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SOME REMARKS ON DAVIE'S UNIQUENESS THEOREM

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Abstract We present a new approach to Davie's theorem on the uniqueness of solutions to the equation

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}W_t$$

for almost all Brownian paths. A generalization of this result and a discussion of some closely related problems are given.

Keywords: Brownian motion; stochastic differential equation; pathwise uniqueness

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1. Introduction

In this paper we consider the stochastic differential equation

$$X_t = x + W_t + \int_0^t b(s, X_s)\,\mathrm{d}s. \tag{1.1}$$

When the drift coefficient b is a Borel measurable bounded mapping, the uniqueness of the strong solution follows from the well-known result by Veretennikov [**10**]. Later, the result of Veretennikov was extended to the case of locally unbounded measurable drift by Gyongy and Martinez [**6**] and Krylov and Röckner [**8**] (see also [**2, 3**] and references therein). After Veretennikov had proved his result, Krylov suggested the problem of showing the uniqueness of the solution in a stronger sense. Namely, that for almost every Wiener trajectory the solution of the corresponding integral equation is unique.

In [**1**] Davie proved the following theorem.

Theorem 1.1. *Let $b\colon [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a Borel measurable bounded mapping. Then for almost all Brownian paths, (1.1) has exactly one solution.*

The proof of Davie is quite self-contained, but rather technically complicated. In particular, it does not rely on the uniqueness of strong solutions. It turns out that in some cases the pathwise uniqueness can be proved with a slightly simpler approach. The main idea is to use the Hölder regularity of the flow generated by the strong solution proved in [**3**] and a modification of the van Kampen uniqueness theorem for ordinary differential equations with a Lipschitz flow and continuous coefficients (see [**9**]). This approach also enables us to extend Davie's result to some other classes of irregular drifts.

2. Auxiliary results

The proof of Davie uses the following estimate.

Proposition 2.1. *Let $b \in C([0,1], C_b^1(\mathbb{R}^d, \mathbb{R}^d))$, $\|b\|_\infty \leqslant 1$. There exist positive constants C, α (that do not depend on b) such that the following inequality holds:*

$$\mathbb{E} \exp\left(\alpha \left| \int_0^1 b'_x(t, W_t)\,\mathrm{d}t \right|^2\right) \leqslant C.$$

An interesting discussion of this inequality and some similar problems can be found in [**4**]. The original proof of Davie is quite long and relies on some explicit computations for the Gaussian kernel. Since our approach to Davie's theorem in the case of a Borel measurable drift also uses this estimate, below we present a proof that seems to be less technical than that in [**1**].

Proof. We first prove the desired inequality for $d = 1$. Let

$$Z_s := b(s, W_s).$$

For the quadratic covariation of the processes Z and W we have the following representations (see [**5**]):

$$\begin{aligned}
[Z, W]_1 &= \int_0^1 b'_x(s, W_s)\,\mathrm{d}s, \\
[Z, W]_1 &= \lim \sum (Z_{t_{i+1}} - Z_{t_i})(W_{t_{i+1}} - W_{t_i}), \\
[Z, W]_1 &= \int_0^1 Z_t\,\mathrm{d}^* W_t - \int_0^1 Z_t\,\mathrm{d}W_t,
\end{aligned}$$

where

$$\int_0^1 Z_t\,\mathrm{d}^* W_t = \int_0^1 Z_{1-s}\,\mathrm{d}\tilde{W}_s, \quad \tilde{W}_s = W_{1-s}.$$

The process $\tilde{W}_t$ (the time-reversed Brownian motion) satisfies the integral equality

$$\tilde{W}_t = \tilde{W}_0 + B_t + \int_0^t \frac{-\tilde{W}_s}{1-s}\,\mathrm{d}s,$$

where B is another Brownian motion. Then

$$\begin{aligned}
&\int_0^1 b'_x(t, W_t)\,\mathrm{d}t \\
&\qquad = \int_0^1 b(1-t, W_{1-t})\,\mathrm{d}B_t + \int_0^1 \frac{-W_{1-t} b(1-t, W_{1-t})}{1-t}\,\mathrm{d}t - \int_0^1 b(t, W_t)\,\mathrm{d}W_t \\
&\qquad = I_1 + I_2 + I_3.
\end{aligned}$$

It is easy to notice that the terms I_1 and I_3 can be estimated by means of the Dubins–Schwarz theorem and the well-known formula for the distribution of the maximum of a

Wiener process on the interval $[0,1]$. The assumption that $\|b\|_\infty \leqslant 1$ implies that there exist constants $\alpha_1, C_1 > 0$ such that

$$\mathbb{E}\exp(\alpha_1(I_1^2 + I_3^2)) \leqslant C_1.$$

Let us estimate the term I_2. Applying Jensen's inequality we obtain the following estimates:

$$\begin{aligned}
\mathbb{E}\exp(\tfrac{1}{16}I_2^2) &= \mathbb{E}\exp\left(\frac{1}{4}\biggl(\int_0^1 b(1-t, W_{1-t})\frac{W_{1-t}}{2-2t}\,\mathrm{d}t\biggr)^2\right) \\
&\leqslant \mathbb{E}\int_0^1 \exp\left(\tfrac{1}{4}b^2(1-t, W_{1-t})\left|\frac{W_{1-t}}{\sqrt{1-t}}\right|^2\right)\frac{\mathrm{d}t}{2\sqrt{1-t}} \\
&\leqslant \mathbb{E}\int_0^1 \exp\left(\frac{1}{4}\left|\frac{W_{1-t}}{\sqrt{1-t}}\right|^2\right)\frac{\mathrm{d}s}{2\sqrt{1-t}} \\
&\leqslant C_2 < \infty.
\end{aligned}$$

Now it is trivial to complete the proof in the case $d = 1$.

