The Solutions of Mathieu's Differential Equation: Representation by Contour Integrals, and Asymptotic Expansions.

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1. It is an obvious remark that the Mathieu functions, being the harmonic functions of the elliptic cylinder, must be closely related to the Bessel functions, the harmonic functions of the circular cylinder. Reference has been made to some aspects of this relationship in two earlier communications,* to which the present paper may be regarded as a sequel.

To any theorem in Mathieu functions, there may be expected to correspond a theorem, probably simpler, in Bessel Functions. There will not usually be much difficulty in passing from the general to the special case; the converse problem may be much more formidable. It is a case of the latter kind with which the present paper deals, and in this instance the generalization turns out to be comparatively simple though not, I think, trivial or obvious.

The Bessel Functions can be represented by definite integrals in several well-known ways. One of these representations, which Bessel, Jacobi and Hankel all had a hand in developing, expresses a solution of Bessel's equation

$$\frac{d^2 w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{s^2}{z^2}\right) w = 0$$

in the form †

$$z^s \int e^{izt} (t^2 - 1)^{s - \frac{1}{2}} \, dt,$$

* *The Solution of Mathieu's Differential Equation*, *Proc. Edin. Math. Soc.*, Vol. XXXIV (1915-16);

The Solutions of Mathieu's Differential Equation, and their Asymptotic Expansions, *Proc. Edin. Math. Soc.*, Vol. XLI (1922–23).

These papers will be referred to as I and II; and I (1), *e.g.*, will be used for "Equation (1) of Paper I."

† Gray, Mathews and MacRobert, *Bessel Functions*, 2nd Edition, Chap. V, § 2;

Whittaker and Watson, *Modern Analysis*, 3rd Edition, Chap. XVII, § 17. 3.

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the integral being taken round a suitable contour; and it is this integral which is most convenient, and has been most used, for the purpose of obtaining the asymptotic expansions of the solutions. The object of the analysis given below is to find an analogous representation for the solutions of Mathieu's equation. In the result, the analogy is complete so far as the factors z^s and e^{izt} are concerned; but the remaining factor is a good deal more complex than the simple function $(t^2-1)^{s-\frac{1}{2}}$ which occurs in the above integral; it is found initially in the form of a power series with a finite radius of convergence, and an essential step in the investigation (the only step calling for any sort of ingenuity) is to determine the "continuation" of this function over the whole plane. Once this has been done it is a simple matter to get contour integrals for the solutions defined in the earlier papers. These reduce to the integrals for the Bessel Functions as a special case.

The asymptotic series for the solution of the second kind follows at once, and is identical with the one obtained in the 1922 paper.

2. The differential equation is

$$\frac{d^2 u}{d\alpha^2} + (\tfrac{1}{2}\kappa^2 c^2 \cosh 2\alpha - s^2)\, u = 0.$$

When we put $\sigma = \kappa c e^{\alpha} / 2i$, this becomes (with $k = \frac{1}{2}\kappa c$)

$$\frac{d^2 u}{d\sigma^2} + \frac{1}{\sigma}\frac{du}{d\sigma} - \left(1 + \frac{s^2}{\sigma^2} + \frac{k^4}{\sigma^4}\right) u = 0 \quad \text{...........(1)}$$

In the previous papers (*e.g.* II, § 15) solutions of (2) have been defined which, ignoring reference to k and s, we may write

$$\left.\begin{aligned} J(\mu, i\sigma) &= \surd(2/\pi)\, e^{i\mu\pi} \sum_{n=-\infty}^{\infty} \chi(n+\mu)\, \sigma^{2n+2\mu}, \\ J(-\mu, i\sigma) &= \surd(2/\pi)\, e^{-i\mu\pi} \sum_{n=-\infty}^{\infty} \chi(n-\mu)\, \sigma^{2n-2\mu}, \\ G(\mu, i\sigma) &= (\tfrac{1}{2}\pi/\sin 2\mu\pi)\, \{J(-\mu, i\sigma) - e^{-2i\mu\pi} J(\mu, i\sigma)\} \\ &= \surd(\pi/2)\, \frac{e^{-i\mu\pi}}{\sin 2\mu\pi}\, \{\chi(n-\mu)\, \sigma^{2n-2\mu} \\ &\qquad - \chi(n+\mu)\, \sigma^{2n+2\mu}\}. \end{aligned}\right\} \quad \text{...(2)}$$

As pointed out at II, § 24, the Bessel functions $J_s(i\sigma)$, $J_{-s}(i\sigma)$,

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$G_s(i\sigma)$ may be regarded simply as the special cases of the above three functions corresponding to the value 0 for k.

