INFINITE STRINGS AND THEIR LARGE SCALE PROPERTIES

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Abstract. The aim of this paper is to shed light on our understanding of large scale properties of infinite strings. We say that one string α has weaker large scale geometry than that of β if there is color preserving bi-Lipschitz map from α into β with small distortion. This definition allows us to define a partially ordered set of large scale geometries on the classes of all infinite strings. This partial order compares large scale geometries of infinite strings. As such, it presents an algebraic tool for classification of global patterns. We study properties of this partial order. We prove, for instance, that this partial order has a greatest element and also possess infinite chains and antichains. We also investigate the sets of large scale geometries of strings accepted by finite state machines such as Büchi automata. We provide an algorithm that describes large scale geometries of strings accepted by Büchi automata. This connects the work with the complexity theory. We also prove that the quasi-isometry problem is a Σ_2^0-complete set, thus providing a bridge with computability theory. Finally, we build algebraic structures that are invariants of large scale geometries. We invoke asymptotic cones, a key concept in geometric group theory, defined via model-theoretic notion of ultra-product. Partly, we study asymptotic cones of algorithmically random strings, thus connecting the topic with algorithmic randomness.

§1. Introduction.

1.1. Motivation and basic definitions. The goal of this paper is to shed light on our understanding of large scale properties and patterns of infinite strings. Our hope is to answer the questions of the following types. How does one define large scale properties of infinite strings? What does it mean for two strings to possess the same large scale invariants? How can we compare large scale properties of one string with those of the other string? What are the large scale characteristics of (algorithmically) random strings? To answer these and similar types of questions we employ concepts and ideas from coarse geometry and geometric group theory.

In coarse geometry and geometric group theory, a foundational concept is that of quasi-isometry between metric spaces. Suppose that we are given two metric spaces $\mathcal{M}_1 = (M_1, d_1)$ and $\mathcal{M}_2 = (M_2, d_2)$. A map $f : M_1 \to M_2$ is called an (A, B, C)-*quasi-isometry*, where $A \geqslant 1$, $B \geqslant 0$, and $C \geqslant 0$, if the following two properties are satisfied:

1. For all $x, y \in M_1$ we have

$$(1/A) \cdot d_1(x, y) - B \leqslant d_2(f(x), f(y)) \leqslant A \cdot d_1(x, y) + B.$$

2. For every element $y \in M_2$ there is an $x \in M_1$ such that $d_2(y, f(x)) \leqslant C$.

We call the first property *quasi-isometric inequality*, the second one *density property*, and the constants A, B, C will be called *quasi-isometry constants* (of the mapping f).

Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be an (A, B, C)-quasi-isometry. If $B = 0$, the mapping becomes bi-Lipschitz (and hence continuous) and injective. Therefore the quasi-isometry f can be viewed as a bi-Lipschitz map with distortion B between the metric spaces (in case $B > 0$). If $C = 0$, the quasi-isometry f becomes onto.

The quasi-isometry relation between metric spaces is an equivalence relation. Transitivity and reflexivity of this relation is evident. To see that this relation is symmetric, consider an (A, B, C)-quasi-isometry $f : \mathcal{M}_1 \to \mathcal{M}_2$. Define $\bar{f}$ which "reverses" f from $\mathcal{M}_2$ into $\mathcal{M}_1$ as follows. For all $y \in M_2$, if $y \in Range(f)$, then select an x from $f^{-1}(y)$ and set $\bar{f}(y) = x$. Otherwise, select an $x' \in M_1$ such that $d_2(y, f(x')) \leqslant C$ and set $\bar{f}(y) = x'$. It is easy to see that $\bar{f}$ determines a quasi-isometry from metric space $\mathcal{M}_2$ into $\mathcal{M}_1$.

We give two examples. For the first example, consider the spaces $\mathbb{Z}^n$ (the n-dimensional grid of integers) and $\mathbb{R}^n$ with their natural metrics. These are quasi-isometric via the identity map from $\mathbb{Z}^n$ into $\mathbb{R}^n$. Informally, two metric spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are quasi-isometric if these spaces look the same from far away. We will make this comment precise in the section on asymptotic cones.

The second example is from group theory. Let G be a finitely generated group, say with a generator set S. Consider the Cayley graph $\Gamma(G, S)$ of G based on S. The metric in $\Gamma(G, S)$ is given by the shortest-path distance in the graph $\Gamma(G, S)$. By varying the set S of generators of the group, we vary the Cayley graphs, and hence corresponding metric spaces. It turns out that all these metric spaces are quasi-isometric to each other. In other words, the quasi-isometry class of the Cayley graphs $\Gamma(G, S)$ is invariant of the group G even if we change the generators of the group. It turns out that some important properties of groups are quasi-isometry invariants. Examples include being virtually nilpotent, virtually free, hyperbolic, having polynomial growth rate, being finitely presentable, having decidable word problem, having the same asymptotic cones [13, 4, 7]. Studying quasi-isometry invariants of groups proved to be crucial in solving many problems in group theory [5–7]. Nowadays finding quasi-isometry invariants of groups is an important theme in geometric group theory.

Recall that by infinite string we mean a mapping α from ω (the set of natural numbers) into a finite alphabet Σ. So, α is viewed as the infinite sequence $\alpha(0)\alpha(1)\alpha(2)\ldots$. Natural numbers $i \in \omega$ are called positions of the string α. In formal language theory, logic and applications, infinite strings α are one of the main objects of investigation. These objects are discrete and somewhat boring from a geometry point of view. How are the coarse geometry concepts (e.g., quasi-isometry) relevant to the study of large scale properties of infinite strings? How does one introduce the concepts of coarse geometry into understanding large scale properties

of the infinite strings? One possible answer comes from a simple observation that each string α has the natural metric inherited from the set ω of natural numbers. The elements of α are positions i, j in α and the distance $d_\alpha(i, j)$ between the positions is the natural distance $|i - j|$. So, α as a metric space is simply $(\omega; d_\alpha)$; the quasi-isometry type of $(\omega; d_\alpha)$ is the metric space $\mathbb{R}_{\geqslant 0}$ of all positive reals. So, from a quasi-isometry view point, α viewed as a metric space $(\omega; d)$ is still somewhat uneventful. However, one crucial difference between the metric spaces $(\omega; d_\alpha)$ and α from the setting of large scale/coarse geometry is that the metric space α possesses the natural coloring associated with it. Namely, for every position $i \in \omega$ in α, the color of the position i is σ exactly when $\alpha(i) = \sigma$.

The observation above suggests the idea of applying the notions of coarse geometry (such as the concept of quasi-isometry) to colored metric spaces. Precisely this is what we propose in this paper. We investigate quasi-isometries of colored metric spaces with strings α as our primary objects (as colored metric spaces), thus initiating the study of large scale properties of strings.

A colored metric space $\mathcal{M}$ is a tuple $(M; d, Cl)$, where $(M; d)$ is the *underlying metric space* with metric d, and $Cl : M \to 2^\Sigma$ is a color function; the set Σ is a finite set of colors called an *alphabet*. If $\sigma \in Cl(m)$ then we say that the element $m \in M$ has color σ. We allow elements to have several (including none) colors.

EXAMPLE 1.1. As mentioned above, infinite strings α over Σ are colored metric spaces of the form $(\omega; d, \alpha)$, where d is the natural metric (so, for all $i, j \in \omega$ we have $d(i, j) = |i - j|$) and $\alpha : \omega \to \Sigma$ is the color function. Every element in α has a unique color. We always assume that the cardinality $|\Sigma|$ of Σ is at least 2. So, we identify the colored metric space $(\omega; d, \alpha)$ with α.

Denote the set of all infinite strings over Σ, considered as colored metric spaces, by Σ^ω. These metric spaces will be our main objects of study. The set of all finite words over Σ is denoted by $\Sigma^\star$.

EXAMPLE 1.2. Let $G = (V; E, Cl)$ be a colored graph. The graph G has its natural distance d inherited from the path-distance metric on G. Here we assume that the graph (V, E) is symmetric (that is, E is subset of the set of 2-element subsets of V), and $Cl : V \to \Sigma$ is the given coloring function. For d to be the proper metric we need to assume that G is connected.

We now adapt the definition of quasi-isometry on metric spaces (given at the beginning of this section) for colored metric spaces.

DEFINITION 1.3. Let $\mathcal{M}_1 = (M_1; d_1, Cl_1)$ and $\mathcal{M}_2 = (M_2; d_2, Cl_2)$ be two colored metric spaces. A *color preserving* quasi-isometry $f : M_1 \to M_2$ will be called a *quasi-isometry* from $\mathcal{M}_1$ into $\mathcal{M}_2$.

From now on we write $\mathcal{M}_1 \leqslant_{QI} \mathcal{M}_2$ if there is a quasi-isometry from $\mathcal{M}_1$ into $\mathcal{M}_2$. As mentioned above, for metric spaces (with no colors) the relation $\leqslant_{QI}$ is an equivalence relation, and hence the relation is symmetric. In contrast, for colored metric spaces, however, the symmetry property does not hold. For instance, the mapping $f : \omega \to \omega$ (on the metric space $(\omega; d)$) defined as $f(i) = 2i$ is a quasi-isometry from the infinite string $0^\omega = 00000 \ldots$ into the string $(01)^\omega = 010101 \ldots$. There is no quasi-isometric mapping in the opposite direction.

If $\mathcal{M}_1 \leqslant_{QI} \mathcal{M}_2$ and $\mathcal{M}_2 \leqslant_{QI} \mathcal{M}_1$ then we write this by $\mathcal{M}_1 \sim_{QI} \mathcal{M}_2$. The relation $\sim_{QI}$ is an equivalence relation in the class of all colored metric spaces, in particular on the set Σ^ω. Formally, we give the following definition:

DEFINITION 1.4.   The equivalence classes of $\sim_{QI}$ on Σ^ω are called *quasi-isometry types* or, equivalently, *large scale geometries*. From now on we set $\Sigma_{QI}^\omega = \Sigma^\omega / \sim_{QI}$. Denote the large scale geometry of string α by $[\alpha]$. Thus,

$$\Sigma_{QI}^\omega = \{[\alpha] \mid \alpha \in \Sigma^\omega\}.$$

REMARK 1.5.   A simple yet important note is the following. If f is a *color preserving* quasi-isometry from $\mathcal{M}_1$ into $\mathcal{M}_2$ and there is a constant D such that for all $y \in M_2$ there exists an $x \in M_1$ with $d_2(y, f(x)) \leqslant D$ and $Cl_1(x) = Cl_2(y)$, then one can construct a quasi-isometry from $\mathcal{M}_2$ to $\mathcal{M}_1$ by "reversing" the mapping f as we described earlier.

The fact that $\leqslant_{QI}$ on colored metric spaces is not symmetric implies that we can compare large scale geometries of these spaces, and consider the partial order $\leqslant_{QI}$ on the quasi-isometry types. In particular, the partial order $\leqslant_{QI}$ can be restricted to the large scale geometries of colored metric spaces over a fixed underlying metric space such that ω or $\mathbb{Z}$ or $\mathbb{Z}^2$ with their natural metrics. From this viewpoint the quasi-isometry on colored metric spaces is a much refined version of the usual quasi-isometry relation on uncolored metric space. This is because the quasi-isometry type of every uncolored metric space trivializes to a singleton.

1.2. Informal description of results. Introduction of large scale geometries on colored metric spaces of the type α and the quasi-isometry relation $\leqslant_{QI}$ poses many natural questions. Most of these are mathematical translations of the questions that we posed at the beginning of this paper.

The first question is concerned with understanding the partial order $\leqslant_{QI}$ on the set Σ_{QI}^ω of all large scale geometries. The order presents an algebraic tool aimed at classification of global patterns that occur on strings. Among many results, we prove that the order has the greatest large scale geometry, and in Section 3 we show that the partial order has infinite chains and infinite antichains. We also study density property of the partial $\leqslant_{QI}$. Recall that a partial order is dense if for any two distinct comparable elements x and y of the partial order there exists an element z strictly between x and y. We show that the density property is prevalent for the quasi-isometry order $\leqslant_{QI}$. We also prove that there are infinitely many minimal large scale geometries.

The second question is related to understanding the quasi-isometry relation $\sim_{QI}$ on Σ^ω. One way to do it is to restrict the relation $\sim_{QI}$ to various subsets of infinite words and, given such a subset X, understand the partial order $\leqslant_{QI}$ on the set of equivalence classes under $\sim_{QI}$ in X. For instance, in Section 3.4 we restrict the relation $\sim_{QI}$ to eventually periodic strings, and give a full description of quasi-isometry types and the order $\leqslant_{QI}$ on this class of strings. The other way is to simplify or refine the definition of quasi-isometry. Section 4.2 gives an intuitively syntactically easier yet equivalent definition of the relation $\leqslant_{QI}$ that we call component-wise reducibility. We also give a more refined version of $\sim_{QI}$, called color-equivalence. We show that color-equivalence is strictly stronger than $\sim_{QI}$ equivalence relation.

Finally, it is natural to ask if quasi-isometric maps between strings can be replaced with order preserving quasi-isometric embeddings. We give a negative answer to this question; however, we prove that the answer is positive modulo $\sim_{QI}$ equivalence relation.

The third question is related to describing various sets of large scale geometries. So, let L be a set (usually called a language) of infinite strings over an alphabet Σ. One naturally considers the set $[L]$ consisting of all large scale geometries of strings in L. Thus we have: $[L] = \{[\alpha] \mid \alpha \in L\}$. We call the set $[L]$ the *atlas* of L. The question we ask is related to understanding the atlas $[L]$ given a description of L. In particular, a natural algorithmic question is if there exists an algorithm that, given descriptions of languages L_1 and L_2, decides whether the atlases $[L_1]$ and $[L_2]$ coincide. In Section 5, we use Büchi automata as our description tools and provide a full characterization of the atlases defined by Büchi automata. Our characterization theorem implies that there are only finitely many atlases $[L]$ described by Büchi automata. As a consequence of our characterization, we show that for Büchi automata recognizable languages L_1 and L_2 there is a linear time algorithm (in the size of automata) for deciding the equality of the atlases $[L_1]$ and $[L_2]$. This contrasts with the PSPACE completeness of the equality problem for Büchi languages. This part of the work links large scale geometries with automata theory and complexity theory.

The fourth question addresses the complexity of quasi-isometry relation on infinite strings in terms of arithmetical hierarchy, thus connecting the topic with computability theory. Recall that isometry is color preserving and distance preserving map. So, quasi-isometry relation is weaker than isometry relation since each isometry is a quasi-isometry. Hence, one might expect that quasi-isometry is easier to detect than the isometry. We prove that the quasi-isometry relation on computable infinite strings is Σ_2^0-complete. This is in contrast to Π_1^0-completeness of the isometry relation on computable strings. These are explained in Section 6.

The fifth question asks how one encodes large scale geometries of colored metric spaces α into "limit" structures. For this, we define structures obtained from "looking at α from far away". Dries and Wilkie [13], using ultrafilters, formalized this intuitive notion for metric spaces through the concept of asymptotic cone. Their work gave a logical context to Gromov's work in [5, 7], where it is proved that groups of polynomial growth are virtually nilpotent. We invoke the concept of asymptotic cone and study the relationship between large scale geometries of infinite strings and their asymptotic cones. In Section 7 we prove theorems akin to results on asymptotic cones in geometric group theory. For instance, we show that asymptotic cones of Martin-Löf random strings coincide when scaling factors are computable. These results bridge the topic of this paper with algorithmic randomness and model theory.

1.3. Basic properties of the order $(\Sigma_{QI}^{\omega}, \leqslant_{QI})$. The quasi-isometry relation $\leqslant_{QI}$ on the strings set Σ^{ω} naturally defines the partially ordered set $(\Sigma_{QI}^{\omega}, \leqslant_{QI})$ on the set of all large scale geometries $[\alpha]$. We denote this partially ordered set by Σ_{QI}^{ω}, thus identifying it with its domain. We now need simple definitions related to partially ordered sets. Say that $[\beta]$ *covers* $[\alpha]$ if $[\alpha] \neq [\beta]$, $[\alpha] \leqslant_{QI} [\beta]$, and for all $[\gamma]$ such that $[\alpha] \leqslant_{QI} [\gamma] \leqslant_{QI} [\beta]$ we have $[\gamma] = [\alpha]$ or $[\gamma] = [\beta]$. An element is an *atom* if it covers a minimal element in the partial order.

Here is an easy proposition. In this proposition $\sigma \in \Sigma$; also for any finite word $v \in \Sigma^\star$, v^ω denotes the infinite string $vvvv\ldots$. Part (2) of the proposition will be strengthened later.

PROPOSITION 1.6. *The partial order Σ_{QI}^ω has the following properties*:

1. *There exists a greatest element.*
2. *There exist at least $|\Sigma|$ minimal elements.*
3. *There exist at least $|\Sigma| \cdot (|\Sigma| - 1)/2$ atoms.*

PROOF. Let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$. For Part (1), we claim that $\alpha = (\sigma_1 \ldots \sigma_k)^\omega$ is the greatest element in Σ_{QI}. Indeed, take any $\beta \in \Sigma^\omega$. We write α as $v_0 v_1 \ldots$ where each v_i is $\sigma_1 \ldots \sigma_k$. Define $f : \beta \to \alpha$ so that f maps any position n (in β) to the position k_n in the portion v_n of the string α such that $\beta(n) = \alpha(k_n)$. The mapping f is a witness of $\beta \leqslant_{QI} \alpha$. For Part (2), consider the quasi-isometry types $[\sigma_i^\omega]$. These form minimal elements in Σ_{QI}^ω. Indeed, if $\alpha \leqslant_{QI} \sigma_i^\omega$, then each element of α has color σ_i. For Part (3), consider $\sigma_i(\sigma_j)^\omega$, where $i \neq j$. Clearly, $\sigma_j^\omega \leqslant_{QI} \sigma_i(\sigma_j)^\omega$. Now assume that $\beta \neq \sigma_j^\omega$ and $\beta \leqslant_{QI} \sigma_i(\sigma_j)^\omega$. Then finitely many positions of β are colored with σ_i, and all other positions are colored with σ_j. Hence, $\beta \sim_{QI} \sigma_i(\sigma_j)^\omega$. Therefore, each $\sigma_i(\sigma_j)^\omega$ is an atom in the partial order Σ_{QI}^ω. Thus, we have found at least $k \cdot (k-1)/2$ of the atoms. $\dashv$

§2. Small cross over lemma. A quasi-isometry $g : \alpha \to \beta$ can produce *cross-overs*, for example, pairs n and m of integers such that $n < m$ and $g(m) < g(n)$. Note that this definition of cross-over uses the natural linear order present on ω. The definition of quasi-isometry does not obviously prohibit large cross-overs; in other words, for n and m with $n < m$ the distances $g(m) - g(n)$ might potentially be unbounded. For instance, the following two colored metric spaces $\ldots 00000001111111 \ldots$ and $\ldots 11111110000000 \ldots$ with underlining domain $\mathbb{Z}$ are quasi-isometric and have arbitrary large cross-overs. Nevertheless, the next lemma, that we call Small Cross Over Lemma, shows that there is a uniform bound on cross-overs on colored metric space $\alpha \in \Sigma^\omega$. Namely, for the quasi-isometric map $g : \alpha \to \beta$ there exists a positive constant D such that for all $n < m$ such that $g(m) < g(n)$ we have $g(n) - g(m) \leqslant D$. We note that the proof of the lemma does not use the fact that the spaces are colored. In this sense, the lemma is a quasi-isometric property of the metric space ω with its natural distance. Later we will use this lemma sometimes without an explicit reference to it.

