New proofs of certain characterisations of cyclic circumscriptible quadrilaterals

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A convex quadrilateral $ABCD$ is called *circumscriptible* or *tangential* if it admits an incircle, i.e. a circle that touches all of its sides. A typical circumscriptible quadrilateral is depicted in Figure 1, where the incircle of $ABCD$ touches the sides at A', B', C' and D'. Notice that labellings such as $AA' = AD' = a$ are justified by the fact that two tangents from a point to a circle have equal lengths (a, b, c and d in Figure 1 are called tangent lengths). This simple fact also implies that if x, y and z are the angles shown in the figure, then $x = y$. In fact, if AD and BC are parallel, then $x = y = 90°$. Otherwise, the extensions of AD and BC would meet, say at Q, with $QB' = QD'$. Hence $x = y$. Thus $x = y$ in all cases, and $\sin x = \sin y = \sin z$. We shall use this observation freely. Also we shall denote the vertices and vertex angles of a polygon by the same letters, but after making sure that no confusion may arise.

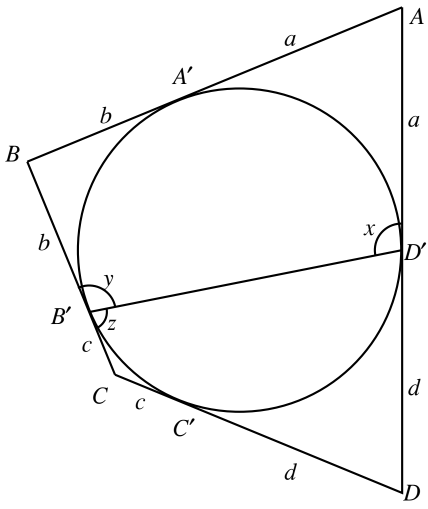

FIGURE 1

Dually, a convex quadrilateral which admits a circumcircle, i.e. a circle that passes through its vertices, is called *cyclic*. Unlike circumscriptible ones, cyclic quadrilaterals appear in Euclid's *Elements*, where several of their properties are proved. These include the property that the convex quadrilateral $ABCD$ shown in Figure 2 is cyclic if, and only if, any of the conditions

(i) $\angle CAD = \angle CBD$ (ii) $A + C = 180°$ (iii) $AE \times EC = BE \times ED$

holds. These conditions are quite well known, and will be freely used.

https://doi.org/10.1017/mag.2019.100 Published online by Cambridge University Press

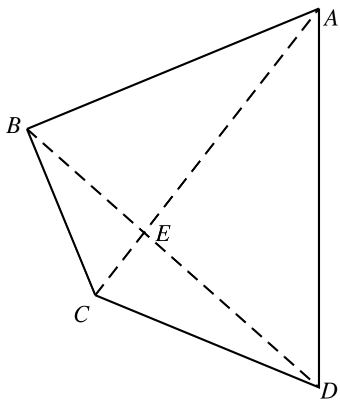

FIGURE 2

There is a fairly large amount of literature on cyclic and circumscriptible quadrilaterals that focuses on their properties; see, for example, [1] and the references therein. Several of these properties take the form of conditions that are necessary and sufficient for a circumscriptible quadrilateral to be cyclic. This note presents new proofs of a few of these properties.

Theorem 1 below appears in [2, page 156] and in [3]. It will be used in the proofs of the other theorems.

Theorem 1: Let $ABCD$ be a circumscriptible quadrilateral as shown in Figure 1. Then AC, BD, $A'C'$ and $B'D'$ are concurrent. Also, if E is the point of intersection, and if

$$u_1 = AE, u_2 = CE, v_1 = BE, v_2 = DE. \tag{1}$$

as shown in Figure 4, then

$$\frac{u_1}{u_2} = \frac{a}{c}, \qquad \frac{v_1}{v_2} = \frac{b}{d}, \tag{2}$$

where a, b, c and d are the tangent lengths.