3. We first express the coefficient $\chi(n+\mu)$, which occurs in (2), as a complex integral.

By II, (14),

$$\chi(n+\mu) = \frac{1}{\Pi(2n+2\mu+\frac{1}{2})} + \frac{B_1}{\Pi(2n+2\mu+\frac{1}{2}+1)} + \dots,$$

which, by means of the properties of the Beta function * is easily expressed in the form

$$\chi(n+\mu) =$$

$$-\frac{1}{2}\frac{e^{-2i\mu\pi}}{\sin(2\mu-\frac{1}{2})\pi}\int_0^{(1+)}\frac{t^{2n}}{\Pi(2n)}\left\{\frac{(1-t)^{2\mu-\frac{1}{2}}}{\Pi(2\mu-\frac{1}{2})} + B_1\frac{(1-t)^{2\mu-\frac{1}{2}+1}}{\Pi(2\mu-\frac{1}{2}+1)} + \dots\right\}dt,$$

the meaning of the notation $\int_0^{(1+)}$ being that the path of integration starts at $t=0$, encircles the point $t=1$ in the positive direction, and returns to the starting point (fig. 1); the initial amplitude of $1-t$ is 0.

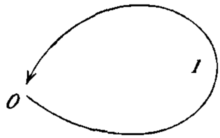

Fig. 1

Denote the function multiplying $t^{2n}/\Pi(2n)$ within the integral sign by $F(t)$, so that

$$F(t) = \sum_{p=0}^{\infty} B_p \frac{(1-t)^{2\mu-\frac{1}{2}+p}}{\Pi(2\mu-\frac{1}{2}+p)}, \quad \dots\dots\dots\dots\dots\dots(3)$$

and

$$\chi(n+\mu) = \frac{1}{2}\frac{e^{-2i\mu\pi}}{\cos 2\mu\pi}\int_0^{(1+)}\frac{t^{2n}}{\Pi(2n)}F(t)\,dt \quad \dots\dots\dots(4)$$

* *Cf.* Whittaker and Watson, *Modern Analysis*, Chap. XII, §§12. 22, 12. 43.

On writing $\mu - \frac{1}{2}$ for z in II (33), we obtain

$$F(t) = \chi(\mu - \tfrac{1}{2}) - t\chi(\mu - 1) + \frac{t^2}{2!}\chi(\mu - \tfrac{3}{2}) - \frac{t^3}{3!}\chi(\mu - 2) + \dots$$
$$= E(t) - \Omega(t), \quad \text{.....(5)}$$

where

$$E(t) = \chi(\mu - \tfrac{1}{2}) + \frac{t^2}{2!}\chi(\mu - \tfrac{3}{2}) + \dots, \quad \text{.....(6)}$$

$$\Omega(t) = t\chi(\mu - 1) + \frac{t^3}{3!}\chi(\mu - 2) + \dots. \quad \text{.....(7)}$$

4. The nature of the convergence of the series (3), (6) and (7) is of essential consequence for what follows. Since, by II (31), as $p \to \infty$,

$$B_p \approx (1/\pi)\cos 2\mu\pi \, 2^{-p}\,\Pi(p - 1), \quad \text{.....(8)}$$

the series (3) has a radius of convergence 2; so that the series defines the function $F(t)$ within the circle $|1 - t| = 2$.

As for (6), the coefficient of $t^{2p}/(2p)!$ is $\chi(\mu - p - \frac{1}{2})$; and, by II (23), when $p \to \infty$,

$$\chi(\mu - p - \tfrac{1}{2}) \approx (2/\pi)\cos 2\mu\pi \, \Pi(2p - 2\mu - \tfrac{1}{2});$$

so that the circle of convergence of (6) is the circle $|t| = 1$.