Let $d > 1$. We have

$$b(t,x) = (b^1(t, x_1, \dots, x_d), \dots, b^d(t, x_1, \dots, x_d)),$$
$$W_t = (W_t^1, \dots, W_t^d).$$

It is easy to see that in this case it suffices to prove the inequality

$$\mathbb{E}\exp\left(\alpha\left|\int_0^1 b'_{x_1}(t, W_t^1, \dots, W_t^d)\,\mathrm{d}t\right|^2\right) \leqslant C$$

for all functions b with $\|b\|_\infty \leqslant 1$. This estimate is a consequence of the following chain of inequalities:

$$\begin{aligned}
&\mathbb{E}\exp\left(\alpha\left|\int_0^1 b'_{x_1}(t, W_t^1, \dots, W_t^d)\,\mathrm{d}t\right|^2\right) \\
&\qquad = \mathbb{E}\biggl[\mathbb{E}\biggl[\exp\left(\alpha\left|\int_0^1 b'_{x_1}(t, W_t^1, \dots, W_t^d)\,\mathrm{d}t\right|^2\right)\biggm| W^2, \dots, W^n\biggr]\biggr] \leqslant \mathbb{E}C = C,
\end{aligned}$$

where the one-dimensional case has been used. □

Corollary 2.2. *There exist constants $C, \alpha > 0$ such that, for any Borel measurable mapping $b \in L^\infty([r,u] \times \mathbb{R}^d, \mathbb{R}^d)$ with $\|b\|_\infty \leqslant 1$, any Borel measurable functions $h_1, h_2 \in L^\infty([r,u], \mathbb{R}^d)$ and any $\lambda \geqslant 0$, the following inequality holds:*

$$P\left[\left|\int_r^u b(s, W_s + h_1(s)) - b(s, W_s + h_2(s))\,\mathrm{d}s\right| \geqslant \lambda l^{1/2}\|h_1 - h_2\|_\infty\right] \leqslant C\exp(-\alpha\lambda^2),$$

where $l = u - r$.

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Proof. Taking into account the scale invariance of the Brownian motion it is easy to notice that we can assume that $r = 0$ and $u = 1$. One can easily show that it is also sufficient to prove the desired estimate just for smooth functions with compact supports. In this case we have

$$\begin{aligned}
&\mathbb{E}\exp\left(\alpha\left|\int_0^1 \frac{b(s,W_s+h_1(s))-b(s,W_s+h_2(s))}{\|h_1-h_2\|_\infty}\,\mathrm{d}s\right|^2\right)\\
&\quad=\mathbb{E}\exp\left(\alpha\left|\int_0^1\int_0^1 b'_x(s,W_s+h_2(s)+\theta(h_1(s)-h_2(s)))\frac{h_1(s)-h_2(s)}{\|h_1-h_2\|_\infty}\,\mathrm{d}\theta\,\mathrm{d}s\right|^2\right)\\
&\quad\leqslant\int_0^1\mathbb{E}\exp\left(\alpha\left|\int_0^1 b'_x(s,W_s+h_2(s)+\theta(h_1(s)-h_2(s)))\frac{h_1(s)-h_2(s)}{\|h_1-h_2\|_\infty}\,\mathrm{d}s\right|^2\right)\mathrm{d}\theta\\
&\quad\leqslant\int_0^1 C\,\mathrm{d}\theta = C.
\end{aligned}$$

In the last inequality, for each θ we have applied Proposition 2.1 to the function

$$\hat{b}(s,x)=b(s,x+h_2(s)+\theta(h_1(s)-h_2(s)))\frac{h_1(s)-h_2(s)}{\|h_1-h_2\|_\infty}.$$

Now the necessary estimate follows by the Chebyshev inequality. □

The next proposition will play the crucial role in the proof of the main results.

Proposition 2.3. *Let*

$$b\in L^q([0,T],L^p(\mathbb{R}^d)),\qquad \frac{d}{p}+\frac{2}{q}<1.$$

Then there exists a Hölder flow of solutions to (1.1). More precisely, for any filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P)$ and a Brownian motion W, there exists a mapping $(s,t,x,\omega)\mapsto\varphi_{s,t}(x)(\omega)$ with values in $\mathbb{R}^d$, defined for $0\leqslant s\leqslant t\leqslant T$, $x\in\mathbb{R}^d$, $\omega\in\Omega$, such that for each $s\in[0,T]$ the following conditions hold:

(1) *for any $x\in\mathbb{R}^d$ the process $X^x_{s,t}=\varphi_{s,t}(x)$ is a continuous $\mathcal{F}_{s,t}$-adapted solution to (1.1);*

(2) *P-almost surely the mapping $x\mapsto\varphi_{s,t}(x)$ is a homeomorphism;*

(3) *P-almost surely for all $x\in\mathbb{R}^d$ and $0\leqslant s\leqslant u\leqslant t\leqslant 1$*

$$\varphi_{s,t}(x)=\varphi_{u,t}(\varphi_{s,u}(x));$$

(4) *P-almost surely for each $\alpha\in(0,1)$ and each positive $N\in\mathbb{R}$ one can find $C(\alpha,N,\omega)<\infty$ such that for all $x,y\in\mathbb{R}^d\colon |x|,|y|<N$ and $s,t\in[0,T]$, $s\leqslant t$,*

$$|\varphi_{s,t}(x)-\varphi_{s,t}(y)|\leqslant C(\alpha,T,N,\omega)|x-y|^\alpha.$$

The existence of a flow possessing properties (1)–(3) is proved in [**3**, Theorem 1.2]. Instead of property (4), Fedrizzi and Flandoli prove (see [**3**, Lemma 5.11]) a slightly weaker assertion that almost surely for any fixed $s,t \in [0,1]$, $s \leqslant t$, the mapping $\varphi_{s,t}$ is Hölder continuous. For the sake of completeness, we present below a sketch of the proof of Proposition 2.3 with necessary references to [**2**, **3**] and the key details of the proof of property (4).

