ for all $\emptyset \neq t \subseteq s,\ d \in c_t$
and that $ran(\tau_s \restriction [|\sigma_s|, |\tau_s|)) \cap z_t \neq \emptyset$ for all $\emptyset \neq t \subseteq s$ with $z_t \neq *$.
(This is possible by the construction of $T^{(s)}$.)
Next, choose $\overline{\sigma}_{s^\frown 0}, \overline{\sigma}_{s^\frown 1} \in T^{(s)}$ with $\overline{\sigma}_{s^\frown 0}, \overline{\sigma}_{s^\frown 1} \supseteq \tau_s$
and $\overline{\sigma}_{s^\frown 0}(|\tau_s|) \neq \overline{\sigma}_{s^\frown 1}(|\tau_s|)$.

Again we take a reference branch and try to construct an opponent (with respect to B_0):
First choose a reference branch $\overline{b} \in [T^{(s)} \restriction \overline{\sigma}_{s^\frown 0}] \cap B_0$.
Check if there exists $f \in A_{fin}$ and $\sigma \in T^{(s)}$, $\sigma \supseteq \overline{\sigma}_{s^\frown 1}$, such that
$f^{-1}[\{1\}] \subseteq (\overline{b} \cup ran(\sigma)) \setminus |\tau_s|$.
If we find such σ, f, define $\sigma^{(0)}_{s^\frown 1} := \sigma$, $f_{0,s} := f$ and $\sigma^{(0)}_{s^\frown 0} := \overline{b} \restriction k$ for a sufficiently large $k \in \omega$.
In this case the construction of the opponent (w.r.t. B_0) stops.

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Otherwise try to construct recursively the sequence $\langle \mu^{(m)} \mid m \in \omega\rangle$ as follows:
$m = 0$:
Case 1: For all $a \in B_1$ we have $a \not\supseteq^* \overline{b}$:
Define $\mu^{(0)} := \overline{\sigma}_{s^\frown 1}$
Case 2: There exists $a \in B_1$ with $a \supseteq^* \overline{b}$:
Case 2.1: $\forall \sigma \in T^{(s)}\ (\sigma \supseteq \overline{\sigma}_{s^\frown 1} \rightarrow f_a^{-1}[\{1\}] \nsubseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau_s|)$:
Define $\mu^{(0)} := \overline{\sigma}_{s^\frown 1}$.
Case 2.2: $\exists \sigma \in T^{(s)}\ (\sigma \supseteq \overline{\sigma}_{s^\frown 1} \wedge f_a^{-1}[\{1\}] \subseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau_s|)$:
Then let $\overline{\mu}^{(0)}$ be a witness.
Case 2.2.1:

$$\begin{aligned} X := \{b \in B_0 \cap [T^{(s)} \restriction \overline{\mu}^{(0)}] :\ & \forall \emptyset \neq t \subseteq s \forall d \in c_t\ (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s\ (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\ & \wedge |b \cap a| = \omega\} \end{aligned}$$

is uncountable:
In this case the construction of the $\mu^{(m)}$ stops.
Define $f_{0,s} := f_a$, $\sigma^{(0)}_{s^\frown 1} := \overline{\mu}^{(0)}$, $\sigma^{(0)}_{s^\frown 0} := \overline{b} \restriction k$ with $k \in \omega$ sufficiently large.
Also define $\overline{T}_0^{(s^\frown 0)} := T^{(s)} \restriction \sigma^{(0)}_{s^\frown 0}$ and $\overline{T}_0^{(s^\frown 1)} := cond(X)$.
Case 2.2.2: The set X of the previous case is countable: Then choose $\mu^{(0)} \in T^{(s)}$, $\mu^{(0)} \supseteq \overline{\mu}^{(0)}$, such that:
$|ran(\mu^{(0)}) \cap \overline{b}^c| \geq 1$ and $|ran(\mu^{(0)}) \cap a^c| \geq 1$.

Remark: Because of the definition of $T^{(s)}$ we can conclude in this case, that $\{\sigma \in T^{(s)} \restriction \overline{\mu}^{(0)} :\ \sigma(|\sigma| - 1) \in a^c\}$ is dense in $T^{(s)} \restriction \overline{\mu}^{(0)}$.
So we can arrange in the further construction of the $\mu^{(m)}$, that the opponent (if it exists) will have an infinite intersection with a^c.
$m \rightsquigarrow m + 1$:
Let $\mu^{(m)}$ be constructed for a $m \in \omega$:
Case 1: $\forall a \in B_{m+2}\ a \not\supseteq^* \overline{b}$:
In this case choose $\mu^{(m+1)} \in T^{(s)}$, $\mu^{(m+1)} \supseteq \mu^{(m)}$, such that
$|ran(\mu^{(m+1)}) \cap \overline{b}^c| \geq m + 2$
and for all $k \leq m$, where we had Case 2.2.2 witnessed by some $\tilde{a} \in B_{k+1}$, we have $|ran(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m + 2 - k$.
Case 2: $\exists a \in B_{m+2}\ a \supseteq^* \overline{b}$:
Case 2.1: $\forall \sigma \in T^{(s)}\ (\sigma \supseteq \mu^{(m)} \rightarrow f_a^{-1}[\{1\}] \nsubseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau_s|)$:
In this case choose $\mu^{(m+1)}$ the same way as in Case 1.
Case 2.2: $\exists \sigma \in T^{(s)}\ (\sigma \supseteq \mu^{(m)} \wedge f_a^{-1}[\{1\}] \subseteq (ran(\sigma) \cup \overline{b}) \setminus |\tau_s|)$
In this case let $\overline{\mu}^{(m+1)}$ be a witness.
Case 2.2.1:

