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Relations between interstellar density and magnetic field fluctuations I. Kinetic theory of fluctuations

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Abstract. Using linear kinetic plasma theory the relation between electron density and magnetic field fluctuations for low-frequency plasma waves for Maxwellian background distribution functions of arbitrary temperatures in a uniform magnetic field is derived. By taking the non-relativistic temperature limit this ratio is calculated for the diffuse intercloud medium in our Galaxy. The diffuse intercloud medium is the dominant phase of the interstellar medium with respect to radio wave propagation, dispersion and rotation measure studies. The differences between the relation of electron density and magnetic field fluctuations from the linear kinetic theory compared with the classical magnetohydrodynamics theory are established and discussed.

1. Introduction

Important input quantities for the quasilinear test-particle description of cosmic ray transport in weakly turbulent astrophysical plasmas are the wavenumber power spectra of magnetic field fluctuations. Within the plasma wave viewpoint, the plasma irregularities are usually modelled as a superposition of linear waves well below the ion cyclotron frequency, such as Alfvén and magnetosonic waves. However, the observed turbulence properties in the more distant interstellar and intergalactic plasmas are obtained from radio propagation measurements such as dispersion measures, rotation measures and interstellar scintillation. These turbulence diagnostics are biased towards the high-density ionized interstellar phases with large volume filling factors, i.e. the diffuse intercloud gas and H II envelopes. In particular, dispersion measure and scintillation data are primarily diagnostics of density and only secondarily of magnetic field. These diagnostics demonstrate the existence of interstellar density irregularities with Kolmogorov-type ($\propto \omega^{-s}$, s = constant) frequency power spectra extending over 11 decades in frequency, much below ($\omega \ll \Omega_p$) the proton gyrofrequency (Rickett 1990; Armstrong et al. 1995).

Often the electromagnetic fluctuations are described within magnetohydrodynamic (MHD) theory (e.g. Sturrock 1994; Goldreich and Sridhar 1995; Hollweg 1999; Lithwick and Goldreich 2002), which is appropriate at large turbulence scalelength

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$l \geqslant l_{MHD}$. However, the plasma parameter of the diffuse intercloud medium $g = \nu_{ee}/\omega_{p,e} \approx 10^{-10}$ is much smaller than unity, so that a kinetic description of the electromagnetic turbulence seems to be necessary. It is the purpose of the present paper to provide the relation between electron density and magnetic field fluctuations on the basis of the linear kinetic plasma theory. In paper II of this series we will use the kinetic turbulence relations to calculate frequency power spectra of electron density fluctuations from anisotropic power spectra of magnetic field fluctuations in the form of Alfvén and magnetosonic waves. Such anisotropic interstellar magnetic field power spectra are required in order to be in accord with the heating/cooling balance of the diffuse intercloud medium (Lerche and Schlickeiser 2001).

2. Diffuse intercloud plasma parameters

According to Rickett (1990), Spangler (1991) and Lerche and Schlickeiser (2001), the diffuse intercloud gas can be well modelled as an isotropic Maxwellian plasma with the following plasma and turbulence parameters: $B_0 = 1.3\,\mu$G, $n_e = 0.08\,\mathrm{cm}^{-3}$, $s = \frac{5}{3}$, $v_e \approx 2 \times 10^7\,\mathrm{cm\,s}^{-1}$, $T_e = T_p$, $v_p = v_e(m_e/m_p)^{1/2} = 4.7 \times 10^5\,\mathrm{cm\,s}^{-1}$, $k_{min} = 2\pi/l_{max} = 2 \times 10^{-18}\,\mathrm{cm}^{-1}$ with $l_{max} = 3 \times 10^{18}\,\mathrm{cm}$, $k_{max} = 2\pi/l_{min} = 8 \times 10^{-8}\,\mathrm{cm}^{-1}$ with $l_{min} = 8 \times 10^7\,\mathrm{cm}$. l_{min} and l_{max} refer to the smallest and largest scale turbulence scale length. We then infer for the Alfvén speed $V_A = 10^6\,\mathrm{cm\,s}^{-1}$, so that the ratio $\chi = V_A/v_e = 0.05$. The non-relativistic electron and proton gyrofrequencies are $\boldsymbol{\Omega}_e = 18\,\mathrm{Hz}$ and $\boldsymbol{\Omega}_p = 0.01\,\mathrm{Hz}$, respectively. For the thermal electron and proton gyroradii we obtain $R_e = v_e/|\boldsymbol{\Omega}_e| = 1.1 \times 10^6\,\mathrm{cm}$ and $R_p = (m_p/m_e)^{1/2} R_e = 43 R_e = 4.7 \times 10^7\,\mathrm{cm}$.