Step 1 (see [3, Theorem 3.3, Lemma 3.4 and Lemma 3.5]). Let

$$L_p^q(T) = L^q([0,T], L^p(\mathbb{R}^d)),$$
$$\mathbb{H}_{\alpha,p}^q(T) = L^q([0,T], W^{\alpha,p}(\mathbb{R}^d)), \qquad \mathbb{H}_p^{\beta,q}(T) = W^{\beta,q}([0,T], L^p(\mathbb{R}^d)),$$
$$H_{\alpha,p}^q(T) = \mathbb{H}_{\alpha,p}^q(T) \cap \mathbb{H}_p^{1,q}(T).$$

Let $U\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a solution to the equation

$$\left.\begin{aligned} \frac{\partial U}{\partial t} + \tfrac{1}{2}\Delta U + b \cdot \nabla U = \lambda U - b,\\ U(T,x) = 0, \end{aligned}\right\} \tag{2.1}$$

for sufficiently large positive λ such that

$$\|U\|_{H_{2,p}^q(T)} = \|D_t U\|_{L_p^q} + \|U\|_{H_{2,p}^q(T)} \leqslant C(d,T,p,q,\lambda)\|b\|_{L_p^q(T)},$$
$$\sup_{t \in [0,T]} \|\nabla U\|_{C_b(\mathbb{R}^d)} \leqslant \tfrac{1}{2}.$$

Then the family of mappings $\psi_t\colon \mathbb{R}^d \to \mathbb{R}^d$ defined by the formula

$$\psi_t(x) = x + U(t,x)$$

possesses the following properties:

(1) for each $t \in [0,T]$ the mappings ψ_t, ψ_t^{-1} are C^1-diffeomorphisms of $\mathbb{R}^d$,

(2) uniformly in $t \in [0,T]$ the mappings ψ_t, ψ_t^{-1} have globally bounded Hölder-continuous derivatives with respect to the space variable,

(3) the mapping $(t,x) \mapsto \psi_t(x)$ belongs locally to the class $H_{2,p}^q(T)$.

Step 2 (see [3, Proposition 4.3]). The next step is transforming the original equation, (1.1) (considered as a stochastic equation with the identity diffusion matrix and a Borel measurable drift), into an equation with more regular coefficients by means of the family of the homeomorphisms constructed at the previous step. Let us apply Itô's formula to the process X_t and the function U (see [**3**, p. 4]):

$$\begin{aligned} \mathrm{d}U(t,X_t) &= \frac{\partial U}{\partial t}(t,X_t)\,\mathrm{d}t + \nabla U(t,X_t)(b(t,X_t)\,\mathrm{d}t + \mathrm{d}W_t) + \tfrac{1}{2}\Delta U(t,X_t)\,\mathrm{d}t \\ &= \lambda U(t,X_t) - b(t,X_t)\,\mathrm{d}t + \nabla U(t,X_t)\,\mathrm{d}W_t. \end{aligned}$$

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Then the process

$$Y_t := \psi_t(t, X_t) = X_t + U(t, X_t)$$

has the stochastic differential

$$\mathrm{d}Y_t = \lambda U(t, \psi_t^{-1}(Y_t))\,\mathrm{d}t + [I + \nabla U(t, \psi_t^{-1}(Y_t))]\,\mathrm{d}W_t = \tilde{b}(t, Y_t)\,\mathrm{d}t + \tilde{\sigma}(t, Y_t)\,\mathrm{d}W_t,$$
$$\tilde{b}(t, y) = \lambda U(t, \psi_t^{-1}(y)), \qquad \tilde{\sigma}(t, y) = I + \nabla U(t, \psi_t^{-1}(Y_t)).$$

Step 3 (see [3, Proposition 5.2] and [2, pp. 13–14]). Taking into account the aforementioned properties of the mappings ψ_t it is not difficult to see that it suffices to prove the existence of a uniformly Hölder-continuous flow for the transformed equation. Below we prove only the uniform Hölder continuity of the desired flow since all other details (for example, the proof of its existence) can be found in [**3**].

We have

$$\mathrm{d}Y_t = \tilde{b}(t, Y_t)\,\mathrm{d}t + \tilde{\sigma}(t, Y_t)\,\mathrm{d}W_t. \tag{2.2}$$

Let us show that for each $a \geqslant 2$ there exists a constant $C(a, T)$ such that for any $x, y \in \mathbb{R}^d$ the following estimate holds:

$$\mathbb{E} \sup_{t \in [0,T]} |Y_t^x - Y_t^y|^a \leqslant C(a, T)(|x - y|^a + |x - y|^{a-1}). \tag{2.3}$$

In this case the existence of a uniformly Hölder-continuous flow will follow from the well-known Kolmogorov continuity theorem. Following [**2**, **3**], let us define an auxiliary process

$$A_t := \int_0^t \frac{\|\tilde{\sigma}(s, Y_s^y) - \tilde{\sigma}(s, Y_s^x)\|^2}{|Y_s^y - Y_s^x|^2} I_{\{Y_s^y \neq Y_s^x\}}\,\mathrm{d}s.$$

Then (see [**3**, Lemma 4.5]) for each $k \in \mathbb{R}$ we have

$$\mathbb{E}[\mathrm{e}^{kA_T}] < \infty \tag{2.4}$$

(in the proof of this inequality the Sobolev regularity of $\tilde{\sigma}$ plays the crucial role).

Let

$$Z_t := Y_t^y - Y_t^x.$$

Applying Itô's formula to the process Z_t and the function $f\colon x \mapsto |x|^a$, where $a \geqslant 2$, we obtain

$$\frac{1}{a}\,\mathrm{d}|Z_t|^a = \big\langle (\tilde{b}(t, Y_t^y) - \tilde{b}(t, Y_t^x))\,\mathrm{d}t, Z_t^{a-1} \big\rangle + \big\langle (\tilde{\sigma}(t, Y_t^y) - \tilde{\sigma}(t, Y_t^x))\,\mathrm{d}W_t, Z_t^{a-1} \big\rangle$$
$$+ \tfrac{1}{2}\operatorname{Tr}([\sigma(t, Y_t^y) - \sigma(t, Y_t^x)][\sigma(t, Y_t^y) - \sigma(t, Y_t^x)]^{\mathrm{T}} D^2 f(Z_t))\,\mathrm{d}t,$$
$$[D^2 f(Z_t)]_{i,j} = \delta_{i,j}|Z_t|^{a-2} + (a-2) Z_t^i Z_t^j |Z_t|^{a-4}.$$

Using the Lipschitz continuity of $\tilde{b}$ and the definition of the process A_t we obtain the inequality

$$\mathrm{d}|Z_t|^a \leqslant C|Z_t|^a\,\mathrm{d}t + C|Z_t|^a\,\mathrm{d}A_t + \big\langle (\tilde{\sigma}(t, Y_t^y) - \tilde{\sigma}(t, Y_t^x))\,\mathrm{d}W_t, Z_t^{a-1} \big\rangle.$$

