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A multi-species chemotaxis system: Lyapunov functionals, duality, critical mass†

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We introduce a multi-species chemotaxis type system admitting an arbitrarily large number of population species, all of which are attracted versus repelled by a single chemical substance. The production versus destruction rates of the chemotactic substance by the species is described by a probability measure. For such a model, we investigate the variational structures, in particular, we prove the existence of Lyapunov functionals, we establish duality properties as well as a logarithmic Hardy–Littlewood–Sobolev type inequality for the associated free energy. The latter inequality provides the optimal critical value for the conserved total population mass.

Key words: Multi-species chemotaxis models, Lyapunov functionals, duality, logarithmic Hardy–Littlewood–Sobolev inequality, Moser–Trudinger inequality
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1 Introduction and motivation

Since the pioneering chemotaxis model of Keller and Segel [24], see also Patlak [35], several models have been introduced in order to describe the chemotactic movement of motile species, such as the slime mold *Dictyostelium discoideum*. In particular, much attention has been devoted in recent years to derive multi-species chemotactic models, see [8–10, 13, 18, 44, 49, 50] and the references therein.

Our aim in this note is to introduce and to analyze, particularly from the variational point of view, a new multi-species parabolic–parabolic chemotaxis system involving an arbitrarily large number of population species ρ_α, depending on the index $\alpha \in [-1, 1]$, and a single chemical v. Such a continuously varying index will turn out to be useful in order to efficiently formulate, in terms of a probability distribution $\mathcal{P}(d\alpha)$ defined on

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the index range $[-1,1]$, the variational structures of the system, as well as to describe relevant quantities, such as the conserved total population mass and the overall chemical production rate. We assume that ρ_α and v are defined on a two-dimensional domain, which is a natural setting for species raised in a cell-culture dish. In our model, some of the population species are attracted by the substance v, while others are repelled by it, with different (normalized) intensities given by the value $\alpha \in [-1,1]$, where positive values of α correspond to attraction, whereas negative values correspond to repulsion. In turn, the substance is self-produced by those species it attracts, and destroyed by those species it repels. In particular, this model fits the 'absence of conflicts' definition introduced in [49]. Birth and death rates are neglected.

We are particularly interested in the limit case where the dynamics of the population species is significantly faster than the dynamics of the chemical. In this case, our system may be written as an evolution problem for the chemical substance v only. We further assume that the total mass of *all* the population species, is conserved in time. Such a situation could be of interest when the different species are produced by a cell differentiation process as occurs, e.g., in the early aggregation stages of the Dictyostelium during mound formation [13,47].

More precisely, we consider the following system:

$$\begin{cases} \delta_\alpha \dfrac{\partial \rho_\alpha}{\partial t} = \Delta \rho_\alpha - \alpha \operatorname{div}(\rho_\alpha \nabla v), & \text{in } \Omega \times (0,T),\ \alpha \in [-1,1] \\ \varepsilon \dfrac{\partial v}{\partial t} = \Delta v + \displaystyle\int_{[-1,1]} \alpha \rho_\alpha \, \mathcal{P}(d\alpha), & \text{in } \Omega \times (0,T) \\ \nu \cdot (\nabla \rho_\alpha - \alpha \rho_\alpha \nabla v) = 0,\ v = 0, & \text{on } \partial\Omega \times (0,T) \\ \rho_\alpha(x,0) = \rho_\alpha^0(x) \geqslant 0, \quad v(x,0) = v^0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, ν denotes the outer unit normal vector on $\partial\Omega$, $T > 0$ stands for the maximum existence time for (1.1), $\alpha \in [-1,1]$, $\mathcal{P} \in \mathcal{M}([-1,1])$ is a probability measure, $v^0 \in H_0^1(\Omega)$ and the constants $\varepsilon, \delta_\alpha$ satisfy $\varepsilon > 0$, $\delta_\alpha \geqslant \delta_0$ for some $\delta_0 > 0$. We observe that if $\operatorname{supp} \mathcal{P} \subset [0,1]$, namely if $\mathcal{P}$ is positively supported (see (2.2) below for the precise definition of $\operatorname{supp} \mathcal{P}$), then $v^0 \geqslant 0$ implies $v \geqslant 0$ by the maximum principle. On the other hand, if $\operatorname{supp} \mathcal{P} \cap [-1,0) \neq \emptyset$, the function v is not necessarily non-negative. In this case, v is interpreted as 'chemical potential', see [18].

The evolution equation for ρ_α, together with the no-flux boundary condition in system (1.1), implies the conservation in time of the population mass, for each population ρ_α separately:

$$\int_\Omega \rho_\alpha(x,t)\, dx = \int_\Omega \rho_\alpha^0(x)\, dx \qquad \text{for all } \alpha \in [-1,1]. \tag{1.2}$$

Moreover, (weak) solutions to system (1.1) satisfy $\rho_\alpha \geqslant 0$ almost everywhere in $\Omega \times (0,T)$, see, e.g., [4], Proposition 1, and the references therein.

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We observe that for $\mathcal{P} = \delta_1(d\alpha)$, system (1.1) reduces to the classical Keller–Segel system for a single population, denoted by ψ:

$$\begin{cases} \delta \dfrac{\partial \psi}{\partial t} = \Delta\psi - \mathrm{div}(\psi\nabla v), & \text{in } \Omega \times (0,T) \\ \varepsilon \dfrac{\partial v}{\partial t} = \Delta v + \psi, & \text{in } \Omega \times (0,T) \\ v \cdot (\nabla\psi - \psi\nabla v) = 0, \quad v = 0, & \text{on } \partial\Omega \times (0,T) \\ \psi(x,0) = \psi^0(x), \quad v(x,0) = v^0(x), \quad \psi^0, v^0 \geqslant 0, & \text{in } \Omega. \end{cases} \tag{1.3}$$

For the sake of future reference, we also explicitly note the two-species case $\mathcal{P}(d\alpha) = \tau\delta_{\alpha_1}(d\alpha) + (1-\tau)\delta_{\alpha_2}$, $0 < \tau < 1$, $\alpha_1, \alpha_2 \in [-1,1]$. In this case, system (1.1) takes the form:

$$\begin{cases} \delta_1 \dfrac{\partial \rho_1}{\partial t} = \Delta\rho_1 - \mathrm{div}(\alpha_1\rho_1\nabla v), & \text{in } \Omega \times (0,T), \\ \delta_2 \dfrac{\partial \rho_2}{\partial t} = \Delta\rho_2 - \mathrm{div}(\alpha_2\rho_2\nabla v), & \text{in } \Omega \times (0,T), \\ \varepsilon \dfrac{\partial v}{\partial t} = \Delta v + \tau\alpha_1\rho_1 + (1-\tau)\alpha_2\rho_2, & \text{in } \Omega \times (0,T) \\ v \cdot (\nabla\rho_1 - \alpha_1\rho_1\nabla v) = 0 = v \cdot (\nabla\rho_2 - \alpha_2\rho_2\nabla v), & \text{on } \partial\Omega \times (0,T) \\ v = 0, & \text{on } \partial\Omega \times (0,T) \\ \rho_1(x,0) = \rho_1^0(x) \geqslant 0, \quad \rho_2(x,0) = \rho_2^0(x) \geqslant 0 & \text{in } \Omega \\ v(x,0) = v^0(x), & \text{in } \Omega. \end{cases} \tag{1.4}$$

System (1.1) admits the following relevant limit cases.

Slow population dynamics limit: $\delta_\alpha > 0$, $\varepsilon = 0$

In this case, system (1.1) reduces to the following parabolic-elliptic system:

$$\begin{cases} \delta_\alpha \dfrac{\partial \rho_\alpha}{\partial t} = \Delta\rho_\alpha - \alpha\,\mathrm{div}(\rho_\alpha\nabla v), & \text{in } \Omega \times (0,T),\ \alpha \in [-1,1] \\ -\Delta v = \displaystyle\int_{[-1,1]} \alpha\rho_\alpha\,\mathcal{P}(d\alpha), & \text{in } \Omega \times (0,T) \\ v \cdot (\nabla\rho_\alpha - \alpha\rho_\alpha\nabla v) = 0, \quad v = 0 & \text{on } \partial\Omega \times (0,T) \\ \rho_\alpha(x,0) = \rho_\alpha^0(x) \geqslant 0, & \text{in } \Omega. \end{cases} \tag{1.5}$$

Systems of the form (1.5) also appear in statistical mechanics (where they are sometimes called Smoluchowski–Poisson systems) as well as in the theory of semi-conductors, see [4,6,15] and the references therein. In the context of chemotaxis, concentration phenomena for (1.5) were obtained in [19]. We note that system (1.5) decouples, in the sense that it may be written as an integro-differential system for the populations ρ_α, $\alpha \in [-1,1]$:

$$\delta_\alpha \frac{\partial \rho_\alpha}{\partial t} = \Delta\rho - \mathrm{div}\left(\alpha\rho_\alpha\nabla \iint_{\Omega\times[-1,1]} G(x,y)\beta\rho_\beta(y)\,dy\mathcal{P}(d\beta)\right), \qquad \alpha \in [-1,1], \tag{1.6}$$

where G denotes the Green's function for $-\Delta$, see (3.8) below for the precise definition.

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Fast population dynamics limit: $\delta_\alpha = 0$ *for all* $\alpha \in [-1,1]$*,* $\varepsilon = 1$

As already mentioned, we are particularly interested in this case. Under this limit, we obtain the following elliptic-parabolic system:

$$\begin{cases} \Delta\rho_\alpha - \alpha\,\mathrm{div}(\rho_\alpha\nabla v) = 0, & \text{in } \Omega\times(0,T),\ \alpha\in[-1,1] \\ \dfrac{\partial v}{\partial t} = \Delta v + \displaystyle\int_{[-1,1]} \alpha\rho_\alpha\,\mathcal{P}(d\alpha), & \text{in } \Omega\times(0,T) \\ \nu\cdot(\nabla\rho_\alpha - \alpha\rho_\alpha\nabla v) = 0,\ v = 0, & \text{on } \partial\Omega\times(0,T) \\ v(x,0) = v^0(x), & \text{in } \Omega. \end{cases} \tag{1.7}$$

In this case, it is not difficult to check (see the proof of Theorem 2.1-(iii) in Section 3 below) that

$$\rho_\alpha(x,t) = C_\alpha(t)e^{\alpha v(x,t)}$$

for some $C_\alpha(t) > 0$ independent of $x \in \Omega$. Therefore, system (1.7) decouples into the following semi-linear parabolic non-local equation for the chemical substance v:

$$\frac{\partial v}{\partial t} = \Delta v + \int_{[-1,1]} \alpha C_\alpha(t)e^{\alpha v}\,\mathcal{P}(d\alpha). \tag{1.8}$$