Proof: Referring to Figure 3(a), we let G be the point of intersection of AC and $B'D'$, $s_1 = AG$, $s_2 = CG$, and we let $G = \angle AGD' = \angle CGB'$. Applying the law of sines to triangles AGD' and CGB', we obtain

$$\frac{s_1}{\sin x} = \frac{a}{\sin G}, \qquad \frac{s_2}{\sin z} = \frac{c}{\sin G}.$$

Dividing, we obtain

$$\frac{s_1}{s_2} = \frac{a}{c}. \tag{3}$$

Similarly, let F be the point of intersection of AC and $A'C'$, and let $t_1 = AF$ and $t_2 = CF$, as in Figure 3(b). Applying the same argument to triangles $AA'F$ and $CC'F$, we obtain

$$\frac{t_1}{t_2} = \frac{a}{c}. \tag{4}$$

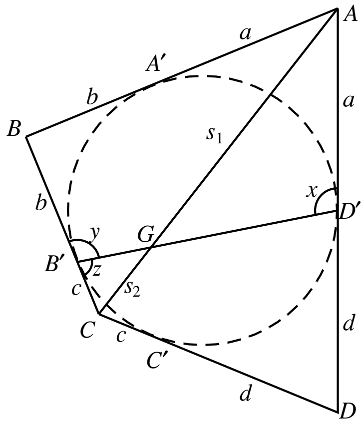

FIGURE 3(a)

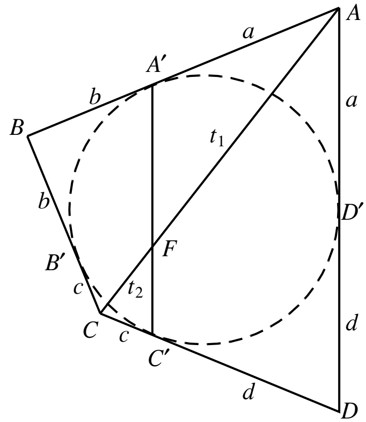

FIGURE 3(b)

It follows from (3) and (4) that $F = G$, and that $A'C'$ and $B'D'$ intersect AC at the same point G, and that the first equation in (2) holds.

By symmetry, $A'C'$ and $B'D'$ intersect BD also at the same point (so $E = G = F$), and that

$$\frac{v_1}{v_2} = \frac{b}{d}. \tag{5}$$

The following properties, which we have not found any reference to, will be freely used.

Theorem 2: Let $ABCD$ be a circumscriptible quadrilateral, as in Figure 4. Then the following are equivalent.

(i) $ABCD$ is cyclic.

(ii) $\dfrac{u_1}{v_1} = \dfrac{a}{b}$, i.e. $A'C'$ bisects $\angle AEB$.

(iii) $\dfrac{u_2}{v_2} = \dfrac{c}{d}$, i.e. $A'C'$ bisects $\angle AEB$.

(iv) $\dfrac{u_1}{v_2} = \dfrac{a}{d}$, i.e. $B'D'$ bisects $\angle AED$.

(v) $\dfrac{u_2}{v_1} = \dfrac{c}{b}$, i.e. $B'D'$ bisects $\angle AED$.

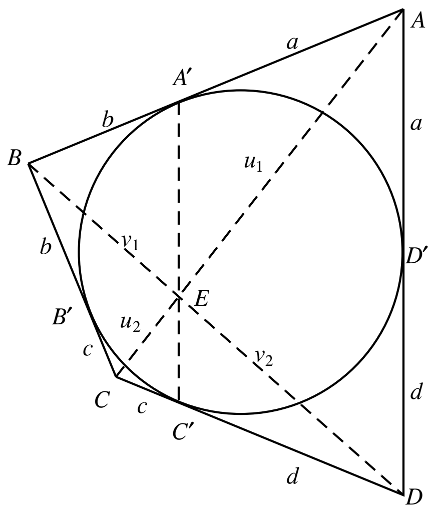

FIGURE 4

Proof: If AB and CD are parallel, i.e. $ABCD$ is a trapezium, then $ABCD$ is cyclic if, and only if, $A = B$, both being supplementary to C, and hence if, and only if, $A'C'$ is an axis of symmetry of $ABCD$. This is equivalent to saying that $A'C'$ bisects $\angle AEB$. Thus (i) is equivalent to each of (ii) and (iii). If AB and CD are not parallel, then we may assume that AB and DC meet, when produced, say at Q, with $QA' = QC'$. Then $\angle AA'E = \angle DC'E$, and hence

$$ABCD \text{ is cyclic} \Leftrightarrow \angle A'AE = \angle C'DE \Leftrightarrow \angle AEA' = \angle DEC'$$
$$\Leftrightarrow A'C' \text{ bisects } \angle AEB.$$

Thus (i) is equivalent to each of (ii) and (iii).

