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Sharp affine Trudinger–Moser inequalities: A new argument

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Abstract. We set up the sharp Trudinger–Moser inequality under arbitrary norms. Using this result and the L_p Busemann-Petty centroid inequality, we will provide a new proof to the sharp affine Trudinger–Moser inequalities without using the well-known affine Pólya–Szegö inequality.

1 Introduction

Geometric and functional inequalities together with their sharp constants and extremal functions have been the main subject of a lot of research. They play important roles in several problems arising in the calculus of variations, partial differential equations, geometry, etc. Among those inequalities, the Sobolev-type inequalities are probably one of the most important and interesting, and there is a vast literature.

In [39], J. Moser proved a sharp limiting Sobolev inequality, namely, the embedding $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$, where $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp(\alpha|t|^{N/(N-1)}) - 1$ for some $\alpha > 0$, on any finite-measure domain Ω in the Euclidean space $\mathbb{R}^N$. This inequality was studied without its best form independently by Pohozaev [43], Trudinger [48], and Yudovich [50]. This embedding can be considered as the sharp border-line Sobolev inequality, because it provides information on the optimal target space for the Sobolev embedding in the limiting case. Indeed, it is well known that $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \le q \le p^* = \frac{pN}{N-p}$ when $p < N$ and $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \le q < \infty$ when $p = N$ but ${W_0}^{1,p}(\Omega) \notin L^\infty(\Omega)$.

Using the symmetrization arguments, J. Moser proved the following theorem.

Theorem A *Let Ω be a domain with finite measure in Euclidean N-space $\mathbb{R}^N$, $N \ge 2$. Then the following holds:*

$$\sup_{u \in W_0^{1,N}(\Omega): \int_\Omega |\nabla u|^N dx \le 1} \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega \left[\exp(\alpha_N |u|^{N'}) - 1\right] dx < \infty, \tag{1.1}$$

with $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} is the area of the surface of the unit N-ball. Here, $N' = \frac{N}{N-1}$.

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Moser also constructed the following Moser sequence to show that the constant $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ is optimal in the sense that if we replace α_N by any number $\alpha > \alpha_N$, then the above supremum is infinite:

$$u_n(x) = \begin{cases} \left(\frac{1}{\omega_{N-1}}\right)^{1/N}\left(\frac{n}{N}\right)^{(N-1)/N}, & 0 \le |x| \le e^{-(n/N)}, \\ \left(\frac{N}{\omega_{N-1}n}\right)^{1/N}\log\left(\frac{1}{|x|}\right), & e^{-(n/N)} < |x| < 1, \\ 0, & |x| \ge 1. \end{cases} \tag{1.2}$$

Moreover, it can be checked that the constant α_N is indeed sharp in the following sense:

$$\sup_{u \in W_0^{1,N}(\Omega): \int_\Omega |\nabla u|^N\,dx \le 1} \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega \exp(\alpha_N |u|^{N/(N-1)})|u|^a\,dx = \infty, \quad \forall a > 0.$$

Moser used the following classical Schwarz rearrangement argument: every function $u \ge 0$ is associated with a radially symmetric function $u^\#$ such that the sublevel-sets of $u^\#$ are balls with the same volume as the corresponding sublevel-sets of u. Moreover, $u^\#$ is a positive and non-increasing function defined on $B_R(0)$ where $\mathrm{vol}\left(B_R(0)\right) = \mathrm{vol}(\Omega)$. Hence, by the layer cake representation, we can have that

$$\int_\Omega f(u)\,dx = \int_{B_R(0)} f\left(u^\#\right)dx$$

for any function f that is the difference of two monotone functions. In particular, we obtain

$$\|u\|_p = \|u^\#\|_p,$$
$$\int_\Omega \exp\left(\alpha |u|^{n/(n-1)}\right)dx = \int_{B_R(0)} \exp\left(\alpha |u^\#|^{n/(n-1)}\right)dx.$$

In particular, the well-known Pólya–Szegö inequality

$$\int_{B_R(0)} \left|\nabla u^\#\right|^p dx \le \int_\Omega |\nabla u|^p\,dx \tag{1.3}$$

plays a crucial role in the approach of J. Moser.

The sharp Trudinger–Moser inequalities can be improved and extended in many ways. We refer the interested reader to, for instance, [1, 4, 9, 18, 27, 29, 33, 38, 44, 45, 47, 49]. In particular, the authors proved the following affine Trudinger–Moser inequality in [11] where the standard L^N energy of gradient $\left(\int_{\mathbb{R}^N} |\nabla u|^N\,dx\right)^{1/N}$ is replaced by the smaller quantity, namely, the affine energy $\mathcal{E}_N(u)$:

$$\sup_{u \in W_0^{1,N}(\Omega): \mathcal{E}_N(u) \le 1} \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega \left[\exp(\alpha_N |u|^{N/(N-1)}) - 1\right]dx < \infty.$$

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Here, for $p \geq 1$,

$$\mathcal{E}_p(u) = (N\omega_N)^{1/N}\left(\frac{N\omega_N\omega_{p-1}}{2\omega_{N+p-2}}\right)^{1/p}\left(\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}|\nabla u(x)\cdot\sigma|^p\,dx\right)^{-N/p}d\sigma\right)^{-(1/N)}.$$

Indeed, by the Hölder inequality and Fubini's theorem, it is easy to verify that $\mathcal{E}_p(u) \leq \left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{1/p}$. Moreover, the ratio $\mathcal{E}_p(u)/\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{1/p}$ can be made arbitrary small. Hence, the sharp affine Trudinger–Moser inequalities are really stronger than the standard Trudinger–Moser inequalities. It is worth mentioning that the argument in [11] was again based on the Schwarz rearrangement. Indeed, using the L^p Brunn–Minkowski theory, the authors in [11] were able to show the affine Pólya–Szegö inequality

$$\mathcal{E}_p(u^*) \leq \mathcal{E}_p(u).$$

For the applications of such inequalities to geometric analysis and nonlinear PDEs, we refer the reader to the papers of Chang and Yang [10] and of de Figueiredo, do Ó, and Ruf [16].