LEMMA 2.1 (*Small Cross Over Lemma*). *For any quasi-isometry constants (A, B, C) there are constants $D \leqslant 0$ and $D' \leqslant 0$ such that for all (A, B, C)-quasi-isometry maps $g : \alpha \to \beta$ we have the following*:

1. *For all $n, m \in \omega$ if $n < m$ and $g(m) < g(n)$ we have $g(m) - g(n) \geqslant D$.*
2. *For all $n, m \in \omega$ if $n < m$ and $g(m) < g(n)$ then $n - m \geqslant D'$.*

PROOF. For the first part of the lemma, suppose that for some integers $n < m$ we have $g(m) - g(n) = D$ that provides a cross-over for a constant $D \leqslant 0$. Our goal is to find a lower bound for the constant D. We refer the reader to Figure 1 in order to follow the arguments below.
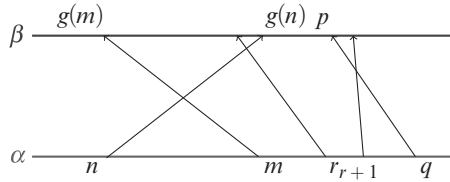
FIGURE 1. Small Cross Over Lemma.

In the list $g(m+1), g(m+2), \ldots,$ select the minimal integer p such that $p \geqslant g(n) + 1$, that is $p = \min(g([m+1, \infty)) \cap [g(n)+1, \infty))$. Define $q = \min(g^{-1}(p) \cap [m+1, \infty))$. So, we have $q \geqslant m+1$ and $p = g(q)$. Then, since g is an (A, B, C)-quasi-isometry we obtain

$$d_\beta(g(n), p) \geqslant \frac{1}{A} \cdot d_\alpha(n, q) - B \geqslant \frac{1}{A} d_\alpha(n, m) - B$$
$$\geqslant \frac{1}{A} \cdot \frac{1}{A}(d_\beta(g(n), g(m)) - B) - B$$
$$= -\frac{D}{A^2} - \frac{(A^2 + 1)}{A^2} B.$$

We have $g([m, q]) \subset [0, g(n)] \cup [p, \infty)$ by our definition of p and q. Note $g(m) \in [0, g(n)]$ and $g(q) \in [p, \infty)$. Hence, there exists $r \in [m, q-1]$ such that $g(r) \in [0, g(n)]$ and $g(r+1) \in [g(p), \infty)$, which means

$$d_\beta(g(n), p) \leqslant d_\beta(g(r), g(r+1)) \leqslant A + B.$$

From these two systems of inequalities we have the inequality:

$$-\frac{D}{A^2} - \frac{(A^2+1)}{A^2} B \leqslant A + B.$$

So we have

$$D \geqslant -(A + B)A^2 - \frac{A^2 + 1}{A^2} B$$

that was desired to be proved.

Note that the lemma states that the distance $g(m) - g(n)$ is uniformly bounded from below when n and m form a cross-over pair. Since g is a quasi-isometry, for all pairs $n < m$ that form a cross-over, the distance between $n - m$ is also universally bounded from below. This proves the second part of the lemma.

We note that the claimed constants D and D' are independent on any particular (A, B, C)-quasi-isometry f; namely, D and D' depend on (A, B, C) only. ⊣

COROLLARY 2.2. *There exists a computable function that, given quasi-isometry constants A, B, C, computes constants D and D' stated in the lemma.*

Consider an (A, B, C)-quasi-isometry $f : \alpha \to \beta$. Let $[x, y]$ be an interval in β, where $x \leqslant y$, and consider two values $M_{xy} = \max f^{-1}([x, y])$ and $m_{xy} =$

min $f^{-1}([x, y])$. We would like to measure the size of the interval between m_{xy} and M_{xy} in terms of the length of the interval $[x, y]$.

LEMMA 2.3. *For the* (A, B, C)-*quasi-isometry* $f : \alpha \to \beta$ *there exist constants* A' *and* B' *such that for all positions* $x, y \in \beta$ *with* $x \leqslant y$ *we have*

$$(1/A') \cdot d_\beta(x, y) - B' \leqslant d_\alpha(m_{xy}, M_{xy}) \leqslant A' \cdot d_\beta(x, y) + B'.$$

PROOF. The inequality in the definition of (A, B, C)-quasi-isometry for function f implies:

$$(1/A) \cdot d_\beta(f(n), f(m)) - (B/A) \leqslant d_\alpha(n, m)$$
$$\leqslant A \cdot d_\beta(f(n), f(m)) + AB.$$

Therefore, we have

$$d_\alpha(m_{xy}, M_{xy}) \leqslant A \cdot d_\beta(f(m_{xy}), f(M_{xy})) + AB$$
$$\leqslant A \cdot d_\beta(x, y) + AB.$$

For the lower bound we have:

$$(1/A) \cdot d_\beta(f(m_{xy}), f(M_{xy})) - (B/A) \leqslant d_\alpha(m_{xy}, M_{xy}).$$

So, we need to rewrite the lower bound in terms of the length of the interval x and y. For this we use the quasi-isometry inequality twice:

$$d_\beta(f(m_{xy}), f(M_{xy})) \geqslant (1/A)d_\alpha(m_{xy}, M_{xy}) - B$$
$$\geqslant (1/A)d_\alpha(\min f^{-1}(x), \max f^{-1}(y)) - B$$
$$\geqslant (1/A)((1/A)d_\beta(x, y) - B) - B$$
$$= (1/A^2)d_\beta(x, y) - ((A + 1)/A)B.$$

All these give us the desired constants A' and B'.   ⊣

In terms of the last lemma, we note that we can tune our constants A' and B' using the Small Cross-Over Lemma. Indeed, consider the values $M'_{xy} = \max([x, y] \cap f(\alpha))$ and $m'_{xy} = \min([x, y] \cap f(\alpha))$. Then due to the density condition of quasi-isometricity it holds that $d_\beta(y, M'_{xy}) \leqslant 2C$ and $d_\beta(x, m'_{xy}) \leqslant 2C$. Also from Lemma 2.1 we have

$$d_\beta(f(M_{xy}), M'_{xy}) \leqslant -D \quad \text{and} \quad d_\beta(f(m_{xy}), m'_{xy}) \leqslant -D.$$

Hence, by the triangular inequality, $d_\beta(f(m_{xy}), f(M_{xy})) \geqslant d_\beta(x, y) - 4C + 2D$.

In summary, we have the following:

$$(1/A) \cdot d_\beta(x, y) - (4C + B - 2D)/A \leqslant d_\alpha(m_{xy}, M_{xy}) \leqslant A \cdot d_\beta(x, y) + AB.$$

Set $A' = A$ and $B' = \max\{(4C + B - 2D)/A, AB\}$.

We have the next immediate corollary (that can also be proved easily without the use of the lemmas above):

COROLLARY 2.4. *Let* $f : \alpha \to \beta$ *be a quasi-isometry. There is a $C > 0$ such that* $|f^{-1}(y)| < C$ *for all $y \in \beta$.* ⊣

§3. Algebraic properties of the order $(\Sigma_{QI}^{\omega}, \leqslant_{QI})$**.** The partially ordered set $(\Sigma_{QI}^{\omega}, \leqslant_{QI})$ is an algebraic tool that allows us to study large scale properties of infinite strings and compare them. Proposition 1.6 already describes some of the elementary properties of this partial order. In this section we further embrace this partial order and derive some nontrivial properties of it.

3.1. Infinite chains and antichains. A natural question is if the partially ordered set $(\Sigma_{QI}^{\omega}; \leqslant_{QI})$ is infinite. We positively answer this question in two ways. On the one hand we show that the partial order has infinite chains, and on the other hand we show that the order has infinite antichains.

THEOREM 3.1 (Infinite Chain Theorem). *The order Σ_{QI}^{ω} has a chain of the type of integers, that is, a collection $\{\alpha_n\}_{n \in \mathbb{Z}}$ of strings such that $[\alpha_n <_{QI} \alpha_{n+1}]$ for all $n \in \mathbb{Z}$.*

PROOF. Take two distinct elements in Σ, say 0 and 1, respectively. Define the following sequence $\{\alpha_n\}_{n \in \mathbb{Z}}$ of infinite strings from Σ^{ω}:

$$\alpha_n = \begin{cases} (01)^{2^n}(011)^{2^n}...(01^{2^k})^{2^n}... & (n \geqslant 0), \\ (01)(01^{2^{2^{-n}}})...(01^{(2^k)^{2^{-n}}})... & (n < 0). \end{cases}$$

The idea behind this definition is the following. The string α_0 is of the form:

$$(01)(011)(01111)...(01^{2^k})(01^{2^{k+1}})...$$

Let us call the substrings (01^{2^k}) blocks of α_0. The above definition tells us that α_1 is obtained from α_0 by doubling each block of α_0. Similarly, α_{-1} is obtained from α_0 by removing every other block. This process is propagated to all α_n's.

We show that $\alpha_n \leqslant_{QI} \alpha_{n+1}$ for all $n \geqslant 0$. For negative n, the proof is similar. In α_n consider the segment of the form

$$\underbrace{(01^{2^{k+1}})(01^{2^{k+1}})...(01^{2^{k+1}})}_{2^n \text{ times}}.$$

We map this segment of α_n into the following segment of the string α_{n+1}:

$$\underbrace{(01^{2^k})(01^{2^k})(01^{2^k})(01^{2^k})...(01^{2^k})(01^{2^k})}_{2^{n+1} \text{ times}}.$$

The mapping sends the block $(01^{2^{k+1}})$ in α_n into $(01^{2^k})^2$ in α_{n+1} in an obvious injective way. This mapping can easily be seen to be a quasi-isometry from α_n into α_{n+1}. We refer the reader to Figure 2.

Now we need to prove that there is no quasi-isometric mapping from α_{n+1} into α_n. For the sake of contradiction assume that g is an (A, B, C)-quasi-isometry from α_{n+1} into α_n. Then using Lemma 2.3 and Lemma 2.1, one sees that g needs to

$$\alpha_{n+1} = \ \ldots 01\ldots 101\ldots 10\ldots$$

$$\alpha_n = \ \ldots 01\ldots 11\ldots 10\ldots$$

FIGURE 2. The mapping from $(01^{2^{k+1}})$-block in α_n into $(01^{2^k})(01^{2^k})$-block in α_{n+1}.

be strictly monotone on all but finitely many positions of α_{n+1} with color 0; that is, we need to have $M \in \omega$ such that for all $m, m' > M$ and $m > m'$ we have if $\alpha_{n+1}(m) = \alpha_{n+1}(m') = 0$ then $g(m) > g(m')$.

We now show that we have a contradiction. Let $m_{i,n}$ and $m_{j,n+1}$ be the positions of the ith 0 in α_n and jth 0 in α_{n+1}, respectively. Now assume that $m_{i,n+1} > M$ and $g(m_{i,n+1}) = m_{j,n}$. Then from monotonicity and injectivity of g on all positions colored 0 in α_{n+1}, we should have

$$d_{\alpha_n}(g(m_{i,n+1}), g(m_{i+k,n+1})) \geqslant m_{j+k,n} - m_{j,n}.$$

Together with the (A, B, C)-quasi-isometry condition, this implies $m_{j+k,n} - m_{j,n} \leqslant A(m_{i+k,n+1} - m_{i,n+1}) + B$ and hence

$$r(k) = \frac{m_{j+k,n} - m_{j,n}}{m_{i+k,n+1} - m_{i,n+1}} \leqslant A + \frac{B}{m_{i+1,n+1} - m_{i,n+1}} \quad (k \geqslant 1),$$

which in particular says the value $r(k)$ is uniformly bounded with respect to k. On the other hand, from the construction of α_n and α_{n+1} one can observe that the growth rate of $m_{k,n}$ with respect to k is much faster than that of $m_{k,n+1}$; more precisely, one can verify that $m_{2^n \cdot k,n} = \mathcal{O}(2^k)$ holds for each $n \in \omega$, and thus $m_{2^n \cdot k,n+1} = m_{2^{n+1} \cdot \frac{k}{2}, n+1} = \mathcal{O}(2^{\frac{k}{2}})$ holds when k is even. This implies $r(k)$ is unbounded, which is a contradiction.   ⊣

THEOREM 3.2 (Infinite Antichain Theorem).   *The partial order Σ_{QI}^ω has a countable antichain $\{\beta_n\}_{n\in\omega}$.*

PROOF.   As above we assume that $0, 1 \in \Sigma$. Consider the following sequence of strings β_n, $n \in \omega$:

$$\beta_n = 010^{2^n} 1^{2^n} 0^{3^n} 1^{3^n} \ldots 0^{k^n} 1^{k^n} \ldots$$

We claim that this sequence forms an antichain in Σ_{QI}^ω. Take any $n, m \in \omega$ and suppose $\beta_n \leqslant_{QI} \beta_m$ via an (A, B, C)-quasi-isometry f. We will show that $n = m$.

Let $A_{s,t}$ be the "tth block" of zeros in β_s. We can represent $A_{s,t}$ as the interval

$$A_{s,t} = [2 \sum_{i=0}^{t-1} i^s, 2 \sum_{i=0}^{t-1} i^s + (t)^s - 1].$$

Since f is a quasi-isometry, an easy argument shows that there exists $k, l \in \omega$ such that (1) $f(A_{n,k}) \subseteq A_{m,l}$, and (2) for every $k' \in \omega$, we have $f(A_{n,k'+k}) \subseteq A_{m,k'+l}$. Intuitively, these say that from some point the quasi-isometry f maps β_n in the "block by block" manner without vacancy—one can easily verify that, without this property f generates unbounded distortions (and thus is not a quasi-isometry). Now

note that we have:

$$(1/A)|A_{n,k'+k}| - B \leqslant |A_{m,k'+l}| \leqslant A|A_{n,k'+k}| + B.$$

Therefore, we should have a positive upper and lower bounds on the ratio

$$\frac{|A_{n,k'+k}|}{|A_{m,k'+l}|} = \frac{(k'+k)^n}{(k'+l)^m}$$

with respect to k'. The only case that satisfies this condition is $n = m$. $\dashv$

The trivial large scale geometries $[0^\omega]$ and $[1^\omega]$, as we noted, are minimal elements of Σ_{QI}^ω. The next theorem shows one way to build nontrivial minimal large scale geometries.

THEOREM 3.3 (Minimal Elements Theorem). *Let* $\{a_n\}_{n\in\omega}$ *be a sequence such that* $\lim_{n\to\infty} a_n = \infty$. *Then the large scale geometry of the string* $\alpha = 0^{a_0} 1^{a_1} 0^{a_2} 1^{a_3} ... 0^{a_{2k}} 1^{a_{2k+1}} ...$ *is a minimal element in any partial order* Σ_{QI}^ω *with* $0, 1 \in \Sigma$.

PROOF. Suppose that $\beta \leqslant_{QI} \alpha$ via an (A, B, C)-quasi-isometry f. Our goal is to show that $\alpha \sim_{QI} \beta$. It suffices to show that there exists a constant D such that for every sufficiently large $y \in \alpha$ (i.e., for every $y \geqslant Y$ with some fixed constant Y) there exists an $x \in \beta$ with $|y - f(x)| \leqslant D$ and $Cl(x) = Cl(y)$. Indeed, assuming such a condition we can construct a quasi-isometry $g : \alpha \to \beta$ as follows; for each $y \in \alpha$,

1. if there exists $y' \in \alpha$ such that $|y' - y| \leqslant D$ for which there is $x' \in \beta$ with $f(x') = y'$, then let $g(y) = x'$;
2. otherwise, let $x \in \beta$ be any position such that $Cl(x) = Cl(y)$, and let $g(y) = x$.

Here $g(y)$ is well-defined in case 2, as β clearly contains both of 0 and 1 as its color. It is easy to verify that g is a quasi-isometry—recall the argument in Remark 1.5, and observe that the case 2 happens only for finitely many y's.

Now we prove existence of the constant D. Enumerate all elements of the image of f as $\{y_n\}_{n\in\omega}$, where $y_n < y_{n+1}$ for all n. Then by definition of (A, B, C)-quasi-isometry, we have $y_{n+1} - y_n \leqslant 2C + 1$ (as otherwise there is y between y_n and y_{n+1} that violates the density property). On the other hand, it is clear from the construction of α that every sufficiently large $y \in \alpha$ is a member of a single-colored interval I in α with the size greater than $2C + 1$. Thus for such a y, one can find $x \in \beta$ such that $|y - f(x)| \leqslant 2C + 1$ and $Cl(x) = Cl(y)$. $\dashv$

By this result, the strings β_n constructed in the Infinite Antichain Theorem are all minimal. Hence, we have the following corollary:

COROLLARY 3.4. *The partially ordered set* Σ_{QI}^ω *possesses infinitely many minimal elements.* $\dashv$

3.2. Density theorem. Recall that a partial order $\leqslant$ is dense if for all x, y the condition $x \leqslant y$ and $x \neq y$ implies existence of z such that $x \leqslant z \leqslant y$ and $z \notin \{x, y\}$. Since Σ_{QI}^ω has atoms, this order is not dense. However, the goal of this section is to show the density property for $\leqslant_{QI}$ is prevalent.

We start with the lemma showing that all (A, B, C)-quasi-isometries between strings are bounded by one linear function; the function depends on the constants A, B, C only and not the quasi-isometries.

LEMMA 3.5. *For given $A, B, C \in \omega$, there is a constant $D' \in \omega$ such that for all (A, B, C)-quasi isometries $f : \alpha \to \beta$ and $n \in \omega$ we have $f(n) \leqslant An + D'$.*

PROOF. Let $f : \alpha \to \beta$ be an (A, B, C)-quasi-isometry. The constant C tells us that there is $n_0 \in \omega$ such that $f(n_0) \in [0, C]$. Thus, by the Small Cross Over Lemma there is a bound D (which is independent of the choice of f) such that $f(0) \leqslant D$. Since f is an (A, B, C)-quasi-isometry, we have

$$f(n) \leqslant An + B + f(0) \leqslant An + (B + D).$$

So, we let $D' = B + D$.   ⊣

For the density theorem we use the following standard notation. For string α, let $\alpha\!\restriction_k$ be the string $\alpha(0) \dots \alpha(k)$, where $k \in \omega$. Since we will be handling finite approximations to quasi-isometric mappings, we need the following definition.

DEFINITION 3.6. Let $A, B, C \in \omega$ be quasi-isometry constants, and let $D \leqslant 0$ be the largest cross over bound for (A, B, C)-quasi-isometries that is given by Lemma 2.1. A function $f : \alpha\!\restriction_n \to \beta$ is *possibly extendable to* (A, B, C)-*quasi-isometry* if it satisfies the following conditions:

1. f is color preserving.
2. For all $k, m \in \text{dom}(f)$, we have

$$(1/A) \cdot d(k, m) - B < d(f(k), f(m)) < A \cdot d(k, m) + B.$$

3. Let $M_n = \max\{f(x) \mid x \in \text{dom}(f)\}$. If $M_n > C - D$, then for any $y < M_n - (C - D)$ there exists an $x < n$ such that $d(y, f(x)) \leqslant C$.