Thus in all cases (i), (ii), and (iii) are equivalent. Similarly, (i), (iv), and (v) are equivalent, and hence all of them are equivalent.

Theorems 3 and 4 are taken from [4]. These theorems first appeared, with rather lengthy proofs, in [5]. Much shorter proofs were given in [4], and the one for Theorem 4 was also reproduced in [6, p. 115].

Theorem 3: Let $ABCD$ be a circumscriptible quadrilateral with $AC = u$ and $BD = v$. Then $ABCD$ is cyclic if, and only if,

$$\frac{u}{v} = \frac{a + c}{b + d}.$$

Proof: We refer to Figure 4. By Theorem 1, we have

$$\frac{u_1}{u_2} = \frac{a}{c}, \qquad \frac{v_1}{v_2} = \frac{b}{d}.$$

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Adding 1 to each side, we obtain

$$\frac{u}{u_2} = \frac{a + c}{c}, \qquad \frac{v}{v_2} = \frac{b + d}{d}.$$

Hence

$$\frac{u}{v} = \left(\frac{a + c}{b + d}\right)\left(\frac{d}{c}\right)\left(\frac{u_2}{v_2}\right).$$

By Theorem 2, $ABCD$ is cyclic if, and only if, $\frac{u_2}{v_2} = \frac{c}{d}$. Therefore $ABCD$ is cyclic if, and only if,

$$\frac{u}{v} = \frac{a + c}{b + d},$$

as desired.

Theorem 4: Let $ABCD$ be a circumscriptible quadrilateral with tangent lengths a, b, c and d. Then $ABCD$ is cyclic if, and only if,

$$ac = bd.$$

Proof: If $ABCD$ is cyclic, then by (iv) and (v) of Theorem 2, we have

$$\frac{u_1}{v_2} = \frac{a}{d}, \qquad \frac{u_2}{v_1} = \frac{c}{b}.$$

Multiplying, and using the fact that in cyclic quadrilaterals $u_1 u_2 = v_1 v_2$, we obtain $ac = bd$, as desired.

For the converse, we assume that $ac = bd$, and we show that $ABCD$ is cyclic. Since $\frac{a}{d} = \frac{b}{c}$, it follows that there is a point Q on AC such that QD' is parallel to CD and QB' is parallel to AB; see Figure 5.

Therefore

$$\frac{QD'}{c + d} = \frac{a}{a + d} \text{ and } \frac{QB'}{a + b} = \frac{c}{c + b}.$$

Hence

$$\begin{aligned}\frac{QD'}{QB'} &= \frac{a(c + d)(b + c)}{c(b + a)(a + d)} \\ &= \frac{acb\left(1 + \frac{d}{c}\right)\left(1 + \frac{c}{b}\right)}{cab\left(1 + \frac{a}{b}\right)\left(1 + \frac{d}{a}\right)} \\ &= 1, \text{ because } \frac{d}{c} = \frac{a}{b} \text{ and } \frac{c}{b} = \frac{d}{a}.\end{aligned}$$

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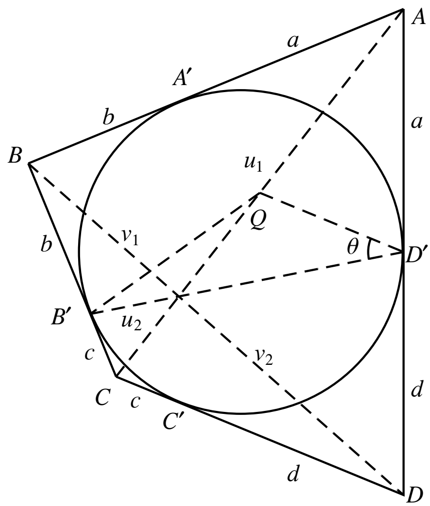

FIGURE 5

Therefore $QD' = QB'$, and hence $\angle B'D'Q = \angle D'B'Q\,(= \theta$, say). Therefore

$$D + B = \angle AD'Q + \angle CB'Q$$
$$= (\angle AD'B' - \theta) + (\angle D'B'C + \theta) = \angle AD'B' + \angle D'B'C = 180^\circ.$$

Therefore *ABCD* is cyclic, as claimed.