$$\begin{aligned} X := \{b \in B_0 \cap [T^{(s)} \restriction \overline{\mu}^{(m+1)}] :\ & \forall \emptyset \neq t \subseteq s \forall d \in c_t\ (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s\ (z_t \neq * \rightarrow (|b \cap z_t| = \omega) \\ & \wedge |b \cap a| = \omega\} \end{aligned}$$

is uncountable:
In this case the construction of the $\mu^{(m)}$ stops.

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Define $f_{0,s} := f_a$, $\sigma^{(0)}_{s\frown 1} := \overline{\mu}^{(m+1)}$, $\sigma^{(0)}_{s\frown 0} := \overline{b} \restriction k$ with $k \in \omega$ sufficiently large. Also define $\overline{T}^{(s\frown 0)}_0 := T^{(s)} \restriction \sigma^{(0)}_{s\frown 0}$ and $\overline{T}^{(s\frown 1)}_0 := cond(X)$.

Case 2.2.2: The set X of the previous case is countable: In this case choose $\mu^{(m+1)} \in T^{(s)}$, $\mu^{(m+1)} \supseteq \overline{\mu^{(m+1)}}$, such that
$|ran(\mu^{(m+1)}) \cap \overline{b}^c| \geq m+2$,
$|ran(\mu^{(m+1)}) \cap a^c| \geq 1$,
and for all $k \leq m$, where we had Case 2.2.2 witnessed by some $\tilde{a} \in B_{k+1}$ we have $|ran(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m+2-k$.

REMARK: In this case we also have that $\{\sigma \in T^{(s)} \restriction \overline{\mu}^{(m+1)} :\ \sigma(|\sigma|-1) \in a^c\}$ is dense in $T^{(s)} \restriction \overline{\mu}^{(m+1)}$.

Now, if the construction of the $\mu^{(m)}$ is successful, we can define our opponent by $y := \bigcup_{m\in\omega} \mu^{(m)}$. Again it is true, that for all $f \in A$ we have $f^{-1}[\{1\}] \cap ((\omega \setminus (y \cup \overline{b})) \cup |\tau_s|) \neq \emptyset$ or $|dom(f) \cap (\overline{b} \cup y)| = \omega$. As before we can find $h \in Si$ with $h \perp A$. Hence A is not maximal.

If on the other hand the construction of the opponent stops, we try to construct opponents with respect to $B_{|t|}$ for all $t \subseteq s$ with $t \neq \emptyset$ and $z_t = *$ in the following sense:

Assume that for some $i < |s|$ and for all $j \leq i$ the $f_{j,s}$, $\sigma^{(j)}_{s\frown 0}$, and $\sigma^{(j)}_{s\frown 1}$ have been constructed together with $\overline{T}^{(s\frown 0)}_j = T^{(s)} \restriction \sigma^{(j)}_{s\frown 0}$ and $\overline{T}^{(s\frown 1)}_j = cond(X)$ for uncountable

$$\begin{aligned} X = \{b \in B_0 \cap [\overline{T}^{(s\frown 1)}_{j-1} \restriction \sigma^{(j)}_{s\frown 1}] : {} & \forall \emptyset \neq t \subseteq s \forall d \in c_t\ (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s\ (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\ & \wedge \forall 1 \leq l \leq j\ (z_{s\restriction l} = * \rightarrow |b \cap dom(f_{l,s})^c| = \omega)\} \end{aligned}$$

and assume that $z_{s\restriction i+1} = *$. (Otherwise just set $\sigma^{(i+1)}_{s\frown k} := \sigma^{(i)}_{s\frown k}$ and define $\overline{T}^{(s\frown k)}_{i+1}$ accordingly.)
We know because of the fact that $z_{s\restriction i+1} = *$ and the respective remark that for all $e \in B_{i+1}$ the set $\{\sigma \in \overline{T}^{(s\frown 1)}_i :\ \sigma(|\sigma|-1) \in e^c\}$ is dense in $\overline{T}^{(s\frown 1)}_i$.
Pick some reference branch $\overline{b} \in B_0 \cap [\overline{T}^{(s\frown 0)}_i]$ and check if there exist $f \in A_{fin}$, $\sigma \in \overline{T}^{(s\frown 1)}_i$ with $\{k \in \omega :\ f(k) \neq x_{i+1}(k)\} \subseteq (\overline{b} \cup ran(\sigma)) \setminus |\tau_s|$.
If such f, σ exist define $f_{i+1,s} := f$, $\sigma^{(i+1)}_{s\frown 1} := \sigma$ and $\sigma^{(i+1)}_{s\frown 0} := \overline{b} \restriction k$ for $k \in \omega$ sufficiently large, as well as $\overline{T}^{(s\frown 0)}_{i+1} := \overline{T}^{(s\frown 0)}_i \restriction \sigma^{(i+1)}_{s\frown 0}$ and $\overline{T}^{(s\frown 1)}_{i+1} := \overline{T}^{(s\frown 1)}_i \restriction \sigma^{(i+1)}_{s\frown 1}$.
If such f, σ do not exist we try again to construct an opponent sequence $\langle \mu^{(m)} |\ m \in \omega \rangle$ by recursion:

Before we start the recursion we consider two different cases:

(i) $\forall e \in B_{i+1}\ e \not\supseteq^* \overline{b}$,
(ii) $\exists! e \in B_{i+1}\ e \supseteq^* \overline{b}$. (remember that B_{i+1} is an a.d.-family)

These two cases will be handled slightly differently in the following recursion.
Also pick some bijection $\phi : \omega \rightarrow \omega \setminus \{0, i+1\}$.