The main purpose of this note is to provide a different point of view for the sharp affine Trudinger–Moser inequalities without using the affine Pólya–Szegö inequality. Our aim is to provide another option to study the affine Trudinger–Moser inequalities in frameworks where the well-known Pólya–Szegö inequality is not available. It is worth noting that there are many settings where the Schwarz rearrangement is not applicable. Thus, it is interesting and challenging to study the Trudinger–Moser inequalities in such cases. For instance, when there is the presence of weights on both sides of the inequalities, the classical Schwarz procedure is not available, even on the Euclidean spaces. In this direction, Dong, Lam, and Lu [19] considered a suitable quasiconformal transform to reduce the singular case to the nonweighted case and proved, among other results, that for any $\alpha \leq \alpha_N$ and $0 \leq \beta < N$,

$$\int_\Omega \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)\frac{dx}{|x|^\beta} \lesssim \int_\Omega \frac{dx}{|x|^\beta}. \tag{1.4}$$

Again the constant α_N is sharp. This extends the results in [3] where the authors used the Schwarz rearrangement to study an interpolation between Moser's inequality and the Hardy inequality in the singular case $0 \leq \beta < N$:

$$\sup_{u\in W_0^{1,N}(\Omega):\int_\Omega|\nabla u|^N\,dx\leq 1}\frac{1}{\mathrm{vol}(\Omega)^{1-(\beta/N)}}\int_\Omega \exp\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{N/(N-1)}\right)\frac{dx}{|x|^\beta} \leq c_0 \tag{1.5}$$

for any $\alpha \leq \alpha_N$. Indeed, it is clear that $\int_\Omega \frac{dx}{|x|^\beta} \leq \mathrm{vol}(\Omega)^{1-(\beta/N)}$. Hence, (1.4) is an improvement of (1.5). In fact, (1.4) is essentially stronger than (1.5), since the ratio $\int_\Omega \frac{dx}{|x|^\beta}/\mathrm{vol}(\Omega)^{1-(\beta/N)}$ is not uniformly bounded from below by any positive constant, as Ω ranges in the set of finite-volume domains. Another interesting situation where we cannot use the rearrangement approach is the higher order Trudinger–Moser inequalities, namely, the Adams inequalities. In this case, in [2], the author used a lemma of O'neil [41] and the Fourier transform to control functions with compact support by Riesz potential with explicit constants to overcome the lack of Pólya–Szegö

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type inequalities on higher order Sobolev spaces. The ideas in [2] were also employed to study the sharp Trudinger–Moser inequalities on other settings like, for example, Riemannian manifolds [21], sub-Riemannian manifolds [6, 7, 12, 13, 14, 28, 30], and measure spaces [22].

In this paper, we will follow the strategy in [17, 26, 40]. First, we will investigate the sharp Trudinger–Moser inequalities with arbitrary norms. Then, for each $f \in C_0^\infty(\mathbb{R}^N)$, we will apply these results for a suitable norm that depends on f and use the L^p Busemann–Petty centroid inequality to derive the sharp affine Trudinger–Moser inequalities.

To state our main results, let us now introduce some notation. Let $C : \mathbb{R}^N \to \mathbb{R}^+$ be an even strictly convex function. We suppose that C is q-homogeneous for some $q > 1$, that is

$$C(\lambda x) = \lambda^q C(x) \quad \forall \lambda \geq 0, \quad \forall x \in \mathbb{R}^N.$$

Then C^*, the Legendre transform of C, defined by

$$C^*(x) = \sup_y \left\{ \langle x, y \rangle - C(y) \right\},$$

is even, strictly convex function and is p–homogeneous with $p = \frac{q}{q-1}$.

We have that $\langle X, Y \rangle \leq C^*(X) + C(Y)$ for all X, Y. Hence, $\langle X, Y \rangle \leq \lambda^p C^*(X) + \lambda^{-q} C(Y)$ for all $\lambda > 0$, X, Y. Minimizing the right-hand side with respect to λ gives the Cauchy–Schwarz inequality

$$X \cdot Y \leq [qC(Y)]^{1/q} [pC^*(X)]^{1/p}.$$

By Young's inequality, we have

$$X \cdot Y \leq [qC(Y)]^{1/q} [pC^*(X)]^{1/p} \leq C^*(x) + C(y).$$

Hence, we also have that

$$[pC^*(X)]^{1/p} = \sup_Y \frac{X \cdot Y}{[qC(Y)]^{1/q}}.$$

In other words,

$$C^*(X) = \sup_Y \frac{|X \cdot Y|^p}{p[qC(Y)]^{p/q}}.$$

We will assume that for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that

$$x \cdot x^* = qC(x) \text{ and } C^*(x^*) = (q-1)C(x) = \frac{q}{p} C(x).$$

In other words, for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that the equality in the Cauchy-Schwarz inequality happens.

Our first aim of this article is to prove the following theorem.

Theorem 1.1 *Let C be N'-homogeneous. There exists $C_N > 0$ such that for every $f \in W^{1,N}(\mathbb{R}^N)$ with $0 < \operatorname{vol}\big(\operatorname{supp}(f)\big) < \infty$ and $\int_{\mathbb{R}^N} NC^*(\nabla f)\,dx \le 1$, we have*

$$\frac{1}{\operatorname{vol}\big(\operatorname{supp}(f)\big)}\int_{\mathbb{R}^N}\big[\exp\big(\gamma_C|f(x)|^{N/(N-1)}\big)-1\big]\,dx \le C_N.$$

Here, the constant $\gamma_C = N^{N/(N-1)}\kappa_C^{1/(N-1)}$, $\kappa_C = \operatorname{vol}(\{N'C(x)\le 1\})$, is optimal.