The limit system (1.7) no longer implies the total mass conservation (1.2). Therefore, we cannot *a priori* exclude the dependence of C_α on the time t and on the index α. On the other hand, the explicit value of $C_\alpha(t)$ is irrelevant to the dynamics of ρ_α, which only involves ∇v by the first equation of (1.7). Therefore, we *assume* a suitable form of mass conservation. In this limit, we focus our attention on the following *average* mass conservation property with respect to $\mathcal{P}$:

$$\iint_{\Omega\times[-1,1]} \rho_\alpha(x,t)\,dx\mathcal{P}(d\alpha) = \lambda, \qquad \text{for all } t\in(0,T). \tag{1.9}$$

As already mentioned, such a 'generalized' mass conservation property may be of interest in the situation where the single species ρ_α are produced by a cell differentiation process. From (1.8)–(1.9), we finally obtain the following non-local evolution problem for v:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + \lambda\displaystyle\int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\iint_{\Omega\times[-1,1]} e^{\beta v(y,t)}\,dy\mathcal{P}(d\beta)}\,\mathcal{P}(d\alpha), & \text{in } \Omega\times(0,T) \\ v = 0, & \text{on } \partial\Omega\times(0,T) \\ v(x,0) = v^0(x), & \text{in } \Omega. \end{cases} \tag{1.10}$$

Interestingly, the exponential type non-linearity in (1.10) is exactly the non-linearity contained in the mean field equation derived by Neri [29] in the context of the statistical mechanics description of 2D turbulence, extending Onsager's approach [31], see also [5]. In other words, (1.10) corresponds to the parabolic flow associated to the stochastic hydrodynamic equilibrium equation derived in [29]. Such a flow is also known as 'relaxation equation' associated to the elliptic problem, and is of interest even in the cases where it does not describe the actual dynamics, since it provides, at least in principle, an algorithm to obtain numerical simulations for the elliptic problem, see [7], in particular the Remark at the end of p. 97.

The steady states for (1.10) received a considerable attention in recent years, see, e.g., [11, 17, 32, 33, 38, 41] and the references therein, particularly in relation the existence and qualitative properties of solutions. Thus, by analyzing (1.10), we provide further insight for the mean field equation derived in [29]. We finally note that results for the evolution problems of the 'mean field' form (1.10), in the 'standard' case $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$ are obtained in [1, 2, 23, 48]. Some related non-local evolution problems have also been analyzed in connection with the modelling of shear banding and Ohmic heating, see [21, 25, 26] and the references therein.

From the mathematical point of view, we are interested in the variational structures associated to the multi-species chemotaxis system (1.1), which are a key tool in the study of stability and global existence of solutions [1, 16, 18, 34]. We note that whether or not the 'critical mass' for boundedness from below of the Lyapunov functionals provides a threshold for global existence versus finite time blow-up of solutions for (1.1) significantly depends on the distribution $\mathcal{P}$, as recently shown in [39], where the existence of stationary solutions for (1.1) beyond the critical mass is shown to depend on $\mathcal{P}$ even in the two-species case (1.4). In this article, we rigorously establish the existence of a Lyapunov functional and we establish a duality principle for ρ_α and v. Some of these results are stated and justified heuristically in [45]. The rigorous proof however requires some care, since the natural functional space for $(\rho_\alpha)_{\alpha\in[-1,1]}$ is the logarithmic space $L^1([-1,1], L\log L(\Omega); \mathcal{P})$, which is known to be non-reflexive, see, e.g., [36, 37]. To this end, we adapt some ideas from [4, 37]. Finally, in the fast population dynamics limit, we determine the critical mass for the global existence of solutions versus chemotactic collapse [12, 19], in the form of an optimal logarithmic Hardy–Littlewood–Sobolev type inequality which is in the spirit of [3, 43], although with different constraints. In view of the duality principle, our inequality is equivalent to the sharp Moser–Trudinger type inequality, [28, 46], obtained in [40] and thus provides a new proof for it.

This article is organized as follows. In Section 2, we state our main results. In Section 3, we obtain the Lyapunov functionals for (1.1)–(1.5)–(1.7). Section 4 is devoted to the proof of the duality principle. In Section 5, we prove the logarithmic HLS inequality and thus we derive the critical mass for global existence. Section 6 contains some necessary technical estimates. Finally, in Section 7, we provide some concluding remarks on the steady states of (1.1). In particular, we observe that the two stationary mean field problems of [29] and [42], which have been extensively analyzed in recent years, see [11, 17, 20, 30, 40, 41, 45] and the references therein, may both be obtained as steady states of (1.1) in the fast population dynamics limit, by assuming different conserved population mass constraints. Hence, we provide a unified point of view for such stationary problems.

1.1 Notation

In what follows, all integrals are taken in the sense of Lebesgue. When the integration variable is clear from the context, we may omit it.

2 Statement of the main results

In order to state our main results, we define the following functionals:

$$\begin{aligned}\mathcal{L}(\oplus\rho_\alpha, v) &:= \iint_{\Omega\times[-1,1]} \rho_\alpha(\log\rho_\alpha - 1)\,dx\mathcal{P}(d\alpha) + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx \\ &\qquad - \iint_{\Omega\times[-1,1]} \alpha\rho_\alpha v\,dx\mathcal{P}(d\alpha), \\ \mathcal{F}(\oplus\rho_\alpha) &:= \iint_{\Omega\times[-1,1]} \rho_\alpha(\log\rho_\alpha - 1)\,dx\mathcal{P}(d\alpha) \\ &\qquad - \frac{1}{2}\iint_{[-1,1]^2} \alpha\beta\,\mathcal{P}(d\alpha)\mathcal{P}(d\beta)\iint_{\Omega^2} G(x,y)\rho_\alpha(x)\rho_\beta(y)\,dxdy, \\ \mathcal{J}_\lambda(v) &:= \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \lambda\log\left(\iint_{\Omega\times[-1,1]} e^{\alpha v}\,dx\mathcal{P}(d\alpha)\right) + \lambda(\log\lambda - 1),\end{aligned} \tag{2.1}$$

defined for $\oplus\rho_\alpha \in L^1([-1,1], L\log L(\Omega);\mathcal{P})$, $\rho_\alpha \geqslant 0$ for all $\alpha \in [-1,1]$ and for $v \in H_0^1(\Omega)$, where, following [45], we denote $\oplus\rho_\alpha := \oplus_{\alpha\in[-1,1]}\rho_\alpha = (\rho_\alpha)_{\alpha\in[-1,1]}$.

We recall that the space $L\log L(\Omega)$ is defined as

$$L\log L(\Omega) = \left\{\psi \in L^1(\Omega)\,:\, \int_\Omega |\psi\log|\psi|| < +\infty\right\},$$

and that it may be structured as an Orlicz space with Young function $\Phi(s) = (s+1)\log(s+1) - s$, see, e.g., [16, 18, 36]; however, we shall not need this point of view.

For all $\lambda > 0$, we define the following set of admissible functions:

$$\widetilde{\Gamma}_\lambda := \left\{\oplus\rho_\alpha \in L^1([-1,1], L\log L(\Omega);\mathcal{P})\,:\, \begin{array}{c}\rho_\alpha \geqslant 0\ \forall\alpha\in[-1,1], \\ \iint_{\Omega\times[-1,1]} \rho_\alpha\,dx\mathcal{P}(d\alpha) = \lambda\end{array}\right\}.$$

With this notation, our main results may be summarized as follows.

Theorem 2.1 (Variational structures) *The following properties hold true.*

(i) *The functional $\mathcal{L}$ is a Lyapunov functional for* (1.1), *in the sense that the function*

$$g_0(t) := \mathcal{L}(\oplus\rho_\alpha(x,t), v(x,t))$$

decreases along solutions $(\oplus\rho_\alpha(x,t), v(x,t))$ to (1.1). *Moreover, g_0 decreases strictly unless $\rho_\alpha(x,t) = C_\alpha(t)e^{\alpha v(x,t)}$ for some $C_\alpha(t) > 0$ independent of $x \in \Omega$.*

(ii) *The functional $\mathcal{F}$ is a Lyapunov functional for the Smoluchowski–Poisson system* (1.6), *in the sense that the function*

$$h_0(t) := \mathcal{F}(\oplus\rho_\alpha(x,t))$$

decreases along solutions $\oplus\rho_\alpha(x,t)$ to (1.6). *Moreover, h_0 decreases strictly away from stationary solutions.*

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(iii) *The semi-linear parabolic problem* (1.10) *is the gradient flow for* $\mathcal{J}_\lambda$.

(iv) *The following duality property holds true:*

$$\inf_{\widetilde{\Gamma}_\lambda \times H_0^1(\Omega)} \mathcal{L} = \inf_{\widetilde{\Gamma}_\lambda} \mathcal{F} = \inf_{H_0^1(\Omega)} \mathcal{J}_\lambda.$$

We note that Lyapunov functionals are a key tool in establishing the global existence of solutions, see [12, 16]. Although property (iv) is derived heuristically in [45], a rigorous proof is rather delicate due to the non-reflexivity of the Orlicz space $L \log L(\Omega)$. Here, we overcome this difficulty by some *ad hoc* truncation arguments, in the spirit of [37].

Our next result is a sharp logarithmic HLS inequality for the functional $\mathcal{F}$ of the type derived in [3,43], which provides the critical total population mass threshold for the global existence of solutions, see [12, 16, 22].

Theorem 2.2 (Sharp logarithmic HLS type inequality) *Suppose that* $\operatorname{supp}\mathcal{P} \cap \{-1, 1\} \neq \emptyset$. *Then, the functional* $\mathcal{F}$ *is bounded from below on* $\widetilde{\Gamma}_\lambda$ *if and only if* $\lambda \leqslant 8\pi$.

Here, $\operatorname{supp}\mathcal{P}$ denotes the support of $\mathcal{P}$, namely

$$\operatorname{supp}\mathcal{P} := \{\alpha \in [-1, 1] \,:\, \mathcal{P}(U) > 0 \text{ for all open neighborhoods } U \text{ containing } \alpha\}. \quad (2.2)$$

We observe that in view of the duality property stated in Theorem 2.1-(iv), the inequality stated in Theorem 2.2 is equivalent to the Moser–Trudinger type inequality [28,46] derived in [40] and given by

$$\inf_{H_0^1(\Omega)} \mathcal{J}_\lambda > -\infty \qquad \text{if and only if } \lambda \leqslant 8\pi. \quad (2.3)$$

The proof of Theorem 2.2 is independent of the results in [40]; hence, here we also provide an alternative proof of (2.3).

The remaining part of this article is devoted to the proofs of Theorem 2.1 and of Theorem 2.2.