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$m = 0$:
Case 1: $\forall a \in B_{\phi(0)}\ a \not\supseteq^* \overline{b}$:
In this case just set $\mu^{(0)} := \sigma^{(i)}_{s\frown 1}$.
Case 2: $\exists a \in B_{\phi(0)}\ a \supseteq^* \overline{b}$:
Case 2.1:
$\forall \sigma \in \overline{T}^{(s\frown 1)}_i, \sigma \supseteq \sigma^{(i)}_{s\frown 1}\ (\{k \in \omega :\ f_a(k) \neq x_{i+1}(k)\} \nsubseteq (\overline{b} \cup ran(\sigma)) \setminus |\tau_s|)$:
In this case set $\mu^{(0)} := \sigma^{(i)}_{s\frown 1}$.
Case 2.2:
Otherwise let $\overline{\mu}^{(0)} \in \overline{T}^{(s\frown 1)}_i$, $\overline{\mu}^{(0)} \supseteq \sigma^{(i)}_{s\frown 1}$ with
$\{k \in \omega :\ f_a(k) \neq x_{i+1}(k)\} \subseteq (\overline{b} \cup ran(\overline{\mu}^{(0)})) \setminus |\tau_s|$.
Case 2.2.1:

$$\begin{aligned} X := \{b \in B_0 \cap [\overline{T}^{(s\frown 1)}_i \restriction \overline{\mu}^{(0)}] : &\forall \emptyset \neq t \subseteq s \forall d \in c_t\ (|b \cap d| = \omega) \\ &\wedge \forall \emptyset \neq t \subseteq s\ (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\ &\wedge \forall 1 \leq l \leq i\ (z_{s\restriction l} = * \rightarrow (|b \cap (dom(f_{l,s}))^c| = \omega) \\ &\wedge |b \cap a| = \omega\} \end{aligned}$$

is uncountable:

Then define $f_{i+1,s} := f_a$, $\sigma^{(i+1)}_{s\frown 1} := \overline{\mu}^{(0)}$, $\sigma^{(i+1)}_{s\frown 0} := \overline{b} \restriction k$ for $k \in \omega$ sufficiently large, such that $\{k \in \omega :\ f_a(k) \neq x_{i+1}(k)\} \subseteq (ran(\sigma^{(i+1)}_{s\frown 0}) \cup ran(\sigma^{(i+1)}_{s\frown 1})) \setminus |\tau_s|$.
Also define $\overline{T}^{(s\frown 0)}_{i+1} := \overline{T}^{(s\frown 0)}_i \restriction \sigma^{(i+1)}_{s\frown 0}$ and $\overline{T}^{(s\frown 1)}_{i+1} := cond(X)$.
The construction of the opponent stops in this case.
Case 2.2.2: The set X of the previous case is countable:
Then choose $\mu^{(0)} \in \overline{T}^{(s\frown 1)}_i$, $\mu^{(0)} \supseteq \overline{\mu}^{(0)}$ with
$|ran(\mu^{(0)}) \cap \overline{b}^c| \geq 1$, $|ran(\mu^{(0)}) \cap a^c| \geq 1$ (and $ran(|\mu^{(0)}) \cap e^c| \geq 1$ in case of (ii)).
We also obtain our usual density property.

$m \rightsquigarrow m+1$:
Let $\mu^{(m)}$ be constructed for some $m \in \omega$:
Case 1: $\forall a \in B_{\phi(m+1)}\ a \not\supseteq^* \overline{b}$:
In this case choose $\mu^{(m+1)} \in \overline{T}^{s\frown 1}_i$, $\mu^{(m+1)} \supseteq \mu^{(m)}$
with
$|ran(\mu^{(m+1)}) \cap \overline{b}^c| \geq m+2$ (and $|ran(\mu^{(m+1)}) \cap e^c| \geq m+2$ in the case of (ii))
and for each $k \leq m$, where we had 2.2.2 witnessed by $\tilde{a} \in B_{\phi(k)}$, we have: $|ran(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m+2-k$.
Case 2: There exists $a \in B_{\phi(m+1)}$ with $a \supseteq^* \overline{b}$:
Case 2.1:
$\forall \sigma \in \overline{T}^{(s\frown 1)}_i\ (\sigma \supseteq \mu^{(m)} \rightarrow \{k \in \omega :\ f_a(k) \neq x_{i+1}(k)\} \nsubseteq (\overline{b} \cup ran(\sigma)) \setminus |\tau_s|)$:
Pick $\mu^{(m+1)}$ as in Case 1.
Case 2.2:
$\exists \sigma \in \overline{T}^{(s\frown 1)}_i\ (\sigma \supseteq \mu^{(m)} \wedge \{k \in \omega :\ f_a(k) \neq x_{i+1}(k)\} \subseteq (\overline{b} \cup ran(\sigma)) \setminus |\tau_s|)$:
Then let $\overline{\mu}^{(m+1)}$ be a witness.