In (7), however, the coefficient of $t^{2n-1}/(2n - 1)!$ is $\chi(\mu - n)$, which is simply a multiple of the coefficient of $\sigma^{-2n+2\mu}$ in the series (2) which defines $J(\mu, i\sigma)$. The function $\Omega(t)$ is therefore holomorphic.

The asymptotic value of $\chi(\mu - n)$ as $n \to \infty$, is easily proved (n being a positive integer) by I, § 5, and II, § 2, to be

$$\frac{\chi(\mu)}{\chi(-\mu)} k^{4n} \frac{1}{\Pi\, 2n - 2\mu + \frac{1}{2})}.$$

5. From (3) it follows that $t = 1$ is a singular point of $F(t)$; and we can prove in a moment that $t = -1$ is another. For, by (5) and (3),

$$F(t) = \{E(t) + \Omega(t)\} - 2\Omega(t)$$
$$= \sum_{p=0}^{\infty} B_p \frac{(1 + t)^{2\mu - \frac{1}{2} + p}}{\Pi(2\mu - \frac{1}{2} + p)} - 2\Omega(t); \quad \text{.....(9)}$$

this formula continues $F(t)$ over the area of the circle $|t + 1| = 2$.

The formulae (3), (9) and (5) respectively represent $F(t)$ over the three circles shown in fig. 2, having their centres at 1, -1 and 0.

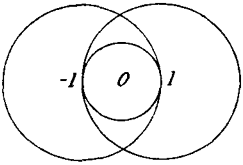

Fig. 2

It will be seen presently that $F(t)$ has no singularity at a finite distance besides $t = 1$ and $t = -1$.

6. With a view to finding a formula representing $F(t)$ over the region outside the circle $|t| = 1$, consider the complex integral.

$$\int_{\epsilon+\infty i}^{\epsilon-\infty i} \frac{(te^{-i\pi/2})^z \chi(\mu - \frac{1}{2} - \frac{1}{2}z)}{\Pi(z)\sin\pi z \sin(\frac{1}{2}z - 2\mu + \frac{1}{2})\pi} \pi dz, \quad \ldots\ldots(10)$$

the path being straight, parallel to the imaginary axis, and crossing the real axis where $z = \epsilon$; ϵ is taken so that $-1 < \epsilon < 0$, and so that the path avoids any zero of $\sin(\frac{1}{2}z - 2\mu + \frac{1}{2})\pi$.

We shall show, provided

$$0 < \operatorname{amp} t < \pi, \text{ and } |t| > 0,$$

that the integral converges, and that the form of the integrand at infinity is such that we can equate the integral to (i) $2i\pi$. sum of residues at poles to right of path, when $|t| < 1$, (ii) $2i\pi$. sum of residues at poles to left of path, when $|t| > 1$.
In the first place, for any large z to the left of the path we have (II (12) and II, §7)

$$\chi(\mu - \tfrac{1}{2} - \tfrac{1}{2}z) \approx 1/\Pi(2\mu - \tfrac{1}{2} - z);$$

and, to the right of the path (II (23)),

$$\chi(\mu - \tfrac{1}{2} - \tfrac{1}{2}z)$$
$$\approx (2/\pi)\sin(2\mu - \tfrac{1}{2} - \tfrac{1}{2}z)\pi \ \sin(-\tfrac{1}{2} - \tfrac{1}{2}z)\pi\Pi(-2\mu - \tfrac{1}{2} + z).$$

Hence, for the asymptotic value of the integrand when $z \to \infty$, we have

(i) to left of path:

$$(te^{-i\pi/2})^z\,(-)\,\frac{\Pi(-z-1)}{\Pi(2\mu-\frac{1}{2}-z)}\,\frac{1}{\sin(\frac{1}{2}z-2\mu+\frac{1}{2})\pi};$$

(ii) to right of path:

$$(te^{-i\pi/2})^z\,\frac{\Pi(-2\mu-\frac{1}{2}+z)}{\Pi(z)}\,\frac{1}{\sin\frac{1}{2}z}.$$

In both cases the ratio of the two Π functions is given at once by the asymptotic formula

$$\Pi(u+a) \approx u^a\,\Pi(u),$$

which holds if $u \to \infty$ in any direction except the negative direction of the real axis; and, if $te^{-i\pi/2} = re^{i\theta}$, and $z = x + iy$, where r, θ, x, y are real, then

$$(te^{-i\pi/2})^z = (re^{i\theta})^{x+iy} = r^{iy}\,e^{ix\theta}\,r^x\,e^{-y\theta}.$$

The factor $e^{-y\theta}$ is taken care of by the sine in the denominator of the integrand, if $|\theta| < \frac{1}{2}\pi$; and the factor r^x by the proviso that $r > 1$ when $x < 0$, and $r < 1$ when $x > 0$; and one, or both, of these exponential factors will outweigh any factor of the form z^a arising from the ratio of the Π functions.