3 Variational structures and proof of Theorem 2.1-(i)–(iii)

Henceforth, it will be convenient to denote $I := [-1, 1]$ and to adopt the product space notation introduced in [29]. Namely, let

$$\widetilde{\Omega} := \Omega \times I, \qquad \tilde{x} := (x, \alpha), \qquad d\tilde{x} := dx\mathcal{P}(d\alpha).$$

We denote

$$\rho(\tilde{x}) = \rho(x, \alpha) := \rho_\alpha(x).$$

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The full system (1.1) *and the proof of Theorem 2.1-(i)*

In product space notation system (1.1) takes the form:

$$\begin{cases} \delta_\alpha \dfrac{\partial \rho}{\partial t} = \Delta\rho - \alpha\, \mathrm{div}(\rho\nabla v), & \text{in } \widetilde{\Omega} \times (0,T) \\ \varepsilon \dfrac{\partial v}{\partial t} = \Delta v + \displaystyle\int_I \alpha\rho\, \mathcal{P}(d\alpha), & \text{in } \Omega \times (0,T) \\ \nu \cdot (\nabla\rho - \alpha\rho\nabla v) = 0,\ v = 0, & \text{on } \partial\Omega \times I \times (0,T) \\ \rho(\tilde{x},0) = \rho^0(\tilde{x}) \geqslant 0, & \text{in } \widetilde{\Omega} \\ v(x,0) = v^0(x), & \text{in } \Omega. \end{cases} \tag{3.1}$$

For $\rho \in L\log L(\widetilde{\Omega})$, $\rho \geqslant 0$ a.e. in $\widetilde{\Omega}$, and $v \in H_0^1(\Omega)$, the functional $\mathcal{L}$ defined in (2.1) takes the form:

$$\mathcal{L}(\rho, v) = \int_{\widetilde{\Omega}} \rho(\tilde{x})(\log\rho(\tilde{x}) - 1)\, d\tilde{x} + \frac{1}{2}\int_\Omega |\nabla v|^2\, d\tilde{x} - \int_{\widetilde{\Omega}} \alpha\rho(\tilde{x})v(x)\, d\tilde{x}. \tag{3.2}$$

A formal proof of Theorem 2.1-(i) is easily obtained by straightforward differentiation. Indeed, for any $\varphi \in L^\infty(\widetilde{\Omega})$, we note that formally (and rigorously, if the strict inequality $\rho > 0$ holds true)

$$\langle \mathcal{L}_\rho(\rho,v), \varphi\rangle_{L^2(\widetilde{\Omega})} = \int_{\widetilde{\Omega}} (\log\rho - \alpha v)\varphi\, d\tilde{x}, \tag{3.3}$$

where $\langle \mathcal{L}_\rho(\rho,v), \varphi\rangle_{L^2(\widetilde{\Omega})} = \frac{d}{ds}\mathcal{L}(\rho + s\varphi, v)|_{s=0}$ denotes the usual Gâteaux derivative. In particular, along a solution $(\rho(\tilde{x},t), v(x,t))$ to (1.1), we formally have

$$\begin{aligned} \langle \mathcal{L}_\rho(\rho,v), \rho_t\rangle &= \int_{\widetilde{\Omega}} (\log\rho - \alpha v)\rho_t\, d\tilde{x} = \int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}(\log\rho - \alpha v)\, \mathrm{div}(\rho\nabla(\log\rho - \alpha v))\, d\tilde{x} \\ &= \int_I \frac{\mathcal{P}(d\alpha)}{\delta_\alpha} \int_\Omega (\log\rho - \alpha v)\, \mathrm{div}(\rho\nabla(\log\rho - \alpha v))\, dx \\ &= -\int_{\widetilde{\Omega}} \frac{\rho}{\delta_\alpha} |\nabla(\log\rho - \alpha v)|^2\, d\tilde{x} \leqslant 0. \end{aligned} \tag{3.4}$$

Similarly, for $\xi \in H_0^1(\Omega)$, we compute:

$$\langle \mathcal{L}_v(\rho,v), \xi\rangle = \int_{\widetilde{\Omega}} (\nabla v \cdot \nabla\xi - \alpha\rho\xi)\, d\tilde{x} = -\int_{\widetilde{\Omega}} (\Delta v + \alpha\rho)\xi\, d\tilde{x}.$$

In particular, along a solution $(\rho(\tilde{x},t), v(x,t))$ to (1.1), we have

$$\begin{aligned} \langle \mathcal{L}_v(\rho,v), v_t\rangle &= -\frac{1}{\varepsilon}\int_{\widetilde{\Omega}} (\Delta v + \alpha\rho)\left(\Delta v + \int_I \alpha'\rho\, \mathcal{P}(d\alpha')\right) d\tilde{x} \\ &= -\frac{1}{\varepsilon}\int_\Omega \left(\Delta v + \int_I \alpha\rho\, \mathcal{P}(d\alpha)\right)^2 dx \leqslant 0. \end{aligned}$$

Thus, along solutions of (1.1), we formally have the *non-increase of* $\mathcal{L}$:

$$\frac{d}{dt}\mathcal{L}(\rho(\tilde{x},t), v(x,t)) \leqslant 0 \qquad \text{for all } t \in (0,T). \tag{3.5}$$

We now provide a rigorous proof of Theorem 2.1-(i), by adapting an argument in [4].

Proof of Theorem 2.1-(i) Let $(\rho(x,t), v(x,t))$ be a fixed classical solution for (1.1) and for $\delta > 0$ let

$$g_\delta(t) := \mathcal{L}(\rho(x,t)+\delta, v(x,t)).$$

Then,

$$g_\delta(t) - g_\delta(0) = \int_0^t \left\{\langle \mathcal{L}_\rho(\rho+\delta, v), \rho_t\rangle + \langle \mathcal{L}_v(\rho+\delta, v), v_t\rangle\right\}.$$

We compute, recalling that in product space notation $\rho = \rho(\tilde{x}) = \rho(x,\alpha)$:

$$\begin{aligned}
\langle \mathcal{L}_\rho(\rho+\delta, v), \rho_t\rangle &= \int_{\widetilde{\Omega}} (\log(\rho+\delta) - \alpha v)\,\rho_t\,d\tilde{x} = \int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}(\log(\rho+\delta) - \alpha v)\,\mathrm{div}(\nabla\rho - \alpha\rho\nabla v)\\
&= -\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}\nabla(\log(\rho+\delta) - \alpha v)\cdot(\nabla\rho - \alpha\rho\nabla v)\\
&= -\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}\nabla(\log(\rho+\delta) - \alpha v)\cdot(\nabla\rho - \alpha(\rho+\delta)\nabla v + \alpha\delta\nabla v)\\
&= -\int_{\widetilde{\Omega}} \frac{\rho+\delta}{\delta_\alpha}|\nabla(\log(\rho+\delta) - \alpha v)|^2 - \delta\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}\nabla(\log(\rho+\delta) - \alpha v)\cdot\alpha\nabla v.
\end{aligned}$$

Using the elementary identity

$$|\nabla\log(\rho+\delta)|^2 = |\nabla(\log(\rho+\delta) - \xi)|^2 + |\nabla\xi|^2 + 2\nabla(\log(\rho+\delta) - \xi)\cdot\nabla\xi, \tag{3.6}$$

with $\xi = \alpha v$, we may write

$$\nabla(\log(\rho+\delta) - \alpha v)\cdot\alpha\nabla v = \frac{1}{2}\{|\nabla\log(\rho+\delta)|^2 - |\nabla(\log(\rho+\delta) - \alpha v)|^2 - |\alpha\nabla v|^2\}.$$

We deduce that

$$\begin{aligned}
\langle \mathcal{L}_\rho(\rho+\delta, v), \rho_t\rangle = &-\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}(\rho + \frac{\delta}{2})|\nabla(\log(\rho+\delta) - \alpha v)|^2 - \frac{\delta}{2}\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}|\nabla\log(\rho+\delta)|^2\\
&+ \frac{\delta}{2}\int_{\widetilde{\Omega}} \frac{\alpha^2}{\delta_\alpha}|\nabla v|^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\langle \mathcal{L}_v(\rho+\delta, v), v_t\rangle &= \int_{\widetilde{\Omega}} \nabla v\cdot\nabla v_t - \int_{\widetilde{\Omega}} \alpha(\rho+\delta)v_t = -\int_{\widetilde{\Omega}} (\Delta v + \alpha\rho)v_t - \delta\int_{\widetilde{\Omega}} \alpha v_t\\
&= -\int_\Omega (\Delta v + \int_I \alpha\rho\,\mathcal{P}(d\alpha))v_t - \delta\int_I \alpha\,\mathcal{P}(d\alpha)\int_\Omega v_t\\
&= -\frac{1}{\varepsilon}\int_\Omega (\Delta v + \int_I \alpha\rho\,\mathcal{P}(d\alpha))^2 - \delta\int_I \alpha\,\mathcal{P}(d\alpha)\int_\Omega v_t.
\end{aligned}$$

It follows that

$$
\begin{aligned}
g_\delta(t)-g_\delta(0) = &-\int_0^t\int_{\widetilde{\Omega}}\frac{1}{\delta_\alpha}\left(\rho+\frac{\delta}{2}\right)|\nabla(\log(\rho+\delta)-\alpha v)|^2-\frac{\delta}{2}\int_0^t\int_{\widetilde{\Omega}}\frac{1}{\delta_\alpha}|\nabla\log(\rho+\delta)|^2\,d\tilde{x}\\
&+\frac{\delta}{2}\int_0^t\int_I\frac{\alpha^2}{\delta_\alpha}\mathcal{P}(d\alpha)\int_\Omega|\nabla v|^2-\frac{1}{\varepsilon}\int_0^t\int_\Omega\left(\Delta v+\int_I\alpha\rho\,\mathcal{P}(d\alpha)\right)^2\\
&-\delta\int_0^t\int_I\alpha\mathcal{P}(d\alpha)\int_\Omega v_t.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&g_\delta(t)-g_\delta(0)+\int_0^t\int_{\widetilde{\Omega}}\frac{1}{\delta_\alpha}\left(\rho+\frac{\delta}{2}\right)|\nabla(\log(\rho+\delta)-\alpha v)|^2\\
&\leqslant\frac{\delta}{2}\int_0^t\int_I\frac{\alpha^2}{\delta_\alpha}\mathcal{P}(d\alpha)\int_\Omega|\nabla v|^2-\delta\int_0^t\int_I\alpha\mathcal{P}(d\alpha)\int_\Omega v_t.
\end{aligned}
$$

By continuity of the function $s\mapsto s\log s$ at 0, we have

$$\lim_{\delta\to 0^+} g_\delta(t)=\mathcal{L}(\rho(\tilde{x},t),v(x,t)).$$

Therefore, letting $\delta\to 0^+$, we obtain

$$\mathcal{L}(\rho(\tilde{x},t),v(x,t))-\mathcal{L}(\rho(\tilde{x},0),v(x,0))+\limsup_{\delta\to 0^+}\int_0^t\int_{\widetilde{\Omega}}\frac{1}{\delta_\alpha}(\rho+\frac{\delta}{2})|\nabla(\log(\rho+\delta)-\alpha v)|^2\leqslant 0.$$

Hence, the asserted decreasing properties of $\mathcal{L}$ are established. □

The case $\delta_\alpha>0$, $\varepsilon=0$ and the proof of Theorem 2.1-(ii)