Because one is dealing with waves representing departures from a homogenous system, there are fluctuations in particle number density and magnetic field fluctuations that are correlated. The basic procedure to identify these variations was originally given for the solar wind plasma by Wu and Huba (1975). However, their analysis deals with a specific range of parameter values; in particular, they consider the case of $v_e \ll V_A$, opposite to ours. Here we calculate anew the relation between electron number density fluctuations, δn_e, and magnetic field fluctuations, δB_i, for a more general range of plasma parameter values.

3. Kinetic derivation of fluctuations in the electron number density

3.1. *General reduction for isotropic background distribution functions*

Expanding the Fourier–Laplace-transformed Vlasov equation and Maxwell's equations for collisionless plasmas to first order in perturbed quantities one obtains the fluctuations, δf_a, to the particle distribution function of plasma species a and the magnetic field fluctuations $\delta\mathbf{B}$ in terms of the fluctuations $\mathbf{E}$ in the electric field for any wave type and isotropic equilibrium distribution function $f_a^{(0)}(p)$ (see, e.g., Schlickeiser 2002, (B.16) and (8.3.18)),

$$\delta f_a(\mathbf{k}, \omega, p) = -\frac{q_a\,\mathrm{sgn}(q_a)}{\boldsymbol{\Omega}_a}\frac{\gamma}{p}\frac{\partial f_a^{(0)}(p)}{\partial p} \times \int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha\,\{p_\perp[E_x\cos(\phi-\alpha) + E_y\sin(\phi-\alpha)] + E_z p_\parallel\} \times \exp\left[\frac{i\gamma}{\boldsymbol{\Omega}_a}\{(k_\parallel v_\parallel - \omega)\alpha - k_\perp v_\perp[\sin(\phi-\alpha) - \sin\phi]\}\right] \tag{3.1}$$

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and

$$\delta B(\mathbf{k},\omega) = \frac{c}{\omega}\mathbf{k}\times\mathbf{E}, \tag{3.2}$$

where we have adopted a cartesian coordinate system with the wave vector $\mathbf{k} = (k_\perp, 0, k_\parallel) = k(\sin\theta, 0, \cos\theta)$ and the uniform background magnetic field $\mathbf{B}_0 = (0, 0, B_0)$ lying parallel to the z-axis.

By integrating δf_a over all particle momenta $\mathbf{p} = \gamma m_a \mathbf{v}$, one obtains the fluctuations in number density, δn_a, for each particle species, thereby directly relating the number density fluctuations to the electric field fluctuations,

$$\begin{aligned}
\delta n_a(\mathbf{k},\omega) &= \int d^3p\, \delta f_a(\mathbf{k},\omega,p) = \int_0^{2\pi} d\phi \int_0^{\pi} d\Theta \sin\Theta \int_0^{\infty} dp\, p^2 \delta f_a \\
&- \frac{q_a\, \mathrm{sgn}(q_a)(m_a c)^2}{\Omega_a}\int_0^\infty dx\,\sqrt{1+x^2}x^2\frac{\partial f_a^{(0)}(x)}{\partial x} \\
&\times \int_{-\infty\,\mathrm{sgn}(q_a)}^0 d\alpha \exp\left[-\frac{i\,\mathrm{sgn}(q_a)\omega\sqrt{1+x^2}\alpha}{\Omega_a}\right] \\
&\times [(E_x\cos\alpha - E_y\sin\alpha)\sin\Theta\cos\phi \\
&+ (E_x\sin\alpha + E_y\cos\alpha)\sin\Theta\sin\phi + E_z\cos\Theta] \\
&\times \exp\left\{-\frac{icx\,\mathrm{sgn}(q_a)}{\Omega_a}[k_\parallel\alpha\cos\Theta + k_\perp(1-\cos\alpha)\sin\Theta\sin\phi\right. \\
&\left. + k_\perp\sin\alpha\sin\Theta\cos\phi]\right\},
\end{aligned} \tag{3.3}$$

where we have introduced the normalized momentum $x = p/(m_a c)$ so that $\gamma = \sqrt{1+x^2}$.

Applying the general integral (Gradshteyn and Ryzhik 1965, (4.624))

$$\begin{aligned}
J(A,B,D) &\equiv \int_0^\pi d\Theta \int_0^{2\pi} d\phi\, F(A\cos\Theta + B\sin\Theta\cos\phi + D\sin\Theta\sin\phi) \\
&= 2\pi\int_{-1}^1 dt\, F(Rt) = 4\pi\sinh(R)/R
\end{aligned} \tag{3.4}$$

to the function $F(t) = \exp(t)$, where $R = \sqrt{A^2+B^2+D^2}$, and derivatives of (3.4) with respect to A, B and D, respectively, we reduce (3.3) to

$$\begin{aligned}
\delta n_a &= -\frac{4\pi i c q_a (m_a c)^2}{\Omega_a^2}\int_0^\infty dx\,\sqrt{1+x^2}x^3\frac{\partial f_a^{(0)}(x)}{\partial x} \\
&\times \int_{-\infty\,\mathrm{sgn}(q_a)}^0 d\alpha \exp\left[-\frac{i\,\mathrm{sgn}(q_a)\omega\sqrt{1+x^2}\alpha}{\Omega_a}\right] \\
&\times\{k_\parallel E_z\alpha + k_\perp[E_x\sin\alpha + E_y(\cos\alpha - 1)]\}\left(\frac{\sin Gx}{G^3x^3} - \frac{\cos Gx}{G^2x^2}\right),
\end{aligned} \tag{3.5}$$