On the path itself, since x is constant, obviously we need only have $|t| > 0$. The assertion about the integral (10) is therefore proved.

7. We can now continue $F(t)$ into the region $|t| > 1$, by applying the residue theorem to the integral (10).

(*a*) If $|t| < 1$,

integral $/2i\pi$ = sum of residues at poles to right: and

(i) at $z = 0, 1, 2, \ldots$ residues are

$$\frac{1}{\sin(-2\mu+\frac{1}{2})\pi}\left\{\chi(\mu-\tfrac{1}{2})+\frac{t^2}{1\cdot 2}\chi(\mu-\tfrac{3}{2})+\ldots\right\}$$

$$+\quad\frac{i}{\sin 2\mu\pi}\quad\left\{t\chi(\mu-1)+\frac{t^3}{3\,!}\chi(\mu-2)+\ldots\right\};$$

(ii) at $\frac{1}{2}z = 2\mu - \frac{1}{2} + n$, or $z = 4\mu - 1 + 2n$, where n is any integer, residue is

$$\frac{t^{4\mu - 1 + 2n} e^{(1 - 4\mu) i\pi/2} e^{-in\pi}}{\Pi(4\mu - 1 + 2n) \sin(4\mu - 1)\pi} \frac{\chi(-n-\mu)}{\frac{1}{2}\cos n\pi}$$
$$= \frac{t^{4\mu - 1 + 2n}}{\Pi(4\mu - 1 + 2n)} \chi(-n-\mu) \frac{(-)2ie^{-2i\mu\pi}}{\sin 4\mu\pi}.$$

On the right of the path, residues of the form (ii) occur for values of n such that $4\mu - 1 + 2n > \epsilon$; and it is clear from the remark at the end of § 4 that this series of residues converges whether $|t| < 1$ or not.

(*b*) If $|t| > 1$,
integral $/2i\pi = (-)$ sum of residues at poles to left: and these residues are as in (*a*) (ii), but with $4\mu - 1 + 2n < \epsilon$; and in this case we must have $|t| > 1$, the fraction $\chi(-n-\mu)/\Pi(4\mu - 1 + 2n)$ being proportional to a power of n, asymptotically, when $n \to -\infty$.

Thus the function represented by the integral (10) for all values of t such that $0 < \arg t < \pi$, and $|t| > 0$, is given by the series of (*a*), (i) and (ii) when $|t| < 1$; and by the series (*b*), when $|t| > 1$. It follows that the function of t given by (i) + (ii) of (*a*) when $|t| < 1$ continues, *through the semicircle above the real axis* which has $t = 1$, $t = -1$ for ends of a diameter, into the function given by (*b*), when $|t| > 1$. Further, as has been noted, the form (*a*) (ii) holds in both regions, so that this statement about continuation will still be true when we subtract this series (*a*) (ii) from both sides. Hence, finally, in the notation of § 3 the function

$$\frac{1}{\cos 2\mu\pi} E(t) + \frac{i}{\sin 2\mu\pi} \Omega(t), \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

given in the space $|t| < 1$ by (6) and (7), continues, through the upper half of the circle on 1, -1 as diameter, into the space $|t| > 1$, in the form

$$\sum_{n=-\infty}^{\infty} \frac{t^{4\mu - 1 + 2n}}{\Pi(4\mu - 1 + 2n)} \chi(-n-\mu) \frac{2ie^{-2i\mu\pi}}{\sin 4\mu\pi}. \quad \ldots\ldots\ldots(12)$$

Since $\Omega(t)$ is holomorphic, this result gives also at once the continuation of $E(t)$ and of $F(t)$ (§ 3).