In product space notation, system (1.5) takes the form

$$
\begin{cases}
\delta_\alpha\dfrac{\partial\rho}{\partial t}=\Delta\rho-\alpha\,\mathrm{div}(\rho\nabla v), & \text{in } \widetilde{\Omega}\times(0,T)\\
-\Delta v=\displaystyle\int_I\alpha\rho\,\mathcal{P}(d\alpha), & \text{in } \Omega\times(0,T)\\
\nu\cdot(\nabla\rho_\alpha-\alpha\rho_\alpha\nabla v)=0,\ v=0, & \text{on } \partial\Omega\times(0,T),\ \alpha\in[-1,1]\\
\rho(\tilde{x},0)=\rho^0(\tilde{x})\geqslant 0, & \text{in } \widetilde{\Omega}.
\end{cases}
\tag{3.7}
$$

We first recall that the Green function $G(\cdot,\cdot)$ for $-\Delta$ in Ω with Dirichlet boundary conditions is defined for $x,y\in\Omega$, $x\neq y$, by

$$
\begin{cases}
-\Delta_x G(x,y)=\delta_y, & \text{in } \Omega\\
G(\cdot,y)=0, & \text{on } \partial\Omega.
\end{cases}
\tag{3.8}
$$

By means of G, we may define a symmetric kernel $\widetilde{G}(x, y, \alpha, \beta)$ for $(x, y, \alpha, \beta) \in \widetilde{\Omega} \times \widetilde{\Omega}$, $x \neq y$, with corresponding convolution operator defined by

$$(\widetilde{G} * \rho)(x, \alpha) = \int_{\widetilde{\Omega}} G(x, y)\rho(y, \beta)\, dy\mathcal{P}(d\beta). \tag{3.9}$$

We note that, we may write:

$$\begin{aligned}\int_{\widetilde{\Omega}} \alpha\rho\, \widetilde{G} * (\alpha\rho)\, d\tilde{x} &= \int_{\widetilde{\Omega}} \alpha\rho(x, \alpha) \int_{\widetilde{\Omega}} G(x, y)\beta\rho(y, \beta)\, dy\mathcal{P}(d\beta) \\ &= \iint_{\widetilde{\Omega}^2} \alpha\beta\, G(x, y)\rho(x, \alpha)\rho(y, \beta)\, dxdy\mathcal{P}(d\alpha)\mathcal{P}(d\beta).\end{aligned}$$

Therefore, the functional $\mathcal{F}$ may be equivalently written in the form:

$$\mathcal{F}(\rho) := \int_{\widetilde{\Omega}} \rho(\log\rho - 1) - \frac{1}{2}\int_{\widetilde{\Omega}} \alpha\rho\, \widetilde{G} * (\alpha\rho).$$

For later use, we observe that we may also write:

$$\int_{\widetilde{\Omega}} \alpha\rho\, \widetilde{G} * (\alpha\rho)\, d\tilde{x} = \int_{\Omega} \left(\int_I \alpha\rho\, \mathcal{P}(d\alpha)\right) G * \left(\int_I \alpha\rho\, \mathcal{P}(d\alpha)\right)\, dx. \tag{3.10}$$

From (3.7) we deduce that

$$v = \widetilde{G} * (\alpha\rho) = G * \left(\int_I \alpha\rho\, \mathcal{P}(d\alpha)\right).$$

Proof of Theorem 2.1-(ii) Similarly as above, for $\delta > 0$ let

$$h_\delta(t) := \mathcal{F}(\rho(\tilde{x}, t) + \delta).$$

Then, using the symmetry of $\widetilde{G}$, we compute

$$\begin{aligned}h_\delta'(t) &= \int_{\widetilde{\Omega}} \left\{\log(\rho + \delta) - \alpha\widetilde{G} * (\alpha(\rho + \delta))\right\} \rho_t \\ &= \int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha} \left\{\log(\rho + \delta) - \alpha\widetilde{G} * (\alpha(\rho + \delta))\right\} \operatorname{div}(\nabla\rho - \alpha\rho\nabla\widetilde{G} * (\alpha\rho)) \\ &= -\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}\nabla \left\{\log(\rho + \delta) - \alpha\widetilde{G} * (\alpha\rho)\right\} \cdot \{\nabla\rho - \alpha(\rho + \delta)\nabla\widetilde{G} * (\alpha\rho) + \alpha\delta\nabla\widetilde{G} * (\alpha\rho)\} \\ &\quad - \delta\int_{\widetilde{\Omega}} \frac{\alpha}{\delta_\alpha}\nabla\widetilde{G} * \alpha \cdot \{\nabla\rho - \alpha\rho\nabla\widetilde{G} * (\alpha\rho)\} \\ &= -\int_{\widetilde{\Omega}} \frac{\rho + \delta}{\delta_\alpha}|\nabla\{\log(\rho + \delta) - \alpha\widetilde{G} * (\alpha\rho)\}|^2 - I - II\end{aligned}$$

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where

$$I := \int_{\widetilde{\Omega}} \frac{\delta}{\delta_\alpha} \nabla\{\log(\rho+\delta) - \alpha\widetilde{G} * (\alpha\rho)\} \cdot \alpha\nabla\widetilde{G} * (\alpha\rho),$$

$$II := \delta \int_{\widetilde{\Omega}} \frac{\alpha}{\delta_\alpha} \nabla\widetilde{G} * \alpha \cdot \{\nabla\rho - \alpha\rho\nabla\widetilde{G} * (\alpha\rho)\}.$$

Using (3.6) with $\xi = \alpha\widetilde{G} * (\alpha\rho)$, we have

$$\nabla\{\log(\rho+\delta) - \alpha\widetilde{G} * (\alpha\rho)\} \cdot \alpha\nabla\widetilde{G} * (\alpha\rho) = \frac{1}{2}|\nabla\log(\rho+\delta)|^2 - \frac{1}{2}|\nabla(\log(\rho+\delta) - \alpha\widetilde{G} * (\alpha\rho)|^2 - \frac{1}{2}|\nabla\alpha\widetilde{G} * (\alpha\rho)|^2.$$

Therefore,

$$h_\delta'(t) = -\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}\left(\rho + \frac{\delta}{2}\right)|\nabla(\log(\rho+\delta) - \alpha\widetilde{G} * (\alpha\rho))|^2 - \frac{\delta}{2}\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}|\nabla\log(\rho+\delta)|^2 + \frac{\delta}{2}\int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}|\nabla\alpha\widetilde{G} * (\alpha\rho)|^2 - II$$

We conclude that

$$h_\delta(t) - h_\delta(0) + \int_0^t \int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}\left(\rho + \frac{\delta}{2}\right)|\nabla(\log(\rho+\delta) - \alpha\widetilde{G} * (\alpha\rho))|^2 \leqslant \frac{\delta}{2}\int_0^t \int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}|\nabla\alpha\widetilde{G} * (\alpha\rho)|^2 - \delta\int_0^t \int_{\widetilde{\Omega}} \alpha\nabla\widetilde{G} * \alpha \cdot \{\nabla\rho - \alpha\rho\nabla\widetilde{G} * (\alpha\rho)\}.$$

Now, we observe that $\lim_{\delta\to 0^+} h_\delta(t) = \mathcal{F}(\rho(\tilde{x}, t))$. Therefore, letting $\delta \to 0^+$, we obtain

$$\mathcal{F}(\rho(\tilde{x}, t)) - \mathcal{F}(\rho(\tilde{x}, 0)) + \limsup_{\delta\to 0^+} \int_0^t \int_{\widetilde{\Omega}} \frac{1}{\delta_\alpha}\left(\rho + \frac{\delta}{2}\right)|\nabla(\log(\rho+\delta) - \alpha\widetilde{G} * (\alpha\rho))|^2 \leqslant 0,$$

and the asserted monotonicity property for $\mathcal{F}(\rho(\tilde{x}, t))$ follows.

If the decrease is not strict, then $\nabla(\log\rho - \alpha\widetilde{G} * (\alpha\rho)) \equiv 0$. In view of (1.6), we conclude that the solution is stationary. □

The case $\delta_\alpha = 0$, $\varepsilon = 1$ and the proof of Theorem 2.1-(iii)

In product space notation system (1.7) takes the form

$$\begin{cases} \Delta\rho - \alpha\,\mathrm{div}(\rho\nabla v) = 0, & \text{in } \widetilde{\Omega} \times (0, T) \\ \dfrac{\partial v}{\partial t} = \Delta v + \displaystyle\int_I \alpha\rho\,\mathcal{P}(d\alpha), & \text{in } \Omega \times (0, T) \\ \nu \cdot (\nabla\rho - \alpha\rho\nabla v) = 0,\ v = 0, & \text{on } \partial\Omega \times I \times (0, T) \\ v(x, 0) = v^0(x), & \text{in } \Omega. \end{cases} \tag{3.11}$$

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Proof of Theorem 2.1-(iii) We observe that for every fixed $\alpha \in I$, $t \in (0, T)$ we may write

$$\nabla\rho - \alpha\rho\nabla v = e^{\alpha v}\nabla(e^{-\alpha v}\rho). \tag{3.12}$$

Multiplying the first equation in (3.11) by $e^{-\alpha v}\rho$ and integrating, in view of the no-flux boundary condition, we have:

$$0 = \int_{\partial\Omega} e^{-\alpha v}\rho\nu\cdot(\nabla\rho - \alpha\rho\nabla v) - \int_\Omega e^{\alpha v}|\nabla(e^{-\alpha v}\rho)|^2 = -\int_\Omega e^{\alpha v}|\nabla(e^{-\alpha v}\rho)|^2.$$

We deduce that $\nabla(e^{-\alpha v}\rho) = 0$ a.e. in Ω, and consequently

$$\rho(x,\alpha,t) = C_\alpha(t)e^{\alpha v(x,t)} \tag{3.13}$$

for some $C_\alpha(t) \geqslant 0$. We shall assume that $C_\alpha(t)$ is independent of α. We note that such an assumption does not affect the dynamics of the population species ρ, which only depends on ∇v. Assuming the mass conservation (1.9), we derive from (3.11)–(3.13) the following evolution problem for v:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + \lambda\dfrac{\int_I \alpha e^{\alpha v}\,\mathcal{P}(d\alpha)}{\iint_{\Omega\times I} e^{\alpha v}\,\mathcal{P}(d\alpha)dx}, & \text{in } \Omega\times(0,T)\\ v(x,t) = 0, & \text{on } \partial\Omega\times(0,T)\\ v(x,0) = v^0(x), & \text{in } \Omega. \end{cases} \tag{3.14}$$

We recall from (2.1) that

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \lambda\log\int_{\widetilde{\Omega}} e^{\alpha v}\,d\tilde{x} + \lambda(\log\lambda - 1), \qquad v \in H_0^1(\Omega).$$

It is readily checked that (3.14) is the gradient flow for $\mathcal{J}_\lambda$. □

4 Duality and proof of Theorem 2.1-(iv)

We recall from (2.1) that $\mathcal{L}$ is defined for $\rho \in L\log L(\widetilde{\Omega})$, $\rho \geqslant 0$, and $v \in H_0^1(\Omega)$ by

$$\mathcal{L}(\rho, v) := \int_{\widetilde{\Omega}} \rho(\log\rho - 1)\,d\tilde{x} + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_{\widetilde{\Omega}} \alpha\rho v\,d\tilde{x}$$

and

$$\tilde{\Gamma}_\lambda := \left\{\rho \in L\log L(\widetilde{\Omega}) :\ \rho \geqslant 0 \text{ a.e. in } \widetilde{\Omega},\ \int_{\widetilde{\Omega}} \rho(\tilde{x})\,d\tilde{x} = \lambda\right\}.$$

The main properties needed to establish Theorem 2.1-(iv) are contained in the following statement.