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where

$$G \equiv \frac{c}{\Omega_a}\sqrt{k_\parallel^2\alpha^2 + 2k_\perp^2(1-\cos\alpha)}. \tag{3.6}$$

3.2. *Maxwellian background distribution functions*

Adopting the thermal equilibrium distribution

$$f_a^{(0)} = \frac{\mu_a n_a}{4\pi(m_a c)^3 K_2(\mu_a)}\exp(-\mu_a\sqrt{1+x^2}) \tag{3.7}$$

with the temperature parameter

$$\mu_a = \frac{m_a c^2}{k_B T_a}. \tag{3.8}$$

Equation (3.5) becomes

$$\delta n_a = \frac{iq_a\mu_a^2 n_a}{\Omega_a^2 m_a K_2(\mu_a)}\int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha\, G^{-3}\{k_\parallel E_z\alpha + k_\perp[E_x\sin\alpha + E_y(\cos\alpha - 1)]\}$$
$$\times\int_0^\infty dx\,\exp(-Q\sqrt{1+x^2})(x\sin Gx - Gx^2\cos Gx), \tag{3.9}$$

where

$$Q \equiv \mu_a + \frac{i\,\mathrm{sgn}(q_a)\omega\alpha}{\Omega_a}. \tag{3.10}$$

The x-integrations can be readily performed using (Gradshteyn and Ryzhik 1965, (3.914))

$$J(G) = \int_0^\infty dx\, e^{-Q\sqrt{1+x^2}}\cos Gx = Q\frac{K_1(\sqrt{Q^2+G^2})}{\sqrt{Q^2+G^2}} \tag{3.11}$$

and its first and second derivative with respect to G, yielding

$$\int_0^\infty dx\,\exp(-Q\sqrt{1+x^2})(x\sin Gx - Gx^2\cos Gx)$$
$$= -\frac{\partial J}{\partial G} + G\frac{\partial^2 J}{\partial G^2} = QG^3\frac{K_3(\sqrt{Q^2+G^2})}{(Q^2+G^2)^{3/2}} \tag{3.12}$$

so that

$$\delta n_a = \frac{iq_a\mu_a^2 n_a}{\Omega_a^2 m_a K_2(\mu_a)}\int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha\,\{k_\parallel E_z\alpha + k_\perp[E_x\sin\alpha + E_y(\cos\alpha - 1)]\}$$
$$\times Q\frac{K_3(\sqrt{Q^2+G^2})}{(Q^2+G^2)^{3/2}}, \tag{3.13}$$

which is exact for all temperature values μ_a.

3.3. *Non-relativistic temperature limit $\mu_a \gg 1$*

For non-relativistic temperatures $\mu_a \gg 1$ and low frequencies $\omega \ll \Omega_a \ll \mu_a\Omega_a$ we use the approximations $K_\nu(x) \approx \sqrt{\pi/2x}\exp(-x)$ and

$$G^2 \ll \mu_a^2 \tag{3.14}$$

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to derive for (3.13) the compact expression

$$\frac{\delta n_a}{n_a} \approx \frac{iq_a}{\Omega_a^2 m_a}(k_\parallel E_z H_1 + k_\perp E_x H_2 + k_\perp E_y H_3). \tag{3.15}$$

It remains to solve the three α-integrals

$$H_1 = \int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha\, \alpha \exp\left[-\frac{i\,\mathrm{sgn}(q_a)\omega\alpha}{\Omega_a} - \frac{c^2 k_\parallel^2 \alpha^2}{2\mu_a \Omega_a^2} + \rho_a(\cos\alpha - 1)\right], \tag{3.16}$$

$$H_2 = \int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha\, \sin\alpha \exp\left[-\frac{i\,\mathrm{sgn}(q_a)\omega\alpha}{\Omega_a} - \frac{c^2 k_\parallel^2 \alpha^2}{2\mu_a \Omega_a^2} + \rho_a(\cos\alpha - 1)\right] \tag{3.17}$$

and

$$\begin{aligned} H_3 &= \int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha\, (\cos\alpha - 1) \exp\left[-\frac{i\,\mathrm{sgn}(q_a)\omega\alpha}{\Omega_a} - \frac{c^2 k_\parallel^2 \alpha^2}{2\mu_a \Omega_a^2} + \rho_a(\cos\alpha - 1)\right] \\ &= \frac{\partial}{\partial \rho_a} \int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha \exp\left[-\frac{i\,\mathrm{sgn}(q_a)\omega\alpha}{\Omega_a} - \frac{c^2 k_\parallel^2 \alpha^2}{2\mu_a \Omega_a^2} + \rho_a(\cos\alpha - 1)\right], \end{aligned} \tag{3.18}$$

where we have introduced the kineticity

$$\rho_a = \frac{c^2 k_\perp^2}{\mu_a \Omega_a^2} = \frac{k_\perp^2 v_{th,a}^2}{3\Omega_a^2} = \frac{1}{3} k_\perp^2 R_a^2 \tag{3.19}$$

where $R_a = v_{th,a}/|\Omega_a|$ denotes the gyroradius of thermal particles of species a.