Clearly the function $F(t)$ is now completely defined. Thus, within the unit circle $|t| < 1$, (6), (7) and (5) give initial determinations of $E(t)$ and $F(t)$ without ambiguity; and the effect of a turn round $t = 1$, or $t = -1$, is seen at once from (3), or (9). By means of combinations of these turns, in the positive or negative direction, and the continuation defined at (11) and (12), we can follow the values of $F(t)$ for any path of the variable, starting from the initial determination, say $F_0(t)$, within the unit circle.

8. We are now in a position to construct the definite integral representations we set out to obtain.

We have to transform

$$\sum_{n=-\infty}^{\infty} \chi(n+\mu)\,\sigma^{2n+2\mu};$$

as a first step (4) will allow us to form a convenient expression for the part of this sum arising from positive values of n. Thus

$$\sum_{n=0}^{\infty} \chi(n+\mu)\,\sigma^{2n+2\mu}$$

$$= \tfrac{1}{4}\sigma^{2\mu}\,\frac{e^{-2i\mu\pi}}{\cos 2\mu\pi}\int_0^{(1+)} (e^{\sigma t} + e^{-\sigma t})\,F(t)\,dt.$$

This integral may be divided into two, each involving $e^{-\sigma t}$, to the exclusion of $e^{\sigma t}$; for, on changing t into $-t$,

$$\int_0^{(1+)} e^{\sigma t}\,F(t)\,dt$$

becomes, by (9)

$$-\int_0^{(-1+)} e^{-\sigma t}\left\{\frac{(1+t)^{2\mu-\frac{1}{2}}}{\Pi(2\mu-\frac{1}{2})} + B_1\frac{(1+t)^{2\mu-\frac{1}{2}+1}}{\Pi(2\mu-\frac{1}{2}+1)} + \dots\right\}dt$$

$$= -\int_0^{(-1+)} e^{-\sigma t}\,\{F(t) + 2\Omega(t)\}\,dt,$$

$$= -\int_0^{(-1+)} e^{-\sigma t}\,F(t)\,dt,$$

since $\Omega(t)$ is holomorphic.

Hence

$$\sum_{n=0}^{\infty} \chi(n+\mu)\,\sigma^{2n+2\mu}$$

$$= \tfrac{1}{4}\sigma^{2\mu}\,\frac{e^{-2i\mu\pi}}{\cos 2\mu\pi}\int_0^{(-1-,\,1+)} e^{-\sigma t} F(t)\,dt, \quad \text{.........(13)}$$

the notation meaning* that the path of integration starts at 0, encircles the point -1 in the negative direction and returns to 0, then encircles the point 1 in the positive direction and returns to 0 (fig. 3); also the argument of $1-t$ is 0 at the middle of the path, *i.e.* when the moving variable returns to 0 for the first time; in other words, each of the powers of $1-t$ has then the value 1, so that $F(t) = \chi(\mu - \frac{1}{2})$.

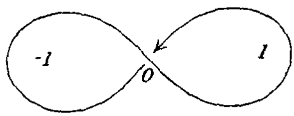

Fig. 3

9. We have still to supplement (13) by an expression for

$$\sum_{n=-1}^{-\infty} \chi(n+\mu)\,\sigma^{2n+2\mu}.$$

This we can obtain by prolonging the path in fig. 3 to infinity from both ends, as in fig. 4, leaving the right hand side of (13) otherwise unaltered; the direction of the path at a great distance being taken so that σt is real and positive.

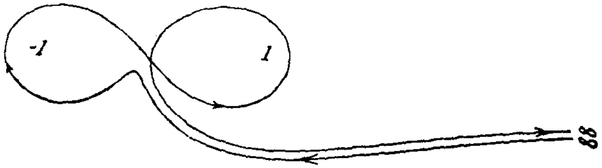

Fig. 4

* WHITTAKER and WATSON, *Modern Analysis*, § 12. 43.

To prove this, consider what $F(t)$, starting as $F_0(t)$ (end of § 7) at the middle of the path in fig. 3, becomes at the ends of that path.