Let $\alpha, \beta \in \Sigma^\omega$ and $A, B, C \in \omega$ be quasi-isometry constants. Our goal is to construct a rooted tree $T(\alpha, \beta, A, B, C)$ with the following properties:

1. The tree $T(\alpha, \beta, A, B, C)$ is finitely branching and computable in α and β. In particular, if α and β are computable then so is $T(\alpha, \beta, A, B, C)$.
2. For any n one can effectively compute, with an oracle for α and β, the number of nodes in the tree at distance n from the root. Note that Part (1) does not always imply Part (2).
3. The tree $T(\alpha, \beta, A, B, C)$ is infinite if and only if there exists an (A, B, C)-quasi-isometric map from α into β.
4. There is a bijection between (A, B, C)-quasi-isometric maps from α to β and the set $[T(\alpha, \beta, A, B, C)]$ of all infinite paths of the tree $T(\alpha, \beta, A, B, C)$.

The nodes of $T(\alpha, \beta, A, B, C)$ are finite partial functions that can possibly be extended to (A, B, C)-quasi-isometric maps from α to β. Here is a formal construction of the tree $T(\alpha, \beta, A, B, C)$:

1. The root of the tree $T(\alpha, \beta, A, B, C)$ is the empty set $\emptyset$, that is, nowhere defined partial function.
2. Let x be a leaf of the tree $T(\alpha, \beta, A, B, C)$ constructed so far, and f_x be the partial function associated with x. We identify x and f_x. We assume that

(a) x is at distance n from the root, and $f_x = \{(0, i_0), \ldots, (n, i_n)\}$.

(b) The function f_x is possibly extendable to (A, B, C)-quasi-isometry.

(c) $i_n \leqslant An + D'$, where D' is from Lemma 3.5.

3. List all possibly extendable extensions ϕ of f_x such that the domain of ϕ is $\{0, 1, \ldots, n, n+1\}$. The children of f_x are all these functions ϕ.

In Part 2 of the construction the number of extensions ϕ of f_x is finite since these extensions must satisfy the quasi-isometry condition. If such extensions ϕ of f_x do not exist, then the node x becomes a leaf. Finally, Lemma 3.5 guarantees that the root of the tree has finitely many children. Hence, the tree is finitely branching.

LEMMA 3.7. *The following are true*:

1. *There is a bijection between the set of all infinite paths of $T(\alpha, \beta, A, B, C)$ and the set of all (A, B, C)-quasi-isometric maps from α to β.*

2. *The tree $T(\alpha, \beta, A, B, C)$ is infinite if and only if there is an (A, B, C)-quasi-isometric map from α into β.*

PROOF. Let $f : \alpha \to \beta$ be an (A, B, C)-quasi-isometry and let $f_n = f\!\restriction_n$. We show that f_n satisfies the last condition of possibly-extendable function (i.e., the condition 3 in Definition 3.6); Conditions 1 and 2 of Definition 3.6 clearly hold. Assume the contrary, letting $y < M_n - (C - D)$ be an element such that no $x \in \{0, \ldots, n\}$ satisfies $f_n(x) \in \{y - C, \ldots, y + C\}$. As f is an (A, B, C)-quasi-isometry, there should be an $x' > n$ that satisfies $f(x') \in \{y - C, \ldots, y + C\}$. But then, for $x'' \in f_n^{-1}(M_n)$, we have $f(x') - f(x'') \leqslant (y + C) - M_n < D$. This contradicts the Small Cross Over Lemma.

If $x_1, x_2, \ldots$ is an infinite path through the tree $T(\alpha, \beta, A, B)$ then the mapping f defined as the limit of the sequence $f_{x_1}, f_{x_2}, \ldots$ is an (A, B, C)-quasi-isometry from α to β. ⊣

From Lemma 3.7 above we immediately have the following.

COROLLARY 3.8. *Let $\alpha, \beta \in \Sigma^\omega$ be given. If for each $k \in \omega$ there is a function $f : \alpha\!\restriction_k \to \beta$ that is possibly extendable to (A, B, C)-quasi-isometry, then there is an (A, B, C)-quasi-isometry from α to β.*

Let $\alpha, \beta \in \Sigma^\omega$, and let $Cl(\beta)$ be the set of all colors of the string β. Also let w, v be finite strings over $Cl(\beta)$ respectively of lengths $n, m \in \omega$. Define the following two strings obtained by changing the initial segments of α and β:

$$\alpha' = w\alpha(n)\alpha(n+1)\ldots \quad \text{and} \quad \beta' = v\beta(m)\beta(m+1)\ldots.$$

Also, assume that every letter present in one of these two strings occurs in both strings infinitely often. With these conditions, we have the following simple lemma that informally states that in quasi-isometries initial segments of strings do not matter.

LEMMA 3.9. *Under assumptions above $\alpha \leqslant_{QI} \beta$ if and only if $\alpha' \leqslant_{QI} \beta'$.*

PROOF. If $\alpha \leqslant_{QI} \beta$ via $f : \alpha \to \beta$, then take $n' \geqslant n$ such that for any $i \geqslant n'$ we have $f(i) \geqslant m$. Define $f' : \alpha' \to \beta'$ as follows. For all $j \leqslant n'$, make f' color preserving. For all $i > n$ set $f'(i) = f(i)$. This is a quasi-isometry. The converse can be shown in a similar way. ⊣

THEOREM 3.10 (Density Theorem). *Let $\alpha, \beta \in \Sigma^{\omega}$ be given. Assume $\alpha <_{QI} \beta$, and every letter in α or β occurs in both α and β infinitely often. Then there exists $\gamma \in \Sigma^{\omega}$ such that $\alpha <_{QI} \gamma <_{QI} \beta$.*

PROOF. Note that the assumption of the theorem (that letters that occur in α or β must occur in both of these strings infinitely often) is important. For instance, no γ exists such that $0^{\omega} <_{QI} \gamma <_{QI} 10^{\omega}$.

Let $\alpha <_{QI} \beta$ via f. We can assume (if necessary, by taking QI-equivalent strings $\alpha' \sim_{QI} \alpha$ and $\beta' \sim_{QI} \beta$) that f is injective and monotonic. The reader can verify this (or the reader can refer to Theorem 4.11). So, β is of the form

$$\beta = \alpha(0)w_0\alpha(1)w_1\ldots,$$

where the length of finite strings w_k are all uniformly bounded by a constant. By $p(n)$ we denote the position of $\alpha(n)$ in β, that is, $p(n) = n + \sum_{i=0}^{n-1} |w_i|$.

An informal idea of constructing γ is the following. Initially γ starts copying α until an initial segment of α such that no mapping from β can possibly be extended to a $(1, 1, 1)$-quasi-isometry into γ built. From that point on, γ switches to β and copies β until an initial segment of γ is built such that no mapping from γ can possibly be extended to a $(2, 2, 2)$-quasi-isometry into α built. Then γ switches to α and continues on copying α. Below is a formal description.

In what follows we inductively define the sequence γ_k of infinite strings, where $k \in \omega$. The strings γ_k will always be of the form $\gamma_k = \alpha(0)u_0^{(k)}\alpha(1)u_1^{(k)}\ldots$, where for each i and k, the string $u_i^{(k)}$ is either w_i or an empty string.

Let $p_k(n)$ be the position of $\alpha(n)$ in the string γ_k, that is, $p_k(n) = n + \sum_{i=0}^{n-1} |w_i^{(k)}|$. Call a string δ an extension of a finite string w if it has w as its prefix (i.e., $\delta = w\delta'$ for some δ'). Now we define the sequence $\{\gamma_k\}_{k \in \omega}$ as follows:

- Set $\gamma_0 := \alpha$.
- For given $k \geq 1$ we proceed as follows.
  - Assume $k - 1$ is even. Find $n_k \in \omega$ such that no mapping $f : \beta\restriction_{n_k} \to \gamma_{k-1}$ exists that can possibly be extendable to (k, k, k)-quasi-isometry. Such n_k exists because $\beta \not\leq_{QI} \gamma_{k-1}$ (by inductive hypothesis). Take a sufficiently large s, say $s = kn_k + D_k$, where D_k is a constant in Lemma 3.5 for (k, k, k)-quasi-isometry constant, and set

    $$\gamma_k = \gamma_{k-1}\restriction_s \beta(s + 1)\beta(s + 2)\ldots.$$

    Note that any map from $\beta\restriction_{n_k}$ to any extension of $\gamma_{k-1}\restriction_s$ is not possibly extendable to a (k, k, k)-quasi-isometry. Note that, by Lemma 3.9, we also have $\gamma_k \not\leq_{QI} \alpha$.
  - Assume that $k - 1$ is odd. Find $n_k \in \mathrm{range}(p_{k-1})$ such that no map $f : \gamma_{k-1}\restriction_{n_k} \to \alpha$ is possibly extendable to (k, k, k)-quasi-isometry; this is possible because $\gamma_{k-1} \not\leq_{QI} \alpha$. Let $s = p_{k-1}^{-1}(n_k)$ (thus we have $\gamma_{k-1}\restriction_{n_k} = \alpha(0)u_0^{(k-1)}\alpha(1)u_1^{(k-1)}\ldots\alpha(s)$), and just as above, set

    $$\gamma_k = \gamma_{k-1}\restriction_{n_k} \alpha(s + 1)\alpha(s + 2)\ldots.$$

Note that, by Lemma 3.9, we have $\beta \not\leq_{QI} \gamma_k$.

Let $\gamma = \lim_{k \to \infty} \gamma_k$. Now we show that $\gamma \nleqslant_{QI} \alpha$. Assume that $\gamma \leqslant_{QI} \alpha$ via a quasi-isometry. Select the smallest even k such that $\gamma \leqslant_{QI} \alpha$ via a (k, k, k)-quasi-isometry g. Consider the construction of γ_k. The string γ_k has the prefix $\gamma_{k-1}\!\restriction_{n_k}$ such that no map $f : \gamma_{k-1}\!\restriction_{n_k} \to \alpha$ is possibly extendable to (k, k, k)-quasi-isometry. By construction, $\gamma_{k-1}\!\restriction_{n_k}$ is a prefix of γ, that is, $\gamma\!\restriction_{n_k} = \gamma_{k-1}\!\restriction_{n_k}$. Hence, g restricted to $\gamma_{k-1}\!\restriction_{n_k}$ is possibly extendable to a (k, k, k)-quasi-isometry. Contradiction. Similarly, one can show that $\gamma \ngeqslant_{QI} \beta$. Finally note that γ is of the form $\alpha(0)u_0\alpha(1)u_1 ...$, where u_i is either w_i or an empty string for each i. This implies that $\alpha \leqslant_{QI} \gamma \leqslant_{QI} \beta$. ⊣

COROLLARY 3.11. *Let $\alpha, \beta \in \Sigma^\omega$ be given. Assume $\alpha <_{QI} \beta$, and every letter in α or β occurs in both α and β infinitely many often. Then there is a countable antichain $\{\gamma^{(n)}\}$ such that $\alpha <_{QI} \gamma^{(n)} <_{QI} \beta$ for each $n \in \omega$.*

PROOF. Assume we have constructed $\gamma^{(0)}, ..., \gamma^{(n)}$ such that $\alpha <_{QI} \gamma^{(i)} <_{QI} \beta$ for each $i \leqslant n$. Then construct $\gamma^{(n+1)}$ via the same construction of γ in the proof of Theorem 3.10, but n_k is a number such that

1. no map $f : \beta\!\restriction_{n_k} \to \gamma_{k-1}$ or $f : \gamma^{(i)}\!\restriction_{n_k} \to \gamma_{k-1} (i \leqslant n)$ is possibly extendable to (k, k, k)-quasi-isometry; and
2. if $k - 1$ is odd, then the finite string $\beta(n_{k-1} + 1) ... \beta(n_k)$ contains every color in $Cl(\beta)$.

Notice that the condition 2 is imposed to assure that any color in $\gamma^{(n)}$ occurs in it infinitely often. This property of $\gamma^{(n)}$ is required to apply Lemma 3.9 when we compare γ_k and $\gamma^{(n)}$ in the construction of $\gamma^{(m)}$, $m > n$. ⊣

The reader can easily notice that the proof of the Density Theorem can be carried out effectively. We state this as the next corollary.

COROLLARY 3.12. *Let α and β be computable strings such that $\alpha <_{QI} \beta$ via a computable quasi-isometry. Then there exists a computable string γ such that $\alpha <_{QI} \gamma <_{QI} \beta$.* ⊣

3.3. Ideals and filters in Σ_{QI}^ω. Recall that a *filter* in the partially ordered set Σ_{QI}^ω is any set X such that for all $x \in X$ and $x \leqslant y$ we have $y \in X$. A filter X is *principal* if X has the least element. Similarly, a set Y is called an *ideal* if for all $x \in Y$ and $y \leqslant x$ we have $y \in Y$. Our goal is to give several natural examples of filters and ideals in the partial order Σ_{QI}^ω that give some insight into algebraic properties of the partial order $\leqslant_{QI}$.

In this section the alphabet Σ is $\{0, 1\}$. For alphabets of larger sizes the concepts and results of this section can easily be extended. Consider the string

$$\alpha = 0^{n_0}1^{m_0}0^{n_1}1^{m_1} ...$$

from $\{0, 1\}^\omega$, where $n_i, m_i \geqslant 1$. Call the substrings 0^{n_i} and 1^{m_i} the 0-*blocks and* 1-*blocks*, respectively. Define:

- $\mathcal{F}(0) = \{[\alpha] \mid$ there exists a constant B such that the lengths of 0-blocks in α are bounded by $B\}$,
- $\mathcal{F}(1) = \{[\alpha] \mid$ there exists a constant B such that the lengths of 1-blocks in α are bounded by $B\}$.

THEOREM 3.13. *The sets $\mathcal{F}(0)$, $\mathcal{F}(1)$ are principal filters.*

PROOF. We prove that $\mathcal{F}(0)$ is a filter. Let $\alpha \in \mathcal{F}(0)$ and $\alpha \leqslant_{QI} \beta$ through an (A, B, C)-quasi-isometry f. We need to show that $\beta \in \mathcal{F}(0)$. Assume, for the sake of contradiction, that the lengths of 0-blocks in β are unbounded. Let B be the length of the largest 0-block in α. Now, take a 0-block $\beta(i) \dots \beta(i + n_0)$ in β, where n_0 is sufficiently large. We give bound on the value of n_0 below. Since f is a quasi-isometry, there exists a $\beta(i_0)$ that belongs to the block $\beta(i) \dots \beta(i + n_0)$ such that $\beta(i_0)$ has a preimage $\alpha(j_0)$ and $\beta(i_0)$ is within distance C from the center of the 0-block $\beta(i) \dots \beta(i + n_0)$. The length of the block in α that contains $\alpha(j_0)$ is bounded by B. We conclude that the string

$$\alpha(j_0 - B - 1) \dots \alpha(j_0) \dots \alpha(j_0 + B + 1)$$

must contain 1. The length of this string is $2B + 2$. There exists a constant D such that f-image of all intervals of length $2B + 2$ is contained in the intervals in β of length at most D. So, if $n_0 = 2D$ then the f-image of the interval

$$\alpha(j_0 - B - 1) \dots \alpha(j_0) \dots \alpha(j_0 + B + 1)$$

must be inside of $\beta(i) \dots \beta(i + n_0)$. This implies that the original 0-block $\beta(i) \dots \beta(i + n_0)$ contains 1 since f preserves colors. It must be the case that $n_0 \leqslant 2D$. Hence, $\beta \in \mathcal{F}(0)$. Finally, it is clear that $[1^\omega]$ is the least element in $\mathcal{F}(0)$. Hence, $\mathcal{F}(0)$ is a principal filter. The same proof shows that the set $\mathcal{F}(1)$ is a principal filter. ⊣

Now we define the following two sets:

- $\mathcal{I}(u) = \{[\alpha] \mid$ there is no universal bound on the lengths of both 0-blocks and 1-blocks in $\alpha\}$.
- $\mathcal{F}(b) = \{[\alpha] \mid$ there is a constant B such that the lengths of both 0-blocks and 1-blocks in α are bounded by $B\}$.

COROLLARY 3.14. *The set $\mathcal{I}(u)$ is an ideal.*

PROOF. Let $\alpha \in \mathcal{I}(u)$ and $\beta \leqslant_{QI} \alpha$. If there is a universal bound on the lengths of all 0-blocks in β then from the theorem above we would have a universal bound on the lengths of 0-blocks in α. This is a contradiction. Similarly, there could not be a universal bound on the lengths of 1-blocks in β. ⊣

COROLLARY 3.15. *The set $\mathcal{F}(b)$ is the singleton $\{[(01)^\omega]\}$.*

PROOF. Assume that $\alpha = 0^{n_0} 1^{m_0} 0^{n_1} 1^{m_1} \dots$ and there is a constant B such that $1 \leqslant n_i, m_i \leqslant B$ for all i. Consider a mapping $f : (01)^\omega \to \alpha$ that maps the ith occurrence of 0 and 1 in $(01)^\omega$ to any position of the blocks 0^{n_i} and 1^{m_i} in α, respectively. Then f witnesses $(01)^\omega \leqslant_{QI} \alpha$. Also the string $(01)^\omega$ is the greatest element of $\{0, 1\}^\omega_{QI}$, as shown in the proof of Proposition 1.6. Thus we have $\alpha \leqslant_{QI} (01)^\omega$. ⊣

The results above imply that the element $(01)^\omega$ is the least upper bound of any two elements $\alpha \in \mathcal{F}(0)$ and $\beta \in \mathcal{F}(1)$.

COROLLARY 3.16. *For all $\alpha \in \mathcal{F}(0)$, $\beta \in \mathcal{F}(1)$, $\gamma \in \Sigma^\omega$, if $\alpha \leqslant_{QI} \gamma$ and $\beta \leqslant_{QI} \gamma$ then $\gamma \sim_{QI} (01)^\omega$.*

One of the natural algebraic operations on strings is the join operation $\oplus$. The join operation $\oplus$, given α and β, produces the string $\alpha \oplus \beta$ defined as $\alpha(0)\beta(0)\alpha(1)\beta(1) \ldots$. This operation is used in computability theory and complexity theory to produce the least upper bounds for various computability and complexity theoretic reducibilities on sets.

Note that $\alpha \leqslant_{QI} \alpha \oplus \beta$ and $\beta \leqslant_{QI} \alpha \oplus \beta$. So $[\alpha \oplus \beta]$ is an upper bound for all $[\alpha]$ and $[\beta]$. However, the operation $\oplus$ does not always produce the least upper bound. Indeed, consider the strings

$$\alpha = 01 \ldots (0)^{2^n}(1)^{2^n} \ldots \quad \text{and} \quad \beta = 10 \ldots (1)^{2^n}(0)^{2^n} \ldots .$$

We can construct a quasi-isometry from α to β by mapping the $(0)^{2^n}(1)^{2^n}$-block in α to the $(0)^{2^{n+1}}(1)^{2^{n+1}}$-block in β; a quasi-isometry from β to α can be also constructed in a similar way. So we have $\alpha \sim_{QI} \beta$, and hence the least upper bound for $[\alpha]$ and $[\beta]$ is $[\alpha]$. However, $[\alpha \oplus \beta] = [(01)^{\omega}]$, and $[(01)^{\omega}] \neq [\alpha]$, $[(01)^{\omega})] \neq [\beta]$, and $[\alpha] = [\beta]$.