Proposition 4.1 *For every fixed $v \in H_0^1(\Omega)$ there exists $\rho_v \in \widetilde{\Gamma}_\lambda$ such that*

$$\inf_{\widetilde{\Gamma}_\lambda} \mathcal{L}(\cdot, v) = \mathcal{L}(\rho_v, v).$$

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Moreover, ρ_v satisfies

$$\rho_v = \lambda \frac{e^{\alpha v}}{\int_{\widetilde{\Omega}} e^{\alpha v}\, d\tilde{x}}, \qquad \text{a.e. in } \widetilde{\Omega}. \tag{4.1}$$

Before we proceed further with the proof of Proposition 4.1, we need to state and prove two auxiliary results. We first point out that a minimizing sequence $\rho_n \in \widetilde{\Gamma}_\lambda$ for $\mathcal{L}(\cdot, v)$ may be taken uniformly bounded in $L^\infty(\widetilde{\Omega})$ and moreover the minimizer ρ_v satisfies $\rho_v > 0$ a.e. in $\widetilde{\Omega}$, following an approach established in [37]. The underlying idea is that, since the non-linearity

$$f(t) = t(\log t - 1) \tag{4.2}$$

blows up at infinity and attains a strictly negative minimum given by $\min f = f(1) = -1$, the minimizing sequence ρ_n may be modified so that $0 \leqslant \rho_n \leqslant M$ for some $M > 0$ independent of n, a.e. in $\widetilde{\Omega}$, without increasing the value of $\mathcal{L}(\cdot, v)$, and the minimizer ρ_v satisfies $\rho_v > 0$ a.e. in $\widetilde{\Omega}$. Then, the proof of Proposition 4.1 easily follows.

Lemma 4.1 *For any fixed $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ there exists $M > 0$ depending only on $\widetilde{\Omega}, \lambda$ and v such that for any $\rho \in \widetilde{\Gamma}_\lambda$ there exists $\rho^* \in \widetilde{\Gamma}_\lambda$ such that $0 \leqslant \rho^* \leqslant M$ and*

$$\mathcal{L}(\rho^*, v) \leqslant \mathcal{L}(\rho, v).$$

Proof For a fixed $M > 2\lambda/|\Omega|$ we define:

$$\widetilde{A} := \{\tilde{x} \in \widetilde{\Omega} : \rho \geqslant M\}, \qquad \widetilde{E} := \left\{\tilde{x} \in \widetilde{\Omega} : \rho \leqslant \frac{2\lambda}{|\Omega|}\right\}, \qquad k^M := \int_{\widetilde{A}} (\rho - M).$$

We claim that

$$|\widetilde{E}| \geqslant \frac{|\Omega|}{2}. \tag{4.3}$$

Indeed, we have:

$$\lambda = \int_{\widetilde{\Omega}} \rho\, d\tilde{x} = \int_{\widetilde{E}} \rho\, d\tilde{x} + \int_{\widetilde{\Omega}\setminus\widetilde{E}} \rho\, d\tilde{x} \geqslant \frac{2\lambda}{|\Omega|}(|\widetilde{\Omega}| - |\widetilde{E}|) = 2\lambda \left(1 - \frac{|\widetilde{E}|}{|\widetilde{\Omega}|}\right),$$

where we used the fact $|\widetilde{\Omega}| = \mathcal{P}(I)|\Omega| = |\Omega|$. This implies (4.3).

We also note that $k^M \leqslant \lambda$ and therefore, in view of (4.3):

$$\frac{k^M}{|\widetilde{E}|} \leqslant \frac{2\lambda}{|\Omega|}. \tag{4.4}$$

We define:

$$\rho^* := M\chi_{\widetilde{A}} + \rho\chi_{\widetilde{\Omega}\setminus(\widetilde{A}\cup\widetilde{E})} + \left(\rho + \frac{k^M}{|\widetilde{E}|}\right)\chi_{\widetilde{E}}. \tag{4.5}$$

It is readily checked that $\rho^* \in \widetilde{\Gamma}_\lambda$, indeed we have:

$$\begin{aligned}\int_{\widetilde{\Omega}} \rho^* =& M|\widetilde{A}| + \int_{\widetilde{\Omega}\setminus(\widetilde{A}\cup\widetilde{E})} \rho\, d\tilde{x} + \int_{\widetilde{E}} \rho\, d\tilde{x} + k^M \\ =& M|\widetilde{A}| + \int_{\widetilde{\Omega}\setminus\widetilde{A}} \rho\, d\tilde{x} + \int_{\widetilde{A}} (\rho - M)\, d\tilde{x} = \int_{\widetilde{\Omega}} \rho\, d\tilde{x} = \lambda.\end{aligned}$$

We write:

$$\mathcal{L}(\rho^*, v) - \mathcal{L}(\rho, v) = \int_{\widetilde{A}} [f(M) - f(\rho)] + \int_{\widetilde{E}} \left[f\left(\rho + \frac{k^M}{|\widetilde{E}|}\right) - f(\rho) \right] - \int_{\widetilde{\Omega}} \alpha(\rho^* - \rho) v.$$

Using the Mean Value Theorem, we estimate:

$$\int_{\widetilde{A}} [f(\rho) - f(M)] = \int_{\widetilde{A}} f'(M + \theta(x)(\rho - M))(\rho - M) \geqslant \log M \int_{\widetilde{A}} (\rho - M) = k^M \log M,$$

where $0 \leqslant \theta(x) \leqslant 1$. Similarly, we have

$$\int_{\widetilde{E}} \left[f\left(\rho + \frac{k^M}{|\widetilde{E}|}\right) - f(\rho) \right] \leqslant k^M C(f, \lambda), \tag{4.6}$$

where $C(f, \lambda) = \max_{1/2 \leqslant s \leqslant 4\lambda/|\Omega|} |f'(s)|$. Indeed, since f is decreasing on $[0, 1]$, if $k^M/|\widetilde{E}| \leqslant 1/2$, we readily have

$$\int_{\widetilde{E}\cap\{0\leqslant\rho\leqslant 1/2\}} \left[f(\rho) - f\left(\rho + \frac{k^M}{|\widetilde{E}|}\right) \right] \geqslant 0.$$

If $k^M/|\widetilde{E}| \geqslant 1/2$, then $0 \leqslant \rho + k^M/|\widetilde{E}| - 1/2 \leqslant k^M/|\widetilde{E}|$ and therefore

$$\begin{aligned}&\int_{\widetilde{E}\cap\{0\leqslant\rho\leqslant 1/2\}} \left[f(\rho) - f\left(\rho + \frac{k^M}{|\widetilde{E}|}\right) \right] \geqslant \int_{\widetilde{E}\cap\{0\leqslant\rho\leqslant 1/2\}} \left[f\left(\frac{1}{2}\right) - f\left(\rho + \frac{k^M}{|\widetilde{E}|}\right) \right] \\ &= \int_{\widetilde{E}\cap\{0\leqslant\rho\leqslant 1/2\}} f'\left(\frac{1}{2} + \theta(x)\left(\rho + \frac{k^M}{|\widetilde{E}|} - \frac{1}{2}\right)\right)\left(\rho + \frac{k^M}{|\widetilde{E}|} - \frac{1}{2}\right) \\ &\geqslant -k^M \max_{1/2\leqslant s\leqslant 1/2+2\lambda/|\Omega|} |f'(s)|.\end{aligned}$$

Hence, (4.6) is established. Finally, we have

$$\begin{aligned}\left|\int_{\widetilde{\Omega}} (\rho^* - \rho)\alpha v\right| \leqslant \left|\int_{\widetilde{A}} (\rho^* - \rho)\alpha v\right| + \left|\int_{\widetilde{E}} (\rho^* - \rho)\alpha v\right| &\leqslant \int_{\widetilde{A}} |\rho - M| \|v\|_\infty + k^M \|v\|_\infty \\ &\leqslant 2k^M \|v\|_\infty.\end{aligned}$$

We conclude that

$$\mathcal{L}(\rho^*, v) - \mathcal{L}(\rho, v) \leqslant (-\log M + C(f, \lambda) + 2\|v\|_\infty) k^M = (-\log M + O(1)) k^M$$

and the asserted statement follows by letting $M \to +\infty$. □

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For $\rho \in \widetilde{\Gamma}_\lambda$, we define

$$\widetilde{A} := \left\{ \tilde{x} \in \widetilde{\Omega} : \rho(\tilde{x}) \geqslant \frac{\lambda}{2|\Omega|} \right\} \qquad \text{and} \qquad \widetilde{E} := \{ \tilde{x} \in \widetilde{\Omega} : \rho(\tilde{x}) = 0 \}.$$

Lemma 4.2 *Fix $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Suppose that $|\widetilde{E}| > 0$. Then, there exists $\rho_* \in \widetilde{\Gamma}_\lambda$ such that $\rho_* > 0$ a.e. in $\widetilde{\Omega}$ and*

$$\mathcal{L}(\rho_*, v) - \mathcal{L}(\rho, v) < 0.$$

Proof We claim that $|\widetilde{A}| > 0$. Indeed, if it is not the case, we have $\rho \leqslant \lambda/(2|\Omega|)$ a.e. in $\widetilde{\Omega}$. It follows that

$$\lambda = \int_{\widetilde{\Omega}} \rho \, d\tilde{x} \leqslant |\widetilde{\Omega}| \frac{\lambda}{2|\Omega|} = \frac{\lambda}{2},$$

a contradiction. Thus, we may define

$$\varphi := \frac{\chi_{\widetilde{E}}}{|\widetilde{E}|} - \frac{\chi_{\widetilde{A}}}{|\widetilde{A}|} = \begin{cases} |\widetilde{E}|^{-1}, & \text{in } \widetilde{E} \\ 0, & \text{in } \widetilde{\Omega} \setminus (\widetilde{A} \cup \widetilde{E}) \\ -|\widetilde{A}|^{-1}, & \text{in } \widetilde{A}. \end{cases}$$