Let

$$M_t := \int_0^t \big\langle (\tilde\sigma(t,Y_t^y) - \tilde\sigma(t,Y_t^x))\,\mathrm{d}W_t, Z_t^{a-1}\big\rangle.$$

Since the coefficient $\tilde\sigma$ is bounded and all moments of the random variable $|Z_t|$ are finite (see [**3**, Proposition 2.7]), the process M_t is a square-integrable continuous martingale. Then we have

$$\begin{aligned}\mathrm{d}(\mathrm{e}^{-CA_t}|Z_t|^a) &= -C\mathrm{e}^{-CA_t}|Z_t|^a\,\mathrm{d}A_t + \mathrm{e}^{-CA_t}\,\mathrm{d}|Z_t|^a \\ &\leqslant -C\mathrm{e}^{-CA_t}|Z_t|^a\,\mathrm{d}A_t + \mathrm{e}^{-CA_t}C|Z_t|^a\,\mathrm{d}t + \mathrm{e}^{-CA_t}C|Z_t|^a\,\mathrm{d}A_t + \mathrm{e}^{-CA_t}\,\mathrm{d}M_t \\ &= C\mathrm{e}^{-CA_t}|Z_t|^a\,\mathrm{d}t + \mathrm{e}^{-CA_t}\,\mathrm{d}M_t.\end{aligned}$$

Consequently, the following estimate holds:

$$\mathbb{E}\mathrm{e}^{-CA_t}|Z_t|^a \leqslant |x-y|^a + C\int_0^t \mathbb{E}\mathrm{e}^{-CA_t}|Z_t|^a\,\mathrm{d}t.$$

Applying Gronwall's inequality we obtain the estimate

$$\mathbb{E}\mathrm{e}^{-CA_t}|Z_t|^a \leqslant |x-y|^a\mathrm{e}^{CT}.$$

Taking into account Hölder's inequality and estimate (2.4) we have

$$\mathbb{E}|Z_t|^a = \mathbb{E}\mathrm{e}^{CA_t}\mathrm{e}^{-CA_t}|Z_t|^a \leqslant [\mathbb{E}\mathrm{e}^{2CA_t}]^{1/2}[\mathbb{E}\mathrm{e}^{-2CA_t}|Z_t|^{2a}]^{1/2} \leqslant C(a,T)|x-y|^a.$$

The next chain of inequalities easily follows from Doob's martingale inequality and the boundedness of $\tilde\sigma$:

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\mathrm{e}^{-2CA_t}|Z_t|^{2a} \\ &\quad\leqslant 4|Z_0|^{2a} + 4\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t C\mathrm{e}^{-CA_s}|Z_s|^a\,\mathrm{d}s\right|^2 + 4\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \mathrm{e}^{-CA_s}\,\mathrm{d}M_s\right|^2 \\ &\quad\leqslant 4|x-y|^{2a} + 4C^2T\mathbb{E}\int_0^T \mathrm{e}^{-2CA_t}|Z_t|^{2a}\,\mathrm{d}t \\ &\qquad + 16\mathbb{E}\int_0^T \mathrm{e}^{-2CA_t}\|\tilde\sigma(t,Y_t^y)-\tilde\sigma(t,Y_t^x)\|^2|Z_t|^{2a-2}\,\mathrm{d}t \\ &\quad\leqslant K|x-y|^{2a} + K\mathbb{E}\int_0^t \mathrm{e}^{-2CA_s}|Z_s|^{2a}\,\mathrm{d}s + K\mathbb{E}\int_0^t \mathrm{e}^{-2CA_s}|Z_s|^{2a-2}\,\mathrm{d}s \\ &\quad\leqslant K(a,T)(|x-y|^{2a} + |x-y|^{2a-2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}|Z_t|^a \leqslant \mathbb{E}\mathrm{e}^{CA_T}\sup_{t\in[0,T]}\mathrm{e}^{-CA_t}|Z_t|^a &\leqslant [\mathbb{E}\mathrm{e}^{2CA_T}]^{1/2}\Big[\mathbb{E}\sup_{t\in[0,T]}\mathrm{e}^{-2CA_t}|Z_t|^{2a}\Big]^{1/2} \\ &\leqslant K'(a,T)(|x-y|^a + |x-y|^{a-1}).\end{aligned}$$

It is now easy to complete the proof.

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3. Main results

To illustrate the main idea let us prove Davie's theorem for some (possibly unbounded) drift coefficient b possessing Hölder continuity with respect to the space variable. It is worth noting that the reasoning from [**1**] can not be directly applied in this case, since they essentially use the global boundedness of the drift.

Theorem 3.1. *Assume that the coefficient b satisfies the following conditions:*

(1) *there exists $M_1 \in L^{q_1}([0,T],\mathbb{R})$ such that*

$$|b(t,x)| \leqslant M_1(t), \quad t \in [0,T],\ x \in \mathbb{R}^d;$$

(2) *there exist $M_2 \in L^{q_2}([0,T],\mathbb{R})$ and $\beta > 0$ such that*

$$|b(t,x) - b(t,y)| \leqslant M_2(t)|x-y|^{\beta}, \quad t \in [0,T],\ x,y \in \mathbb{R}^d;$$

(3) *one has*

$$q_1 \geqslant q_2 > 2, \quad \beta > 0, \quad \frac{\beta}{p_1} + \frac{1}{p_2} > 1, \quad \text{where } \frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Then there exists a set Ω' with $P(\Omega') = 1$ such that for each $\omega \in \Omega'$ (1.1) has exactly one solution.

Proof. Let Y_t be a solution to (1.1) for a fixed Brownian trajectory W. Then the estimate

$$\max_{t\in[0,T]} |Y_t| \leqslant |x| + \max_{t\in[0,T]} |W_t| + T^{1/p_1}\|M_1\|_{L^{q_1}[0,T]} =: M(x,W)$$

holds, so without loss of generality we can assume that $b(t,x) = b(t,x)I_{\{|x|<N\}}$ for some $N > 0$. Then Proposition 2.3 (it is clear that one can take q_1 for q and any sufficiently large positive number for p) yields that P-almost surely (1.1) has a Hölder-continuous flow of solutions, which will be denoted by $X(s,t,x,W)$, $s \leqslant t$, $x \in \mathbb{R}^d$.