Using the identity in terms of the modified Bessel function $I_n(\rho)$ of the first kind,

$$e^{\rho_a \cos\alpha} = \sum_{n=-\infty}^{\infty} I_n(\rho_a) e^{in\alpha} \tag{3.20}$$

we obtain

$$\begin{aligned} H_1 = -\frac{\mu_a \Omega_a^2 e^{-\rho_a}}{c^2 k_\parallel^2} \sum_{n=-\infty}^{\infty} I_n(\rho_a) \Bigg[1 + i\sqrt{\frac{\pi \mu_a \Omega_a^2}{2c^2 k_\parallel^2}} \exp\left[-\frac{\mu_a \Omega_a^2}{2c^2 k_\parallel^2}\left(\frac{\omega}{|\Omega_a|} - n\right)^2\right] \\ \times \left\{\mathrm{sgn}(q_a) + \mathrm{erf}\left[i\sqrt{\frac{\mu_a \Omega_a^2}{2c^2 k_\parallel^2}}\left(\frac{\omega}{|\Omega_a|} - n\right)\right]\right\}\Bigg]. \end{aligned} \tag{3.21}$$

Likewise,

$$\begin{aligned} H_2 &= \frac{e^{-\rho_a}}{2i} \sum_{n=-\infty}^{\infty} I_n(\rho_a) \int_{-\infty\,\mathrm{sgn}(q_a)}^{0} d\alpha\, (e^{i\alpha} - e^{-i\alpha}) \exp\left[-\frac{c^2 k_\parallel^2 \alpha^2}{2\mu_a \Omega_a^2} - i\alpha\left(\frac{\omega}{|\Omega_a|} - n\right)\right] \\ &= \frac{e^{-\rho_a}}{2i} \sqrt{\frac{\pi \mu_a \Omega_a^2}{2c^2 k_\parallel^2}} \sum_{n=-\infty}^{\infty} I_n(\rho) \Bigg[\mathrm{sgn}(q_a)\Bigg\{\exp\left[-\frac{\mu_a \Omega_a^2}{2c^2 k_\parallel^2}\left(\frac{\omega}{|\Omega_a|} - n - 1\right)^2\right] \\ &\quad - \exp\left[-\frac{\mu_a \Omega_a^2}{2c^2 k_\parallel^2}\left(\frac{\omega}{|\Omega_a|} - n + 1\right)^2\right]\Bigg\} \end{aligned}$$

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$$+\exp\left[-\frac{\mu_a\Omega_a^2}{2c^2k_\parallel^2}\left(\frac{\omega}{|\Omega_a|}-n-1\right)^2\right]\mathrm{erf}\left[i\sqrt{\frac{\mu_a\Omega_a^2}{2c^2k_\parallel^2}}\left(\frac{\omega}{|\Omega_a|}-n-1\right)\right]$$

$$-\exp\left[-\frac{\mu_a\Omega_a^2}{2c^2k_\parallel^2}\left(\frac{\omega}{|\Omega_a|}-n+1\right)^2\right]\mathrm{erf}\left[i\sqrt{\frac{\mu_a\Omega_a^2}{2c^2k_\parallel^2}}\left(\frac{\omega}{|\Omega_a|}-n+1\right)\right]\Bigg]. \quad (3.22)$$

Finally,

$$H_3=\sum_{n=-\infty}^{\infty}\frac{\partial[e^{-\rho_a}I_n(\rho_a)]}{\partial\rho_a}\int_{-\infty\,\mathrm{sgn}(q_a)}^{0}d\alpha\exp\left[-\frac{c^2k_\parallel^2\alpha^2}{2\mu_a\Omega_a^2}-i\alpha\left(\frac{\omega}{|\Omega_a|}-n\right)\right]$$

$$=\sqrt{\frac{\pi\mu_a\Omega_a^2}{2c^2k_\parallel^2}}\sum_{n=-\infty}^{\infty}\frac{\partial[e^{-\rho_a}I_n(\rho_a)]}{\partial\rho_a}\exp\left[-\frac{\mu_a\Omega_a^2}{2c^2k_\parallel^2}\left(\frac{\omega}{|\Omega_a|}-n\right)^2\right]$$

$$\times\left\{\mathrm{sgn}(q_a)+\mathrm{erf}\left[i\sqrt{\frac{\mu_a\Omega_a^2}{2c^2k_\parallel^2}}\left(\frac{\omega}{|\Omega_a|}-n\right)\right]\right\}. \quad (3.23)$$