For $t > 0$ sufficiently small we set

$$\rho_* := \rho + t\varphi.$$

We note that since $\int_{\widetilde{\Omega}} \varphi \, d\tilde{x} = 0$, we have $\rho_* \in \widetilde{\Gamma}_\lambda$. Using the identity

$$\int_{\widetilde{\Omega}} (\rho + t\varphi)(\log(\rho + t\varphi) - 1) - \int_{\widetilde{\Omega}} \rho(\log \rho - 1) = \int_{\widetilde{\Omega}} \rho(\log(\rho + t\varphi) - \log \rho) + \int_{\widetilde{\Omega}} t\varphi(\log(\rho + t\varphi) - 1),$$

we may write:

$$\begin{aligned}
\mathcal{L}(\rho_*, v) - \mathcal{L}(\rho, v) &= \int_{\widetilde{\Omega}} \rho(\log(\rho + t\varphi) - \log \rho) + \int_{\widetilde{\Omega}} t\varphi(\log(\rho + t\varphi) - 1) - \int_{\widetilde{\Omega}} \alpha t \varphi v \\
&= \int_{\widetilde{A}} \rho \left(\log\left(\rho - \frac{t}{|\widetilde{A}|} \right) - \log \rho \right) + \int_{\widetilde{E}} \frac{t}{|\widetilde{E}|} \left(\log \frac{t}{|\widetilde{E}|} - 1 \right) \\
&\quad - \int_{\widetilde{A}} \frac{t}{|\widetilde{A}|} \left(\log\left(\rho - \frac{t}{|\widetilde{A}|} \right) - 1 \right) + \int_{\widetilde{A}} \alpha \frac{t}{|\widetilde{A}|} v - \int_{\widetilde{E}} \alpha \frac{t}{|\widetilde{E}|} v \\
&= t \left\{ \left(\log \frac{t}{|\widetilde{E}|} - 1 \right) + \frac{1}{|\widetilde{A}|} \int_{\widetilde{A}} \frac{\rho |\widetilde{A}|}{t} \log\left(1 - \frac{t}{\rho |\widetilde{A}|} \right) \right. \\
&\quad \left. - \frac{1}{|\widetilde{A}|} \int_{\widetilde{A}} \left[\log\left(\rho - \frac{t}{|\widetilde{A}|} \right) - 1 \right] - \frac{1}{|\widetilde{E}|} \int_{\widetilde{E}} \alpha v + \frac{1}{|\widetilde{A}|} \int_{\widetilde{A}} \alpha v \right\} \\
&= t\{\log t + O(1)\}
\end{aligned}$$

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as $t \to 0^+$, where in order to derive the last line we used the fact

$$\frac{|\widetilde{A}|}{t}\int_{\widetilde{A}} \rho\left(\log\left(\rho - \frac{t}{|\widetilde{A}|}\right) - \log\rho\right) = \int_{\widetilde{A}} \frac{|\widetilde{A}|\rho}{t}\log\left(1 - \frac{t}{|\widetilde{A}|\rho}\right) = O(1).$$

We conclude that $\mathcal{L}(\rho_*, v) - \mathcal{L}(\rho, v) < 0$ for sufficiently small values of $t > 0$. □

Proof of Proposition 4.1 In view of Lemma 4.1, we may assume that the minimizing sequence ρ_n is uniformly bounded in $L^\infty(\widetilde{\Omega})$. In particular, it is uniformly bounded in $L^p(\widetilde{\Omega})$ for all $1 < p < +\infty$. Consequently, there exists $\rho_v \in L^p(\widetilde{\Omega})$ such that, up to subsequences, $\rho_n \rightharpoonup \rho_v \in \tilde{\Gamma}_\lambda$ weakly in $L^p(\widetilde{\Omega})$, for all $1 < p < +\infty$. By convexity of $\mathcal{L}(\cdot, v)$, ρ_v is the desired minimizer. We are left to establish (4.1). To this end, for every $\delta > 0$, we define $\Lambda_\delta := \{\rho_v > \delta\}$ and $U_\delta := \{\varphi \in L^\infty(\widetilde{\Omega}) : \|\varphi\|_{L^\infty(\widetilde{\Omega})} < \delta/2\}$. We can differentiate the function $\mathcal{L}(\rho_v + t\chi_{\Lambda_\delta}\varphi, v)$ with respect to t with constraint $\int_{\widetilde{\Omega}} \chi_{\Lambda_\delta}\varphi\, d\tilde{x} = 0$ at $t = 0$. We thus obtain that

$$\log\rho_v - \alpha v = C \qquad \text{a.e. in } \Lambda_\delta,$$

where C is a Lagrange multiplier. Since for $\delta' < \delta$, we have $\Lambda_{\delta'} \supset \Lambda_\delta$, we conclude that C does not depend on δ. Hence, (4.1) holds true in $\bigcup_{\delta>0}\Lambda_\delta$. In view of Lemma 4.2, we have $|\widetilde{\Omega} \setminus \bigcup_{\delta>0}\Lambda_\delta| = 0$.

Since $\rho_v \in \tilde{\Gamma}_\lambda$, we conclude that

$$\rho_v = \lambda\frac{e^{\alpha v}}{\int_{\widetilde{\Omega}} e^{\alpha v}\, d\tilde{x}} \qquad \text{a.e. in } \widetilde{\Omega}. \tag{4.7}$$

Now the proof of Proposition 4.1 is complete. □

Proof of Theorem 2.1-(iv) We claim that

$$\inf_{\widetilde{\Gamma}_\lambda}\mathcal{L}(\cdot, v) = \mathcal{L}(\rho_v, v) = \mathcal{J}_\lambda(v). \tag{4.8}$$

Indeed, from (4.7) we derive that

$$\log\rho_v = \alpha v - \log\int_{\widetilde{\Omega}} e^{\alpha v} + \log\lambda.$$

We compute

$$\begin{aligned}\mathcal{L}(\rho_v, v) &= \int_{\widetilde{\Omega}} \rho_v(\log\rho_v - 1) + \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\widetilde{\Omega}} \alpha\rho_v v\\ &= \int_{\widetilde{\Omega}} \rho_v\left(\alpha v - \log\int_{\widetilde{\Omega}} e^{\alpha v} + \log\lambda - 1\right) + \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\widetilde{\Omega}} \alpha\rho_v v\\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda\log\int_{\widetilde{\Omega}} e^{\alpha v} + \lambda(\log\lambda - 1) = \mathcal{J}_\lambda(v),\end{aligned}$$

where we used $\int_{\widetilde{\Omega}} \rho_v = \lambda$ to derive the last line. Thus, (4.8) is established.

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Similarly, we claim that for every fixed $\rho \in \widetilde{\Gamma}_\lambda$ there holds

$$\inf_{H_0^1(\Omega)} \mathcal{L}(\rho, \cdot) = \mathcal{F}(\rho). \tag{4.9}$$

Indeed, it is standard to check that $\inf_{H_0^1(\Omega)} \mathcal{L}(\rho, \cdot)$ is attained at the solution $v_\rho \in H_0^1(\Omega)$ of the following:

$$-\Delta v_\rho = \int_I \alpha\rho\, \mathcal{P}(d\alpha) \quad \text{in } \Omega, \qquad v_\rho = 0 \quad \text{on } \partial\Omega.$$

We observe that

$$\int_\Omega |\nabla v_\rho|^2 = \int_\Omega (-\Delta v_\rho) v_\rho = \int_\Omega \int_I \alpha\rho\, \mathcal{P}(d\alpha) v_\rho\, dx = \int_\Omega \int_I \alpha\rho\, \mathcal{P}(d\alpha)\, G * \int_I \alpha\rho\, \mathcal{P}(d\alpha).$$

In view of the above and (3.10) we deduce:

$$\begin{aligned} \mathcal{L}(\rho, v_\rho) &= \int_{\widetilde{\Omega}} \rho(\log\rho - 1) - \frac{1}{2} \int_{\widetilde{\Omega}} \int_I \alpha\rho\mathcal{P}(d\alpha) v_\rho \\ &= \int_{\widetilde{\Omega}} \rho(\log\rho - 1) - \frac{1}{2} \int_\Omega \int_I \alpha\rho\mathcal{P}(d\alpha) G * \int_I \alpha\rho\mathcal{P}(d\alpha). \end{aligned}$$

Now the proof of Theorem 2.1-(iv) follows from (4.8) and (4.9). □

5 Critical mass and proof of Theorem 2.2

In order to prove Theorem 2.2 we set

$$\Gamma_\lambda = \left\{ \psi \in L\log L(\Omega) : \psi \geqslant 0 \text{ a.e. in } \Omega, \int_\Omega \psi = \lambda \right\}.$$

We recall that $f(t) = t(\log t - 1)$ for $t \geqslant 0$, see (4.2). For $\psi \in \Gamma_\lambda$ let

$$\mathcal{F}_0(\psi) = \int_\Omega \psi(\log\psi - 1)\, dx - \frac{1}{2} \int_\Omega \psi\, G * \psi\, dx.$$

The following sharp logarithmic Hardy–Littlewood–Sobolev inequality is due to Beckner.

Lemma 5.1 ([3]) *The functional $\mathcal{F}_0$ is bounded from below on Γ_λ if and only if $\lambda \leqslant 8\pi$.*

We shall need the following slightly more general result, which follows directly from Lemma 5.1.

Corollary 5.1 *There holds:*

$$\inf \left\{ \mathcal{F}_0(\psi) : \psi \in \bigcup_{\lambda \leqslant 8\pi} \Gamma_\lambda \right\} > -\infty.$$

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Proof Let $\psi \in \Gamma_\lambda$ and let $0 \leqslant t \leqslant 1$. We compute:

$$\begin{aligned}\mathcal{F}_0(t\psi) &= \int_\Omega t\psi(\log(t\psi)-1) - \frac{t^2}{2}\int_\Omega \psi\, G*\psi = \int_\Omega t\psi(\log\psi + \log t - 1) - \frac{t^2}{2}\int_\Omega \psi\, G*\psi \\ &= t\int_\Omega \psi(\log\psi - 1) + t\log t\int_\Omega \psi - \frac{t^2}{2}\int_\Omega \psi\, G*\psi \\ &= t\left\{\int_\Omega \psi(\log\psi-1) - \frac{t}{2}\int_\Omega \psi\, G*\psi\right\} + \lambda\, t\log t.\end{aligned}$$

Since $\int_\Omega \psi\, G*\psi \geqslant 0$, and using the fact $t\log t \geqslant -e^{-1}$, we deduce that

$$\mathcal{F}_0(t\psi) \geqslant t\mathcal{F}_0(\psi) - \frac{\lambda}{e} \geqslant \min\left\{\inf_{\Gamma_\lambda}\mathcal{F}_0, 0\right\} - \frac{\lambda}{e}.$$