The following theorem was proved using the angle sum of a quadrilateral [7, pp. 123-124].

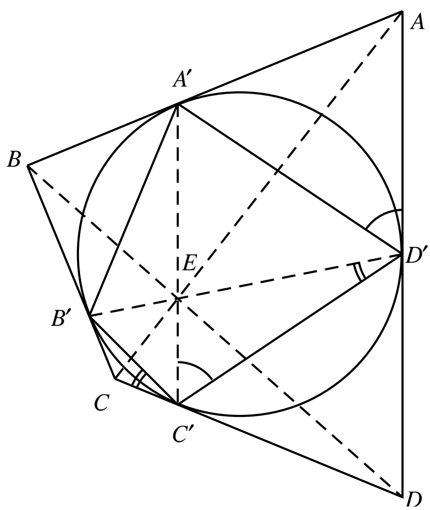

FIGURE 6

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Theorem 5: Let $ABCD$ be a circumscriptible quadrilateral, as shown in Figure 6. Then $ABCD$ is cyclic if, and only if, $A'C'$ and $B'D'$ are orthogonal.

Proof: We use Proposition 32 of Book III of Euclid's *Elements*, which says that if UV and UW are chords in a circle and if UZ is a tangent such that UV lies between UW and UZ, then $\angle VUZ = \angle VWU$; see, for example, Proposition 7.19 (p. 184) of [8], Cor. 4.14 (p. 71) of [9], or Cor. 1.24 (p. 31) of [10]. Thus $\angle AD'A' = \angle A'C'D'$ and $\angle CC''B' = \angle C'D'B'$. Therefore

$$2\angle AD'A' + A + 2\angle CC'B' + C = 2\angle A'C'D' + 2\angle C'D'B' + A + C.$$

$$360^\circ = 2(180^\circ - E) + A + C$$

$$A + C = 2E$$

Since $ABCD$ is cyclic if, and only if, $A + C = 180^\circ$, it follows that $ABCD$ is cyclic if, and only if, $E = 90^\circ$, i.e. $A'C'$ and $B'D'$ are orthogonal.

Remark: On page 157 of [2], the author made the slip that $A'C'$ and $B'D'$ are orthogonal if, and only if, AC and BD are orthogonal. This is obviously incorrect, because if $ABCD$, as in Figure 1, is a rhombus that is not a square, then it is not cyclic. Hence AC and BD are orthogonal while $A'C'$ and $B'D'$ are not orthogonal.

Acknowledgement: The authors would like to thank the anonymous referee for the many suggestions that improved the paper considerably.

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10.1017/mag.2019.100

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Nemo, continued from page 400.

The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd February 2020.

1. "No," said Tom, opening his pocket-knife and holding it over the puff, with his head on one side in a dubitative manner. (It was a difficult problem to divide that very irregular polygon into two equal parts.)

2. He lived in a place called the Polygon, in Somers Town, where there were at that time a number of poor Spanish refugees walking about in cloaks, smoking little paper cigars.

3. "There's nothing she can do so well. But you're of course so many-sided."
"If one's two-sided it's enough," said Isabel.
"You're the most charming of polygons!" her companion broke out.

4. He spoke to Mr. Dawkins in the most respeckful and flatrin manner – agread in every think he said – prazed his taste, his furniter, his coat, his classick nolledge, and his playin on the floot; you'd have thought, to hear him, that such a polygon of exlens as Dawkins did not breath – that such a modist, sinsear, honrabble genlmn as Deuceace was to be seen nowhere xcept in Pump Cort.

5. I walked miles and miles; till at last I found him residing in a very old-fashioned house in the Polygon, at Somers Town.

6. But in the work of a real mystic the triangle is a hard mathematical triangle not to be mistaken for a cone or a polygon.