Even though the operation $\oplus$ is not well-behaved on the $\sim_{QI}$-equivalence classes, the operation can still be useful in constructing counter-examples as shown in the next corollary.

COROLLARY 3.17. *The filters $\mathcal{F}(0)$ and $\mathcal{F}(1)$ have infinite chains and infinite antichains.*

PROOF. The chain constructed in Theorem 3.1 is in $\mathcal{F}(0)$. Hence, both $\mathcal{F}(0)$ and $\mathcal{F}(1)$ have countable chains. Let β_n, $n \in \omega$, be the sequence from the proof of Theorem 3.2:

$$\beta_n = 010^{2^n}1^{2^n}0^{3^n}1^{3^n} \ldots 0^{k^n}1^{k^n} \ldots .$$

Using this sequence, construct the following new sequence $\{\beta_n \oplus 1^{\omega}\}_{n \in \omega}$. Each string $\beta_n \oplus 1^{\omega}$ belongs to the filter $\mathcal{F}(0)$. In the same manner as in the proof of Theorem 3.2 one can show that $\{\beta_n \oplus 1^{\omega}\}_{n \in \omega} \subset \mathcal{F}(1)$ is an antichain. Similarly $\{\beta_n \oplus 0^{\omega}\}_{n \in \omega} \subset \mathcal{F}(0)$ is an antichain. $\dashv$

COROLLARY 3.18. *The ideal $\mathcal{I}(u)$ has an infinite antichain consisting of minimal elements. Moreover, the ideal does not have maximal elements.*

PROOF. The minimal elements constructed in Theorem 3.3 are clearly in the ideal $\mathcal{I}(u)$. For the second part we adapt the proof of the Density Theorem. Assume that α is a maximal element. We can represent α as follows:

$$\alpha = 0^{n_0}u_01^{n_1}u_10^{n_2}u_21^{n_3}u_30^{n_4}u_41^{n_4} \ldots ,$$

where $n_0 < n_1 < n_2 < \cdots$. Note that $\alpha <_{QI} (01)^{\omega}$. If we apply the proof of the Density Theorem directly, then we might produce γ such that $\alpha <_{QI} \gamma <_{QI} (01)^{\omega}$ with $\gamma \notin \mathcal{I}(u)$. To guarantee that $\gamma \in \mathcal{I}(u)$ we need to slightly change the proof of the Density Theorem. Namely, every time when γ copies a part of α, it should include a 0-block and also a 1-block whose lengths are greater that the lengths of all 0-blocks and 1-blocks in γ that has been built so far. It is clear that this condition can be satisfied. This puts γ into the ideal $\mathcal{I}(u)$. $\dashv$

3.4. Quasi-isometry on eventually periodic strings. In this section we describe the isomorphism type of the partial order $\leqslant_{QI}$ restricted to the set of all eventually periodic strings of Σ^ω. These are the strings that have a particularly simple structure:

DEFINITION 3.19. A string $\alpha \in \Sigma^\omega$ is *eventually periodic* if there are finite words $x, u \in \Sigma^\star$ such that $\alpha = xuuu \dots$. We write such α as xu^ω. Call u the period of α. If x is the empty string then α is *periodic*.

Let α be a string (that can possibly be finite). Recall that $Cl(\alpha)$ denotes the set of all colors that appear in α, that is,

$$Cl(\alpha) = \{\sigma \in \Sigma \mid \exists i(\alpha(i) = \sigma)\}.$$

One obvious quasi-isometry invariant of the string $\alpha \in \Sigma^\omega$ is the set of all colors that appear in α. Namely, if $f : \alpha \to \beta$ is a quasi-isometry then $Cl(\alpha) \subseteq Cl(\beta)$. This implies that if $\alpha \sim_{QI} \beta$ then $Cl(\alpha) = Cl(\beta)$.

THEOREM 3.20. *For eventually periodic words* α, β *we have* $\alpha \leqslant_{QI} \beta$ *iff there are* $x, y, u, v \in \Sigma^\star$ *such that* $\alpha = xu^\omega$, $\beta = yv^\omega$, $Cl(x) \subseteq Cl(y)$, *and* $Cl(u) \subseteq Cl(v)$.

PROOF. If $\alpha \leqslant_{QI} \beta$ then it is easy to select finite strings x, y, u, v that satisfy the statement of the theorem.

Assume that there are $x, y, u, v \in \Sigma^\star$ such that $\alpha = xu^\omega$, $\beta = yv^\omega$, $Cl(x) \subseteq Cl(y)$ and $Cl(u) \subseteq Cl(v)$. Construct a quasi-isometry f from α to β as follows. First, map the prefix x into the prefix y so that colors are preserved. Secondly, we consider the set $X = \{x_1, \dots, x_k\}$ of all distinct colors that appear in u. Now map every x_i-colored position in the kth copy of u in the string α into an x_i colored position in the kth copy of v in the string β. Set $A = \max\{|x|, |y|, |u|, |v|\}$ and $A = B = C$. It is clear that f preserves colors and the quasi-isometry inequality between the distances $d(x, y)$ and $d(f(x), f(y))$ with quasi-isometry constants A, B, C as required. $\dashv$

Consider the following two sets:

$$E_P = \{[\alpha] \mid \alpha \text{ is eventually periodic string over } \Sigma\}, \text{ and}$$
$$P = \{[\alpha] \mid \alpha \text{ is periodic string over } \Sigma\}.$$

So, we have two partially ordered sets $\mathcal{E}_P$ and $\mathcal{P}$ obtained by restricting the order $\leqslant_{QI}$ on the sets E_P and P, respectively.

Let $\mathcal{X}$ be the partially ordered set of all nonempty subsets of Σ with respect to set-theoretic inclusion. Let $\mathcal{Y}$ be the partially ordered set $\mathcal{X} \times \mathcal{X}$, where the order is component-wise order. We have the following corollary that fully characterize the isomorphism types of $\mathcal{E}_P$ and $\mathcal{P}$.

COROLLARY 3.21. *The following is true*:

1. *The partially ordered set* $\mathcal{E}_P$ *is isomorphic to* $\mathcal{X} \times \mathcal{X}$.
2. *The partially ordered set* $\mathcal{P}$ *is isomorphic to* $\mathcal{X}$.

PROOF. Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_n\}$ be elements of $\mathcal{X}(\Sigma)$. Associate with the pair (A, B) the infinite string $a_1 \dots a_k (b_1 \dots b_n)^\omega$. This establishes the mapping from $\mathcal{X} \times \mathcal{X}$ into $\mathcal{E}_P$. By Theorem 3.20 this is an isomorphism from $\mathcal{X} \times \mathcal{X}$ onto $\mathcal{E}_P$. The second part is proved similarly. $\dashv$

Let $f : \alpha \to \beta$ be a quasi-isometric map., and let Φ be a property. Say that f is *eventually*Φ if there is an index i such that f restricted to the suffix $\alpha_i = \alpha(i)\alpha(i+1)\alpha(i+2)\ldots$ satisfies Φ. A quasi-isometry f does not need to be eventually order preserving map; neither f needs to be eventual embedding (that is, eventually injective map). Now we show that if α and β are eventually periodic words, then f can be computable eventually order preserving injective map.

THEOREM 3.22. *For eventually periodic strings α and β such that $\alpha \leqslant_{QI} \beta$ there exists a computable quasi-isometric map $f_\alpha : \alpha \to \beta$ which is eventually order preserving and eventually injective.*

PROOF. Let $\alpha = xuuu\ldots$ and $\beta = yvvv\ldots$. Assume that $Cl(x) \subseteq Cl(y)$ and $Cl(u) \subseteq Cl(v)$. Our desired mapping f_α maps x into y by preserving colors. This is where f_α need not be order preserving embedding. So, it suffices to construct a computable quasi-isometric embedding from $\alpha_1 = u^\omega$ to $\beta_1 = v^\omega$. Let $X = \{x_1, \ldots, x_k\}$ be the set of all colors that appear in u, and hence in v. Let a_i be the number of times that color x_i appears in u. Clearly $a_i \geqslant 1$. Consider the new string v_1 obtained by writing v exactly $a_1 + \ldots + a_n$ times, that is, $v_1 = (v)^{a_1+a_2+\ldots+a_n}$. now it is easy to see that there exists an embedding f' from u into v_1 that preserves colors and the order. We now easily propagate this mapping f' to a color preserving embedding f_α from α_1 into β_1. The map $f_\alpha : \alpha \to \beta$ thus built is a desired quasi-isometry. $\dashv$

§4. Component-wise reducibility.

In this section we recast the definition of quasi-isometry, and give an equivalent but intuitively clearer description of the $\leqslant_{QI}$-reducibility. Our intuitively clearer description is called component-wise reducibility. A string α is *component-wise reducible* to string β if we can partition $\alpha = u_1 u_2 \ldots$ and $\beta = v_1 v_2 \ldots$ so that the sizes of the segments u_i, v_j are uniformly bounded, and for each i all colors present in u_i are also present in v_i. It is easy to see that if α is component-wise reducible to β then there is a quasi-isometry from α to β. We will show that the converse is also true. Our tool will be the decomposition theorem (which is interesting in its own right) that decomposes any quasi-isometric function $f : \alpha \to \beta$ into a finite sequence consisting of (1) more elementary quasi-isometric functions and (2) monotonic quasi-isometric embeddings.

4.1. Decomposing quasi-isometries.

In this section we want to understand the quasi-isometric mappings syntactically. For instance, we want to know to what extend quasi-isometric mapping can be replaced by monotonic and/or injective quasi-isometries.

LEMMA 4.1. *Let $f : \alpha \to \beta$ be a quasi-isometry. We can decompose this map into a sequence $\alpha \xrightarrow{f_1} \gamma \xrightarrow{f_2} \beta$ of quasi-isometries such that f_1 is an injection and f_2 is a monotonic surjection. Moreover, γ is quasi-isometric to β.*

PROOF. We define γ using β and the quasi-isometry f. For each position i in string β consider $f^{-1}(i)$. If $f^{-1}(i) = \emptyset$ then replace i with the position $(i, \emptyset)$ and color this position with $\beta(i)$. Otherwise, let $j_{i,1} < \cdots < j_{i,k}$ be all f-preimages of i in the string α listed in their natural order. Replace the position i in β with positions

$$(i, 1) < (i, 2) < \cdots < (i, k),$$

and color positions in this interval by color $\beta(i)$. Thus, we have defined the new string γ. Let us map elements $(i,1)$, $(i,2)$, ..., (i,k) in γ into the position i of β. If i has no f-preimage then map $(i,\emptyset)$ on i. This determines the function f_2 stated in the lemma. Clearly, f_2 is a surjective quasi-isometry. The function f_1 is obtained from f as follows. For every $i \in \beta$ if $f^{-1}(i) \neq \emptyset$, then map elements $j_{i,1}, \dots, j_{i,k}$ on the elements $(i,1)$, $(i,2)$, ..., (i,k), respectively. The mapping is an injective quasi-isometry from α into γ. Finally, it is also easy to see that $\beta \leqslant_{QI} \gamma$ through the map $i \to (i,\emptyset)$ if $f^{-1}(i) = \emptyset$, and $i \to (i,1)$ if $f^{-1}(i) \neq \emptyset$. ⊣

It is not too hard to see that the process described in the proof can be carried out uniformly on α, β and f.

COROLLARY 4.2. *There exists a procedure that given strings α, β and a quasi-isometry $f : \alpha \to \beta$ produces the decomposition $\alpha \xrightarrow{f_1} \gamma \xrightarrow{f_2} \beta$ as in Lemma 4.1. In particular, if α, β and f are computable then so is the decomposition.* ⊣

LEMMA 4.3. *Let $f : \alpha \to \beta$ be a quasi-isometry. We can decompose this map into a sequence $\alpha \xrightarrow{f_1} \gamma \xrightarrow{f_2} \beta$ of quasi-isometries such that f_1 is a surjection and f_2 is a monotonic injection. Moreover, γ is quasi-isometric to β.*

PROOF. Let γ be a string that is obtained by eliminating each position of β that is not in the image of f, and construct f_1 and f_2 in an obvious way. It is clear that $\beta \leqslant_{QI} \gamma$. ⊣

We note that just as the corollary above, the uniform version of this lemma is also true.

DEFINITION 4.4. We say that a map $h : \alpha \to \beta$ is an *atomic crossing map* if the following holds. We have a sequence (possibly finite) $(a_0, b_0), (a_1, b_1), \dots$ of pairs of positions in α such that for some C and all i we have $(a_i < b_i < a_{i+1})$ and $(b_i - a_i \leqslant C)$ and

$$h(a) = \begin{cases} b_i & \text{if } a = a_i, \\ a_i & \text{if } a = b_i, \\ a & \text{otherwise.} \end{cases}$$

Clearly, every atomic crossing map is bijective.

DEFINITION 4.5. Let $f : \alpha \to \gamma$ be a quasi-isometry. The *cross-over constant* of f is the maximal D_f that witnesses a cross-over. In other words, the value of D_f equals to

$$\max\{D \mid \exists n, m(n < m \ \& \ f(m) \leqslant f(n) \ \& \ D = f(n) - f(m))\}.$$

Note that if the cross-over constant D_f equals 0 then the quasi-isometry f is a monotonic nondecreasing function.

LEMMA 4.6. *Let $f : \alpha \to \gamma$ be an injective quasi-isometry such that $D_f > 0$. We can decompose this map into a sequence $\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma$ of quasi-isometries such that f_1 is an injection, $D_{f_1} < D_f$, f_2 is a finite composition of atomic crossing maps, and γ is quasi-isometric to γ_1.*

PROOF. let us list all pairs $(a_0, b_0), (a_1, b_1), \dots$ that constitute maximal crossings via f, that is, $a_i < b_i$ and $f(a_i) - f(b_i) = D_f$. We list these pairs in the way that

$b_i < a_{i+1}$ holds for each i, which is possible as we prove below. Without loss of generality let $a_i \leqslant a_{i+1}$ for each i. First we have $a_i \neq a_j$ for every $i \neq j$, as otherwise we have $f(b_i) = f(b_j) = f(a_i) - D_f$, which contradicts injectivity of f; in a similar way we can show $b_i \neq b_j$ for every $i \neq j$. Next we have $f(a_i) < f(a_j)$ for each $i < j$, as otherwise $f(a_i) - f(b_j) > f(a_j) - f(b_j) = D_f$ while $a_i < a_j < b_j$, which contradicts the fact that D_f is the cross-over constant of f; in a similar way we can show $f(b_i) < f(b_j)$ for each $i < j$. Lastly we have $b_i < a_{i+1}$ for every i, as otherwise we have $f(a_{i+1}) - f(b_i) > f(a_i) - f(b_i) = D_f$ while $a_{i+1} < b_i$, again which contradicts the fact that D_f is the cross-over constant of f.

Now we construct f_1 by "undoing" each maximal crossing in f. It is done by a function g that swaps the positions $f(a_i)$ and $f(b_i)$ in γ for each i. Formally, we define g as the following function:

$$g(a) = \begin{cases} f(b_i) & \text{if } a = f(a_i), \\ f(a_i) & \text{if } a = f(b_i), \\ a & \text{otherwise.} \end{cases}$$

Notice that g satisfies all conditions of atomic crossing maps except that the sequence that defines g, namely $(f(b_0), f(a_0)), (f(b_1), f(a_1)), \ldots$, may not satisfy $f(a_i) < f(b_{i+1})$ for some i. In what follows we decompose g into a finite composition of atomic crossing maps.

If the sequence $(f(a_0), f(b_0)), (f(a_1), f(b_1)), \ldots$ is finite, then an aforementioned decomposition of g is clearly possible. If it is infinite, then for each $k = 0, 1, \ldots, D_f - 1$ define a sequence $(a_0^{(k)}, b_0^{(k)}), (a_1^{(k)}, b_1^{(k)}), \ldots$ by

$$(a_i^{(k)}, b_i^{(k)}) = (f(b_{iD_f + k}), f(a_{iD_f + k})).$$

As we have $f(b_{i+1}) > f(b_i)$ for each i, we have $a_{i+1}^{(k)} \geqslant a_i^{(k)} + D_f = b_i^{(k)}$; we also have $a_{i+1}^{(k)} \neq b_i^{(k)}$ because f is injective. Thus we have $a_{i+1}^{(k)} > b_i^{(k)}$ for each i, and the sequence $(a_0^{(k)}, b_0^{(k)}), (a_1^{(k)}, b_1^{(k)}), \ldots$ defines an atomic crossing map, say g_k. Clearly, we have $g = g_{D_f - 1} \circ \cdots \circ g_0$.

Now let γ_1 be a string that is obtained by swapping each color of γ at the positions $f(a_i)$ and $f(b_i)$, and let $f_1 = g \circ f$ and $f_2 = g^{-1} = g$. Below we prove $D_{f_1} < D_f$ holds; all other desired properties of f_1, f_2 and γ_1 are easy to derive from their construction.

Below we prove $f_1(a) - f_1(b) < D_f$ for each pair (a, b) of positions such that $a < b$. The inequality holds if $(a, b) = (a_i, b_i)$ for some i, as every maximal crossing in f is undone by g. Otherwise we have $f(a) - f(b) < D_f$, so it suffices to show $f_1(a) - f_1(b) < D_f$ for each (a, b) such that $a < b$ and $f_1(a) - f_1(b) > f(a) - f(b)$. Notice that we have $f_1(a) = f(b_i) < f(a)$ if $a = a_i$, $f_1(a) = f(a_i) > f(a)$ if $a = b_i$, or $f_1(a) = f(a)$ otherwise. From this observation, it suffices to verify the cases in which $b = a_j$ holds for some j, or $a = b_i$ holds for some i. We verify the former case as follows; the latter can be checked in a similar way.

- If $a = a_i$ for some i, then we have $i < j$ and hence $f_1(a) - f_1(b) = f(b_i) - f(b_j) < 0$.
- If $a = b_i$ for some i, then $f_1(a) - f_1(b) = f(a_i) - f(b_j) < f(a_i) - f(b_i) = D_f$.
- Otherwise, we have $f_1(a) - f_1(b) = f(a) - f(b_j)$. Notice that $a < b = a_j < b_j$, and hence $f(a) - f(b_j) \leqslant D_f$ holds; we also have $f(a) - f(b_j) \neq D_f$, as the equality holds only if $(a, b_j) = (a_i, b_i)$ for some i. Thus we have $f_1(a) - f_1(b) < D_f$. $\dashv$

COROLLARY 4.7. *Let* $f : \alpha \to \gamma$ *be an injective quasi-isometry such that* $D_f > 0$. *We can decompose this map into a sequence* $\alpha \xrightarrow{f_1} \alpha_1 \xrightarrow{f_2} \gamma$ *of quasi-isometries such that* f_1 *is a finite composition of atomic crossing maps,* f_2 *is an injection,* $D_{f_2} < D_f$, *and* α *is quasi-isometric to* α_1.