Now, we define the general L^p affine energy for $f \in W^{1,p}(\mathbb{R}^N)$ by

$$\begin{aligned}\mathcal{E}_{\lambda,p}(f) = c_{N,p}\Bigg(\int_{\mathbb{S}^{N-1}}\Bigg(\int_{\mathbb{R}^N}\big[(1-\lambda)\big(\nabla f(x)\cdot\sigma\big)_+^p \\ + \lambda\big(\nabla f(x)\cdot\sigma\big)_-^p\big]\,dx\Bigg)^{-(N/p)}d\sigma\Bigg)^{-(1/N)} \\ c_{N,p} = (N\omega_N)^{1/N}\left(\frac{N\omega_N\omega_{p-1}}{\omega_{N+p-2}}\right)^{1/p}.\end{aligned}$$

Here, $0\le\lambda\le 1$. It can be noted that when $\lambda = \frac{1}{2}$, $\mathcal{E}_{1/2,p}(f)$ is the L^p affine energy $\mathcal{E}_p(f)$ introduced in [11], while $\mathcal{E}_{0,p}(f)$ is exactly the asymmetric L^p affine energy $\mathcal{E}_p^+(f)$ studied in [24].

Using Theorem 1.1 and the L^p Busemann–Petty centroid inequality, we establish the following result.

Theorem 1.2 *For every $f \in W^{1,N}(\mathbb{R}^N)$ with $0 < \operatorname{vol}(\operatorname{supp}(f)) < \infty$ and $\mathcal{E}_{\lambda,N}(f)\le 1$, we have*

$$\frac{1}{\operatorname{vol}(\operatorname{supp}(f))}\int_{\mathbb{R}^N}\big[\exp\big(\alpha_N|f(x)|^{N/(N-1)}\big)-1\big]\,dx \le C_N. \tag{1.6}$$

Here, $\alpha_N = N^{N/(N-1)}\omega_N^{1/(N-1)}$ is optimal.

As far as the existence of extremal functions of Moser's inequality, the first breakthrough was due to the celebrated work of Carleson and Chang [8] in which they proved that the supremum

$$\sup_{u\in W_0^{1,N}(\Omega),\int_\Omega|\nabla u|^N dx\le 1}\frac{1}{\operatorname{vol}(\Omega)}\int_\Omega\big[\exp\big(\alpha_N|u|^{N/(N-1)}\big)-1\big]\,dx \tag{1.7}$$

can be achieved when Ω is an Euclidean ball. This result came as a surprise, because it was known that the Sobolev inequality does not have extremal functions supported on any finite ball. Subsequently, the existence of extremal functions has been established on arbitrary domains in [15, 20, 34], and on Riemannian manifolds in [31, 32], to name just a few.

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Now, since $\mathcal{E}_{\lambda,N}(f) \le \|\nabla f\|_N$, we get that

$$\frac{1}{\mathrm{vol}(\mathrm{supp}(f))}\int_{\mathbb{R}^N}\left[\exp\left(\alpha_N\left|\frac{f(x)}{\|\nabla f\|_N}\right|^{N/(N-1)}\right)-1\right]dx \le \frac{1}{\mathrm{vol}(\mathrm{supp}(f))}\int_{\mathbb{R}^N}\left[\exp\left(\alpha_N\left|\frac{f(x)}{\mathcal{E}_{\lambda,N}(f)}\right|^{N/(N-1)}\right)-1\right]dx.$$

Also, when f is radial, we have $\mathcal{E}_{\lambda,N}(f) = \|\nabla f\|_N$. Hence, from the attainability of the standard Trudinger–Moser inequality (1.7), we can deduce that optimal functions for the affine Trudinger–Moser inequality (1.6) exist as well. Moreover, by the properties of the affine energy $\mathcal{E}_{\lambda,N}$, composing of optimizers for the standard Trudinger–Moser inequality (1.7), and any element in $GL(N)$ also gives extremizers for the affine Trudinger–Moser inequality (1.6). This phenomenon has been already observed in [11].

2 Preliminaries

2.1 Brunn–Minkowski Theory

We recall here some background material from the Brunn–Minkowski theory of convex bodies. The interested reader is referred to [35, 36, 46] and the references therein.

A convex body $K \subset \mathbb{R}^N$ is a convex compact subset of $\mathbb{R}^N$ with nonempty interior. Its support function is defined as

$$h_K(x) = \sup_{y\in K}\langle x, y\rangle.$$

We also define the gauge $\|\cdot\|_K$ and radial $r_K(\cdot)$ functions as

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\},$$
$$r_K(x) = \sup\{\lambda > 0 : \lambda x \in K\}.$$

It is obvious that $\|x\|_K = \frac{1}{r_K(x)}$.

For $K \subset \mathbb{R}^N$, we also define its polar body by

$$K^o = \left\{x \in \mathbb{R}^N : \langle x, y\rangle \le 1 \text{ for all } y \in K\right\}.$$

Hence, $h_K = r_{K^o}$. We also have that

$$\mathrm{vol}(K) = \frac{1}{N}\int_{S^{N-1}} r_K^N(x)\,dx = \frac{1}{N}\int_{S^{N-1}}\|x\|_K^{-N}\,dx.$$

For each $\lambda \in [0,1]$, the general L^p centroid body of K is the convex body

$$\Gamma_{\lambda,p}K = (1-\lambda)\Gamma_p^+K + \lambda\Gamma_p^-K,$$

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where $\Gamma_p^- K = \Gamma_p^+(-K)$. Here, $\Gamma_p^+ K$ is the asymmetric L^p centroid body of K that is defined by

$$h^p_{\Gamma_p^+ K}(x) = \frac{1}{\alpha_{N,p}\,\mathrm{vol}(K)} \int_K \langle x, y\rangle_+^p \, dy,$$
$$\alpha_{N,p} = \frac{\omega_{N+p-2}}{(N+p)\omega_N \omega_{p-1}}.$$

Hence, we get that

$$h^p_{\Gamma_{\lambda,p} K}(x) = \frac{1}{\alpha_{N,p}\,\mathrm{vol}(K)} \int_K \left[(1-\lambda)\langle x, y\rangle_+{}^p + \lambda \langle x, y\rangle_-^p\right] dy.$$

One of our main tools is the following general affine L^p Busemann–Petty centroid inequality.