The claim follows. □

Proof of Theorem 2.2, 'if' part Setting

$$\psi_\rho(x) := \left|\int_I \alpha\rho(x,\alpha)\,\mathcal{P}(d\alpha)\right|,$$

we find that

$$0 \leqslant \psi_\rho(x) \leqslant \int_I \rho(x,\alpha)\,\mathcal{P}(d\alpha), \tag{5.1}$$

and therefore

$$\int_\Omega \psi_\rho \leqslant \int_{\widetilde{\Omega}} \rho\, d\tilde{x} = \lambda.$$

In particular, we have

$$\psi_\rho \in \bigcup_{\lambda\leqslant 8\pi} \Gamma_\lambda. \tag{5.2}$$

In view of (3.10) and (4.2), we may write

$$\mathcal{F}(\rho) = \int_{\widetilde{\Omega}} f(\rho) - \frac{1}{2}\int_\Omega \left(\int_I \alpha\rho\right)\widetilde{G}*\left(\int_I \alpha\rho\right).$$

Consequently, we have

$$\mathcal{F}(\rho) \geqslant \int_{\widetilde{\Omega}} f(\rho) - \frac{1}{2}\int_\Omega \psi_\rho\, G*\psi_\rho = \int_{\widetilde{\Omega}} f(\rho) - \int_\Omega f(\psi_\rho) + \mathcal{F}_0(\psi_\rho).$$

In view of (5.2) and Corollary 5.1, we are thus reduced to show that

$$\inf_{\widetilde{\Gamma}_\lambda}\left\{\int_{\widetilde{\Omega}} f(\rho)\,d\tilde{x} - \int_\Omega f(\psi_\rho)\,dx\right\} > -\infty. \tag{5.3}$$

Since f is convex and $\mathcal{P}(I)=1$, in view of Jensen's inequality we have, for every *fixed* $x \in \Omega$, that

$$f\left(\int_I \rho(x,\alpha)\,\mathcal{P}(d\alpha)\right) \leqslant \int_I f(\rho(x,\alpha))\,\mathcal{P}(d\alpha).$$

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Integrating over Ω we deduce that

$$\int_\Omega f\left(\int_I \rho(x,\alpha)\,\mathcal{P}(d\alpha)\right)dx \leqslant \int_{\widetilde{\Omega}} f(\rho)\,d\tilde{x}.$$

In order to complete the proof, we observe that from (5.1) and some elementary properties of the non-linearity f, in particular the fact $f(t) \geqslant -1$ for all $t \geqslant 0$, we obtain

$$f(\psi_\rho) \leqslant f\left(\int_I \rho_\alpha\,\mathcal{P}(d\alpha)\right) + 1.$$

This concludes the proof of the 'if part' of Theorem 2.2. □

For the proof of the 'only if' part we may use the same test functions as may be found, e.g., in [40]. For $\epsilon > 0$, let U_ϵ be the radial 'Liouville bubble' defined by

$$U_\epsilon(x) := \log\frac{8\epsilon^2}{(\epsilon^2 + |x|^2)^2}. \tag{5.4}$$

It is well known that the functions U_ϵ satisfy

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U < +\infty, \end{cases} \tag{5.5}$$

and moreover there holds

$$\int_{\mathbb{R}^2} e^{U_\epsilon} = 8\pi, \qquad \text{for all } \epsilon > 0.$$

Without loss of generality, we assume that $0 \in \Omega$. Let

$$\psi_\epsilon := \lambda\frac{e^{U_\epsilon}}{\int_\Omega e^{U_\epsilon}}. \tag{5.6}$$

Clearly, $\psi_\epsilon \in \Gamma_\lambda$ for all $\epsilon > 0$. We first establish a lemma for the functions ψ_ϵ defined in (5.6).

Lemma 5.2 *The following expansions hold true.*

(i) $\int_\Omega \psi_\epsilon \log\psi_\epsilon = \lambda\log\frac{1}{\epsilon^2} + O(1)$;

(ii) $\int_\Omega \psi_\epsilon\, G * \psi_\epsilon = \frac{\lambda^2}{8\pi + o(1)}\log\frac{1}{\epsilon^4} + O(1)$.

The proof of Lemma 5.2 is straightforward; the details are provided in the appendix.

Now we can conclude the proof of Theorem 2.2.

Proof of Theorem 2.2, 'only if' part Assuming that $\lambda > 8\pi$, we provide a family of functions $\rho_\epsilon \in \widetilde{\Gamma}_\lambda$ such that

$$\mathcal{F}(\rho_\epsilon) \to -\infty \quad \text{as } \epsilon \to 0^+. \tag{5.7}$$

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We assume that $\operatorname{supp}\mathcal{P} \ni 1$, the remaining case being completely analogous. Let $0 < \eta < 1$. Then, $\mathcal{P}([1-\eta, 1]) > 0$. For all $\epsilon > 0$ we define

$$\rho_\epsilon(\tilde{x}) = \rho_\epsilon(x,\alpha) := \lambda \frac{\chi_{[1-\eta,1]}(\alpha)}{\mathcal{P}([1-\eta,1])} \frac{e^{U_\epsilon(x)}}{\int_\Omega e^{U_\epsilon}} = \frac{\chi_{[1-\eta,1]}(\alpha)}{\mathcal{P}([1-\eta,1])} \psi_\epsilon(x).$$

Clearly, $\int_{\widetilde{\Omega}} \rho_\epsilon = \lambda$ for all $\epsilon > 0$.

We claim that

$$\int_{\widetilde{\Omega}} \alpha\rho_\epsilon \, \widetilde{G} * (\alpha\rho_\epsilon) \, d\tilde{x} = \left(\frac{\int_{[1-\eta,1]} \alpha \, \mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \right)^2 \int_\Omega \psi_\epsilon \, G * \psi_\epsilon. \tag{5.8}$$

Indeed, we have

$$\begin{aligned}
\int_{\widetilde{\Omega}} \alpha\rho_\epsilon \, \widetilde{G} * (\alpha\rho_\epsilon) \, d\tilde{x} &= \int_{[1-\eta,1]} \frac{\alpha\mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \int_\Omega \psi_\epsilon(x) \, dx \int_{\widetilde{\Omega}} G(x,y)\beta\rho_\epsilon(y) \, \mathcal{P}(d\beta) dy \\
&= \int_{[1-\eta,1]} \frac{\alpha\mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \int_\Omega \psi_\epsilon(x) \, dx \int_{[1-\eta,1]} \frac{\beta\mathcal{P}(d\beta)}{\mathcal{P}([1-\eta,1])} \int_\Omega G(x,y)\psi_\epsilon(y) \, dy \\
&= \left(\frac{\int_{[1-\eta,1]} \alpha \, \mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \right)^2 \int_\Omega \psi_\epsilon \, G * \psi_\epsilon.
\end{aligned}$$

We claim that

$$\int_{\widetilde{\Omega}} \rho_\epsilon(\tilde{x}) \log \rho_\epsilon(\tilde{x}) \, d\tilde{x} = \int_\Omega \psi_\epsilon(x) \log \psi_\epsilon(x) \, dx. \tag{5.9}$$

Indeed, we have

$$\begin{aligned}
\int_{\widetilde{\Omega}} \rho_\epsilon(\tilde{x}) \log \rho_\epsilon(\tilde{x}) \, d\tilde{x} &= \int_{[1-\eta,1]} \frac{\mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \int_\Omega \psi_\epsilon \log(\chi_{[1-\eta,1]}(\alpha)\psi_\epsilon(x)) \, dx \\
&= \int_{[1-\eta,1]} \frac{\mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \int_\Omega \psi_\epsilon \log \psi_\epsilon(x) \, dx \\
&= \int_\Omega \psi_\epsilon(x) \log \psi_\epsilon(x) \, dx.
\end{aligned}$$

In view of (5.8) and (5.9), we may write

$$\mathcal{F}(\rho_\epsilon) = \int_\Omega \psi_\epsilon \log \psi_\epsilon - \frac{1}{2} \left(\frac{\int_{[1-\eta,1]} \alpha \, \mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \right)^2 \int_\Omega \psi_\epsilon \, G * \psi_\epsilon - \lambda.$$

In view of Lemma 5.2, we deduce the expansion

$$\mathcal{F}(\rho_\epsilon) = \lambda \left\{ 1 - \left(\frac{\int_{[1-\eta,1]} \alpha \, \mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \right)^2 \frac{\lambda}{8\pi + o(1)} \right\} \log \frac{1}{\epsilon^2} + O(1),$$

as $\epsilon \to 0^+$. Since $\lambda > 8\pi$, by taking $0 < \eta \ll 1$, we may assume that

$$\lambda > \left(\frac{\mathcal{P}([1-\eta,1])}{\int_{[1-\eta,1]} \alpha\,\mathcal{P}(d\alpha)} \right)^2 8\pi.$$

It follows that for some suitably small $\epsilon_0 > 0$, we have

$$1 - \left(\frac{\int_{[1-\eta,1]} \alpha\,\mathcal{P}(d\alpha)}{\mathcal{P}([1-\eta,1])} \right)^2 \frac{\lambda}{8\pi + o(1)} < 0$$

for all $0 < \epsilon < \epsilon_0$, and the desired asymptotic behaviour (5.7) follows.

The proof of Theorem 2.2 is now complete. □

6 Appendix: proof of Lemma 5.2

We recall from Section 5 that

$$\psi_\epsilon = \lambda \frac{e^{U_\epsilon}}{\int_\Omega e^{U_\epsilon}},$$

where U_ϵ is the Liouville bubble defined in (5.4). In what follows we define:

$$\Omega_\epsilon := \{y \in \mathbb{R}^2 : \epsilon y \in \Omega\}. \tag{6.1}$$

We compute:

$$\int_\Omega \psi_\epsilon \log \psi_\epsilon = \int_\Omega \frac{\lambda}{\int_\Omega e^{U_\epsilon}} e^{U_\epsilon} \log\left(\frac{\lambda}{\int_\Omega e^{U_\epsilon}} e^{U_\epsilon}\right) = \frac{\lambda}{\int_\Omega e^{U_\epsilon}} \int_\Omega e^{U_\epsilon} U_\epsilon + \lambda \log\left(\frac{\lambda}{\int_\Omega e^{U_\epsilon}}\right). \tag{6.2}$$

Moreover,

$$\int_\Omega \psi_\epsilon\, G * \psi_\epsilon = \left(\frac{\lambda}{\int_\Omega e^{U_\epsilon}}\right)^2 \int_\Omega e^{U_\epsilon}\, G * e^{U_\epsilon}. \tag{6.3}$$

Lemma 6.1 *The following expansion holds, as $\epsilon \to 0^+$:*

$$\int_\Omega e^{U_\epsilon} = 8\pi + o(1).$$

Proof We have, recalling (6.1):

$$\int_\Omega e^{U_\epsilon} = \int_\Omega \frac{8\epsilon^2}{(\epsilon^2 + |x|^2)^2}\, dx = 8 \int_{\Omega_\epsilon} \frac{dy}{(1+|y|^2)^2}.$$

Let $0 < r_1 < r_2$ be such that $B_{r_1} \subset \Omega \subset B_{r_2}$. We have, for $j = 1, 2$:

$$\int_{B_{r_j/\epsilon}} \frac{dy}{(1+|y|^2)^2} = \pi\left(1 - \frac{1}{1 + (\frac{r_j}{\epsilon})^2}\right)$$

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so that

$$8\pi\left(1-\frac{1}{1+(\frac{r_1}{\epsilon})^2}\right)\leqslant\int_\Omega e^{U_\epsilon}\leqslant 8\pi\left(1-\frac{1}{1+(\frac{r_2}{\epsilon})^2}\right)$$

and the claim follows. □

Lemma 6.2 *The following expansion holds, as $\epsilon\to 0^+$:*

$$\int_\Omega e^{U_\epsilon}U_\epsilon=\log\left(\frac{1}{\epsilon^2}\right)\int_\Omega e^{U_\epsilon}+O(1),$$

uniformly for $\epsilon\to 0^+$.