PROOF. Use a construction similar to the one from the proof of Lemma 4.6, but swapping colors of α rather than γ. ⊣

From the last lemma, we can construct a uniform procedure that decomposes an injective quasi-isometry into monotonic injective and bijective ones. Unlike Corollary 4.2, the procedure also depends on quasi-isometry constants.

COROLLARY 4.8. *There exists a uniform procedure that given strings* α *and* γ, *an injective quasi-isometry* $f : \alpha \to \gamma$ *and the quasi-isometry constants* (A, B, C) *of* f, *produces a decomposition of* f *into a sequence* $\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma$ *of quasi-isometries such that* f_1 *is a monotonic injection,* f_2 *is a finite composition of atomic crossing maps, and* γ *is quasi-isometric to* γ_1.

PROOF. First, compute an upper bound $\hat{D}_f$ of the cross-over constant D_f. It can be computed from the quasi-isometry constants (A, B, C) via Lemma 2.1. Then let $\gamma = \gamma_{\hat{D}_f}$, $f = f_{\hat{D}_f}$, and for each $k = \hat{D}_f, \dots, 1$, decompose f_k into a sequence $\alpha \xrightarrow{f_{k-1}} \gamma_{k-1} \xrightarrow{g_{k-1}} \gamma_k$ as in the proof of Lemma 4.6 that is modified in the following way: at the beginning of the procedure, we list all pairs $(a_0, b_0), (a_1, b_1), \dots$ such that $a_i < b_i$ and $f(a_i) - f(b_i) = k$ rather than $f(a_i) - f(b_i) = D_f$. As $k \geqslant D_{f_k}$ holds for each k, the resulting list is either empty, or the same as the one via the original procedure. Notice that in the former case the modified procedure returns $\gamma_{k-1} = \gamma_k$, $f_{k-1} = f_k$ and $g_{k-1} = \mathrm{id}$. Through this we obtain the sequence $\alpha \xrightarrow{f_0} \gamma_0 \xrightarrow{g} \gamma$, where $g = g_{\hat{D}_f - 1} \circ \cdots \circ g_0$; this is a decomposition with the desired properties. ⊣

From Corollary 4.7, we can obtain a decomposition procedure that corresponds to Corollary 4.8.

COROLLARY 4.9. *There exists a uniform procedure that given strings* α *and* γ, *an injective quasi-isometry* $f : \alpha \to \gamma$ *and the quasi-isometry constants* (A, B, C) *of* f, *produces a decomposition of* f *into a sequence* $\alpha \xrightarrow{f_1} \alpha_1 \xrightarrow{f_2} \gamma$ *of quasi-isometries such that* f_1 *is a finite composition of atomic crossing maps,* f_2 *is a monotonic injection, and* α *is quasi-isometric to* α_1. ⊣

In Corollary 4.7, we have $f_1 = \mathrm{id}$ if f is bijective. Thus we have the following yet another corollary.

COROLLARY 4.10. *Any bijective quasi-isometry can be decomposed into a finite sequences of atomic crossing maps.* ⊣

We combine the results above into the following theorem.

THEOREM 4.11 (Decomposition Theorem). *There exists a procedure that given* (A, B, C)-*quasi-isometry* $f : \alpha \to \beta$ *produces a decomposition of* f *into quasi-isometries* $\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma_2 \xrightarrow{f_3} \beta$ *with one of the following sets of properties* (*and each set of properties can be realized*):

1. f_1 *is a bijection, f_2 is a monotonic injection, and f_3 is a monotonic surjection.*
2. f_1 *is a monotonic injection, f_2 is a bijection, and f_3 is a monotonic surjection.*
3. f_1 *is a bijection, f_2 is a monotonic surjection, and f_3 is a monotonic injection.*

PROOF. The case 1 and 2 are derived from Corollary 4.2, 4.8 and 4.7. To show the case 3, first apply Lemma 4.3 and then Corollary 4.2 to the surjection; notice, from the construction, that Lemma 4.1 decomposes a surjection into a bijection and a monotonic surjection.   ⊣

From all of the above, we thus have the following corollary.

COROLLARY 4.12. *If $\alpha \leqslant_{QI} \beta$, then there is $\beta' \sim_{QI} \beta$ so that $\alpha \leqslant_{QI} \beta'$ via strictly monotonic quasi-isometry.*

PROOF. Recall that the second decomposition in Theorem 4.11 is derived from Corollary 4.2 and 4.8. As a property given from these corollaries, we have $\gamma_1 \sim_{QI} \gamma_2$ and $\gamma_2 \sim_{QI} \beta$. Now let $\beta' = \gamma_1$.   ⊣

4.2. Component-wise reducibility. In this section we formulate an equivalent more intuitive condition for quasi-isometry between colored metric spaces. Recall that $Cl(u)$ denotes the set of all colors present in the string u.

DEFINITION 4.13. Say α is *component-wise reducible* to β, written $\alpha \leqslant_{CR} \beta$, if we can partition α and β as

$$\alpha = u_1 u_2 \dots \text{ and } \beta = v_1 v_2 \dots$$

so that $Cl(u_i) \subseteq Cl(v_i)$ for all i and $|u_j|, |v_j|$ are uniformly bounded by a constant C. Call these representations of α and β *witnessing partitions* and intervals u_i and v_i *partitioning intervals*. If the sequence $u_1, u_2, \dots$ is computable then we call this a computable partitioning of α.

It is clear that $\alpha \leqslant_{CR} \beta$ implies $\alpha \leqslant_{QI} \beta$: any color-preserving map that maps each interval u_i in α to v_i in β is a quasi-isometry. Now our goal is to show that $\leqslant_{QI}$ implies $\leqslant_{CR}$. One of the key lemmas is the following.

LEMMA 4.14. *Suppose $\alpha \leqslant_{QI} \beta$ via an atomic crossing map $f : \alpha \to \beta$ and $\beta \leqslant_{CR} \gamma$. Then $\alpha \leqslant_{CR} \gamma$.*

PROOF. Let $\beta = v_1 v_2 \dots$ and $\gamma = w_1 w_2 \dots$ be witnessing partitions for $\beta \leqslant_{CR} \gamma$. We initially partition α in the following way:

$$\alpha = u_1 u_2 \dots,$$

where $|u_n| = |v_n|$ for all n. In what follows we show that if we merge some u_n's and v_n's in a careful way, then we obtain witnessing partitions of α and β. From these partitions we can then derive witnessing partitions of α and γ.

Let $(a_1, b_1), (a_2, b_2), \dots$ be the sequence of crossing pairs that define the quasi-isometric atomic crossing map $f : \alpha \to \beta$. We can assume that no interval u_i contains one of these crossing pairs. Indeed, if u_i contains a crossing pair (a, b) then we can simply change α by interchanging colors at positions a and b. Similarly, if (a, b) is a crossing pair and $\alpha(a) = \alpha(b)$ we can change the map f be sending a to

$$\beta = \quad 0\,1\ \big|\ 0\,2\ \big|\ 0\,2\ \big|\ 0\,2\ \ldots$$

$$\alpha = \quad 0\,0\ \big|\ 1\,0\ \big|\ 2\,0\ \big|\ 2\,0\ \ldots$$
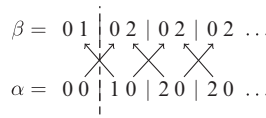
FIGURE 3. With the condition 1 only, we may have a problem.

a and b to b. The output of the process described is a new atomic crossing map $f' : \alpha' \to \beta$, where $\alpha' \sim_{QI} \alpha$ is obtained from α by switching colors, and f' is obtained from f by switching images in every crossing pair (a, b) that belongs to some u_i.

Assume that (a, b) is a crossing pair such that $a \in u_i$ and $b \in u_{i+j}$. In this case we say that u_i contains the left-end of the crossing pair (a, b) and u_{i+j} contains the right-end of the crossing pair. If $j > 1$, we can replace u_i with the segment $u_i u_{i+1} \ldots u_{i+j-1}$ and v_i with $v_i v_{i+1} \ldots v_{i+j-1}$. This changes the partition of α and β but does not change the quasi-isometry map f. Also, the sizes of the new partitioning intervals are uniformly bounded. Thus, we can always assume that the partition $\alpha = u_1 u_2 u_2 \ldots$ is such that $|u_i| = |v_i|$ for all i, all crossing pairs occur between neighboring intervals only, and $\alpha(a) \neq \alpha(b)$ for all crossing pairs (a, b).

Starting from $n = 1$, we incrementally check if we should merge u_n and u_{n+1} (and thus v_n and v_{n+1}). To assure that they can be kept separated, we need to check the following conditions. Note that the second condition below is indispensable. Figure 3 illustrates a situation in which the lack of the condition causes a problem.

1. Let $m < n$ be the largest number that is declared as a separated partition (i.e., u_m and u_{m+1} are kept separated and $u_{m+1} \ldots u_n$ are all merged). Then $Cl(u_{m+1} \ldots u_n) \subseteq Cl(v_{m+1} \ldots v_n)$.
2. If a crossing pair (a_k, b_k) belongs to $u_n u_{n+1}$, then $Cl(b_k)$ appears in v_{n+1+i} for some sufficiently small i.

This idea is realized as the following procedure which generates a merging policy $\varphi : \omega \to \{0, 1\}$: here, φ indicates that u_n and u_{n+1} should be merged for each $n \in \varphi^{-1}(0)$, and should be kept separated for each $n \in \varphi^{-1}(1)$. For instance, $\varphi(n) = 0$, $\varphi(n + 1) = 0$ and $\varphi(n + 2) = 1$ means that u_n, u_{n+1}, u_{n+2} are merged and u_{n+2} is the last interval merged with u_n.

1. Set $n = 1$, $i = 0$, and let $\varphi : \omega \to \{0, 1\}$ be a currently undefined function. Here n and i represent the following situation of the procedure: we have declared that u_{n-1} and u_n should be kept separated, and $u_n, \ldots, u_{n+i}$ should be merged. Formally, through the procedure it always holds that $\varphi(n - 1) = 1$ and $\varphi(n) = \ldots = \varphi(n + i - 1) = 0$, and φ is defined over $\{1, \ldots n + i - 1\}$.
2. If no crossing pair exists between u_{n+j} and u_{n+j+1} for some $j \in \{i, \ldots, i + |\Sigma|\}$, then let $\varphi(k) = 0$ for each k such that $n + i \leqslant k \leqslant n + j - 1$, $\varphi(n + j) = 1$, and $n = n + j + 1$, $i = 0$. In other words, we declare that $u_{n+i}, \ldots, u_{n+j}$ should be merged (together with $u_n, \ldots, u_{n+i-1}$, which have been already declared to be merged) while u_{n+j} and u_{n+j+1} should be kept separated.
3. Otherwise, we check if

$$Cl(u_n \ldots u_{n+j}) \subseteq Cl(v_n \ldots v_{n+j}) \text{ for all } j \in \{i, \ldots, i + |\Sigma|\} \quad (\star).$$

4. If ($\star$) holds, then find a letter that occurs twice or more in positions $b_k, \ldots, b_{k+|\Sigma|}$, where b_k belongs to u_{n+i+1}. Let $b_{k+\ell}$ be the earliest position among them, and set

$$\varphi(n+i) = \ldots = \varphi(n+i+\ell-1) = 0, \; \varphi(n+i+\ell) = 1,$$

and $n = n+i+\ell+1$, $i = 0$.

5. If ($\star$) does not hold, then take (say, the largest) $j \in \{i, \ldots, i+|\Sigma|\}$ such that $Cl(u_n \ldots u_{n+j}) \nsubseteq Cl(v_n \ldots v_{n+j})$, let $\varphi(n+i) = \ldots = \varphi(n+i+j) = 0$ and let $i = j+1$.

6. Go to 2.

Notice that, in step 4 of the procedure, the pigeonhole principle guarantees that it is always possible to find such multiple occurrences. Now collect all positions for which φ in the above procedure eventually assigns 1, and let $\{n_k\}$ be its ordered enumeration. The following shows that the merging policy via φ indeed gives us witnessing partitions.

CLAIM. *The sequence $\{n_k\}$ is an infinite sequence $\{n_k\}_{k \in \omega}$, and there is a bound D such that $n_{k+1} - n_k \leqslant D$ for all $k \in \omega$. Furthermore we have $Cl(u_{n_{k-1}+1} \ldots u_{n_k}) \subseteq Cl(v_{n_{k-1}+1} \ldots v_{n_k})$ for all $k \in \omega$, where we assume $n_{-1} = 1$.* ⊣

The color inclusion part of the claim is immediate to show, as it is assured in the step 3 of the procedure. Thus it suffices to show that the condition checked in 3 is always true if $i \geqslant D'$ for some constant D'.

For its proof, let n and i be given, where $n = n_k + 1$ for some k, a crossing pair (a_{l+j}, b_{l+j}) lies between u_{n+j} and u_{n+j+1} for each $j \in \{i, \ldots, i+|\Sigma|\}$, and assume $Cl(u_n \ldots u_{n+j}) \nsubseteq Cl(v_n \ldots v_{n+j})$ for some $j \in \{i, \ldots, i+|\Sigma|\}$. This holds only if either of the following is true: (1) there is a crossing pair (a_h, b_h) that lies between u_{n-1} and u_n, and $\alpha(b_h)$ does not occur in $v_n \ldots v_{n+j}$; or (2) $\alpha(a_{l+j})$ does not occur in $v_n \ldots v_{n+j}$.

The case (1) occurs only if $j < |\Sigma| - 1$, as the choice of n_k in step 4 of the procedure assures that $\alpha(b_h) = \alpha(b_{h+h'})$ for some $h' \leqslant |\Sigma| - 1$. The case (2) occurs only for finitely many j's (indeed it's up to $|\Sigma| - 1$ times), as in such a case $v_n \ldots v_{n+j+1}$ should contain strictly more colors than $v_n \ldots v_{n+j}$. The whole argument shows that the condition ($\star$) in step 3 of the procedure above is always true if $i \geqslant |\Sigma| \cdot (|\Sigma| + 1)$ holds; the case (1) does not occur if $i \geqslant |\Sigma| - 1$, and after that the case (2) occurs only up to $|\Sigma| - 1$ times, where each occurrence increases i to up to $|\Sigma| + 1$.

THEOREM 4.15. $\alpha \leqslant_{QI} \beta$ *implies* $\alpha \leqslant_{CR} \beta$.

PROOF. Let $f : \alpha \to \beta$ be a quasi-isometry. By Theorem 4.11 and Corollary 4.10, it can be decomposed into quasi-isometry maps $\alpha \xrightarrow{f_1} \gamma \xrightarrow{f_2} \beta$, where f_1 is a finite concatenation of atomic crossing quasi-isometries and f_2 is monotonic. For any monotonic quasi-isometric map $g : \alpha' \to \beta'$ we easily have $\alpha' \leqslant_{CR} \beta'$: Indeed, enumerate all elements of the image of g as $\{b_n\}_{n \in \omega}$, where $b_n < b_{n+1}$ for all n, let $u_n = g^{-1}(b_n)$ and $v_n = [b_n, b_{n+1} - 1]$. Then $\alpha' = u_1 u_2 \ldots$ and $\beta' = v_1 v_2 \ldots$ are witnessing partitions. Hence we have $\gamma \leqslant_{CR} \beta$, and Lemma 4.14 completes the proof. ⊣

From the proofs of Theorem 4.15 and the Decomposition Theorem 4.11, we obtain the following corollary:

COROLLARY 4.16. *Let α and β be computable strings. Then there exists a computable quasi-isometry $f : \alpha \to \beta$ if and only if there exists a computable partition of α and β witnessing component-wise reducibility of α to β.*

As an application of Theorem 4.15, we can provide an alternative proof for constructing minimal elements in the partial order Σ_{QI}^{ω}:

COROLLARY 4.17. *Let $\{a_n\}_{n\in\omega}$ be a sequence such that $\lim_{n\to\infty} a_n = \infty$. Then the large scale geometry of the string $\alpha = 0^{a_0} 1^{a_1} 0^{a_2} 1^{a_3} ... 0^{a_{2k}} 1^{a_{2k+1}} ...$ is a minimal element in any partial order Σ_{QI}^{ω} with $0, 1 \in \Sigma$.*

PROOF. Suppose $\beta \leqslant_{CR} \alpha$, and let $\alpha = u_1 u_2 ...$ and $\beta = v_1 v_2 ...$ be witnessing partitions. Then for sufficiently large $N \in \omega$ we have the following: for each $i > N$, if $Cl(u_i) = \{0, 1\}$ then $Cl(u_{i-1}) = \{0\}$ and $Cl(u_{i+1}) = \{1\}$, or vice versa. Now modify the witnessing partitions as follows: for each $i > N$ such that $Cl(u_i) = \{0, 1\}$, if $Cl(v_i) = \{0\}$ or $Cl(v_i) = \{1\}$, then either move the boundary between v_{i-1} and v_i (respectively, between u_{i-1} and u_i) one position to the left, or move the boundary between v_i and v_{i+1} (respectively between u_i and u_{i+1}) one position to the right, depending on whether v_{i-1} or v_{i+1} contains the color missing in v_i. Concatenating $u_1, ... , u_{N-1}$ and $v_1, ... , v_{N-1}$ we have witnessing partitions of α and β.   ⊣

4.3. Refinement of component-wise reducibility.
One could attempt to go further, namely say α and β are *color-equivalent* if they are component-wise reducible to each other via the *same* witnessing partitions. A formal definition is this:

DEFINITION 4.18. We say that α and β are *color-equivalent*, written $\alpha \sim_{cl} \beta$, if we can partition α and β as

$$\alpha = u_1 u_2 ... \text{ and } \beta = v_1 v_2 ...$$

such that $Cl(u_i) = Cl(v_i)$ for all i and $|u_j|, |v_j|$ are uniformly bounded by a constant C. Call these presentations of α and β *witnessing partitions* and intervals u_i and v_i *partitioning intervals*.

It is clear that $\alpha \sim_{cl} \beta$ implies that $\alpha \sim_{CR} \beta$ (and hence $\alpha \sim_{QI} \beta$). A natural question arises if $\alpha \sim_{QI} \beta$ implies their color-equivalence. Below, we give a negative answer to this question.

PROPOSITION 4.19. *Suppose $|\Sigma| \geqslant 3$. Then there are sequences α and β such that α and β are quasi-isometric via monotonic embeddings but α and β are not color-equivalent.*

PROOF. Consider a 3 letter alphabet $\{0, 1, *\}$. We define strings $\alpha, \beta \in \Sigma^{\omega}$ as follows. We first define β by:

$$\beta = 010 * \ \ 0 * 1 * 0 * * * \ \ ... \ 0(*)^n 1(*)^n 0(*)^{2n+1} \ ...$$

Thus, the nth-part of β is the string $0(*)^n 1(*)^n 0(*)^{2n+1}$. The string α is obtained from β by omitting the second occurrence of 0 in the nth-substring of β. So, α can be written as

$$\alpha = 01 * \ \ 0 * 1 * * * * \ \ ... \ 0(*)^n 1(*)^n (*)^{2n+1} \ ...$$

Thus, the nth-substring of α is $0(*)^n 1(*)^n (*)^{2n+1}$.