Theorem 2.1 *Let $p \geq 1$ and K be a convex body containing the origin in its interior. We have*

$$\mathrm{vol}(\Gamma_{\lambda,p} K) \geq \mathrm{vol}(K)$$

with the equality occurs if and only if K is an origin-centered ellipsoid.

This affine isoperimetric inequality was investigated in [23] to strengthen the affine L^p Busemann–Petty centroid inequality in [37] and generalize the classical Busemann–Petty centroid inequality by Petty [42].

2.2 Affine Sobolev Inequality

To help explain our method, in this subsection, we will recall a proof of the sharp affine Sobolev inequalities [51] using the Sobolev type inequalities with arbitrary norm and the L^p Busemann–Petty centroid inequality. This proof could be found in [25, 26].

Let $H : \mathbb{R}^N \to [0, \infty)$ be a convex function satisfying the homogeneity property $H(tx) = |t| H(x)$, $\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$. Futhermore, we assume that $\alpha|x| \leq H(x) \leq \beta|x|$, $\forall x \in \mathbb{R}^N$ for some positive constants $\alpha \leq \beta$. Hence, we can assume without loss of generality that the convex closed set $K = \{H(x) \leq 1\}$ has volume $\mathrm{vol}(K) = \omega_N$. Denote

$$H^o(x) = h_K(x) = \sup_{y \in K} \langle x, y\rangle.$$

Then it is clear that H^o is a convex homogeneous function, and

$$H^o(x) = \sup_{y \neq 0} \frac{\langle x, y\rangle}{H(y)}; \quad H(x) = \sup_{y \neq 0} \frac{\langle x, y\rangle}{H^o(y)}.$$

Of course, H^o is the gauge of $K^o = \{H^o(x) \leq 1\}$. We say that K and K^o are polar to each other. Denote $\kappa_N = \mathrm{vol}(K^o)$. In [5, Corollary 3.2], using the convex symmetrization, the following Sobolev type inequality was proved.

Theorem 2.2 *We have that* $\|f\|_{p^*} \le \frac{\omega_N^{1/N}}{\kappa_N^{1/N}} S_{N,p} \big(\int_{\mathbb{R}^n} H^p(\nabla f)\,dx\big)^{1/p}$, *where* $S_{N,p}$ *is the best constant in the Sobolev inequality with the standard euclidean norm.*

Now, for $f \in C_0^\infty(\mathbb{R}^N)$, we will use the following p-homogeneous function:

$$H^p(x) = C_f^*(x) = \frac{1}{p}\int_{S^{N-1}} \|y\|_{p,\lambda,f}^{-N-p}\big[(1-\lambda)\langle x,y\rangle_+^p + \lambda\langle x,y\rangle_-^p\big]\,dy$$

$$\|x\|_{p,\lambda,f} = \left[\int_{\mathbb{R}^N}(1-\lambda)\langle x,\nabla f(y)\rangle_+^p + \lambda\langle x,\nabla f(y)\rangle_-^p\,dy\right]^{1/p}.$$

Then

$$\int_{\mathbb{R}^N} H^p(\nabla f)\,dx = \left(\frac{\mathcal{E}_{\lambda,p}(f)}{c_{N,p}}\right)^{-N}.$$

We also set

$$K_{\lambda,p}(f) = \Big\{x : C_f(x) \le \frac{1}{q}\Big\},$$
$$L_{\lambda,p}(f) = \Big\{x : \|x\|_{p,\lambda,f} \le 1\Big\}$$

By using the L^p Busemann–Petty centroid inequality $\mathrm{vol}(\Gamma_{\lambda,p}L_{\lambda,p}(f)) \ge \mathrm{vol}(L_{\lambda,p}(f))$, we get that

$$\kappa_C = \mathrm{vol}\big(K_{\lambda,p}(f)\big) \ge \big[(N+p)\alpha_{N,p}\big]^{N/p}\left(\frac{1}{N}\left(\frac{\mathcal{E}_{\lambda,p}(f)}{c_{N,p}}\right)^{-N}\right)^{(N+p)/p}.$$

Hence,

$$\begin{aligned}
\|f\|_{p^*} &\le \frac{\omega_N^{1/N}}{\kappa_N^{1/N}} S_{N,p}\left(\int_{\mathbb{R}^n} H^p(\nabla f)\,dx\right)^{1/p} \\
&\le \omega_N^{1/N} S_{N,p}\left(\big[(N+p)\alpha_{N,p}\big]^{N/p}\left(\frac{1}{N}\left(\frac{\mathcal{E}_{\lambda,p}(f)}{c_{N,p}}\right)^{-N}\right)^{(N+p)/p}\right)^{-(1/N)} \\
&\quad\times\left(\frac{\mathcal{E}_{\lambda,p}(f)}{c_{N,p}}\right)^{-(N/p)} \\
&= S_{N,p}\mathcal{E}_{\lambda,p}(f),
\end{aligned}$$

which is the sharp affine Sobolev inequality.

3 Proof of Theorem 1.1

Let $H^o(x) = [N'C(x)]^{1/N'}$ and $H(x) = [NC^*(x)]^{1/N}$.