Now let us prove that, for each trajectory W such that there exists the aforementioned Hölder-continuous flow, (1.1) has exactly one solution. Let us fix $t \in [0,T]$ and define an auxiliary function f by the formula

$$f(s) = X(s,t,Y_s,W) - X(0,t,x,W), \quad s \in [0,t].$$

From the definition of f and the Hölder continuity of the flow $X(s,t,x,W)$ we obtain that for all $u,r\colon 0 \leqslant u \leqslant r \leqslant t$ one has

$$\begin{aligned}
|f(r) - f(u)| &= |X(r,t,Y_r,W) - X(u,t,Y_u,W)| \\
&= |X(r,t,Y_r,W) - X(r,t,X(u,r,Y_u,W),W)| \\
&\leqslant C(\alpha,T,M(x,W),\omega)|Y_r - X(u,r,Y_u,W)|^{\alpha}.
\end{aligned}$$

Let us estimate $|Y_r - X(u,r,Y_u,W)|$. It is clear that we have the following trivial bound:

$$\begin{aligned}|Y_r - X(u,r,Y_u,W)| &\leqslant \int_u^r |b(s,Y_s) - b(s,X(u,s,Y_u,W))|\,\mathrm{d}s\\ &\leqslant 2\int_u^r M_1(s)\,\mathrm{d}s\\ &\leqslant 2\|M_1\|_{L^{q_1}[0,T]}|r-u|^{1/p_1}.\end{aligned}$$

The previous estimate can be improved if we take into account the Hölder continuity of the coefficient b:

$$\begin{aligned}|Y_r - X(u,r,Y_u,W)| &\leqslant \int_u^r |b(s,Y_s) - b(s,X(u,s,Y_u,W))|\,\mathrm{d}s\\ &\leqslant \int_u^r M_2(s)|Y_s - X(u,s,Y_u,W)|^{\beta}\,\mathrm{d}s\\ &\leqslant K'\int_u^r M_2(s)|r-u|^{\beta/p_1}\,\mathrm{d}s\\ &\leqslant K'\|M_2\|_{L^{q_2}[0,T]}|r-u|^{\beta/p_1+1/p_2}.\end{aligned}$$

Let us pick $\alpha \in (0,1)$ such that

$$\frac{\alpha\beta}{p_1} + \frac{\alpha}{p_2} = 1+\delta, \quad \delta > 0.$$

Then we have

$$|f(r) - f(u)| \leqslant C(\alpha,T,M(x,W),\omega)|r-u|^{1+\delta}.$$

Consequently, $f \equiv 0$ (here we have also used the fact that $f(0) = 0$, which is clear from the definition of f). Finally, $Y_t = X(0,t,x,W)$ and we obtain the desired assertion, since $t \in [0,T]$ was arbitrary. □

Now we show how to prove the original result of Davie (his Theorem 1.1) for the case in which b is merely Borel measurable. Similarly to the proof of Theorem 3.1, it is readily seen that without loss of generality we can assume that $b(t,x) = b(t,x)I_{\{|x|<N\}}$ and $\|b\|_\infty \leqslant 1$. In this case for each $\alpha \in (0,1)$ (1.1) P-almost surely possesses a Hölder-continuous flow of solutions that will be denoted by $X(s,t,x,W)$. The main aim of the reasoning below is to find a substitute for the Hölder condition on the coefficient b that allows us to repeat the proof of Theorem 3.1 with minor changes.

We will need the following set of functions:

$$\begin{aligned}&\mathrm{Lip}_N([r,u],\mathbb{R}^d)\\ &\quad := \Big\{h \in C([r,u],\mathbb{R}^d)\ \Big|\ |h(t)-h(s)| \leqslant |t-s|s,\ t \in [r,u],\ \max_{s\in[r,u]}|h(s)| \leqslant N\Big\}\end{aligned}$$

with the uniform metric $\varrho(h_1,h_2) = \|h_1 - h_2\|_\infty$.

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Lemma 3.2. *There exist constants $C, \gamma > 0$ such that for all $N, \varepsilon > 0$ the set $\mathrm{Lip}_N([r,u],\mathbb{R}^d)$ contains an ε-net $\mathcal{N}_\varepsilon$ with no more than*

$$C\left(\frac{N}{\varepsilon}\right)^d \exp\left(\gamma\frac{u-r}{\varepsilon}\right)$$

elements.

Proof. This estimate can be easily obtained from [**7**, § 2, (7)]. □

Now let us temporarily fix $N > 0$ and $r, u \in [0,T]$ such that $l = u - r \leqslant \frac{1}{2}$. Let

$$\varphi(h, W) := \int_r^u b(s, W_s + h(s))\,\mathrm{d}s.$$

Lemma 3.3. *There exist constants $C, \zeta > 0$, independent of $l = u - r$, a countable dense subset $\mathcal{N}$ in $\mathrm{Lip}_N([r,u],\mathbb{R}^d)$, independent of b, and a set Ω' such that*

$$P(\Omega \setminus \Omega') \leqslant C\exp(-l^{-\zeta}),$$

and for any $h_1, h_2 \in \mathcal{N}$ with $\|h_1 - h_2\|_\infty \leqslant 3l$ and $W \in \Omega'$ the following inequality holds:

$$|\varphi(h_1, W) - \varphi(h_2, W)| \leqslant Cl^{4/3}.$$

Proof. Let α and γ be positive constants from Corollary 2.2 and Lemma 3.2, respectively. Let us define sequences $\{\varepsilon_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 0}$ as follows:

$$\varepsilon_k = l^{1+k/4}, \quad \lambda_k = \mu l^{-1/6-k/6}, \quad \text{where } \mu^2 = \frac{\gamma+1}{\alpha}.$$