3.4. Electron density fluctuations

According to Sec. 2 we obtain for the electron kineticity (3.19) in the diffuse intercloud medium

$$\rho_e=\frac{1}{3}R_e^2k_\perp^2=\frac{\sin^2\theta}{3}(R_ek)^2\leqslant\frac{1}{3}(R_ek_{max})^2=2.6\times10^{-3}\ll 1 \quad (3.24)$$

values much smaller than unity at all wavenumbers and propagation angles. For electron fluctuations it is therefore justified to take the limit $\rho_e\to 0$ of (3.21)–(3.23).

It is convenient to introduce Dawson's integral (Lebedev 1972, 19ff.)

$$D[x]\equiv e^{-x^2}\int_0^x dt\,e^{t^2}=-\frac{i\pi^{1/2}}{2}e^{-x^2}\,\mathrm{erf}(ix) \quad (3.25)$$

and the parameters

$$\psi_e^2\equiv\frac{\mu_e\omega^2}{2c^2k_\parallel^2}=\frac{3\omega^2}{2v_e^2k_\parallel^2} \quad (3.26)$$

and

$$\alpha_e\equiv\frac{|\Omega_e|\psi_e}{\omega}=\sqrt{\frac{3}{2}}\frac{1}{R_ek\cos\theta}\gg 1. \quad (3.27)$$

In the limit $\rho_e\to 0$ we obtain

$$H_1(\rho_e=0)=\alpha_e^2\left[2\psi_eD[\psi_e]+i\pi^{1/2}\psi_ee^{-\psi_e^2}-1\right] \quad (3.28)$$

$$H_2(\rho_e=0)=\frac{|\Omega_e|}{\omega}\psi_e\left[D\left[\psi_e\left(1-\frac{|\Omega_e|}{\omega}\right)\right]-D\left[\psi_e\left(1+\frac{|\Omega_e|}{\omega}\right)\right]\right.$$

$$\left.-i\frac{\pi^{1/2}}{2}\left\{\exp\left[-\psi_e^2\left(1+\frac{|\Omega_e|}{\omega}\right)^2\right]-\exp\left[-\psi_e^2\left(1-\frac{|\Omega_e|}{\omega}\right)^2\right]\right\}\right]$$

$$=-\alpha_e\left[D[\alpha_e-\psi_e]+D[\alpha_e+\psi_e]-i\pi^{1/2}\sinh(2\alpha_e\psi_e)\right.$$

$$\left.\times\exp\left(-\psi_e^2-\alpha_e^2\right)\right] \quad (3.29)$$

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and

$$H_3(\rho_e = 0) = i\frac{|\Omega_e|}{\omega}\left\{D\left[\psi_e\left(1+\frac{|\Omega_e|}{\omega}\right)\right] + D\left[\psi_e\left(1-\frac{|\Omega_e|}{\omega}\right)\right] - 2D[\psi_e]\right\}$$

$$+\frac{\pi^{1/2}}{2}\frac{|\Omega_e|}{\omega}\left\{2e^{-\psi_e^2} - \exp\left[-\psi_e^2\left(1+\frac{|\Omega_e|}{\omega}\right)^2\right]\right.$$

$$\left.-\exp\left[-\psi_e^2\left(1-\frac{|\Omega_e|}{\omega}\right)^2\right]\right\}$$

$$= i\frac{\alpha_e}{\psi_e}[D[\alpha_e+\psi_e] - D[\alpha_e-\psi_e] - 2D[\psi_e]]$$

$$+\frac{\pi^{1/2}}{2}\frac{\alpha_e}{\psi_e}e^{-\psi_e^2}\left[1-\cosh(2\alpha_e\psi_e)\exp\left(-\alpha_e^2\right)\right]. \quad (3.30)$$

These equations can be approximated further using the known properties of Dawson's integral. Dawson's integral satisfies the linear differential equation

$$\frac{dD(x)}{dx} = 1 - 2xD(x), \quad (3.31)$$

has a maximum $D_m = 0.541$ for $x = x_M = 0.924$ and an inflection point at $x = x_w = 1.502$, where $D = D_w = 0.428$. At large arguments (Schlickeiser and Mause 1995)

$$D(x \gg 1) \approx \frac{1}{2x}\left(1+\frac{1}{2x^2}+\frac{3}{4x^4}\right), \quad (3.32)$$

whereas at small arguments

$$D(x \ll 1) \approx x - \frac{2}{3}x^3 + \frac{4}{15}x^5. \quad (3.33)$$

To proceed we have to calculate the parameter ψ_e from the dispersion relations of Alfvén and magnetosonic waves in the diffuse intercloud medium.