We will prove the following result.

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Theorem 3.1 *For $1 < p < \infty$, there is a constant $A = A(p)$ such that for all $f \in L^p(\mathbb{R}^N)$ with support contained in Ω, $\mathrm{vol}(\Omega) < \infty$,*

$$\int_\Omega \exp\left(\frac{1}{\kappa_C}\left|\frac{1/((H^o)^{N-1} * f(x))}{\|f\|_p}\right|^{p'}\right) dx \le A\mathrm{vol}(\Omega).$$

We will need the following result [2, Lemma 1].

Lemma 3.2 *Let $a(s,t)$ be a nonnegative measurable function on $[0,\infty) \times [0,\infty)$ such that*

$$a(s,t) \le 1 \quad \text{for} \quad 0 \le s \le t,$$

and

$$\sup_{t>0}\left(\int_t^\infty a(s,t)^{N/(N-1)} ds\right)^{(N-1)/N} = b < \infty.$$

Then there exists a constant $c_0 = c_0(N,b)$ such that for $\varphi(s) \ge 0$ and

$$\int_0^\infty \varphi(s)^N ds \le 1,$$

it follows that

$$\int_0^\infty \exp\left[\left(\int_0^\infty a(s,t)\varphi(s)ds\right)^{N/(N-1)} - t\right] dt \le c_0.$$

We also recall the non-increasing rearrangement g^* of a function g by

$$g^*(t) = \inf\left\{s > 0 : \mathrm{vol}\left(\{|g(x)| > s\}\right) \le t\right\}.$$

We also denote

$$g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)ds.$$

Then we have the following O'Neil lemma [41].

Lemma 3.3 *For $h = f * g$, we have*

$$h^{**}(t) \le tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)ds.$$

Proof of Theorem 3.1 Let $g(x) = \left(\frac{1}{\kappa_C}\right)^{(N-1)/N}\frac{1}{(H^o(x))^{N-1}}$ and $h = g * f$. Then $g^*(t) = t^{-(1/N')}$ and $g^{**}(t) = Ng^*(t)$. Hence, by Lemma 3.3, we obtain

$$h^*(t) \le h^{**}(t) \le Nt^{-(1/N')}\int_0^t f^*(s)ds + \int_t^{\mathrm{vol}(\Omega)} f^*(s)s^{-(1/N')} ds.$$

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We note that

$$\int_{\Omega} \exp\left(\frac{1}{\kappa_C}\left|\frac{1/((H^o)^{N-1} * f(x))}{\|f\|_N}\right|^{N/(N-1)}\right) dx = \int_0^{\mathrm{vol}(\Omega)} \exp\left([h^*(t)]^{N/(N-1)}\right) dt.$$

By changing of variables $t = \mathrm{vol}(\Omega)e^{-\tau}$, we obtain

$$\int_{\Omega} \exp\left(\frac{1}{\kappa_C}\left|\frac{1/((H^o)^{N-1} * f(x))}{\|f\|_N}\right|^{N/(N-1)}\right) dx \le \mathrm{vol}(\Omega) \int_0^{\infty} \exp\left[h^{**}\left(\mathrm{vol}(\Omega)e^{-\tau}\right)^{N/(N-1)} - \tau\right] d\tau.$$

By setting $\varphi(s) = \mathrm{vol}(\Omega)^{1/N} f^*\left(\mathrm{vol}(\Omega)e^{-s}\right)e^{-(s/N)}$, we get

$$h^{**}\left(\mathrm{vol}(\Omega)e^{-\tau}\right) \le Ne^{\tau/N'} \int_{\tau}^{\infty} \varphi(s)e^{-(s/N')}\, ds + \int_0^{\tau} \varphi(s) ds.$$

Now, we can apply Lemma 3.2 with

$$a(s,t) = \begin{cases} 1 & 0 \le s \le t, \\ Ne^{(t-s)/N'} & t < s < \infty. \end{cases}$$

∎

Proof of Theorem 1.1 We begin with the well-known formula

$$f(x) = \int_0^{\infty} \nabla f(x - \sigma r) \cdot \sigma dr.$$

As a consequence,

$$\int_{S^{N-1}} f(x) \frac{1}{\left(H^o(\sigma)\right)^N} d\sigma = \int_{S^{N-1}} \int_0^{\infty} \frac{\nabla f(x - \sigma r)}{\left(H^o(\sigma)\right)^N} \cdot \sigma\, dr\, d\sigma = \int_{S^{N-1}} \int_0^{\infty} \frac{\nabla f(x - \sigma r)}{\left(H^o(r\sigma)\right)^N} \cdot r\sigma r^{N-1}\, dr\, d\sigma.$$

Changing from polar coordinates to rectangular coordinate $y = x - r\sigma$, we have

$$f(x) = \frac{1}{N\kappa_C} \int_{\mathbb{R}^N} \frac{\nabla f(y) \cdot (x - y)}{\left(H^o(x - y)\right)^N} dy.$$

Hence,

$$|f(x)| \le \frac{1}{N\kappa_C}\left(\frac{1}{\left(H^o\right)^{N-1}} * H(\nabla f)\right)(x)$$

Hence by Theorem 3.1, we get

$$\int_{\Omega} \exp\left(\frac{1}{\kappa_C}(N\kappa_C)^{N/(N-1)}|f(x)|^{N/(N-1)}\right) \le \int_{\Omega} \exp\left(\frac{1}{\kappa_C}\left|\frac{1}{\left(H^o\right)^{N-1}} * H(\nabla f)\right|^{N/(N-1)}\right) \le C\,\mathrm{vol}(\Omega).$$