Let π_k denote the mapping that sends a function from $\mathrm{Lip}_N([r,u],\mathbb{R}^d)$ to the nearest element in the ε_k-net $\mathcal{N}_{\varepsilon_k}$. For each $g_{k+1} \in \mathcal{N}_{\varepsilon_{k+1}}$ let

$$\Omega_{g_{k+1}} := \{W : |\varphi(g_{k+1}, W) - \varphi(\pi_k(g_{k+1}), W)| \geqslant l^{1/2}\varepsilon_k\lambda_k\},$$
$$\Omega_{k+1} := \bigcup_{g_{k+1}\in\mathcal{N}_{\varepsilon_{k+1}}} \Omega_{g_{k+1}}.$$

Let θ be a positive constant, below we will explain how θ should be chosen. Now for each pair of functions $f_1, f_2 \in \mathcal{N}_{\varepsilon_0}$ with

$$\|f_1 - f_2\|_\infty \leqslant \theta\varepsilon_0$$

we introduce the sets

$$\Omega_{f_1,f_2} := \{W : |\varphi(f_1, W) - \varphi(f_2, W)| \geqslant l^{1/2}\theta\varepsilon_0\lambda_0\},$$
$$\Omega_0 := \bigcup_{f_1,f_2 :\, \|f_1-f_2\|_\infty \leqslant \theta\varepsilon_0} \Omega_{f_1,f_2}.$$

One can observe that for any $g_{k+1} \in \mathcal{N}_{\varepsilon_{k+1}}$ we have

$$\|g_{k+1} - \pi_k(g_{k+1})\|_\infty \leqslant \varepsilon_k.$$

Applying Corollary 2.2 we obtain the following inequalities:

$$P(\Omega_{k+1}) \leqslant \sum_{g_{k+1} \in \mathcal{N}_{\varepsilon_{k+1}}} P(\Omega_{g_{k+1}}) \leqslant C\left(\frac{N}{\varepsilon_{k+1}}\right)^d \exp\left(\frac{\gamma l}{\varepsilon_{k+1}} - \alpha\lambda_k^2\right),$$

$$P(\Omega_0) \leqslant \sum_{f_1, f_2} P(\Omega_{f_1, f_2}) \leqslant C^2\left(\frac{N}{l}\right)^{2d} \exp(\gamma - \alpha\mu^2 l^{-1/3}).$$

Since

$$\frac{N^d}{\varepsilon_{k+1}^d} = N^d l^{-5d/4 - dk} \leqslant N^d l^{-5d(k+1)/4},$$

$$\begin{aligned} \frac{\gamma l}{\varepsilon_{k+1}} - \alpha\lambda_k^2 &= \gamma l^{-1/4 - k/4} - \alpha\mu^2 l^{-1/3 - k/3} \\ &= \gamma l^{-1/4 - k/4} - (\gamma + 1) l^{-1/3 - k/3} \\ &\leqslant -l^{-(k+1)/3}, \end{aligned}$$

it can be easily verified that there exist positive constants ζ and C such that for any $k \geqslant 0$ the following inequalities hold:

$$P(\Omega_{k+1}) \leqslant C \exp(-l^{-\zeta(k+1)}), \qquad P(\Omega_0) \leqslant C \exp(-l^{-\zeta}).$$

Let

$$\Omega' := \Omega \setminus \bigcup_{k=0}^{\infty} \Omega_k, \qquad \mathcal{N} := \bigcup_{k=0}^{\infty} \mathcal{N}_{\varepsilon_k}.$$

Taking into account the reasoning above we have that

$$P(\Omega \setminus \Omega') \leqslant C(T, N) \exp(-l^{-\zeta}),$$
$$\mathcal{N} \text{ is a dense subset of } \mathrm{Lip}_N([r, u], \mathbb{R}^d).$$

Let W be an arbitrary trajectory in Ω' and let h_1, h_2 be two functions in $\mathcal{N}$ with $\|h_1 - h_2\|_\infty \leqslant 3l$. Let us assume that $h_1 \in \mathcal{N}_{\varepsilon_{k_1}}$, $h_2 \in \mathcal{N}_{\varepsilon_{k_2}}$. Then we can construct two sequences of functions:

$$h_{1,k_1} = h_1,\ h_{1,k_1-1} = \pi_{k_1-1}(h_{1,k_1}),\ h_{1,k_1-2} = \pi_{k_1-2}(h_{1,k_1-1}),\ \ldots,\ h_{1,0} = \pi_0(h_{1,1}),$$
$$h_{2,k_2} = h_2,\ h_{2,k_2-1} = \pi_{k_2-1}(h_{2,k_2}),\ h_{2,k_2-2} = \pi_{k_2-2}(h_{2,k_2-1}),\ \ldots,\ h_{2,0} = \pi_0(h_{2,1}).$$

It is not difficult to show that due to our choice of W we can find a positive number K (which does not depend on θ) such that the following inequalities hold:

$$\|h_1 - h_{1,0}\|_\infty \leqslant Kl, \qquad \|h_2 - h_{2,0}\|_\infty \leqslant Kl.$$

Consequently, taking $2K+3$ for θ, we obtain

$$\|h_{1,0} - h_{2,0}\|_\infty \leqslant \theta l;$$

in particular, the set Ω_0 contains $\Omega_{h_{1,0},h_{2,0}}$.

Now since $W \in \Omega'$ and

$$l^{1/2}\varepsilon_k\lambda_k = \mu l^{4/3+k/12},$$

we conclude that there exists a positive constant $C = C(N,T)$ such that the following estimate holds:

$$|\varphi(h_1, W) - \varphi(h_2, W)| \leqslant Cl^{4/3}.$$

□

Lemma 3.4. *For any $\varepsilon, N > 0$ there exists $\delta > 0$ such that for each open set $U \subset [0,1] \times \mathbb{R}^n$ with $\lambda(U) < \delta$ there is a Borel set of Brownian trajectories Ω_ε with $P(\Omega_\varepsilon) \geqslant 1-\varepsilon$ such that for any $W \in \Omega_\varepsilon$, $h \in \mathrm{Lip}_N([0,1], \mathbb{R}^d)$ the following inequality holds:*

$$\int_0^1 I_U(s, W_s + h(s))\,\mathrm{d}s \leqslant \varepsilon.$$

Proof. Assume that we are given $\varepsilon, N > 0$. Let us choose $l > 0$ such that

$$\frac{2}{l} C \exp{(-l^{-\zeta})} \leqslant \frac{\varepsilon}{2}, \qquad \frac{2}{l} C l^{4/3} \leqslant \frac{\varepsilon}{2},$$