Case 2.2.1:

$$\begin{aligned} X := \{b \in B_0 \cap [\overline{T}_i^{(s^\frown 1)} \restriction \overline{\mu}^{(m+1)}] : {}& \forall \emptyset \neq t \subseteq s \forall d \in c_t \, (|b \cap d| = \omega) \\ & \wedge \forall \emptyset \neq t \subseteq s \, (z_t \neq * \rightarrow |b \cap z_t| = \omega) \\ & \wedge \forall 1 \leq l \leq i \, (z_{s\restriction l} = * \rightarrow |b \cap (dom(f_{l,s}))^c| = \omega) \\ & \wedge |b \cap a| = \omega\} \end{aligned}$$

is uncountable:
In this case define $f_{i+1,s} := f_a$, $\sigma^{(i+1)}_{s\frown 1} := \overline{\mu}^{(m+1)}$ and $\sigma^{(i+1)}_{s\frown 0} := \overline{b} \restriction k$ for $k \in \omega$ sufficiently large. Also set $\overline{T}^{(s^\frown 0)}_{i+1} := \overline{T}^{(s^\frown 0)}_i \restriction \sigma^{(i+1)}_{s\frown 0}$ and $\overline{T}^{(s^\frown 1)}_{i+1} := cond(X)$ and stop the construction of the opponent.
Case 2.2.2: The set X of the previous case is countable:
Then choose $\mu^{(m+1)} \in \overline{T}^{(s\frown 1)}_i$, $\mu^{(m+1)} \supseteq \overline{\mu}^{(m+1)}$ with $|ran(\mu^{(m+1)}) \cap \overline{b}^c| \geq m+2$ (and $|ran(\mu^{(m+1)}) \cap e^c| \geq m+2$ in the case of (ii)) and for each $k \leq m$, where had Case 2.2.2 witnessed by $\tilde{a} \in B_{\phi(k)}$, we have $|ran(\mu^{(m+1)}) \cap \tilde{a}^c| \geq m+2-k$, as well as $|ran(\mu^{(m+1)}) \cap a^c| \geq 1$.

Remark: Again we get $\{\sigma \in \overline{T}^{(s^\frown 1)}_i \restriction \overline{\mu}^{(m+1)} : \ \sigma(|\sigma|-1) \in a^c\}$ is dense in $\overline{T}^{(s^\frown 1)}_i \restriction \overline{\mu}^{(m+1)}$.

If the construction of the opponent $y := \bigcup_{m\in\omega} \mu^{(m)}$ is successful, we have the following situation:
$|\overline{b}^c \cap y| = \omega$ and for $d \in B_0$, $d \neq \overline{b}$, we have $|\overline{b} \cap d^c| = \omega$.
Also we either have that
for all $d \in B_{i+1}$ holds $|\overline{b} \cap d^c| = \omega$ (Case (i))
or else $|y \cap e^c| = \omega \wedge \forall d \in B_{i+1} \, (d \neq e \rightarrow |\overline{b} \cap d^c| = \omega)$. (Case (ii)).
Analogously as with the previous opponents we have for all $j \notin \{0, i+1\}$ and $d \in B_j$ that either
$\exists k \in (\omega \setminus (\overline{b} \cup y) \cup |\tau_s|) \cap dom(f_d) \, (f_d(k) \neq x_{i+1}(k))$
or $|dom(f_d) \cap ((\overline{b} \cup y) \setminus |\tau_s|)| = \omega$.
If we define $h_0 \in Si$ by $dom(h_0) := \omega \setminus (\overline{b} \cup y) \cup |\tau_s|$ and
$h_0 \restriction dom(h_0) :\equiv x_{i+1} \restriction dom(h_0)$ and, by Corollary 2.3, choose some $h_1 \in Si((\overline{b} \cup y) \setminus |\tau_s|)$ with $h_1 \perp_{(\overline{b} \cup y)\setminus|\tau_s|} \{f \in A : \ f \parallel h_0\}$, we have $h \perp A$, where $h := h_0 \cup h_1 \in Si$. Hence A is not maximal.

If on the other hand the construction of the $\mu^{(m)}$ stops, then go on with the construction of T_0 (or the construction of the next opponent). If for all $i < s$ with $z_{s\restriction i+1} = *$ the constructions of the opponents (with respect to $B_{|i+1|}$) fail, we have constructed in particular $\sigma^{(|s|)}_{s\frown 0}$, $\sigma^{(|s|)}_{s\frown 1}$, $\overline{T}^{(s\frown 0)}_{|s|}$ and $\overline{T}^{(s\frown 1)}_{|s|}$. In this case set $\sigma_{s^\frown 0} := \sigma^{(|s|)}_{s\frown 0}$, $\sigma_{s\frown 1} := \sigma^{(|s|)}_{s\frown 1}$ and $\overline{T}^{(s\frown 0)} := \overline{T}^{(s\frown 0)}_{|s|}$, $\overline{T}^{(s\frown 1)} := \overline{T}^{(s\frown 1)}_{|s|}$ and define $c_{s\frown 0} := \{\omega\}$ and $c_{s\frown 1} := \{(dom(f_{i,s}))^c : \ (1 \leq i \leq |s| \wedge \ z_{s\restriction i} = *) \vee i = 0\}$.