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The optimality of γ_C can be verified using the following Moser-type sequences

$$u_n(x) = \begin{cases} 0, & H^o(x) \geq 1, \\ \left(\frac{1}{\kappa_C n}\right)^{1/N} \log\left(\frac{1}{H^o(x)}\right), & e^{-(n/N)} < H^o(x) <, 1 \\ \left(\frac{1}{N\kappa_C}\right)^{1/N}\left(\frac{n}{N}\right)^{(N-1)/N}, & 0 \leq H^o(x) \leq e^{-(n/N)}. \end{cases} \tag{3.1}$$

■

4 Proof of Theorem 1.2

Now, for $f \in C_0^\infty(\mathbb{R}^N)$, we will use the following p-homogeneous function:

$$C_f^*(x) = \frac{1}{p}\int_{S^{N-1}} \|y\|_{p,\lambda,f}^{-N-p}\left[(1-\lambda)\langle x,y\rangle_+^p + \lambda\langle x,y\rangle_-^p\right] dy,$$

$$\|x\|_{p,\lambda,f} = \left[\int_{\mathbb{R}^N} (1-\lambda)\big\langle x, \nabla f(y)\big\rangle_+^p + \lambda\big\langle x, \nabla f(y)\big\rangle_-^p \, dy\right]^{1/p}.$$

We also set

$$K_{\lambda,p}(f) = \left\{x : C_f(x) \leq \frac{1}{q}\right\},$$
$$L_{\lambda,p}(f) = \left\{x : \|x\|_{p,\lambda,f} \leq 1\right\}.$$

We note that

$$\begin{aligned}
\int_{\mathbb{R}^N} pC_f^*(\nabla f(x))\, dx &= \int_{\mathbb{R}^N}\int_{S^{N-1}} \|y\|_{p,\lambda,f}^{-N-p}\left[(1-\lambda)\langle \nabla f(x),y\rangle_+^p + \lambda\langle \nabla f(x),y\rangle_-^p\right] dy \\
&= \int_{S^{N-1}} \|y\|_{p,\lambda,f}^{-N-p}\int_{\mathbb{R}^N}\left[(1-\lambda)\langle \nabla f(x),y\rangle_+^p + \lambda\langle \nabla f(x),y\rangle_-^p\right] dx\, dy \\
&= \int_{S^{N-1}} \|y\|_{p,\lambda,f}^{-N}\, dy \\
&= N \operatorname{vol}\big(L_{\lambda,p}(f)\big) = \left(\frac{\mathcal{E}_{\lambda,p}(f)}{c_{N,p}}\right)^{-N}.
\end{aligned}$$

Moreover, from

$$\begin{aligned}
h_{K_{\lambda,p}(f)}^p(x) &= \int_{S^{N-1}} \|y\|_{p,\lambda,f}^{-N-p}\left[(1-\lambda)\langle x,y\rangle_+^p + \lambda\langle x,y\rangle_-^p\right] dy \\
&= (N+p)\int_{S^{N-1}}\int_0^{\|y\|_{p,\lambda,f}^{-1}} \left[(1-\lambda)\langle x,y\rangle_+{}^p + \lambda\langle x,y\rangle_-^p\right] r^{N+p-1}\, dr\, dy \\
&= (N+p)\int_{L_{\lambda,p}(f)} \left[(1-\lambda)\langle x,y\rangle_+{}^p + \lambda\langle x,y\rangle_-^p\right] dy \\
&= (N+p)\alpha_{N,p} \operatorname{vol}\big(L_{\lambda,p}(f)\big) h_{\Gamma_{\lambda,p}L_{\lambda,p}(f)}{}^p(x),
\end{aligned}$$

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we can deduce that

$$K_{\lambda,p}(f) = \left[(N+p)\alpha_{N,p} \operatorname{vol}\left(L_{\lambda,p}(f)\right)\right]^{1/p} \Gamma_{\lambda,p} L_{\lambda,p}(f).$$

Thus, by the L^p Busemann–Petty centroid inequality $\operatorname{vol}(\Gamma_{\lambda,p} L_{\lambda,p}(f)) \geq \operatorname{vol}(L_{\lambda,p}(f))$, we obtain

$$\operatorname{vol}\left(K_{\lambda,p}(f)\right) \geq \left[(N+p)\alpha_{N,p}\right]^{N/p} \left(\frac{1}{N}\left(\frac{\mathcal{E}_{\lambda,p}(f)}{c_{N,p}}\right)^{-N}\right)^{(N+p)/p}. \tag{4.1}$$

The equality occurs when $L_{\lambda,p}(f)$ is an origin-centered ellipsoid.

In our case, $C^*(x) = C_f^*(x)$ with $p = N$. Hence,

$$\kappa_C = \operatorname{vol}\left(K_{\lambda,N}(f)\right) \geq \left[(N+N)\alpha_{N,N}\right]^{N/N} \left(\frac{1}{N}\left(\frac{\mathcal{E}_{\lambda,N}(f)}{c_{N,N}}\right)^{-N}\right)^{(N+N)/N}.$$

Thus, if $\mathcal{E}_{\lambda,N}^N(f) \leq N^N \omega_N$, then we have

$$\begin{aligned}\int_{\mathbb{R}^N} pC_f^*\left(\nabla f(x)\right) dx &= \left(\frac{\mathcal{E}_{\lambda,N}(f)}{c_{N,p}}\right)^{-N} \\ &\leq N^N 2N\alpha_{N,N} \left(\frac{1}{N}\left(\frac{\mathcal{E}_{\lambda,N}(f)}{c_{N,N}}\right)^{-N}\right)^2 \\ &= N^N \kappa_C.\end{aligned}$$

Hence, by Theorem 1.1,

$$\int_{\mathbb{R}^N} \left[\exp\left(|f(x)|^{N/(N-1)}\right) - 1\right] dx \leq C_N \operatorname{vol}\left(\operatorname{supp}(f)\right).$$