The positive turn round $t = 1$ gives at the final point, by (3),

$$F(t) = -e^{4i\mu\pi} F_0(t).$$

As for the turn round $t = -1$, it follows from (9) that at the initial point of the path in fig. 3,

$$F(t) = -e^{4i\mu\pi}\{F_0(t) + 2\Omega(t)\} - 2\Omega(t)$$
$$= -e^{4i\mu\pi} F_0(t) - 2(e^{4i\mu\pi} + 1)\Omega(t).$$

Since we may suppose the two paths from 0 to ∞ in fig. 4 to be indefinitely close, the combined partial integrals* from these two paths give

$$\tfrac{1}{4}\sigma^{2\mu}\frac{e^{-2i\mu\pi}}{\cos 2\mu\pi}\int_0^\infty e^{-\sigma t}\, 2(e^{4i\mu\pi} + 1)\Omega(t)\,dt,$$

i.e. simply $\sigma^{2\mu}\int_0^\infty \Omega(t)\,dt,$

the two continuations of $F_0(t)$ annulling each other when we subtract.

On putting in the series for $\Omega(t)$ from (7), and integrating term by term, this becomes

$$\sigma^{2\mu}\int_0^\infty e^{-\sigma t}\left\{t\chi(\mu - 1) + \frac{t^3}{3!}\chi(\mu - 2) + \dots\right\}dt$$
$$= \sigma^{2\mu}\{\sigma^{-2}\chi(\mu - 1) + \sigma^{-4}\chi(\mu - 2) + \dots\}$$
$$= \sum_{n=-1}^{-\infty} \chi(n + \mu)\,\sigma^{2n+2\mu},$$

as was stated at the beginning of this §.

Instead of (13) we now have

$$\sum_{n=-\infty}^{\infty} \chi(n + \mu)\,\sigma^{2n+2\mu}$$
$$= \tfrac{1}{4}\sigma^{2\mu}\frac{e^{-2i\mu\pi}}{\cos 2\mu\pi}\int e^{-\sigma t} F(t)\,dt, \quad \dots\dots\dots(14)$$

the path being as in fig. 4.

* It is easy to show from (12) that the integrals of $e^{-\sigma t} F(t)$ over the two infinite paths, which were added to fig. 3 to give fig. 4, exist. *Cf. Modern Analysis*, § 5. 32.

On multiplying (14) by $e^{i\mu\pi}\sqrt{(2/\pi)}$, we obtain a contour integral for $J(\mu, i\sigma)$.

It is obvious that the two paths inserted from 0 to ∞ in fig 4. might be varied. An interesting contour is obtained by taking them above, instead of below, the point 1.

10. There is of course a corresponding integral for $J(-\mu, i\sigma)$, *i.e.* practically, for

$$\sum_{n=-\infty}^{\infty} \chi(n-\mu)\,\sigma^{2n-2\mu},$$

over the same path; we have only to change μ into $-\mu$ throughout in (14).

But there is also another quite different integral for this series, in which the integrand of (14) is retained exactly as it stands, but the path is changed; this new integral can be combined readily with (14), so as to give a single integral for the difference

$$\sum_{n=-\infty}^{\infty} \chi(n-\mu)\,\sigma^{2n-2\mu} - \sum_{n=-\infty}^{\infty} \chi(n+\mu)\,\sigma^{2n+2\mu},$$

which is, in effect, the solution $G(\mu, i\sigma)$.
The new path referred to is: from infinity, round both -1 and 1 positively, and back to infinity (fig. 5). Over all this path

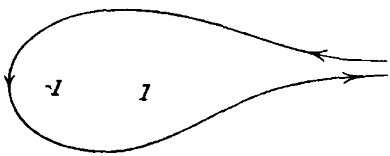

Fig. 5

we can suppose $|t| > 1$, and therefore use the expression for $F(t)$ obtainable at once from the result at (11) and (12) of § 7, which, with (5), gives

$$F(t) = -\Omega(t) - i\cot 2\mu\pi\,\Omega(t) + \frac{ie^{-2i\mu\pi}}{\sin 2\mu\pi}\sum_{n=-\infty}^{\infty}\frac{t^{4\mu-1+2n}}{\Pi(4\mu-1+2n)}\chi(-n-\mu).$$