Also in this case for each $j \in 2$ we have to check two different cases:
Case a:
There is an $a \in B_{|s|+1}$ with:

$$X := \{b \in B_0 \cap [\overline{T}^{(s^\frown j)}] : \forall \emptyset \neq t \subseteq s^\frown j \forall d \in c_t\ (|b \cap d| = \omega) \\ \wedge \forall \emptyset \neq t \subseteq s\ (z_t \neq * \to |b \cap z_t| = \omega) \\ \wedge |b \cap a| = \omega\}$$

is uncountable:
Then define $z_{s^\frown j} := a$ and $T^{(s^\frown j)} := cond(X)$.
Case b:
Otherwise just define $T^{(s^\frown j)} := \overline{T}^{(s^\frown j)}$ and $z_{s^\frown j} := *$.
This finishes the recursion step in the construction of the $\sigma_{s^\frown j},\ T^{(s^\frown j)}$.

If for no $s \in 2^{<\omega}$ the construction of the opponents is successful, then define $T_0 := dwcl(\{\sigma_s : s \in 2^{<\omega}\})$, which is obviously a perfect subtree of T. Let $x \in [T_0]$ be a branch of the tree that is not definable from $A \cup \{A\} \cup T_0 \cup \{T_0\}$. Associated to x is a real $r \in 2^\omega$ by $r := \bigcup\{s : \sigma_s \subseteq x\}$.
Now let $f \in A$ be an arbitrary Silver function of the antichain. We analyze the different classes that f can be a member of:

i) For $f = f_d$ for a $d \in B_k$ with $k \in \omega \setminus \{\emptyset\}$ these are the following subcases:
Case 1: $z_{r \restriction k} = a \in B_k$:
Assume that $f_a^{-1}[\{1\}] \subseteq x$ would hold and choose some $n \in \omega$ with $x \restriction n \supseteq f_a^{-1}[\{1\}]$. We have $x \in [T^{(r \restriction k)}]$ and therefore $x \restriction n \in T^{(r \restriction k)}$, and hence by definition of the tree and the fact that $z_{r \restriction k} = a$ we know, that there exist uncountably many $b \in B_0$ with $b \supseteq x \restriction n \supseteq f_a^{-1}[\{1\}]$ and $|b \cap a| = \omega$. But then $f_a \parallel f_b$, which is clearly a contradiction. So we can conclude $f_a^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$.
On the other hand, we also know by the fact that $x \in [T_0 \restriction \sigma_{r \restriction k}]$ with $z_{r \restriction k} = a$ and the construction of T_0 that $|x \cap a| = \omega$. For any $d \in B_k$ with $d \neq a$ we have $|dom(f_d)^c \cap a| < \omega$, hence we get $|dom(f_d) \cap x| = \omega$.
So in this case we have $|dom(f) \cap x| = \omega$ or $f^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$.
Case 2: $z_{r \restriction k} = *$:
Assume by contradiction, that $\exists a \in B_k\ a \supseteq^* x$. Because x is not definable from $A \cup \{A\} \cup T_0 \cup \{T_0\}$, we know that there exists $N \in \omega$ with $\{z \in [T_0 \restriction \sigma_{r \restriction k}] : N \cup a \supseteq z\}$ being uncountable. In particular, we can find a $\sigma_s \in T_0 \restriction \sigma_{r \restriction k}$ with $|\sigma_s| \geq N$ and $z, z' \in [T_0 \restriction \sigma_{r \restriction k}]$ with $N \cup a \supseteq z, z'$ and $z \supseteq \sigma_{s^\frown 0}$ and $z' \supseteq \sigma_{s^\frown 1}$. Because $z_{r \restriction k} = *$ with $r \restriction k \subseteq s$ we have defined $f_{k,s}$ and $\{n \in \omega : f_{k,s}(n) \neq x_k\} \subseteq (z \cup z') \setminus N \subseteq a$ is true. Also by construction we have $(dom(f_{k,s}))^c \in c_{s^\frown 1}$ and hence $|z' \cap (dom(f_{k,s}))^c| = \omega$ and hence $|a \cap (dom(f_{k,s}))^c| = \omega$. So we can conclude that $f_a \parallel f_{k,s}$ for some $f_{k,s}$ with $(dom(f_{k,s}))^c \notin B_k$ by construction. So there are two different members of the antichain that are compatible, which is a contradiction.
We can conclude that in this case $\forall a \in B_k\ |a^c \cap x| = \omega$, so in particular $|dom(f) \cap x| = \omega$.

ii) $f = f_b$ for $b \in B_0$:
Assume by contradiction that $b \supseteq^* x$.