References

[1] S. Adachi and K. Tanaka, *Trudinger type inequalities in* $\mathbb{R}n^N$ *and their best exponents*. Proc. Amer. Math. Soc. **128**(2000), no. 7, 2051–2057. http://dx.doi.org/10.1090/S0002-9939-99-05180-1

[2] D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives*. Ann. of Math. (2) **128**(1988), no. 2, 385–398. http://dx.doi.org/10.2307/1971445

[3] S. K. Adimurthi, *A singular Moser-Trudinger embedding and its applications*. NoDEA Nonlinear Diff. Equat. Appl. **13**(2007), no. 5–6, 585–603. http://dx.doi.org/10.1007/s00030-006-4025-9

[4] S. K. Adimurthi and Y. Yang, *An interpolation of Hardy inequality and Trudinger-Moser inequality in* $\mathbb{R}^N$ *and its applications*. Int. Math. Res. Not. IMRN 2010, no. 13, 2394–2426. http://dx.doi.org/10.1093/imrn/rnp194

[5] A. Alvino, V. Ferone, G. Trombetti, and P. L. Lions, *Convex symmetrization and applications*. Ann. Inst. H. Poincaré Anal. Non Linéaire **14**(1997), no. 2, 275–293. http://dx.doi.org/10.1016/S0294-1449(97)80147-3

[6] Z. M. Balogh, J. J. Manfredi, and J. T. Tyson, *Fundamental solution for the Q-Laplacian and sharp Moser-Trudinger inequality in Carnot groups*. J. Funct. Anal. **204**(2003), no. 1, 35–49. http://dx.doi.org/10.1016/S0022-1236(02)00169-6

[7] T. P. Branson, L. Fontana, and C. Morpurgo, *Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere*. Ann. of Math. (2) **177**(2013), no. 1, 1–52. http://dx.doi.org/10.4007/annals.2013.177.1.1

[8] L. Carleson, S.-Y. A. Chang, *On the existence of an extremal function for an inequality of J. Moser*. Bull. Sci. Math. (2) **110**(1986), no. 2, 113–127.

[9] D. Cassani, B. Ruf, and C. Tarsi, *Best constants for Moser type inequalities in Zygmund spaces*. Mat. Contemp. **36**(2009), 79–90.

[10] S.-Y. A. Chang and P. C. Yang, *The inequality of Moser and Trudinger and applications to conformal geometry*. Dedicated to the memory of Jürgen K. Moser. Comm. Pure Appl. Math. **56**(2003), no. 8, 1135–1150. http://dx.doi.org/10.1002/cpa.3029

[11] A. Cianchi, E. Lutwak, D. Yang, and G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*. Calc. Var. Partial Diff. Equat. **36**(2009), no. 3, 419–436. http://dx.doi.org/10.1007/s00526-009-0235-4
[12] W. S. Cohn and G. Lu, *Best constants for Moser-Trudinger inequalities on the Heisenberg group*. Indiana Univ. Math. J. **50**(2001), no. 4, 1567–1591. http://dx.doi.org/10.1512/iumj.2001.50.2138
[13] W. S. Cohn and G. Lu, *Best constants of Moser-Trudinger inequalities, fundamental solutions and one-parameter representation formulas on groups of Heisenberg type.* Acta. Math. Sin. (Engl. Ser.) **18**(2002) 375–390. http://dx.doi.org/10.1007/s101140200159
[14] W. S. Cohn and G. Lu, *Sharp constants for Moser-Trudinger inequalities on spheres in complex space* $\mathbb{C}^n$. Comm. Pure Appl. Math. **57**(2004), no. 11, 1458–1493. http://dx.doi.org/10.1002/cpa.20043
[15] G. Csató and P. Roy, *Extremal functions for the singular Moser-Trudinger inequality in 2 dimensions*. Calc. Var. Partial Diff. Equat. **54**(2015), no. 2, 2341–2366. http://dx.doi.org/10.1007/s00526-015-0867-5
[16] D. G. de Figueiredo, J. M. do Ó, and B. Ruf, *Elliptic equations and systems with critical Trudinger-Moser nonlinearities*. Discrete Contin. Dyn. Syst. **30**(2011), no. 2, 455–476. http://dx.doi.org/10.3934/dcds.2011.30.455
[17] P. L. De Nápoli, J. Haddad, C. H. Jiménez, and M. Montenegro, *M. The sharp affine L^2 Sobolev trace inequality and variants*. Math. Ann. **370**(2018), no. 1–2, 287–308. http://dx.doi.org/10.1007/s00208-017-1548-9
[18] J. M. do Ó, *N-Laplacian equations in $\mathbb{R}^N$ with critical growth*. Abstr. Appl. Anal. **2**(1997), no. 3–4, 301–315. http://dx.doi.org/10.1155/S1085337597000419
[19] M. Dong, N. Lam, and G. Lu, *Sharp weighted Trudinger-Moser and Caffarelli-Kohn-Nirenberg inequalities and their extremal functions*. Nonlinear Anal. **173**(2018), 75–98. http://dx.doi.org/10.1016/j.na.2018.03.006
[20] M. Flucher, *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*. Comment. Math. Helv. **67**(1992), 471–497. http://dx.doi.org/10.1007/BF02566514
[21] L. Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*. Comment. Math. Helv. **68**(1993), no. 3, 415–454. http://dx.doi.org/10.1007/BF02565828
[22] L. Fontana and C. Morpurgo, *Adams inequalities on measure spaces*. Adv. Math. **226**(2011), no. 6, 5066–5119. http://dx.doi.org/10.1016/j.aim.2011.01.003
[23] C. Haberl and F. E. Schuster, *General L_p affine isoperimetric inequalities*. J. Diff. Geom. **83**(2009), no. 1, 1–26.
[24] C. Haberl, F. E. Schuster, and J. Xiao, *J. An asymmetric affine Pólya–Szegö principle.* Math. Ann. **352**(2012), 517–542. http://dx.doi.org/10.1007/s00208-011-0640-9
[25] J. Haddad, C. H. Jiménez, and M. Montenegro, *Asymmetric Blaschke-Santaló functional inequalities*. J. Funct. Anal. **278**(2020), no. 2, 108319, 18 pp. http://dx.doi.org/10.1016/j.jfa.2019.108319
[26] J. Haddad, C. H. Jiménez, and M. Montenegro, *Sharp affine Sobolev type inequalities via the L_p Busemann–Petty centroid inequality*. J. Funct. Anal. **271**(2016) 454–473. http://dx.doi.org/10.1016/j.jfa.2016.03.017
[27] N. Lam and G. Lu, *Sharp Adams type inequalities in Sobolev spaces $W^{m,(n/m)}(\mathbb{R}^n)$ for arbitrary integer m*. J. Diff. Equat. **253**(2012), no. 4, 1143–1171. http://dx.doi.org/10.1016/j.jde.2012.04.025
[28] N. Lam and G. Lu, *Sharp Moser-Trudinger inequality on the Heisenberg group at the critical case and applications*. Adv. Math. **231**(2012), no. 6, 3259–3287. http://dx.doi.org/10.1016/j.aim.2012.09.004
[29] N. Lam, G. Lu, and L. Zhang, *Existence and nonexistence of extremal functions for sharp Trudinger-Moser inequalities*. Adv. Math. **352**(2019), 1253–1298. http://dx.doi.org/10.1016/j.aim.2019.06.020
[30] N. Lam and H. Tang, *Sharp constants for weighted Moser-Trudinger inequalities on groups of Heisenberg type*. Nonlinear Anal. **89**(2013), 95–109. http://dx.doi.org/10.1016/j.na.2013.04.017
[31] Y. Li, *Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds*. Sci. China Ser. A **48**(2005), no. 5, 618–648. http://dx.doi.org/10.1360/04ys0050
[32] Y. Li, *Remarks on the extremal functions for the Moser-Trudinger inequality. Acta* Math. Sin. (Engl. Ser.) **22**(2006), no. 2, 545–550. http://dx.doi.org/10.1007/s10114-005-0568-7
[33] Y. Li and B. Ruf, B. *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^n$. Indiana Univ. Math. J. **57**(2008), no. 1, 451–480. http://dx.doi.org/10.1512/iumj.2008.57.3137
[34] K.-C. Lin, *Extremal functions for Moser's inequality*. Trans. Amer. Math. Soc. **348**(1996), 2663–2671. http://dx.doi.org/10.1090/S0002-9947-96-01541-3
[35] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*. J. Diff. Geom. **38**(1993), no. 1, 131–150.