where C, ζ are positive constants from Lemma 3.3. Next let us split the interval $[0,1]$ into a collection of closed subintervals $\Delta_1, \dots, \Delta_M$ of length less than l, $M \leqslant 2/l$. Applying Lemma 3.3 to each interval Δ_k we can find countable sets $\mathcal{N}_1, \dots, \mathcal{N}_M$ (here we also use the fact that these subsets do not depend on b; see Lemma 3.3). Now in each $\mathcal{N}_s$ we take a finite $3l$-net that will be denoted by $\mathcal{N}_s'$. Let us pick $\delta > 0$ such that for each open set U with $\lambda(U) \leqslant \delta$ there exists a set Ω' such that $P(\Omega') \geqslant 1 - \varepsilon/2$ and for any $W \in \Omega'$, $h \in \mathcal{N}_s'$ one has

$$\int_{\Delta_s} I_U(s, W_s + h(s))\,\mathrm{d}s \leqslant \frac{l\varepsilon}{4}$$

(such a δ obviously exists). Let us prove that this δ satisfies the conditions stated above.

Let us fix an open set U with $\lambda(U) \leqslant \delta$. Applying Lemma 3.3 for each s one can find a set Ω_s with

$$P(\Omega \setminus \Omega_s) \leqslant C \exp(-l^{-\zeta})$$

such that for any $h_1, h_2 \in \mathcal{N}_s$ with $\|h_1 - h_2\|_\infty \leqslant 3l$ and $W \in \Omega_s$ the following inequality holds:

$$\left| \int_{\Delta_s} I_U(s, W_s + h_1(s))\,\mathrm{d}s - \int_{\Delta_s} I_U(s, W_s + h_2(s))\,\mathrm{d}s \right| \leqslant Cl^{4/3}.$$

Let

$$\Omega_\varepsilon := \Omega' \cap \bigcap_{s=1}^{M} \Omega_s.$$

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Let us observe that

$$P(\Omega_\varepsilon) \geqslant 1 - \varepsilon$$

and for each $h_s \in \mathcal{N}_s$,

$$\int_{\Delta_s} I_U(s, W_s + h(s))\, \mathrm{d}s \leqslant \frac{l\varepsilon}{4}.$$

Since U is open, applying Fatou's lemma we conclude that the previous inequality is true for all $h \in \mathrm{Lip}_N(\Delta_s, \mathbb{R}^d)$. It is now trivial to complete the proof. □

Lemma 3.5. *Let $b\colon [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a bounded Borel measurable mapping with $\|b\|_\infty \leqslant 1$. Then there exists a set Ω' with $P(\Omega') = 1$ such that for each $W \in \Omega'$ and each sequence of functions $\{h_k\} \subset \mathrm{Lip}_N([0,1], \mathbb{R}^d)$ pointwise converging to a function h the following equality holds:*

$$\lim_{k\to\infty} \int_0^1 b(s, W_s + h_k(s))\, \mathrm{d}s = \int_0^1 b(s, W_s + h(s))\, \mathrm{d}s.$$

Proof. Applying Lemma 3.4 for each $\varepsilon_n = 1/2^n$ we can find the corresponding $\delta_n > 0$. Next, applying Lusin's theorem for each n one can find a function $b_n \in C_b([0,1] \times \mathbb{R}^d, \mathbb{R}^d)$ and an open set $U_n \subset [0,1] \times \mathbb{R}^d$ such that

$$\|b_n\|_\infty \leqslant 1, \quad \lambda(U_n) \leqslant \delta_n, \quad b_n(t,x) = b(t,x) \quad \text{for all } (t,x) \notin U_n.$$

Then there exists a set Ω_n with the following properties:

$$P(\Omega_n) \geqslant 1 - \varepsilon_n$$

and for any $W \in \Omega_n$, $h \in \mathrm{Lip}_N([0,1], \mathbb{R}^d)$,

$$\int_0^1 I_U(s, W_s + h(s))\, \mathrm{d}s \leqslant \varepsilon_n.$$

Next we observe that for any n,

$$\begin{aligned}\int_0^1 b_n(s, W_s + h(s))\, \mathrm{d}s - 2\int_0^1 I_U(s, W_s + h(s))\, \mathrm{d}s \\ \leqslant \int_0^1 b(s, W_s + h(s))\, \mathrm{d}s \\ \leqslant \int_0^1 b_n(s, W_s + h(s))\, \mathrm{d}s + 2\int_0^1 I_U(s, W_s + h(s))\, \mathrm{d}s.\end{aligned}$$

Therefore, for each $W \in \Omega_n$ and each sequence of functions $\{h_k\} \subset \mathrm{Lip}_N([0,1],\mathbb{R}^d)$ pointwise converging to h, the following inequalities hold:

$$\begin{aligned}\int_0^1 b(s, W_s + h(s))\,\mathrm{d}s - 4\varepsilon_n &\leqslant \int_0^1 b_n(s, W_s + h(s))\,\mathrm{d}s - 2\varepsilon_n \\ &\leqslant \liminf_{k\to\infty} \int_0^1 b_n(s, W_s + h_k(s))\,\mathrm{d}s - 2\varepsilon_n \\ &\leqslant \liminf_{k\to\infty} \int_0^1 b(s, W_s + h_k(s))\,\mathrm{d}s, \\ \int_0^1 b(s, W_s + h(s))\,\mathrm{d}s + 4\varepsilon_n &\geqslant \int_0^1 b_n(s, W_s + h(s))\,\mathrm{d}s + 2\varepsilon_n \\ &\geqslant \limsup_{k\to\infty} \int_0^1 b_n(s, W_s + h_k(s))\,\mathrm{d}s + 2\varepsilon_n \\ &\geqslant \limsup_{k\to\infty} \int_0^1 b(s, W_s + h_k(s))\,\mathrm{d}s.\end{aligned}$$

Let

$$\Omega' := \liminf_{n\to\infty} \Omega_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Omega_n.$$

Since

$$P(\Omega_n) \geqslant 1 - \varepsilon_n \quad \text{and} \quad \sum_{n=1}^{\infty} \varepsilon_n < \infty,$$

by the Borel–Cantelli lemma $P(\Omega') = 1$. It is now trivial to complete the proof. □

Lemma 3.6. *There exist constants $C, \zeta > 0$, independent of $l = u - r$, and a set Ω' such that*

$$P(\Omega \setminus \Omega') \leqslant C \exp(-l^{-\zeta}),$$

and for any $h_1, h_2 \in \mathcal{N}$ with $\|h_1 - h_2\|_\infty \leqslant 4l$, $W \in \Omega'$ the following inequality holds:

$$|\varphi(h_1, W) - \varphi(h_2, W)| \leqslant Cl^{4/3}.$$

Proof. This assertion follows directly from Lemmas 3.3 and 3.5. □

We can now proceed to the proof of Theorem 1.1.