4. Dispersion relationships in the diffuse intercloud medium

According to Sec. 2 the plasma parameters of the diffuse intercloud medium are in a range where

$$v_p^2 \ll V_A^2 \ll v_e^2. \quad (4.1)$$

We noted already (see 3.24) that the electron kineticity is much smaller than unity. For the proton kineticity

$$\rho_p = \frac{m_p}{m_e}\rho_e = 4.77\left(\frac{k}{k_{max}}\right)^2 \sin^2\theta \quad (4.2)$$

we also find values much smaller than unity if we limit our discussion to wavenumbers much smaller than $k \ll 0.46k_{max}$. In this limit according to Sitenko (1967, 115ff.) two low-frequency transverse plasma modes exist: the Alfvén wave and the magnetosonic wave.

The Alfvén mode calculated from the dispersion relation $\det\Lambda_{ij} = 0$ obeys

$$\omega^2 = V_A^2 k_\parallel^2 = V_A^2 k^2 \cos^2\theta \qquad \text{Alfvén mode.} \quad (4.3)$$

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The Maxwell operator Λ_{ij} in the relation $\Lambda_{ij}E_j = 0$ also specifies the wave's polarization characteristics

$$\frac{E_y}{E_x} = \frac{\Lambda_{11}\Lambda_{23} + \Lambda_{12}\Lambda_{13}}{\Lambda_{13}\Lambda_{22} - \Lambda_{12}\Lambda_{23}}, \quad \frac{E_z}{E_x} = \frac{\Lambda_{12}^2 + \Lambda_{11}\Lambda_{22}}{\Lambda_{12}\Lambda_{23} - \Lambda_{13}\Lambda_{22}}. \tag{4.4}$$

For obliquely propagating Alfvén waves Sitenko gives

$$\mathbf{E}_A \approx E_x\left(1; -i\frac{\omega}{\Omega_p \tan^2\theta}; -\frac{v_s^2}{V_A^2}\frac{\omega^2}{\Omega_p^2 \sin\theta\cos\theta}\right) \tag{4.5}$$

at frequencies much smaller than the non-relativistic proton gyrofrequency ($\omega \ll \Omega_p$).

The magnetosonic mode obeys the dispersion relation

$$\omega^2 = V_A^2 k_\parallel^2 + \left(V_A^2 + v_s^2\right)k_\perp^2 = V_A^2\left[k_\parallel^2 + (1+\beta)k_\perp^2\right] \quad \text{magnetosonic mode} \tag{4.6}$$

if the velocity of sound is defined by means of the equalities

$$v_s^2 = \frac{2}{3}v_p^2\begin{cases}1 + T_e/2T_p & \text{for } \cos\theta \geqslant \chi\\ 1 + T_e/T_p & \text{for } \cos\theta \leqslant \chi\end{cases} = \begin{cases}v_p^2 & \text{for } \cos\theta \geqslant \chi\\ \frac{4}{3}v_p^2 & \text{for } \cos\theta \leqslant \chi.\end{cases} \tag{4.7}$$

Note that the critical angle $\theta_c = \arccos(\chi) = 87°$. The plasma beta is defined as

$$\beta = \frac{v_s^2}{V_A^2} = \frac{v_p^2}{V_A^2} = \frac{m_e}{m_p\chi^2} = 0.22. \tag{4.8}$$

The polarization vector for obliquely propagating magnetosonic waves is

$$\mathbf{E}_M \approx E_y\left(-i\frac{\omega}{\Omega_p \sin^2\theta}; 1; -i\frac{v_s^2}{V_A^2}\frac{\omega}{\Omega_p}\sin\theta\cos\theta\right) \tag{4.9}$$

at frequencies much smaller than the non-relativistic proton gyrofrequency ($\omega \ll \Omega_p$).

5. Alfvénic electron density fluctuations

Using the Alfvén wave dispersion relation (4.3) we obtain for (3.26)

$$\psi_{e,A} = \sqrt{1.5}\chi = 0.061 \ll 1, \tag{5.1}$$

which is much smaller than unity and much smaller than the parameter α_e (see 3.27).

In the limit $\psi_{e,A} \ll 1 \ll \alpha_e$ (3.28)–(3.30), reduce to

$$H_1^A(\rho_e = 0) \approx -2\alpha_e^2, \quad H_2^A(\rho_e = 0) \approx -1, \quad H_3^A(\rho_e = 0) \approx \pi^{1/2}\alpha_e \tag{5.2}$$

so that according to (3.15)

$$\frac{\delta n_e^A}{n_e} \approx \frac{ie}{\Omega_e^2 m_e}\left(2k_\parallel \alpha_e^2 E_z + k_\perp E_x - k_\perp \pi^{1/2}\alpha_e E_y\right). \tag{5.3}$$

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The dominant term in parentheses in (5.3) comes from the parallel electric field (E_z) term, indicating that the electron density perturbation in oblique Alfvén waves arises in response to their parallel electric fields.