Building the desired monotonic injective quasi-isometry $f : \alpha \to \beta$ is straightforward. The function f maps the nth-part $0(*)^n 1(*)^n(*)^{2n+1}$ of α in injective and monotonic manner into the nth-substring $0(*)^n 1(*)^n 0(*)^{2n+1}$ of β by simply omitting the second 0 from its range.

Defining monotonic and quasi-isometric embedding $g : \beta \to \alpha$ needs a little bit more care. The function g maps the nth-part $0(*)^n 1(*)^n 0(*)^{2n+1}$ of β by "stretching" it along $0(*)^{2n} 1(*)^{2n}(*)^{4n+1}$ $0(*)^{2n+1} 1(*)^{2n+1}(*)^{4n+2+1}$, the $(2n)$th- and $(2n+1)$th-parts of α. More formally, g is defined as follows:

- g maps the initial segment $0(*)^n 1(*)^n$ of the nth-part of β into the $(2n)$th-part $0(*)^{2n} 1(*)^{2n}(*)^{4n+1}$ of α.
- g maps the rest of the nth-part of β, that is $0(*)^{2n+1}$, into the $(2n+1)$th-part $0(*)^{2n+1} 1(*)^{2n+1}(*)^{4n+2+1}$ of α (by omitting 1 from the range of the function g).

Clearly, the function g can be built such that we have $d(i,j) \leqslant d(g(i), g(j)) \leqslant 6d(i,j)$ for all positions i, j in β. Thus g is a witness for quasi-isometric monotonic embedding from β into α.

Now we show that α and β are not color-equivalent. Assuming the contrary, suppose there are partitions $\alpha = u_1 u_2 \ldots$ and $\beta = v_1 v_2 \ldots$ that witness color-equivalence of α and β. Notice that the occurrence of 0 and 1 in α and β become more and more sparse as n increases, while $|u_i|$ and $|v_i|$ are uniformly bounded. Thus for sufficiently large i, it holds that each of u_i and v_i contains at most one 0 and/or 1.

Now suppose $0 \in Cl(u_i) \cap Cl(u_j)$ for $i < j$ with sufficiently large i, which implies $0 \in Cl(v_i) \cap Cl(v_j)$. Then because of the construction of α and the observation above (i.e., each of u_i and u_j contains at most one 0 and/or 1), there exists k such that $i < k < j$ and $1 \in Cl(u_k)$. On the other hand, for such i and j, there can be a case that $1 \notin Cl(v_l)$ for every $i < l < j$; this is clear from the construction of β, as the succession of its nth- and $(n+1)$th-substrings looks like

$$0 * \ldots * 1 * \ldots * \underbrace{0 * \ldots * 0}_{\text{No occurrence of 1}} * \ldots * 1 * \ldots 0 * \ldots * .$$

Here, again notice that each of v_i and v_j contains at most one 0 and/or 1 (in particular it does not contain multiple 0's). Thus for the k that satisfies the property above, there can be a case that $Cl(u_k) \neq Cl(v_k)$. This is a contradiction. ⊣

We say that a mapping $f : \alpha \to \beta$ is an *eventual embedding* if there exists an n such that f restricted to the domain $\{n+1, n+2, \ldots\}$ is an embedding. One can ask if color-equivalence between α and β implies that one can find quasi-isometries $f : \alpha \to \beta$ and $g : \beta \to \alpha$ such that f and g are eventual embedding. The next proposition gives an easy negative answer.

PROPOSITION 4.20. *There are color-equivalent strings α and β such that no eventual embedding witnesses quasi-isometry from α into β.*

PROOF. Consider the following colored metric spaces over the alphabet $\{0, 1\}$:

$$\alpha = 001001^2 001^3 \ldots 001^n \ldots \text{ and } \beta = 011011^2 011^3 \ldots 011^n \ldots .$$

One can construct witnessing partitions of α and β as follows, for example:

$$\alpha = 001|001|1|001|1|1|001|1|1|1\ldots$$
$$\beta = 011|011|1|011|1|1|011|1|1|1\ldots$$

We now prove that no eventual embeddings exist witnessing quasi-isometry from α into β. Let $f : \alpha \to \beta$ be an eventual embedding with quasi-isometry constants A, B, and C. Let i_n and $i_n + 1$ be the sequence of consecutive positions in α with color 0. Since f is a quasi-isometry we have

$$|f(i_n + 1) - f(i_n)| \leqslant A|i_{n+1} - i_n| + B = A + B.$$

There must exist a position i_n such that $f(i_n) < f(i_n + 1)$ and the number of consecutive 1's between $f(i_n)$ and $f(i_n + 1)$ in β is greater than $A + B$. Hence, $f(i_n + 1) - f(i_n) > A + B$. This is a contradiction. $\dashv$

§5. **Büchi automata and large scale geometries.** Let $L \subseteq \Sigma^\omega$ be a language. To simplify our arguments we assume that $\Sigma = \{0, 1\}$. The notion of quasi-isometry leads us to consider the quasi-isometry types of strings from L. Formally, we give the following definition:

DEFINITION 5.1. An *atlas* is a set of quasi-isometry types. In particular, the *atlas of the language L* is the set $[L] = \{[\alpha] \mid \alpha \in L\}$, where $[\alpha]$ is the quasi-isometry type of α.

A question of a general character is concerned with understanding the set $[L]$ given a description of L. In particular, a natural question is if one can decide that $[L_1] = [L_2]$ given descriptions of languages L_1 and L_2. In this section, we study the atlases defined by Büchi automata recognizable languages. We show that for Büchi automata recognizable languages L_1 and L_2 there is an efficient algorithm that decides if $[L_1] = [L_2]$. We recall basic definitions for Büchi automata.

DEFINITION 5.2. A *Büchi automaton* $\mathcal{M}$ is a quadruple (S, ι, Δ, F), where S is a finite set of *states*, $\iota \in S$ is the *initial state*, $\Delta \subset S \times \Sigma \times S$ is the *transition table*, and $F \subseteq S$ is the set of *accepting states*. The triples $(s, \sigma, s') \in \Delta$ are called transitions of $\mathcal{M}$.

A *run* of the automaton $\mathcal{M}$ on string $\alpha = \sigma_0 \sigma_1 \cdots \in \Sigma^\omega$ is a sequence of states $r = s_0, s_1, \ldots$ such that $s_0 = \iota$ and $(s_i, \sigma_i, s_{i+1}) \in \Delta$ for all $i \in \omega$. Note that $\mathcal{M}$ can have several (possibly none) runs on α. The run is *accepting* if the set $Inf(r) = \{s : \exists^\infty i(s_i = s)\}$ has a state from F. The automaton $\mathcal{M}$ accepts the string α if the automaton has an accepting run on α.

DEFINITION 5.3. The *language* accepted by the automaton $\mathcal{M}$, denoted $L(\mathcal{M})$, is the set of all infinite words accepted by $\mathcal{M}$. We call the atlas defined by the language $L(\mathcal{M})$ *Büchi recognizable atlas*.

EXAMPLE 5.4. Here we provide two examples.

1. Let L be the language $\{\alpha \mid \alpha$ has finitely many 1s$\}$. This is Büchi recognizable language. The atlas defined by L is a two-element set $\{[0^\omega], [10^\omega]\}$. The

complement $\bar{L}$ of L is also Büchi recognizable. The atlas defined by $\bar{L}$ is $[\Sigma^\omega] - \{[0^\omega], [10^\omega]\}$.

2. Let L be the language

$$\{\alpha \mid \text{the length of any 0-block in } \alpha \text{ is even}\}.$$

This is Büchi recognizable language. The atlas defined by L is $[\Sigma^\omega]$. The complement $\bar{L}$ of L is Büchi recognizable. The atlas of the complement $\bar{L}$ is also $[\Sigma^\omega]$.

We aim to describe the atlases defined by Büchi recognizable languages. For this we need to do some state space analysis of the automation $\mathcal{M}$.

Let $\mathcal{M} = (S, \iota, \Delta, F)$ be a Büchi automaton. The state space S of the automaton $\mathcal{M}$ can naturally be considered as a directed labeled graph. Namely, we put a directed edge from state s to S' and label the edge with $\sigma \in \Sigma$ if and only if (s, σ, s') is a transition of $\mathcal{M}$. So, we can apply the concepts of graph theory to the space S. A *loop* in $\mathcal{M}$ is a path L of states $s_1, \dots, s_{n+1}$ in the state space S such that $s_1 = s_{n+1}$. We say that a word $\sigma_1 \dots \sigma_n$ *labels* the loop L if $(s_i, \sigma_i, s_{i+1}) \in \Delta$ for $1 \leqslant i \leqslant n$. If all symbols σ_i are 0 only, then we say that the loop is a 0-*loop*. Similarly, if all symbols σ_i are 1 only, then we say that the loop is a 1-*loop*. In the loop L we do not require that the states are pairwise distinct. Any state in a loop is called a *loop state*.

By removing the labels from the edges in the space of transitions of S, we turn the state space S of the automaton $\mathcal{M}$ into a directed graph. Hence, we can write S as a disjoint union of its strongly connected components (scc*s*). Call a strongly connected component $X \subseteq S$ *nontrivial* if the cardinality $|X|$ of X is greater than 1. Thus, we write S as the union

$$T \cup S_1 \cup \dots \cup S_k,$$

where T is the union of trivial strongly connected components and each S_i is a nontrivial strongly connected component. Note that every successful run of the automaton $\mathcal{M}$ on every infinite string α eventually ends up in one of the strongly connected components S_i of the automaton $\mathcal{M}$. Consider Büchi automata $\mathcal{M}_1, \dots, \mathcal{M}_k$ such that the initial states and the state diagrams of $\mathcal{M}_i$ all coincide with that of $\mathcal{M}$ but $F_i = F \cap S_i$. It is clear that

$$(\star) \qquad L(\mathcal{M}) = L(\mathcal{M}_1) \cup \dots \cup L(\mathcal{M}_k).$$

Hence, describing the atlas $[L(\mathcal{M})]$ boils down to describing the atlases of $[L(\mathcal{M}_i)]$, $i = 1, \dots, k$.

Assume that a string α is accepted by $\mathcal{M}_i$. Then we can write α as $v\alpha'$ so that an accepting run of $\mathcal{M}_i$ after reading v stays inside S_i. Therefore, the large scale geometry $[\alpha]$ of α is quasi-isometric to either $[0\alpha']$ or $[1\alpha']$ or both. Based on this observation, we can postulate the following assumption till the end of this section unless otherwise stated.

POSTULATE. The state space S can be written as $\{q_0\} \cup S'$ so that (1) q_0 is the only initial state and $q_0 \notin S'$, (2) S' is the only nontrivial strongly connected component of $\mathcal{M}$, and (3) any transition from q_0 goes into S'.

From now on, our goal is to describe atlases of Büchi recognizable languages accepted by automata that satisfy the postulate above.

Lemma 5.5. *If $\mathcal{M}$ has a 0-loop and a 1-loop, then the atlas $[L(\mathcal{M})]$ coincides with the atlas $[\Sigma^\omega] \setminus X$ for some $X \subseteq \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}$.*

Proof. Assume that L_0 is a 0-loop and L_1 is a 1-loop. Let α be a string with infinitely many 0s and infinitely many 1s. Say, $\alpha = 0^{n_0} 1^{m_0} 0^{n_1} 1^{m_1} \ldots$, where $n_i > 0$ and $m_i > 0$ for all i. We can build a string β of the form

$$u \; 0^{n_1'} v 1^{m_1'} w \; 0^{n_2'} v 1^{m_2'} w \; 0^{n_3'} v 1^{m_3'} w \; \ldots \; 0^{n_i'} v 1^{m_i'} w \; \ldots$$

so that the following properties are satisfied:

1. The string β is accepted by $\mathcal{M}$.
2. For all i we have $n_i = n_i' + c_i$ and $m_i = m_i' + c_i'$, where $c_i < |S|$ and $c_i' < |S|$.
3. The string v labels a path from a state s_0 in L_0 to a state s_1 in L_1.
4. The string w labels a path from s_1 to s_0.
5. The string u is a string that labels a path from the initial state to s_0.

This implies that α and β are $\leqslant_{CR}$-reducible to each other. Hence α and β are quasi-isometric.

Note that none of the strings 0^ω, 1^ω, 10^ω, 01^ω is quasi-isometric to a string with both infinitely many 0s and 1s. Also, these four strings are pairwise not quasi-isometric. Hence, the choice of X is dependent on whether or not $\mathcal{M}$ accepts some (or all) of these four strings. For instance, $0^\omega \in X$ iff 0^ω is not accepted by $\mathcal{M}$. We proved the lemma. ⊣

Lemma 5.6. *If $\mathcal{M}$ has no 0-loop and has no 1-loop, then the atlas $[L(\mathcal{M})]$ equals the singleton atlas $[\{(01)^\omega\}]$.*

Proof. Let $\alpha = \sigma_0 \sigma_1 \ldots$ be a string accepted by the automaton $\mathcal{M}$. Let $\rho = s_1, s_2, \ldots$ be an accepting run of $\mathcal{M}$ on α. Write α as $w_1 w_2 \ldots$ so that

1. the length of each w_i is bounded by $|S|$, and
2. the run ρ along each w_i has a loop; that loop within w_i contains both 0 and 1.

This implies that $(01)^\omega$ and α are $\leqslant_{CR}$-reducible to each other. Hence α and $(01)^\omega$ are quasi-isometric. ⊣

Lemma 5.7. *Suppose that $\mathcal{M}$ contains no 1-loop but has a 0-loop L_0. Then the atlas $[L(\mathcal{M})]$ coincides with one of the following atlases:*

$[\{0^\omega\}], [\{10^\omega\}], [\{0^\omega, 10^\omega\}], [\{0^{n_1} 1 0^{n_2} 1 0^{n_3} \ldots \mid n_i > 0\}] \cup X$, *where $X \subseteq \{[0^\omega], [10^\omega]\}$.*

Proof. The assumption of the lemma implies that if α is accepted by $\mathcal{M}$ then

1. The string α contains infinitely many 0s, and
2. The length of any sequence of consecutive 1s occurring in α is bounded by $|S|$.

If $L(\mathcal{M}) = V 0^\omega$ for some regular language $V \subseteq \Sigma^\star$, then the atlas $[L(\mathcal{M})]$ is either $[\{0^\omega\}]$ or $[\{10^\omega\}]$ or $[\{0^\omega, 10^\omega\}]$.

Thus, we assume that $\mathcal{M}$ accepts a string with infinitely many 1s'. Let α be any string of the form $0^{t_1} 1 0^{t_2} 1 \ldots$, where all $t_i > 0$ for all i. We can build a string β of the form

$$u \; 0^{n_1} v \; 0^{n_2} v \; 0^{n_3} v \; \ldots \; 0^{n_i} v \; \ldots$$

so that the following are true:

1. The string β is accepted by $\mathcal{M}$,
2. $t_i = n_i + c_i$, where $c_i < |S|$ for all i,
3. v is a string that labels a path from a state s_0 in L_0 back to s_0, and
4. The string v contains 1.

The string β is quasi-isometric to the string β' obtained from β by replacing all occurrences of v with 1 and removing the prefix u. In turn β' and α are $\leqslant_{CR}$ reducible to each other. Hence, $\alpha \sim_{QI} \beta$.

Now suppose that β is accepted by $\mathcal{M}$ and β has infinitely many 1's. Then β is of the form $u0^{n_1}1^{m_1}0^{n_2}1^{m_2}\dots$, where we have $n_i, m_i > 0$ for all i. As noted above the numbers m_i are uniformly bounded. The string β is quasi-isometric to the string β' obtained from β by replacing every 1^{m_i} with just 1 and removing the prefix u. Clearly β' is of the form desired.

Note that $\mathcal{M}$ might also accept some (including none) of the strings quasi-isometric to either $[\{0^\omega\}]$, $[\{10^\omega\}]$, or both. Hence, the atlas $[L(\mathcal{M})]$ coincides with

$$[\{0^{n_1}10^{n_2}10^{n_3}\dots \mid n_i > 0\}] \cup X,$$

where X is a subset of $\{[0^\omega], [10^\omega]\}$. $\dashv$

Obviously the next lemma follows from the above:

LEMMA 5.8. *Suppose that $\mathcal{M}$ contains no 0-loop but has a 1-loop L_0. Then the atlas $[L(\mathcal{M})]$ coincides with one of the following atlases*:

$$[\{1^\omega\}], [\{01^\omega\}], [\{1^\omega, 01^\omega\}], [\{1^{n_1}01^{n_2}01^{n_3}\dots \mid n_i > 0\}] \cup X, \text{ where } X \subseteq \{[1^\omega], [01^\omega]\}.$$

We now remove our postulate put on the state space of Büchi automata. Using equality $(\star)$ given above, and the lemmas, we obtain the following characterization result:

THEOREM 5.9 (Characterization Theorem). *Any Büchi recognizable atlas $[L]$ is a finite union of atlases from the following list*:

- $[\Sigma^\omega] \setminus X$, *where* $X \subseteq \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}$,
- $[\{0^{n_1}10^{n_2}\dots \mid n_i > 0\}] \cup X$, *where* $X \subseteq \{[0^\omega], [10^\omega]\}$.
- $[\{1^{n_1}01^{n_2}\dots \mid n_i > 0\}] \cup X$, *where* $X \subseteq \{[1^\omega], [01^\omega]\}$.
- $[\{0^\omega\}], [\{10^\omega\}], [\{0^\omega, 10^\omega\}], [\{(01)^\omega\}]$,
- $[\{1^\omega\}], [\{01^\omega\}], [\{1^\omega, 01^\omega\}]$. $\dashv$

This theorem has a complexity-theoretic implication. Consider the following Atlas Equality Problem. Design an algorithm that given Büchi automata $\mathcal{M}_1$ and $\mathcal{M}_2$ decides if $[L(\mathcal{M}_1)]$ and $[L(\mathcal{M}_2)]$ coincide. Below we prove that the Atlas Equality problem is decidable in linear time. In comparison, the language equality problem for Büchi automata is PSPACE complete.

COROLLARY 5.10. *There exists a linear time algorithm that, given Büchi automata $\mathcal{A}$ and $\mathcal{B}$, decides if the atlases $[L(\mathcal{A})]$ and $[L(\mathcal{B})]$ coincide.*

PROOF. Note that the running time is measured in terms of the sizes of $\mathcal{A}$ and $\mathcal{B}$. So, let $\mathcal{M}$ be a Büchi automaton. We want to find atlases from the list provided

in Theorem 5.9 such that the union of these atlases equals $[L(\mathcal{M})]$. For this, we decompose $\mathcal{M}$ into automata $\mathcal{M}_1, ..., \mathcal{M}_k$ as in the equality $(\star)$, where each $\mathcal{M}_i$ satisfies the postulate. This decomposition can be done in linear time (e.g., using Tarjan's algorithm that finds strongly connected components of directed graphs). Now for each $\mathcal{M}_i$, we check the assumptions of Lemma 5.5 through Lemma 5.8.