[36] E. Lutwak, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*. Adv. Math. **118**(1996), no. 2, 244–294.
[37] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*. J. Diff. Geom. **56**(2000), no. 1, 111–132.
[38] N. Masmoudi and F. Sani, *Trudinger-Moser inequalities with the exact growth condition in $\mathbb{R}^N$ and applications*. Comm. Partial Diff. Equat. **40**(2015), no. 8, 1408–1440. http://dx.doi.org/10.1080/03605302.2015.1026775
[39] J. Moser, *A sharp form of an inequality by Trudinger*. Indiana Univ. Math. J. **20**(1970/71), 1077–1092. http://dx.doi.org/10.1512/iumj.1971.20.20101
[40] V. H. Nguyen, *New approach to the affine Pólya-Szegö principle and the stability version of the affine Sobolev inequality*. Adv. Math. **302**(2016), 1080–1110. http://dx.doi.org/10.1016/j.aim.2016.08.003
[41] R. O'Neil, *Convolution operators and $L(p,q)$ spaces*. Duke Math. J. **30**(1963), 129–142.
[42] C. M. Petty, *Centroid surfaces*. Pacific J. Math. **11**(1961), 1535–1547.
[43] S. I. Pohozhaev, *On the imbedding Sobolev theorem for $pl = n$*. (Russian) Doklady Conference, Section Math. Moscow Power Inst., 1965, pp. 158–170.
[44] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$*. J. Funct. Anal. **219**(2005), no. 2, 340–367. http://dx.doi.org/10.1016/j.jfa.2004.06.013
[45] B. Ruf and F. Sani, *Sharp Adams-type inequalities in $\mathbb{R}^n$*. Trans. Amer. Math. Soc. **365**(2013), no. 2, 645–670. http://dx.doi.org/10.1090/S0002-9947-2012-05561-9
[46] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*. 2nd expanded ed., Encyclopedia of Mathematics and its Applications, 151, Cambridge University Press, Cambridge, 2014.
[47] C. Tarsi, *Adams' inequality and limiting Sobolev embeddings into Zygmund spaces*. Potential Anal. **37**(2012), no. 4, 353–385. http://dx.doi.org/10.1007/s11118-011-9259-4
[48] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*. J. Math. Mech. **17**(1967), 473–483. http://dx.doi.org/10.1512/iumj.1968.17.17028
[49] J. Xiao and Zh. Zhai, *Fractional Sobolev, Moser-Trudinger Morrey-Sobolev inequalities under Lorentz norms*. Problems in Mathematical Analysis, 45, J. Math. Sci. (N.Y.) **166**(2010), no. 3, 357–376. http://dx.doi.org/10.1007/s10958-010-9872-6
[50] V. I. Yudovic, *Some estimates connected with integral operators and with solutions of elliptic equations*. Dokl. Akad.Nauk SSSR, **138**(1961), 805–808; English trans. in Soviet Math. Doklady **2**(1961), 746–749.
[51] G. Zhang, *The affine Sobolev inequality*. J. Diff. Geom. **53**(1999), no. 1, 183–202.

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