Making use of the polarization properties of oblique Alfvén waves (4.5) then yields

$$\frac{\delta n_e^A}{n_e} \approx \frac{ieE_x}{\Omega_e^2 m_e}\left(k_\perp - 2k_\parallel \alpha_e^2 \frac{v_p^2}{V_A^2\Omega_p^2}\frac{\omega^2}{\sin\theta\cos\theta} + i\pi^{1/2}k_\perp \frac{\omega}{\Omega_p \tan^2\theta}\alpha_e\right). \tag{5.4}$$

Using Faraday's induction law (3.2) we can express the magnetic field fluctuations (B_x, B_y, B_z) in terms of the electric field fluctuations (E_x, E_y, E_z)

$$(B_x, B_y, B_z) = \frac{c}{\omega}\mathbf{k}\times(E_x, E_y, E_z), \tag{5.5}$$

yielding for Alfvén waves

$$\begin{aligned} B_y &= \pm\frac{c}{V_A}(1+k^2R_p^2)E_x \approx \pm\frac{c}{V_A}E_x, \quad B_x = i\frac{c}{V_A}\left(\frac{m_p}{m_e}\right)^{1/2}\chi k R_p \frac{\cos\theta}{\tan^2\theta}E_x, \\ B_z &= -i\frac{c}{V_A}\left(\frac{m_p}{m_e}\right)^{1/2}\chi k R_p \frac{\sin\theta}{\tan^2\theta}E_x \end{aligned} \tag{5.6}$$

so that $|B_{x,z}| \ll |B_y|$. Consequently,

$$\frac{\delta n_e^A}{n_e} \approx \pm\frac{iV_A k B_y}{\Omega_e B_0}\left(\sin\theta - 3\frac{m_p}{m_e}\frac{1}{\sin\theta} \pm i\sqrt{\frac{3\pi}{2}}\chi\frac{m_p}{m_e}\frac{\cos\theta}{\tan^2\theta}\right) \approx \mp i3\frac{V_A k B_y}{\Omega_p B_0 \sin\theta}. \tag{5.7}$$

The corresponding wavenumber power spectra are then related by

$$\frac{P_{nn}^A(\mathbf{k})}{n_e^2} = \frac{9V_A^2k^2}{\Omega_p^2\sin^2\theta}\frac{P_{yy}(\mathbf{k})}{B_0^2}. \tag{5.8}$$

6. Magnetosonic electron density fluctuations

Using the magnetosonic wave dispersion relation (4.6) we obtain for (3.26)

$$\begin{aligned} \psi_{e.M} &= \sqrt{\tfrac{3}{2}}\chi[1+(1+\beta)\tan^2\theta]^{1/2} = 0.061\sqrt{1+1.22\tan^2\theta} \\ &\approx \begin{cases} 0.061 & \text{for } \theta \leqslant 42.2 \\ 0.068\tan\theta & \text{for } \theta \geqslant 42.2, \end{cases} \end{aligned} \tag{6.1}$$

which is smaller than unity for propagation angles less than $\theta_c = \arctan(14.78) = 86.1°$.

In the limit $\psi_{e,M} \ll 1 \ll \alpha_e$ (3.28)–(3.30) reduce to

$$H_1^M(\rho_e = 0) \approx -2\alpha_e^2, \quad H_2^M(\rho_e = 0) \approx -1, \quad H_3^M(\rho_e = 0) \approx \pi^{1/2}\alpha_e \tag{6.2}$$

whereas in the limit $1 \ll \psi_{e,M} \ll \alpha_e$

$$H_1^M(\rho_e = 0) \approx \frac{\alpha_e^2}{2\psi_{e,M}^2}, \quad H_2^M(\rho_e = 0) \approx -1, \quad H_3^M(\rho_e = 0) \approx -\frac{\alpha_e}{\psi_{e,M}^2}. \tag{6.3}$$

6.1. Small and intermediate propagation angles $\theta \leqslant \theta_c$

At propagation angles $\theta \leqslant \theta_c$ (3.15) then becomes

$$\frac{\delta n_e^M}{n_e} \approx \frac{ie}{\Omega_e^2 m_e}\left(2k_\parallel \alpha_e^2 E_z + k_\perp E_x - k_\perp \pi^{1/2}\alpha_e E_y\right). \tag{6.4}$$

Using the polarization properties of oblique magnetosonic waves (4.9),

$$\frac{\delta n_e^M}{n_e} \approx -\frac{ieE_y}{\Omega_e^2 m_e}\left(ik_\parallel \alpha_e^2 \frac{v_p^2}{V_A^2}\frac{\omega}{\Omega_p}\sin\theta\cos\theta + ik_\perp \frac{\omega}{\Omega_p \sin^2\theta} + \frac{\pi^{1/2}}{2}k_\perp \frac{\alpha_e}{\psi_{e,M}}\right). \tag{6.5}$$