As an example, the assumptions of Lemma 5.5 can be checked as follows. In order to check if $\mathcal{M}_i$ contains a 0-loop, we remove all transitions labeled with 1 from the state diagram of $\mathcal{M}_i$. This gives us a directed graph whose edges are transitions labelled with 0. This modified graph has a loop iff $\mathcal{M}_i$ has a 0-loop. Similarly, to check if $\mathcal{M}_i$ contains a 1-loop, we remove all transitions labeled with 1. This new directed graph has a loop iff $\mathcal{M}_i$ has a 1-loop. To find the set X from (the statement of) the lemma we just need to check which of the strings $0^\omega, 1^\omega, 10^\omega, 01^\omega$ is accepted by $\mathcal{M}_i$. This is based on standard graph algorithms (such as the path problem that given two vertices s and t in a directed graph determines if there exists a path from s to t). All these can be done in linear time on the size of $\mathcal{M}_i$.

The argument shows that we can find, in linear time on the size of $\mathcal{M}$, the atlases from the list provided in Theorem 5.9 such that the union of these atlases is the atlas $[L(\mathcal{M})]$. Thus, given Büchi automata $\mathcal{A}$ and $\mathcal{B}$ we can decide in linear time if $[L(\mathcal{A})] = [L(\mathcal{B})]$.   ⊣

§6. The quasi-isometry problem. In this section our goal is two fold. The first goal is to give a computability-theoretic treatment of the quasi-isometry problem. The second goal is to study the complexity of quasi-isometric mappings from α into β given that $\alpha \leqslant_{QI} \beta$. We study both questions using the language of the arithmetical hierarchy. In particular we assume that the reader is familiar with basics of computability theory: the halting set, Σ_n^0-sets, Σ_n^0-complete sets, oracle computation, and the low basis theorem [8]. A standard enumeration of all computably enumerable (c.e.) sets is denoted by $W_0, W_1, W_2, ...$. We also use a typical example of a Σ_2^0-complete set, the set of indices of all finite sets; that is $FIN = \{i \mid W_i$ is a finite set$\}$ [12].

Formally, the quasi-isometry problem (over the alphabet Σ) is the set:

$$QIP = \{(\alpha, \beta) \mid \alpha, \beta \in \Sigma^\omega \ \& \ [\alpha] \leqslant_{QI} [\beta]\}.$$

By restricting this set to computable sequences α and β we have:

$$QIP_c = \{(n, m) \mid W_n \text{ and } W_m \text{ enumerate } \alpha, \beta, \text{ respectively, and } \alpha \leqslant_{QI} \beta\}.$$

To achieve the first goal let us give an upper bound (in terms of the arithmetical hierarchy) for the set QIP_c. Consider computable strings α and β. Let (A, B, C) be quasi-isometry constants. Consider the tree $T(\alpha, \beta, A, B, C)$ constructed in Section 3.2. By the Lemma 3.7 the tree is infinite if and only if $\alpha \leqslant_{QI} \beta$. Thus, $\alpha \leqslant_{QI} \beta$ if and only if there exist (A, B, C) such that the tree $T(\alpha, \beta, A, B, C)$ is infinite. Note that if α and β are computable then the tree is also computable. Checking if the tree $T(\alpha, \beta, A, B, C)$ is infinite is a Π_2^0-conditions. This implies that QIP_c is a Σ_3^0-set. At first glance, this bound seems to be a sharp bound. The theorem below, however, shows that QIP_c is a Σ_2^0-complete set. The theorem also (partially) answers our second question posed at the beginning of this section.

THEOREM 6.1. *The following statements are true*:

1. *The quasi-isometry problem for the pairs (α, β) of all computable strings is a Σ_2^0-complete set.*
2. *If $\alpha \leqslant_{QI} \beta$, then there exists a quasi-isometry between α and β computable in the halting set relative to α and β. Furthermore, if α and β are computable then there exists a low quasi-isometry from α into β.*

PROOF. For the first part of the theorem, let us start showing that QIP_c is a Σ_2^0-set. For arbitrary computable strings α and β (given by c.e. indices i and j, respectively), and (A, B, C) quasi-isometry constants, consider the tree $T(\alpha, \beta, A, B, C)$ and a node x at level n of the tree. The node x determines a partial mapping $f_x = \{(0, i_0), \dots, (n, i_n)\}$ that can possibly be extended to an (A, B, C)-quasi-isometry. The crucial point is that we can put a bound on i_n that depends on (A, B, C). The bound is computable from (A, B, C) only. Hence, the tree $T(\alpha, \beta, A, B, C)$ is such that given n we can compute the number of all nodes at level n of the tree. Therefore, the set $L = \{n \mid$ the tree $T(\alpha, \beta, A, B, C)$ has a node at level $n\}$ is computable. Hence, the set

$$\{T(\alpha, \beta, A, B, C) \mid \text{ the tree } T(\alpha, \beta, A, B, C) \text{ is infinite}\}$$

is a Π_1^0 set. Thus, α and β are quasi-isometric if and only if there are constants (A, B, C) such that the tree $T(\alpha, \beta, A, B, C)$ is infinite. This implies that the quasi-isometry problem for computable strings is a Σ_2^0-set.

Consider the following Σ_2^0-complete set $FIN = \{i \mid W_i$ is finite$\}$. We will reduce FIN to QIP_c. For each i we construct two infinite strings α_i and β_i such that W_i is finite if and only if α_i and β_i are quasi-isometric. For this we start enumerating W_i by stages $1, 2, \dots$. At stage 0, we construct $\alpha_{i,0}$ and $\beta_{i,0}$ finite prefixes that can be extended to $(1, 1, 1)$-quasi-isometries and set $A_0 = B_0 = C_0 = 1$. Assume that by stage $s > 0$ we have built finite prefixes $\alpha_{i,s-1}$ (of α_i) and $\beta_{i,s-1}$ (of β_i). At this stage we also assume that the strings $\alpha_{i,s-1}$ and $\beta_{i,s-1}$ have extensions that are $(A_{s-1}, B_{s-1}, C_{s-1})$-quasi-isometric. If at stage s the enumeration of W_i outputs a new element, then we extend $\alpha_{i,s-1}$ to $\alpha_{i,s}$ and $\beta_{i,s-1}$ to $\beta_{i,s}$ so there is no $(A_{s-1}, B_{s-1}, C_{s-1})$-quasi-isometry between $\alpha_{i,s}$ and $\beta_{i,s}$. This can easily be achieved by appending a long finite string of one color at the end of $\alpha_{i,s-1}$ and a long finite string of another color at the end of the string $\beta_{i,s-1}$. Then we select constants (A_s, B_s, C_s) such that the finite strings $\alpha_{i,s}$ and $\beta_{i,s}$ can be extended to an (A_s, B_s, C_s)-quasi-isometry and $A_s > A_{s-1}$, $B_s > B_{s-1}$, and $C_s > C_{s-1}$. If W_i at stage s does not enumerate a new element then we extend $\alpha_{i,s-1}$ to $\alpha_{i,s}$ and $\beta_{i,s-1}$ to $\beta_{i,s}$ such that the new extended finite strings $\alpha_{i,s}$ and $\beta_{i,s}$ have $(A_{s-1}, B_{s-1}, C_{s-1})$-quasi-isometric extensions and we set $A_s = A_{s-1}$, $B_s = B_{s-1}$, $C_s = C_{s-1}$.

Thus, if W_i is infinite then there is an infinite sequence $s_1, s_2, \dots$ of increasing stages at which the enumeration of W_i increases the size of W_i. Hence, from stage s_{i+1} on α and β are not $(A_{s_i}, B_{s_i}, C_{s_i})$-quasi-isometric. This implies that W_i is infinite then α_i and β_i are not quasi-isometric. If W_i is finite then the sequences $\{(A_s, B_s, C_s)\}_{s \in \omega}$ stabilizes and α_i and β_i built are quasi-isometric.

To prove the second part of the theorem, consider strings α and β such that $\alpha \leqslant_{QI} \beta$. The tree $T(\alpha, \beta, A, B, C)$ is computable with oracle for α and β. Hence, there exists a quasi-isometry $f : \alpha \to \beta$ such that f is computable with an oracle

for the halting sets for α and β. Moreover, by the Low Basis theorem, if α and β are computable then f can be chosen to be low. This proves the second part of the theorem.   ⊣

The isometry problem for colored computable metric spaces is a refined version of the quasi-isometry problem. In this sense, somewhat intuitively, one might expect that quasi-isometry is easier to detect than the isometry. However, the opposite is true. The isometry problem for computable colored metric spaces $\alpha \in \Sigma^\omega$ is a Π_1^0-complete set (which is not too hard to see) as opposed to Σ_2^0-completeness of the quasi-isometry problem.

We conclude this section by referring an interesting related fact found by [10]. From Theorem 6.1 we know if $\alpha \leqslant_{QI} \beta$ and both α and β are computable then there exists a quasi-isometry from α to β which is computable in the halting set. Is this bound sharp, that is, are there computable α and β such that $\alpha \leqslant_{QI} \beta$ but no computable function can witness the inequality? It is shown in [10] that the answer is positive.

THEOREM 6.2 [10]. *There exists a pair of strings α and β that satisfies the following properties*:

1. *α and β are computable.*
2. *$\alpha \leqslant_{QI} \beta$ holds.*
3. *There is no computable quasi-isometry from α to β.*   ⊣

§7. Asymptotic cones. Consider a colored metric space $\alpha \in \Sigma^\omega$. Let us zoom out this metric space further and further away. Formally, consider the following sequence of colored metric spaces: $X_{0,\alpha} = (\alpha, d_0)$, $X_{1,\alpha} = (\alpha, d_1)$, $X_{2,\alpha} = (\alpha, d_2)$, ..., where $d_n(i,j) = |i - j|/n$. Intuitively, when n approaches infinity, the sequence produces a limit structure. This limit structure presents large scale properties of α. How do we formally define this limit structure? The notion of asymptotic cone gives one possible formalization of the limit structure.

7.1. Basic facts about asymptotic cones. Let $\mathcal{F}$ be a nonprincipal ultrafilter on ω. Recall that a *nonprincipal ultrafilter* on ω is a nonempty maximal (with respect to the set-theoretic inclusion) subset of $P(\omega)$ that satisfies the following properties:

1. No finite set belongs to $\mathcal{F}$. So, $\mathcal{F}$ does not contain the empty set.
2. For all $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$.
3. For all $A, B \subseteq \omega$ if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

Every ultrafilter $\mathcal{F}$ has the following two properties: (1) For every set $A \subseteq \omega$, either $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$. (2) For all pairwise disjoint sets $A, B \subseteq \omega$ if $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

DEFINITION 7.1. A *scaling factor* is any strictly increasing monotonic function $s : \omega \to \omega$ such that $s(0) = 1$.

Let $\alpha \in \Sigma^\omega$. Define the following sequence of metric spaces:

$$X_{0,\alpha,s} = (\alpha, d_0), \ X_{1,\alpha,s} = (\alpha, d_1), ..., X_{n,\alpha,s} = (\alpha, d_n), ...$$

where $d_n(i, j) = |i - j|/s(n)$. Informally, with the scaling factor $s(n)$, this sequence presents the zooming out process of the string α. For instance, in the metric space $X_{0,\alpha,s}$ the distance from 0 to $s(n)$ equals $s(n)$, while in $X_{n,\alpha,s}$ the distance from 0 to $s(n)$ is scaled down to 1. We assume that the domains of these metric spaces are pairwise disjoint: $X_{i,\alpha,s} \cap X_{j,\alpha,s} = \emptyset$ for all $i \neq j$.

Let $\mathbf{a} = (a_n)_{n \geqslant 0}$ be a sequence, where $a_n \in X_{n,\alpha,s}$ for each n. Call the sequence $\mathcal{F}$-bounded, or bounded for short, if there is a constant L (that depends on $\mathbf{a}$) such that the set $\{n \mid d_n(0, a_n) < L\}$ belong to the ultrafilter $\mathcal{F}$.

We now define $\mathbf{B}(\mathcal{F}, s)$ to be the set of all bounded sequences $\mathbf{a} = (a_n)_{n \geqslant 0}$. Say that two bounded sequences $\mathbf{a}$ and $\mathbf{b}$ are $\mathcal{F}$-equivalent, written $\mathbf{a} \sim_{\mathcal{F}} \mathbf{b}$, if for any $\varepsilon > 0$ the set $\{n \mid d_n(a_n, b_n) \leqslant \varepsilon\}$ belongs to the filter $\mathcal{F}$. This is an equivalence relation on the set $\mathbf{B}(\mathcal{F}, s)$ since $\mathcal{F}$ is an ultrafilter.

DEFINITION 7.2. Define the *asymptotic cone of* α, written $Cone(\alpha, \mathcal{F}, s)$, with respect to the function $s(n)$ and the ultrafilter $\mathcal{F}$ as the quotient set

$$\mathbf{B}(\mathcal{F}, s)/\sim_{\mathcal{F}}$$

equipped with the following metric D and color Cl:

1. $D(\mathbf{a}, \mathbf{b}) = r$ iff $\{n \mid r - \varepsilon \leqslant d_n(a_n, b_n) \leqslant r + \varepsilon\} \in \mathcal{F}$ for all $\varepsilon > 0$.
2. $Cl(\mathbf{a}) = \sigma$ iff $\{n \mid a_n \text{ has color } \sigma\} \in \mathcal{F}$.

The definition implies the following. The metric space $Cone(\alpha, \mathcal{F}, s)$ with the metric D is isometric to the metric space $\mathbb{R}_{\geqslant 0}$ of all positive reals. So, we can simply refer to the elements $\mathbf{a}$ of the asymptotic cone as reals. In addition, since $\mathcal{F}$ is ultrafilter, the color function Cl assigns to every real $\mathbf{a} \in Cone(\alpha, \mathcal{F}, s)$ at least one color. In particular, some elements of the asymptotic cone might have several colors. Also, note that asymptotic cones are dependent on ultrafilters $\mathcal{F}$ and the scaling factors s.

A key property of asymptotic cones construction is that quasi-isometry between α and β implies bi-Lipschitz equivalence between the asymptotic cones $Cone(\alpha, \mathcal{F}, s)$ and $Cone(\beta, \mathcal{F}, s)$. In other words, asymptotic cones are quasi-isometry invariants. The proof is standard with the difference that our metric spaces are colored. However, since our base metric space is a linear order ω, the conclusion of our theorem is a little stronger as it implies order perseverance.

THEOREM 7.3. *If strings* α *and* β *have the same large scale geometry then there are color-preserving and order preserving homeomorphisms*

$$H : Cone(\alpha, \mathcal{F}, s) \rightarrow Cone(\beta, \mathcal{F}, s) \text{ and}$$
$$G : Cone(\beta, \mathcal{F}, s) \rightarrow Cone(\alpha, \mathcal{F}, s),$$

constants C_H *and* C_G *such that for all elements* $\mathbf{a}, \mathbf{b} \in Cone(\alpha, \mathcal{F}, s)$ *and* $\mathbf{c}, \mathbf{d} \in Cone(\beta, \mathcal{F}, s)$ *we have*:

$$(1/C_H) \cdot D(\mathbf{a}, \mathbf{b}) \leqslant D(H(\mathbf{a}), H(\mathbf{b})) \leqslant C_H \cdot D(\mathbf{a}, \mathbf{b}),$$
$$and$$
$$(1/C_G) \cdot D(\mathbf{c}, \mathbf{d}) \leqslant D(G(\mathbf{c}), G(\mathbf{d})) \leqslant C_G \cdot D(\mathbf{c}, \mathbf{d}).$$

PROOF. Let $h : \alpha \to \beta$ and $g : \beta \to \alpha$ be (A_1, B_1, C_1) and (A_2, B_2, C_2) quasi-isometries, respectively. These functions induce quasi-isometries between the colored metric spaces $X_{n,\alpha,s}$ and $X_{n,\beta,s}$. In particular, for the function h and each $X_{n,\alpha,s}$ and $X_{n,\beta,s}$ we have:

$$(1/A_1) \cdot d_n(i, j) - B_1/s(n) \leqslant d_n(h(i), h(j)) \leqslant A_1 \cdot d_n(i, j) + B_1/s(n),$$

where i, j and $h(i), h(j)$ are elements of the spaces $X_{n,\alpha,s}$ and $X_{n,\beta,s}$, respectively. Hence, for $\mathbf{a} = (a_n)_{n \geqslant 0}$ and $\mathbf{b} = (b_n)_{n \geqslant 0}$ we have

$$(1/A_1) \cdot |a_n - b_n|/s(n) - B_1/s(n) \leqslant d_n(h(a_n), h(b_n)) \leqslant A_1 \cdot |a_n - b_n|/s(n) + B_1/s(n).$$

Taking the limits and setting $H(\mathbf{a}) = (h(a_n))_{n \geqslant 0}$, we see that

$$(1/A_1) \cdot D(\mathbf{a}, \mathbf{b}) \leqslant D(H(\mathbf{a}), H(\mathbf{b})) \leqslant A_1 \cdot D(\mathbf{a}, \mathbf{b}).$$

So, the desired C_H is just A_1. The function G is constructed the same way. Note that both H and G preserve colors. One thing that needs to be observed is that H is onto. This follows from the condition put on the quasi-isometry h: for every $a' \in \beta$ there exists an $a \in \alpha$ such that $|a' - h(a)| \leqslant C_1$. Hence for any element $\mathbf{a}' = (a'_n)_{n \geqslant 0}$ in $Cone(\beta, \mathcal{F}, s)$ there exists an element $\mathbf{a} = (a_n)_{n \geqslant 0}$ in $Cone(\alpha, \mathcal{F}, s)$ such that $D(H(a), \mathbf{a}') = 0$, that is, $H(\mathbf{a}) = \mathbf{a}'$. Finally, Small Cross Over Lemma 2.1 guarantees that if $\mathbf{a} \leqslant \mathbf{b}$ then it must be the case that $H(\mathbf{a}) \leqslant H(\mathbf{b})$.   ⊣

The theorem naturally poses several questions. Do there exist nonquasi-isometric strings that have the same asymptotic cones? Do there exist quasi-isometric strings that can produce nonisometric asymptotic cones? Can asymptotic cones determine quasi-isometry types of strings from a specified class (of strings)? The rest of this section will answer all these questions.

LEMMA 7.4.   *The set $\{\mathbf{b} \in Cone(\alpha, \mathcal{F}, s) \mid \sigma \in Cl(\mathbf{b})\}$ is closed for every color $\sigma \in \Sigma$.*

PROOF. For $\sigma \in \Sigma$, consider the set $\{\mathbf{b} \in Cone(\alpha, \mathcal{F}, s) \mid \sigma \in Cl(\mathbf{b})\}$. Let $\mathbf{a} = (a_n)_{n \geqslant 0}$ be a limit point of this set, and let $\mathbf{a_m} = (a_{m,n})_{n \geqslant 0}$ be a point in $Cone(\alpha, \mathcal{F}, s)$ of color σ such that $D(\mathbf{a}, \mathbf{a_m}) \leqslant 2^{-m}$. Without loss of generality we can assume that every point in $\mathbf{a_m}$ has the color σ; if not, the smallest position of $X_{n,\alpha,s}$ with the color σ can be substituted for each differently colored $a_{m,n}$.