As in i) we find z, z' in $[T_0]$ with a common initial segment τ_s, $|\tau_s| \geq N$, and $z \cup z' \subseteq b \cup N$. Then we have $f_{0,s}^{-1}[\{1\}] \subseteq (z \cup z') \setminus N$ and $|z' \cap (dom(f_{0,1}))^c| = \omega$, hence $f_b \parallel f_{0,s}$, which is a contradiction.
We can conclude $|dom(f) \cap x| = \omega$.

iii) $f \in A_{fin}$:
Assume by contradiction $f^{-1}[\{1\}] \cap (\omega \setminus x) = \emptyset$. Then for a sufficiently large $n \in \omega$ we would have $f^{-1}[\{1\}] \subseteq x \restriction n$ and in conclusion $f^{-1}[\{1\}] \subseteq b$ for some $b \in B_0$. This would imply $f \parallel f_b$, a contradiction.
So in this case we can conclude $f^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$.

iv) $|f^{-1}[\{1\}]| = \omega$:
Then $f^{-1}[\{1\}] \cap (\omega \setminus x) \neq \emptyset$ or $|dom(f) \cap x| = \omega$.

So in each case for every $f \in A$ we have $\exists k \in \omega \setminus x \ f(k) = 1 \vee |dom(f) \cap x| = \omega$. This means that if we define $h_0 \in Si$ by $(dom(h_0))^c := x$ and $h_0 \restriction (\omega \setminus x) :\equiv 0$, then the functions of the antichain that are compatible with h_0 form an antichain exclusively consisting of infinite conditions relativized to x. By Corollary 2.3 we can pick some $h_1 \in Si(x)$ that is incompatible to these functions relativized to x. If we define $h := h_0 \cup h_1$, we get $h \perp A$.
Hence A is not a maximal antichain. $\dashv$

We can now use the above theorem to prove the following:

Theorem 2.5. $\mathfrak{d} \leq \mathfrak{a}(Si)$.

Proof. Suppose that we have an antichain A of Silver conditions with $\omega_1 \leq |A| < \mathfrak{d}$. We want to show that A is not maximal. Because of theorem 2.4 we can assume without loss of generality:
$\forall x \in 2^\omega \ |\{f \in A : \ f|x\}| \leq \omega$.
First we define a tree $T \subseteq (2 \cup \{*\})^{<\omega}$ as follows:
$T := \{\sigma \in (2 \cup \{*\})^{<\omega} : \ |\{f \in A : \ \sigma \parallel f \wedge \sigma^{-1}[\{*\}] \subseteq (dom(f))^c\}| \geq \aleph_1\}$.
We will show that there exists a branch in this tree that is a Silver condition that is incompatible with the antichain. Note that obviously $[T] \not\subseteq Si$. Hence, we define $Z^* := [T] \cap Si$. T has the following properties, that are easy to prove:

i) $\forall x \in [T] \forall f \in A_{fin} \exists k \in dom(f) \cap dom(x) \ x(k) \neq f(k)$.
ii) The $*$ are dense in T; that is, $\forall \sigma \in T \exists \tau \in T \ (\tau \supseteq \sigma \wedge \tau^{-1}[\{*\}] \cap [|\sigma|, |\tau|) \neq \emptyset)$.
iii) $\forall \sigma \in T \forall k \in \omega \exists \tau \in T \ (\tau \supseteq \sigma \wedge |\tau| \geq k \wedge \tau^{-1}[\{*\}] \cap [|\sigma|, \tau) = \emptyset)$.
iv*) If $\sigma \in T, f \in A_{inf}$, there are uncountably many conditions of the antichain that are compatible with σ, contain the $*$ of σ in their codomain and are also incompatible with f. This implies the following:
$\forall f \in A_{inf} \forall \sigma \in T \exists \tau \in T \exists k \in \omega \ (\tau \supseteq \sigma \wedge k \in (dom(f) \cap (dom(\tau)) \wedge \tau(k) \neq f(k))) \vee (k \in dom(f) \wedge \tau(k) = *)$.

Because for any $\sigma^\frown * \in T$ we also have $\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$, we can replace iv*) by the following:

iv) $\forall f \in A_{inf} \forall \sigma \in T \exists \tau \in T; \tau \supseteq \sigma \exists k \in dom(\tau) \cap dom(f) \ \tau(k) \neq f(k)$.

Because of ii) and iii) we can pick a function $I : T \to Z^*; \sigma \mapsto x_\sigma$ such that for all σ, σ' with $\sigma \neq \sigma'$ we have that $x_\sigma^{-1}[\{*\}]$ and $x_{\sigma'}^{-1}[\{*\}]$ are almost disjoint. Together with the assumption from the beginning of the proof we get for any $\sigma, \sigma' \in T$ with $\sigma \neq \sigma'$ that the set $B_{\sigma,\sigma'} := \{f \in A_{inf} : \ f|x_\sigma \wedge f|x_{\sigma'}\}$ is countable. Let

$B := \bigcup_{\sigma,\sigma' \in T, \sigma \neq \sigma'} B_{\sigma,\sigma'}$ be the countable set of infinite conditions of the antichain, that do not contradict at least two different $x_\sigma, x_{\sigma'}$. Let
$\langle b_n : n \in \omega \rangle$ be an enumeration of B.
In order to construct our desired branch of T, we need to introduce some auxiliary functions S_f, F_f (for $f \in A_{inf}$), h_n, H_n ($n \in \omega$). They are defined as follows:

$S_f : T \to \omega$

$$S_f(\sigma) := \begin{cases} min\{k \in \omega : k \in dom(x_\sigma) \cap dom(f) \wedge x_\sigma(k) \neq f(k)\} & \text{if } \neg(f | x_\sigma) \wedge f \parallel \sigma, \\ |\sigma| & \text{if } f \perp \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

$F_f : \omega \to \omega$
$F_f(n) := max\{k \in \omega : \exists \sigma \in T \ (k = S_f(\sigma) \wedge |\sigma| \leq n\}$
Note that the F_f are increasing.