Proof. Let us fix a positive number N. Let C, ζ be constants found in Lemma 3.6. For each k we split the interval $[0, 1]$ into $M = 2^k$ closed subintervals

$$\left[0, \frac{1}{M}\right], \ \ldots, \ \left[\frac{M-1}{M}, 1\right].$$

Applying Lemma 3.6 to each interval $[i/M, (i+1)/M]$ we can find the corresponding sets $\Omega_{k,i}$. Let

$$\Omega_k := \bigcap_{i=0}^{M-1} \Omega_{k,i}.$$

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With the help of the Borel–Cantelli lemma it is easy to show that the set

$$\Omega' := \liminf_{k\to\infty} \Omega_k = \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} \Omega_k$$

has probability 1. Removing, if necessary, a set of zero probability from Ω', we can assume that for each $W \in \Omega'$ there exists a Hölder-continuous flow ensured by Proposition 2.3. Let us show that for each $W \in \Omega'$ such that

$$|x| + \max_{t\in[0,1]} |W_t| + 1 \leqslant N,$$

(1.1) has a unique solution. Indeed, let Y_t be a solution to (1.1). It is not difficult to see that $|Y_t| \leqslant N$ for each $t \in [0, 1]$. Due to our choice of Ω' there exists $K = K(\omega)$ such that for all $k \geqslant K$ the Brownian trajectory W belongs to Ω_k. Let

$$M' = 2^{k'}, \quad r = \frac{i}{M'}, \quad \text{where } k' \geqslant K.$$

Let us define an auxiliary function f on the interval $[0, r]$ by the following formula:

$$f(t) := X(0, r, x, W) - X(t, r, Y_t, W).$$

We observe that for any $s \leqslant t$, by the definition of a flow we have

$$\begin{aligned} f(t) - f(s) &= -X(t, r, Y_t, W) + X(s, r, Y_s, W) \\ &= -X(t, r, Y_t, W) + X(t, r, X(s, t, Y_s, W), W). \end{aligned}$$

Hence, there exists a positive constant $C = C(N, W)$ such that

$$|f(t) - f(s)| \leqslant C|Y_t - X(s, t, Y_s, W)|^{4/5}.$$

The difference $Y_t - X(s, t, Y_s, W)$ can be represented as follows:

$$\begin{aligned} Y_t - X(s, t, Y_s, W) &= \int_s^t b\biggl(u, Y_s + W_u - W_s + \int_s^u b(r, Y_r)\,\mathrm{d}r\biggr)\,\mathrm{d}u \\ &\quad - \int_s^t b\biggl(u, Y_s + W_u - W_s + \int_s^u b(r, X_r)\,\mathrm{d}r\biggr)\,\mathrm{d}u \\ &= \int_s^t b(u, W_u + h_1(u))\,\mathrm{d}u - \int_s^t b(u, W_u + h_2(u))\,\mathrm{d}u, \end{aligned}$$

where

$$h_1(u) = Y_s - W_s + \int_s^u b(r, Y_r)\,\mathrm{d}r, \qquad h_2(u) = Y_s - W_s + \int_s^u b(r, X_r)\,\mathrm{d}r.$$

Let $k \geqslant k'$, $M = 2^k$. If we take s, t of the form i/M and $(i + 1)/M$, respectively, then we obtain the following estimate:

$$\left|f\left(\frac{i+1}{M}\right) - f\left(\frac{i}{M}\right)\right| \leqslant \left(\frac{C}{M^{4/3}}\right)^{4/5},$$

and consequently

$$|f(r)| \leqslant \frac{C}{M^{1/15}}.$$

Due to the arbitrariness of k we conclude that

$$f(r) = X(0, r, x, W) - Y_r = 0.$$

Since r was an arbitrary dyadic number in $[0, 1]$ with a sufficiently large denominator, the continuity of Y_t and $X(0, t, x, W)$ implies the equality $Y_t = X(0, t, x, W)$ for each $t \in [0, 1]$. The proof is complete. □

Remark 3.7. We remark that Theorem 1.1 can be generalized to the case in which the drift b is Borel measurable and has linear growth

$$|b(t, x)| \leqslant C + C|x|.$$

Indeed, let X_t be a solution to (1.1). Then we have

$$|X_t| \leqslant |x| + \max_{t\in[0,T]} |W_t| + Ct + C\int_0^t |X_t|\,\mathrm{d}t$$

and Gronwall's inequality yields the bound

$$\max_{t\in[0,T]} |X_t| \leqslant \Big[|x| + \max_{t\in[0,T]} |W_t| + CT\Big]\mathrm{e}^{CT}.$$

The desired statement easily follows from the uniqueness of solutions to the localized equations with bounded drifts of the form $b(t, x)I_{\{|x|<N\}}$.

Remark 3.8. It would be interesting to extend this result to the case of locally unbounded drift considered in [**6**, **8**]. Under the assumptions from [**8**], Hölder-continuous flow still exists but some other ingredients of the proof require refinements. There are two main obstacles here. The first obstacle is that the estimate from Proposition 2.1 essentially uses boundness of b and the author is not aware of any natural counterparts for locally unbounded drifts. The second obstacle is that if b is locally unbounded, then its integral is not Lipschitz continuous anymore. So one should take care of the metric entropy of the corresponding compactum (a good candidate is a subset of Hölder-continuous functions) to find a substitute for Lemma 3.3. The author believes that some progress in this direction is possible but, in the interests of avoiding additional technical complications, no details are presented here.

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