Faraday's induction law yields for magnetosonic waves,

$$B_x = -\frac{c}{\omega}k_\parallel E_y, \quad B_y = \frac{ick\cos\theta}{\Omega_p}\left(\frac{v_p^2}{V_A^2}\sin^2\theta - \frac{1}{\sin^2\theta}\right)E_y, \quad B_z = \frac{c}{\omega}k_\perp E_y \tag{6.6}$$

so that $|B_y| \ll |B_{x,z}|$. Consequently,

$$\frac{\delta n_e^M}{n_e} \approx -\frac{ieB_z}{\Omega_e^2 m_e c}\left(2i\alpha_e^2 \frac{v_p^2}{V_A^2}\frac{\omega^2}{\Omega_p}\cos^2\theta + i\frac{\omega^2}{\Omega_p \sin^2\theta} + \pi^{1/2}\omega\alpha_e\right)$$

$$\approx 3[\cos^2\theta + (1+\beta)\sin^2\theta]\frac{B_z}{B_0} = 3[1 + \beta\sin^2\theta]\frac{B_z}{B_0}. \tag{6.7}$$

The corresponding wavenumber power spectra are then related by

$$\frac{P_{nn}^M(\mathbf{k})}{n_e^2} = \frac{P_{zz}(\mathbf{k})}{B_0^2}9[1 + \beta\sin^2\theta]^2. \tag{6.8}$$

As an aside we note that (6.6) imply

$$P_{zz}(\mathbf{k}) = \tan^2\theta P_{xx}(\mathbf{k}) \tag{6.9}$$

for magnetosonic waves.

6.2. Large propagation angles $\theta \geqslant \theta_c$

Repeating the analysis at large propagation angles using (6.3) yields

$$\frac{\delta n_e^M}{n_e} \approx -\frac{B_z}{B_0}\left[\frac{2}{3}\left(\frac{\cos\theta}{\chi}\right)^2 - (1+\beta)\frac{m_p}{m_e}\chi^2 R_e^2 k^2\right]. \tag{6.10}$$

7. Comparison with classical MHD theory

Equations (5.7), (6.7) and (6.10), and consequently (5.8) and (6.8), are important modifications to the classical MHD results (e.g. Sturrock 1994, Ch. 14.1).

Equation (5.7) contradicts the classical MHD result $\delta n_e = 0$ for Alfvén waves. Only at wavenumbers $k \ll (2\Omega_p/3V_A)$ does the kinetic result yield vanishing density fluctuations in agreement with the classical MHD theory.

For fast magnetosonic waves ($\beta \ll 1$) MHD theory yields

$$\frac{\delta n_e}{n_e} \approx \frac{\delta N}{N} = \frac{\delta B_z}{B_0}\left(1 - \frac{\beta \cos^2\theta}{1+\beta\sin^2\theta}\right)^{-1} \approx \frac{\delta B_z}{B_0}, \tag{7.1}$$

where $N = m_p n_p + m_e n_e$ so that $n_e \approx N/m_p$. Our kinetic result (6.7) contains the additional factor

$$f_{kin}(\theta) = 3(1+\beta\sin^2\theta), \tag{7.2}$$

which increases monotonically from $f_{kin}(0) = 3$ to $f_{kin}(90°) = 3(1+\beta)$.

8. Discussion and conclusions

Using linear kinetic plasma theory we have calculated the relation between electron density and magnetic field fluctuations for low-frequency plasma waves for Maxwellian background distribution functions of arbitrary temperatures in a uniform magnetic field. By taking the non-relativistic temperature limit we determined this ratio for the diffuse intercloud medium in our Galaxy. The diffuse intercloud medium is the dominant phase of the interstellar medium with respect to radio wave propagation, dispersion and rotation measure studies. We have found differences between the relation of electron density and magnetic field fluctuations from the linear kinetic theory compared with the classical MHD theory.

Whereas shear Alfvén waves are incompressible in MHD theory, linear kinetic theory yields the non-zero relation (5.7) even in the limit of vanishing electron kineticity $\rho_e = 0$. Only at very small wavenumbers $k \ll \frac{2}{3}|\Omega_p|\sin\theta/V_A = \frac{2}{3}(\omega_{p,i}/c)\sin\theta$ does the kinetic result agree with the MHD result.

For magnetosonic waves the kinetic ratio of the normalized density and magnetic field fluctuations is modified from the MHD ratio by the factor $f_{kin} = 3(1+\beta\sin^2\theta)$, which is independent of wavenumber and varies within values of 3.0 and $3.0(1+\beta)$.

In the next paper of this series we will use these kinetic turbulence relations to calculate frequency power spectra of electron density fluctuations from anisotropic power spectra of magnetic field fluctuations in the form of Alfvén and magnetosonic waves. Such anisotropic interstellar magnetic field power spectra are required in order to be in accord with the heating/cooling balance of the diffuse intercloud medium.

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