$h_n : T \to \omega$

$$h_n(\sigma) := \begin{cases} min\{k \in \omega : \exists \tau \in T \ (\tau \supseteq \sigma \wedge k \in dom(\tau) \cap dom(b_n) \\ \qquad\qquad \wedge |\tau| = k + 1 \wedge \tau(k) \neq b_n(k))\} & \text{if } b_n \parallel \sigma, \\ |\sigma| & \text{if } b_n \perp \sigma. \end{cases}$$

Note that because of property iv) of our tree h_n is well-defined for any $n \in \omega$.

$H_n : \omega \to \omega$
$H_n(m) := max\{k \in \omega : \exists \sigma \in T \ (k = h_n(\sigma) \wedge |\sigma| \leq m)\}$.
For any $n \in \omega$ we have that H_n is increasing.

$F_f(n)$ gives us, for all x_σ with $|\sigma| \leq n$ that contradict f, an upper bound for the level at which this is witnessed.
$H_n(m)$ gives us, for all nodes of the tree of length at most m, an upper bound for the length of an extension in the tree that contradicts b_n.
We will use this information to construct for each $s \in \omega^{<\omega}$ and $f \in A_{inf}$ a sequence $R_{s,f} := \langle r_k^{(s,f)} : k \in \omega \rangle$ by recursion, that will help us to construct the desired branch of T.

For $k < |s|$ define $r_k^{(s,f)} := s(k)$
and $r_{|s|}^{(s,f)} := F_f \circ H_{|s|-1}(r_{|s|-1}^{(s,f)})$, if $s \neq \emptyset$
or $r_0^{(s,f)} := 0$, if $s = \emptyset$.
Furthermore for $k \in \omega$ define by recursion:
$r_{|s|+k+1}^{(s,f)} := F_f \circ H_{|s|+k}(r_{|s|+k}^{(s,f)})$.

We have $|\{R_{s,f} : s \in \omega^{<\omega}, f \in A_{inf}\}| \leq |A| < \mathfrak{d}$. So we can choose a strictly increasing sequence $R = \langle r_k : k \in \omega \rangle$ in ω with
$\forall s \in \omega^{<\omega} \forall f \in A_{inf} \exists^\infty n \in \omega \ r_n > r_n^{(s,f)}$.
We use this sequence to construct $y \in Z^*$ by recursion:

$n = 0$:
Define $\sigma_0 := \emptyset$.

$n \rightsquigarrow n + 1$:
Let $\sigma_0, .., \sigma_n$ be constructed. Choose $\tilde{\sigma}_{n+1} \in T, \tilde{\sigma}_{n+1} \supseteq \sigma_n$ of minimal length with $\tilde{\sigma}_{n+1} \perp b_n$. Let $k_{n+1} \geq |\tilde{\sigma}_{n+1}|, r_{n+1}$ be sufficiently large such that

$(x_{\tilde{\sigma}_{n+1}} \upharpoonright k_{n+1})^{-1}[\{*\}] \cap [|\sigma_n|, k_{n+1}) \neq \emptyset$. Then define $\sigma_{n+1} := x_{\tilde{\sigma}_{n+1}} \upharpoonright k_{n+1}$.
After ω many steps let $y := \bigcup_{n\in\omega} \sigma_n$. Obviously, by construction we have $y \in Z^* \subseteq Si$.

We claim that $y \perp A$. For all $f \in A_{fin}$ we get $y \perp f$ by property i) . For all $b \in B$ we have $y \perp B$ by construction of y. We still have to show that $y \perp A_{inf} \setminus B$:
Let $f \in A_{inf} \setminus B$. Because $f \notin B$ we can choose $n_0 \in \omega$ sufficiently large such that $\forall |\sigma| \geq |\sigma_{n_0}| \neg(f|x_\sigma)$.
Define $s \in \omega^{n_0+1}$ by:

$$s(k) := \begin{cases} 0 & \text{if } k < n_0, \\ |\sigma_{n_0}| & \text{if } k = n_0. \end{cases}$$

Hence, we have $r^{(s,f)}_{n_0} = |\sigma_{n_0}|$. Now let $k \geq 1$ be minimal with $|\sigma_{n_0+k}| > r^{(s,f)}_{n_0+k}$ (exists by construction of σ_n). Because H_n is increasing and by the fact that $|\sigma_{n_0+k-1}| \leq r^{(s,f)}_{n_0+k-1}$ we can conclude
$|\tilde{\sigma}_{n_0+k}| \leq H_{n_0+k-1}(|\sigma_{n_0+k-1}|) \leq H_{n_0+k-1}(r^{(s,f)}_{n_0+k-1})$, and hence by monotony of F_f we get
$|\sigma_{n_0+k}| > r^{(s,f)}_{n_0+k} = F_f \circ H_{n_0+k-1}(r^{(s,f)}_{n_0+k-1}) \geq F_f(|\tilde{\sigma}_{n_0+k}|)$.
By choice of n_0 and definition of F_f we have $\sigma_{n_0+k} = x_{\tilde{\sigma}_{n_0+k}} \upharpoonright |\sigma_{n_0+k}| \perp f$.
$\Rightarrow y \perp f$.