For each $n \in \omega$, let i_n be a number that satisfies

$$d_n(a_n, a_{i_n,n}) = \min_{m \in \omega} d_n(a_n, a_{m,n}).$$

Now let us set $\mathbf{b} = (b_n)_{n \geqslant 0}$, where $b_n = a_{i_n,n}$. Each b_n has the color σ. We also have $\mathbf{a} \sim \mathbf{b}$. Indeed, for each m we have

$$\{n \mid d_n(a_n, b_n) \leqslant 2^{-m}\} \supseteq \{n \mid d_n(a_n, a_{m+1,n}) \leqslant 2^{-m}\} \in \mathcal{F}.$$

Here, the inclusion part is straightforward from the definition of $\mathbf{b}$; and the membership part is true because $D(\mathbf{a}, \mathbf{a_m}) \leqslant 2^{-m}$ holds, and thus for any $\varepsilon > 0$ we have

$$\{n \mid 2^{-(m+1)} - \varepsilon \leqslant d_n(a_n, a_{m+1,n}) \leqslant 2^{-(m+1)} + \varepsilon\} \in \mathcal{F}.$$

Now because $\mathcal{F}$ is upward closed, we have $\{n \mid d_n(a_n, b_n) \leqslant 2^{-m}\} \in \mathcal{F}$, which proves $\mathbf{a} \sim \mathbf{b}$. Thus $\sigma \in Cl(\mathbf{a})$.   ⊣

7.2. Asymptotic cones and quasi-isometry types. The next theorem states that all asymptotic cones of quasi-isometric eventually periodic strings are isometric. In particular, these asymptotic cones fully characterize quasi-isometry types of eventually periodic strings.

THEOREM 7.5. *For an eventually periodic string* $\alpha = uv^\omega$, *all asymptotic cones* $Cone(\alpha, \mathcal{F}, s)$ *equal the colored metric space* $(\mathbb{R}_{\geqslant 0}; d, Cl)$ *such that every real* $r > 0$ *has colors of* v *and the real* 0 *has colors of both* u *and* v.

PROOF. Let $\mathcal{F}$ be any ultrafilter and $s : \omega \to \omega$ be a scaling factor. It is easy to see that 0 has all colors of both u and v. Let r be a positive real number. We want to show that r has all colors present in v. So, let σ be a color present in v. There is a sequence $(a_n)_{n \geqslant 0}$ such that a_n is an element of $X_{n,\alpha}$ with the color σ, and for every $\varepsilon > 0$ there is an N_ε such that for all $n > N_\varepsilon$ we have $|d_n(a_n, 0) - r| \leqslant \varepsilon$. Indeed, for each $n \in \omega$ such that $|u|/s(n) < r$ let a_n be any element which has the color σ and satisfies $(|u| + i|v|)/s(n) \leqslant d_n(0, a_n) < (|u| + (i+1)|v|)/s(n)$, where i is the unique integer which satisfies $(|u| + i|v|)/s(n) \leqslant r < (|u| + (i+1)|v|)/s(n)$. So, we have $|d_n(a_n, 0) - r| \leqslant |v|/s(n)$. This inequality implies the claim since every ultrafiler contains all co-finite sets. ⊣

We now prove a theorem akin to a known result in geometric group theory stating that there are nonquasi-isometric groups that realize the same cones. For instance, in [1, 2] it is proved that all asymptotic cones of nonelementary hyperbolic group are isometric.

THEOREM 7.6. *There are nonquasi-isometric strings* $\alpha, \beta \in \{0,1\}^\omega$, *a scale factor* $s(n)$, *and an ultrafilter* $\mathcal{F}$ *such that* $Cone(\alpha, \mathcal{F}, s) = Cone(\beta, \mathcal{F}, s)$. *Moreover,* α, $\beta \in \mathcal{F}(1)$.

PROOF. We consider the exponential function as the scaling factor: $s(n) = 2^n$. We need to construct strings α, β and an ultrafilter $\mathcal{F}$ that satisfy the theorem. Let the string α be the characteristic function of the set of powers of 2, that is, $\alpha(i) = 1$ if and only if $i = 2^k$ for some $k \geqslant 0$.

Let $\mathcal{F}$ be any ultrafilter. Consider the asymptotic cone $Cone(\alpha, \mathcal{F}, s)$. Note that every co-finite set belongs to $\mathcal{F}$. This implies that the asymptotic cone coincides with the colored metric space $(\mathbb{R}_{\geqslant 0}; d, Cl)$, where d is the usual metric on reals, all reals have color 0, and real r has also color 1 if and only if r is an integer power of 2 or $r = 0$.

We need to construct a string $\beta \in \{0,1\}^\omega$ and an ultrafilter $\mathcal{F}$ such that α and β are not quasi-isometric but the asymptotic cone $Cone(\beta, \mathcal{F}, s)$ coincides with the above colored metric space $(\mathbb{R}_{\geqslant 0}; d, Cl)$.

Let $n_0, n_1, n_2, ...$ be an increasing sequence of positive integers. Assume that these integers satisfy the following order:

$$n_0 < n_1 < n_1 + 1 < n_2 < n_2 + 1 < n_2 + 2 < \cdots < n_i < \cdots < n_i + i < n_{i+1} < \cdots$$

Let B be the set consisting of all the members of this sequence, that is,

$$B = \{n_0, n_1, n_1 + 1, n_2, n_2 + 1, n_2 + 2, ..., n_i, n_i + 1, ..., n_i + i, n_{i+1}, ...\}.$$

We define $\beta \in \{0,1\}^\omega$ as follows. The string β has one at position i if and only if for some member $s \in B$ we have $i = 2^s$.

Let $\mathcal{F}$ be an ultrafilter that contains the set B. Consider the asymptotic cone $Cone(\beta, \mathcal{F}, s)$. We can view the domain of this cone as the set $\mathbb{R}_{\geqslant 0}$. Since $s(n) = 2^n$ is the exponential scaling factor, in the cone $Cone(\beta, \mathcal{F}, s)$ every dyadic number of the form $q/2^i$ gets color 0. Since dyadic numbers are dense in $R_{\geqslant 0}$, by Lemma 7.4, every real r gets color 0 in the cone $Cone(\beta, \mathcal{F}, s)$. Now we need to show that r gets color 1 if and only if r is an integer power of 2. In what follows we prove the "if" part. The "only if" part is clear from the construction of β; notice that β is obtained by replacing some 1's in α with 0's.

So assume that $r = 2^i$ for some integer i. Consider the colored metric spaces $X_{n_j+k,\beta,s}$ where $i \leqslant j$ and $0 \leqslant k \leqslant j$. Due to the definition of β, the color of the point in $X_{n_j+k,\beta,s}$ at distance 2^i from 0 with respect to d_{n_j+k} is 1. The set

$$\{n_i, n_i + 1, \ldots, n_i + i, n_{i+1}, \ldots, n_{i+k}, n_{i+k} + 1, \ldots, n_{i+k} + (i+k), \ldots\}$$

is obtained by eliminating finite number of elements in B, and hence it belongs to the ultrafilter. Hence, the real 2^i has color 1 in the asymptotic cone $Cone(\beta, \mathcal{F}, s)$.

To finish the proof, we just need to show that we can select the set B so that $\alpha \not\leqslant_{QI} \beta$. This can be done, for instance, by letting $n_{k+1} = 2^{n_k}$. Then one can show that $\alpha \not\leqslant_{QI} \beta$ by comparing frequency of occurrences of 1's in α and β (cf. the proof of Theorem 3.1). As a side note observe that $\beta \leqslant_{QI} \alpha$.   ⊣

In contrast to the theorem above, we show that the same colored metric space can produce two asymptotic cones that are not (even) quasi-isometric. This is similar to the result in geometric group theory where one group can realize two non homeomorphic asymptotic cones [14].

THEOREM 7.7. *There exists a string α and scaling factors s_0, s_1 such that for any ultrafilter $\mathcal{F}$ the asymptotic cones $Cone(\alpha, \mathcal{F}, s_0)$ and $Cone(\alpha, \mathcal{F}, s_1)$ are not quasi-isometric.*

PROOF. We construct α and two scaling factors s_0 and s_1 such that in the asymptotic cone $Cone(\alpha, \mathcal{F}, s_0)$ all reals $r > 1$ have color 0, and in the asymptotic cone $Cone(\alpha, \mathcal{F}, s_1)$ all reals $r > 1$ have color 1 only. These two metric spaces are clearly not quasi-isometric. The idea is to design s_0 and s_1 so that the filter $\mathcal{F}$ sees more and more 0 colors as s_0 scales α down, and more and more 1 colors as s_1 scales α down.

Define the natural numbers p_n and q_n inductively; p_n will indicate the lengths of 1-blocks in α and q_n will indicate the lengths of 0-blocks in α. We start by setting $p_1 = q_1 = 1$.

$$p_{n+1} = n \sum_{i=1}^{n}(p_i + q_i) \text{ and } q_{n+1} = n\left(\sum_{i=1}^{n}(p_i + q_i) + p_{n+1}\right).$$

Now consider the string $\alpha = 0^{p_1}1^{q_1}0^{p_2}1^{q_2}\ldots$, and define the following two scale functions:

- $s_0(n) = \sum_{i=1}^{n}(p_i + q_i)$, and
- $s_1(n) = \left(\sum_{i=1}^{n}(p_i + q_i) + p_{n+1}\right)$.

Consider the sequence $\{X_{n,\alpha,s_j}\}_{n \in \omega}$ of metric spaces. Recall that the metric $d_{j,n}$ in the space X_{n,α,s_j} is defined as $d_{j,n}(x,y) = |y - x|/s_j(n)$ $(j = 0, 1)$. Then all points a in the metric space X_{n,α,s_j} such that $1 < d_{j,n}(0,a) < n$ have the color j. Hence for all $r > 1$ and all ultrafilters $\mathcal{F}$ the color of r is 0 in $Cone(\alpha, \mathcal{F}, s_0)$ and 1 in $Cone(\alpha, \mathcal{F}, s_1)$. ⊣

7.3. Asymptotic cones of algorithmically random strings. In the last two decades the theory of algorithmic randomness has attracted the attention of many experts in computability, logic, complexity, and reverse mathematics. Many variants of algorithmic randomness and related notions for strings have been introduced and investigated. These include Martin-Löf tests, Schnorr tests, prefix-free complexity, 1-generic sets, K-triviality, martingales, connections to differentiability, Solovay and related reducibilities. The monographs by Downey and Hirschfeldt [3] and by Nies [11] expose recent advances in the area.

Martin-Löf tests, that we write ML-tests for short, play a central role in the theory of algorithmic randomness for infinite strings. For completeness, here we define Martin-Löf randomness in the case of infinite binary strings. We view the set $\{0,1\}^\omega$ as an infinite binary tree. This is a Cantor space with the canonical topology generated by the cones $C(w)$ consisting of all infinite binary strings extending the finite string w. We can equip this space with Lebesgue measure induced by the following function $\lambda(C(w)) = 2^{-|w|}$.

Let $V \subseteq \omega \times \{0,1\}^\star$ be a computably enumerable set. Given such V, consider its nth slice $V_n = \{v \in \{0,1\}^\star \mid (n,v) \in V\}$, where $n \in \omega$. Each V_n describes the open set $\mathcal{O}_V(n) = \bigcup_{v \in V_n} C(v)$.

DEFINITION 7.8. The set V is a *Martin-Löf test*, or *ML-test* for short, if the measure of the open set $\mathcal{O}_V(n)$ is bounded by 2^{-n} for all n.

For the ML-test V, the intersection $\bigcap_n \mathcal{O}_V(n)$ is a set of measure zero. We call this intersection set the *effective null-set* generated by V.

DEFINITION 7.9. Let V be an ML-test. An infinite binary sequence $\alpha \in \{0,1\}^\omega$ passes the *ML-test V* if α does not belong to the effective null-set generated by V, that is $\alpha \notin \bigcap_n \mathcal{O}_V(n)$. The string α is *Martin-Löf random*, or *ML-random* for short, if it passes all Martin-Löf tests.

There are only countably many *ML*-tests, hence the union of all null-sets given by tests is a set of measure zero. Therefore, the set of all *ML*-random sequences has Lebesgue measure one.

For our theorem below, we need one more concept. Let $V \subseteq \omega \times \{0,1\}^\star$ be a computably enumerable set. We call V *Solovay test* if the sum of all measures of the open sets $\mathcal{O}_V(n)$ is bounded. String α passes the Solovay test V, if $\alpha \notin \mathcal{O}_V(n)$ except for finitely many n's. It turns out passing all Solovay tests is equivalent to passing all ML-tests. In other words, a string α is ML-random if and only if α is Solovay random (cf. Proposition 3.2.19 in [11]). We will use this fact in our next theorem. The theorem shows that all asymptotic cones of ML-random strings are isometric to each other, and thus they are independent on ultrafilters and scaling factors as long as the scaling functions are computable.

THEOREM 7.10. *If α is Martin-Löf random, then for all computable scaling factors s and ultrafilters $\mathcal{F}$, the asymptotic cone $Cone(\alpha, \mathcal{F}, s)$ coincides with the space $(\mathcal{R}_{\geqslant 0}; d, C)$, where all reals have all colors from alphabet Σ.*

PROOF. Suppose $Cone(\alpha, \mathcal{F}, s)$ does not have some color, say 0, at r. Below we prove that α is not Solovay random. First, we show that there exists $\varepsilon > 0$ for which the following formula is valid:

$$\exists^{\infty} n \in \omega \forall i \in \omega [r - \varepsilon \leqslant \frac{i}{s(n)} \leqslant r + \varepsilon \Rightarrow \alpha(i) \neq 0].$$

Here, $\exists^{\infty} n \in \omega[\varphi(n)]$ means that $\varphi(n)$ is true for infinitely many n's. For its proof, assume the contrary and suppose for every $\varepsilon > 0$ we have

$$\forall^{\infty} n \in \omega \exists i \in \omega [r - \varepsilon \leqslant \frac{i}{s(n)} \leqslant r + \varepsilon \wedge \alpha(i) = 0].$$

Here, $\forall^{\infty} n \in \omega[\varphi(n)]$ means that $\varphi(n)$ is true except for finitely many n's. Then we can construct an element (a_n) of $Cone(\alpha, \mathcal{F}, s)$ such that $\frac{a_n}{s(n)} \xrightarrow{n \to \infty} r$ and $\alpha(a_n) = 0$ for each sufficiently large n. Hence $Cone(\alpha, \mathcal{F}, s)$ has the color 0 at r (recall that any ultrafilter $\mathcal{F}$ contains every co-finite set), which is a contradiction.

Now let ε be a positive real for which the formula above is valid, and consider the set

$$G_n = \{X \in \Sigma^{\omega} \mid s(n)(r - \varepsilon) \leqslant i \leqslant s(n)(r + \varepsilon) \Rightarrow X(i) \neq 0\}.$$

Without loss of generality we assume that r and ε are rational numbers. The set G_n can be described by the set $V_n \subseteq \{0, 1\}^{\star}$ that consists of all finite strings in Σ of length $s(n)(r + \varepsilon)$ whose terminal segments of length $2s(n)\varepsilon$ do not contain 0. Let V be a test of which nth slice is V_n. Then V is computably enumerable, and the sum of the measures of G_n is bounded, as the size of the interval $[s(n)(r - \varepsilon), s(n)(r + \varepsilon)]$ increases at least linearly with respect to n because s is a strictly increasing function that takes values in natural numbers. So, we produced a Solovay test V that the string α fails, that is, $\alpha \in G_n$ for infinitely many n. Hence α is not Solovay random.   ⊣

§8. Open questions. There are several natural open questions and directions of research related to understanding quasi-isometries between infinite strings. One would, for instance, like to understand algebraic properties of the partial order Σ^{ω}_{QI}. For example, one might ask if this partial order is a semi-lattice. Another example is related to automorphisms of this partial order. In the case when $\Sigma = \{0, 1\}$, the structure Σ^{ω}_{QI} has an obvious automorphism that given $\alpha \in \Sigma^{\omega}$ produces β that is obtained by replacing 0 with 1 and 1 with 0 in α. Does the partial order posses any other nontrivial automorphism?

One can also study model-theoretic properties of the partial order Σ^{ω}_{QI}. For instance, one might want to describe relations of this structure definable in the first-order logic. One possible step towards this would be to investigate if the first-order theory of Σ^{ω}_{QI} decidable.

From perspective of computability theory, it is also interesting to investigate the properties of $\leqslant_{QI}$ either by restraining the class of infinite strings (e.g., by

considering computable strings only) or by refining the relation $\leqslant_{QI}$ to computable quasi-isometries only. This direction is pursued by [10], where both of strings and quasi-isometries between strings are restricted to computable ones.

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REFERENCES

[1] A. DIOUBINA and I. POLTEROVICH, *Structures at infinity of hyperbolic space* (*Russian*). **Uspekhi Mat. Nauk**, vol. 53 (1998), no. 5(323), pp. 239–240; translation in **Russian Mathematical Surveys**, vol. 53 (1998), no. 5, pp. 1093–1094.

[2] ———, *Explicit constructions of universal R-trees and asymptotic geometry of hyperbolic spaces*. **Bulletin of the London Mathematical Society**, vol. 33 (2001), no. 6, pp. 727–734.

[3] R. G. DOWNEY and D. R. HIRSCHFELDT, **Algorithmic Randomness and Complexity**, Springer-Verlag, New York, 2010.

[4] C. DRUTU and M. KAPOVICH, **Geometric Group Theory**, Colloquium Publications, vol. 63, American Mathematical Society, Providence, RI, 2018.

[5] M. GROMOV, *Groups of polynomial growth and expanding maps*. **Publications Mathématiques de l'Institut des Hautes Études Scientifiques**, vol. 53 (1981), pp. 53–78.

[6] ———, *Hyperbolic groups*, **Essays in Group Theory** (S. M. Gersten, editor), M.S.R.I. Publications, vol. 8, Springer-Verlag, Berlin, Germany, 1987, pp. 75–263.

[7] ———, *Asymptotic invariants of infinite groups*, **Geometric Group Theory** (G. A. Niblo and M. A. Roller, editors), London Mathematical Society, Lecture Notes Series, vol. 182, Cambridge University Press, Cambridge, 1993, pp. 1–295.

[8] C. G. JOCKUSCH and R. SOARE, Π_1^0-*classes and degrees of theories*. **Transactions of the American Mathematical Society**, vol. 173 (1972), pp. 33–56.

[9] B. KHOUSSAINOV and T. TAKISAKA, *Large scale geometries of infinite strings*. **Proceedings of the LICS** (2017), pp. 1–12.

[10] R. LOU, **Quasi-isometric reductions between K-ary sequences**, Bachelor thesis, National University of Singapore, 2019.

[11] A. NIES, **Computability and Randomness**, Oxford University Press, Oxford, 2012.

[12] H. ROGERS, **Theory of Recursive Functions and Effective Computability**, MIT Press, Cambridge, MA, 1987.

[13] L. VAN DEN DRIES and A. WILKIE, *On Gromov's theorem concerning groups of polynomial growth and elementary logic*. **Journal of Algebra**, vol. 89 (1984), pp. 349–374.

[14] B. VELICKOVIC and T. SIMON, *Asymptotic cones of finitely generated groups*. **Bulletin of the London Mathematical Society**, vol. 32 (2000), pp. 203–220.

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