Proof We have

$$\int_\Omega e^{U_\epsilon}U_\epsilon=\int_\Omega e^{U_\epsilon}\log\frac{8\epsilon^2}{(\epsilon^2+|x|^2)^2}=\int_\Omega e^{U_\epsilon}\log\frac{1}{(\epsilon^2+|x|^2)^2}+\log(8\epsilon^2)\int_\Omega e^{U_\epsilon}.$$

We simplify the first term:

$$\int_\Omega e^{U_\epsilon}\log\frac{1}{(\epsilon^2+|x|^2)^2}\,dx=\int_\Omega e^{U_\epsilon}\log\frac{1}{\epsilon^4(1+|\frac{x}{\epsilon}|^2)^2}\,dx$$
$$\overset{y=x/\epsilon}{=}\log\frac{1}{\epsilon^4}\int_\Omega e^{U_\epsilon}+\int_{\Omega/\epsilon}\frac{8}{(1+|y|^2)^2}\log\frac{1}{(1+|y|^2)^2}\,dy.$$

The asserted expansion follows. □

We note that in view of (5.5), we may write

$$G*e^{U_\epsilon}=PU_\epsilon,$$

where P denotes the projection operator onto $H_0^1(\Omega)$. We recall that

$$PU_\epsilon=U_\epsilon-\log(8\epsilon^2)+8\pi H(x,0)+O(\epsilon^2), \tag{6.4}$$

where $H(x,y)$ is the Robin's function defined by

$$G(x,y)=\frac{1}{2\pi}\log\frac{1}{|x-y|}+H(x,y),$$

see, e.g., [14].

Lemma 6.3 *The following expansion holds:*

$$\int_\Omega e^{U_\epsilon}\,G*e^{U_\epsilon}=\log\frac{1}{\epsilon^4}\int_\Omega e^{U_\epsilon}+O(1).$$

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Proof Using (6.4), we compute:

$$\begin{aligned}\int_\Omega e^{U_\epsilon} G * e^{U_\epsilon} &= \int_\Omega e^{U_\epsilon} P U_\epsilon = \int_\Omega e^{U_\epsilon}(U_\epsilon - \log(8\epsilon^2) + O(1)) \\ &= \log\left(\frac{1}{\epsilon^2}\right) \int_\Omega e^{U_\epsilon} - \log \epsilon^2 \int_\Omega e^{U_\epsilon} + O(1).\end{aligned}$$

The claim follows. □

Proof of Lemma 5.2 Proof of (i). In view of (6.2), Lemma 6.1 and Lemma 6.2, we readily derive the desired expansion.

Proof of (ii). In view of (6.3), Lemma 6.1 and Lemma 6.3, we readily derive the desired expansion. □

7 Concluding remarks: comparison of two mean field equations

We have rigorously established in Theorem 2.1 that the functionals

$$\begin{aligned}\mathcal{L}(\rho, v) &= \int_{\widetilde{\Omega}} \rho(\log \rho - 1)\, d\tilde{x} + \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_{\widetilde{\Omega}} \alpha\rho v\, d\tilde{x}, \\ \mathcal{J}_\lambda(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \lambda \log\left(\int_{\widetilde{\Omega}} e^{\alpha v}\, d\tilde{x}\right) + \lambda(\log\lambda - 1),\end{aligned}$$

where $\rho = \oplus \rho_\alpha \in L\log L(\widetilde{\Omega})$, $v \in H_0^1(\Omega)$, are related by the minimization property

$$\mathcal{J}_\lambda(v) = \min_{\widetilde{\Gamma}_\lambda} \mathcal{L}(\cdot, v) \qquad \text{for all } v \in H_0^1(\Omega),$$

where

$$\widetilde{\Gamma}_\lambda := \left\{\rho \in L\log L(\widetilde{\Omega}) :\ \rho \geqslant 0 \quad \text{a.e.}\ ,\ \int_{\widetilde{\Omega}} \rho\, d\tilde{x} = \lambda\right\}.$$

Moreover, Theorem 2.1–(iv) and Theorem 2.2 imply that the optimal value of $\lambda > 0$, which ensures boundedness from below of $\mathcal{J}_\lambda$ on $H_0^1(\Omega)$ is given by

$$\bar{\lambda} = 8\pi. \tag{7.1}$$

In view of the corresponding results for the case $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$, the value $\bar{\lambda}$ is expected to provide the critical total mass for the occurrence of chemotactic collapse versus the existence of global solutions for (1.1), as well for the evolution problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + \lambda \displaystyle\int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_{\widetilde{\Omega}} e^{\beta v}\, d\tilde{x}} \mathcal{P}(d\alpha), & \text{in } \Omega \times (0, T) \\ v = 0, & \text{on } \partial\Omega \times (0, T) \\ v(x, 0) = v^0(x), & \text{in } \Omega. \end{cases}$$

See [12, 16, 19, 23] and the references therein. The critical value $\bar{\lambda}$ also plays a central role in establishing the existence of the corresponding steady states, i.e., of solutions for the

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non-local semi-linear elliptic problem

$$\begin{cases} -\Delta v = \lambda \int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_{\widetilde{\Omega}} e^{\beta v}\, d\tilde{x}}\, \mathcal{P}(d\alpha), & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

See [11, 27, 32, 38, 41].

It is interesting to compare the properties mentioned above with the corresponding results recently obtained in [37] for the *same* Lyapunov functional $\mathcal{L}$ under a *different* constraint for the conserved population mass. Such conditions were originally motivated by the deterministic model for stationary turbulent flows with variable intensity derived in [42] along the approach introduced by Onsager, see [45] and the references therein.

More precisely, for $\lambda > 0$, we define the functional

$$\mathcal{I}_\lambda(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \lambda \int_{[-1,1]} \log\left(\int_\Omega e^{\alpha v}\, dx\right) \mathcal{P}(d\alpha) + \lambda(\log\lambda - 1).$$

We recall from Section 5 that the set Γ_λ is defined by

$$\Gamma_\lambda := \left\{\psi \in L\log L(\Omega) :\ \psi \geqslant 0 \text{ a.e. in } \Omega,\ \int_\Omega \psi\, dx = \lambda\right\}$$

and we define correspondingly

$$\widetilde{\widetilde{\Gamma}}_\lambda := \oplus_{\alpha\in[-1,1]}\Gamma_\lambda := \{\oplus\rho_\alpha :\ \rho_\alpha \in \Gamma_\lambda \text{ for all } \alpha \in [-1,1]\}.$$

In words, $\widetilde{\widetilde{\Gamma}}_\lambda$ is the admissible set of population densities ρ_α, $\alpha \in I$, *all* of which have total mass λ, i.e., $\int_\Omega \rho_\alpha = \lambda$ for all $\alpha \in I$.

The following duality property was rigorously established in [37] in the same spirit as Theorem 2.1–(iv):

$$\inf_{\widetilde{\widetilde{\Gamma}}_\lambda \times H_0^1(\Omega)} \mathcal{L} = \inf_{\widetilde{\widetilde{\Gamma}}_\lambda} \mathcal{F} = \inf_{H_0^1(\Omega)} \mathcal{I}_\lambda.$$

Moreover,

$$\mathcal{I}_\lambda(v) = \min_{\widetilde{\widetilde{\Gamma}}_\lambda} \mathcal{L}(\cdot, v) \text{ for all } v \in H_0^1(\Omega).$$

This duality property, together with the logarithmic Hardy–Littlewood–Sobolev inequality established in [43], was used to compute the optimal value of λ which ensures boundedness from below of the functional $\mathcal{I}_\lambda$, which is given by

$$\bar{\bar{\lambda}} = \inf\left\{\frac{8\pi\mathcal{P}(K_\pm)}{[\int_{K_\pm} \alpha\, \mathcal{P}(d\alpha)]^2} :\ K_\pm \subset I_\pm \cap \operatorname{supp}\mathcal{P}\right\},$$

where we denote $I_+ := [0,1]$, $I_- := [-1,0)$, and where $K_\pm$ denotes a Borel subset of $I_\pm$. In particular, $\bar{\bar{\lambda}}$ significantly depends on $\mathcal{P}$. The value $\bar{\bar{\lambda}}$ is expected to provide the critical mass for chemotactic collapse versus global existence of solutions for the evolution

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problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + \lambda \displaystyle\int_I \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}\,dx}\,\mathcal{P}(d\alpha) & \text{in } \Omega \times (0,T) \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ v(x,0) = v^0(x), & \text{in } \Omega, \end{cases} \tag{7.3}$$

We note that (7.3) is obtained from (1.8) by assuming the 'individual population mass conservation' constraint:

$$\int_\Omega \rho_\alpha(x,t)\,dx = \lambda \qquad \text{for all } \alpha \in [-1,1]. \tag{7.4}$$

Condition (7.4) is natural when the population species do not evolve from one kind into another. The value $\bar{\bar{\lambda}}$ also yields the first blow-up level for the corresponding steady state problem

$$\begin{cases} -\Delta v = \lambda \displaystyle\int_I \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}\,dx}\,\mathcal{P}(d\alpha), & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

Results for solutions to the stationary problem (7.5) have been obtained in [20, 30]. In particular, the special case $\mathcal{P}(d\alpha) = (\delta_1(d\alpha) - \delta_{1/2}(d\alpha))/2$ was studied in [20] in relation to the Tzitzéica equation in differential geometry.

In short, the steady state analysis for the problems (7.2) and (7.5) shows that, despite of their formal similarity and the fact that they are motivated by the same statistical mechanics problem, the corresponding solution sets exhibit significantly different mathematical properties.

By introducing the new multi-species chemotaxis system (1.1), we have shown that the stationary problems (7.2) and (7.5) may be *both* viewed as steady states for the chemotaxis system (1.1) in the fast population dynamics limit, by imposing *different* conserved population mass constraints given by (1.9) and (7.4), respectively; the former being natural in the situation where the populations ρ_α are produced by a cell differentiation process, the latter in the situation where evolution from one species into another does not occur.

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