Thus the claim is proven and we can conclude that A is not a maximal antichain. $\dashv$

§3. **Additivity of** $I(Si)$. We can use Theorem 2.2 to prove the following theorem, which implies that every reasonable amoeba forcing for Silver forcing adds a dominating real.

THEOREM 3.1. *$add(I(Si)) \leq \mathfrak{b}$.*

PROOF. Let $\kappa < add(I(S))$ and $\langle f_\alpha : \alpha < \kappa \rangle$ be a sequence of functions of ω^ω. We need to show that there exists $g \in \omega^\omega$ that dominates all functions of the sequence. In the following proof we will identify Silver trees with their Silver functions. At any point of the proof it should be clear which representation we are referring to. We will identify a member of $[\omega]^\omega$ with its increasing enumeration.
Note that any Silver tree can be partioned into $2^{\aleph_0}$ subtrees with pairwise disjoint closure. So whenever we have a Silver tree p and some set $Y \subseteq 2^\omega$ with $|Y| < 2^{\aleph_0}$, we can pick some Silver tree $p' \subseteq p$ with $[p'] \cap Y = \emptyset$. It is also obvious that for any $f \in \omega^\omega$ the set $\{p \in Si : (dom(p))^c >^* f\}$ is open dense in Si. Hence, similarly to the proof of Lemma 1.1 of [1], for any $\alpha < \kappa$, we can construct a maximal antichain $A_\alpha \subseteq Si$ with the following two additional properties:

i) $\forall q \in Si\ ([q] \subseteq \bigcup\{[p] : p \in A_\alpha\} \rightarrow \exists B \in [A_\alpha]^{<2^{\aleph_0}}\ [q] \subseteq \bigcup\{[p] : p \in B\})$.
ii) $\forall p \in A_\alpha\ (dom(p))^c >^* f_\alpha$.

The construction is as follows:
Let $\langle q_\xi : \xi < \mathfrak{c} \rangle$ enumerate all Silver trees. We recursively construct a sequence of Silver trees $\langle p_\xi : \xi < \mathfrak{c} \rangle$ and a sequence $\langle x_\xi : \xi < \mathfrak{c} \rangle$ of members of 2^ω as follows:
Let $\langle p_\xi : \xi < \gamma \rangle$ already be constructed for some $\gamma < \mathfrak{c}$. If $\forall \xi < \gamma\ p_\xi \perp q_\gamma$, pick $p_\gamma \subseteq q_\gamma$ with $(dom(p_\gamma))^c >^* f_\alpha$ and such that $x_\xi \notin [p_\gamma]$ for all $\xi < \gamma$. Otherwise just define $p_\gamma := p_0$. In any case check if $[q_\gamma] \subseteq \bigcup_{\xi<\gamma} [p_\xi]$. If this is not the case, let

$x_\gamma \in [q_\gamma] \setminus \bigcup_{\xi<\gamma}[p_\xi]$. Otherwise just let x_γ be the sequence that is constantly 0. If we have constructed the sequences we get by $A_\alpha := \{p_\alpha : \alpha < \mathfrak{c}\}$ the desired antichain.

For each $\alpha < \kappa$ define $X_\alpha := 2^\omega \setminus \bigcup\{[p] : p \in A_\alpha\} \in I(Si)$.
By $\kappa < add(I(Si))$ we have that $X := \bigcup_{\alpha<\kappa} X_\alpha \in I(Si)$. Hence we can pick some $q \in Si$ with $[q] \cap X = \emptyset$. By definition of X we have
$\forall\alpha < \kappa\ [q] \subseteq \bigcup\{[p] : p \in A_\alpha\}$.

We claim: $(*)\ \forall\alpha < \kappa \exists p \in A_\alpha\ (dom(q))^c \subseteq^* (dom(p))^c$.
Assume by contradiction that for some $\alpha < \kappa$ we have
$\forall p \in A_\alpha\ |(dom(q))^c \cap dom(p)| = \omega$.
Because of property i) of A_α we can choose $B \in [A_\alpha]^{<2^{\aleph_0}}$, such that
$[q] \subseteq \bigcup\{[p] : p \in B\}$. Because of our assumption we know that the members of B that are compatible to q form an antichain consisting solely of infinite conditions relativized to $(dom(q))^c$. Hence, by theorem 2.2 we can find a subtree $q' \subseteq q$ with $q' \perp B$. We can conclude that for any $p \in B$ we have $|[q'] \cap [p]| < \omega$ and thus $[q'] \nsubseteq \bigcup\{[p] : p \in B\}$, which is clearly a contradiction.

Now let $\langle b_n : n \in \omega\rangle$ be an enumeration of all $b \in [\omega]^\omega$ with $b =^* (dom(q))^c$. Let $g \in \omega^\omega$ dominate all b_n. We will show that $\forall\alpha < \kappa\ g >^* f_\alpha$:
Let $\alpha < \kappa$ be arbitrary. By $(*)$ we can pick $p \in A_\alpha$ with $(dom(q))^c \subseteq^* (dom(p))^c$. There exists $n \in \omega$ with $b_n \subseteq (dom(p))^c$. Hence by property ii) of A_α we get $g >^* b_n >^* (dom(p))^c >^* f_\alpha$. ⊣

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