Now, as at (14), multiply by $\frac{1}{4}\sigma^{2\mu} \dfrac{e^{-2i\mu\pi}}{\cos 2\mu\pi} e^{-\sigma t}$, and integrate term by term on the right, over the path of fig. 5 (*cf.* § 9). Since $\Omega(t)$ is holomorphic, the terms involving this function contribute nothing. Also, since the result of § 7 was deduced on the understanding that $0 < \arg t < \pi$, the argument of t at the initial part of the path must be taken as 0; hence

$$\int e^{-\sigma t} \frac{t^{4\mu - 1 + 2n}}{\Pi(4\mu - 1 + 2n)} dt$$
$$= \left\{e^{(4\mu - 1)2i\pi} - 1\right\} / \sigma^{4\mu + 2n}$$
$$= e^{4i\mu\pi} 2i \sin 4\mu\pi / \sigma^{4\mu + 2n}.$$

We thus obtain, on changing the sign of n in the summation,

$$\tfrac{1}{4}\sigma^{2\mu} \frac{e^{-2i\mu\pi}}{\cos 2\mu\pi} \int e^{-\sigma t} F(t)\, dt$$
$$= - \sum_{n=-\infty}^{\infty} \chi(n - \mu)\, \sigma^{2n - 2\mu}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

the path of integration being as in fig. 5. This is the second integral we proposed to find for the J function.

11. Now add (14) and (15). The two paths of integration, figs. 4 and 5, are shown side by side in fig. 6. Over the parts

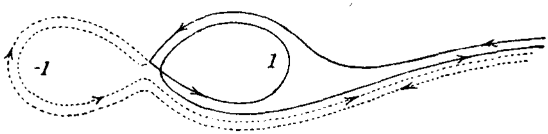

Fig. 6

dotted the contributions from the two separate integrals annul each other, and we are left with the path in fig. 7. Thus, changing

Fig. 7

signs, we have

$$\sum_{n=-\infty}^{\infty} \chi(n-\mu)\,\sigma^{2n-2\mu} - \sum_{n=-\infty}^{\infty} \chi(n+\mu)\,\sigma^{2n+2\mu}$$

$$= -\tfrac{1}{4}\sigma^{2\mu}\,\frac{e_{-2i\mu\pi}}{\cos 2\mu\pi}\int e^{-\sigma t}\,F(t)\,dt, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

over the path in fig. 7; 0 being the initial argument of σt, and $-\pi$ that of $\sigma(1-t)$.

The integral for $G(\mu, i\sigma)$ is obtained from (16) by multiplying both sides by $\surd(\pi/2)\,e^{-i\mu\pi}/\sin 2\mu\pi$.

The path might obviously be reduced at once to one with a single turn round 1, with a mere change of coefficient; indeed, since the particular value we take for μ has not been defined, we can always take μ so that, in (3), the real part of $2\mu-\frac{1}{2}$ is greater than -1, and we can then reduce the integral to an ordinary integral from 1 to ∞.

12. Formula (16) leads at once to the asymptotic formula for the G function; and corresponding formulae for the J functions follow at once (II, § 21). So far as the mechanical details are concerned, the method is simply to substitute for $F(t)$ in (16) its expansion (3) round the point $t=1$, and then integrate term by term. It need scarcely be said that, as an analytical process, this is quite illegitimate; but Poincaré showed that the resulting series, though divergent, has a valuable interpretation; he called it an *asymptotic* series for the function.*

Now, in fig. 7, supposing to fix ideas that σ is real and positive, the argument of $1-t$ is initially $-\pi$, and finally 3π; so that

$$\int e^{-\sigma t}\,\frac{(1-t)^{2\mu-\frac{1}{2}+p}}{\Pi(2\mu-\frac{1}{2}+p)}\,dt,$$

over the path of fig. 7,

$$= -(-1)^p\,2e^{2i\mu\pi}\sin 4\mu\pi\,.\,\sigma^{-2\mu-\frac{1}{2}-p}\,e^{-\sigma}.$$

* *Cf. e.g.* A. R. Forsyth, *Theory of Differential Equations*, Part III, Vol. IV, §§ 101–107.

Thus the right hand member of (16), when treated as described above, gives the series

$$\sin 2\mu\pi\, e^{-\sigma} \sum_{p=0}^{\infty} (-1)^p B_p \sigma^{-p-\frac{1}{2}},$$

so that, by (2), we obtain

$$G(\mu, i\sigma) \approx \sqrt{(\pi/2)}\, e^{-i\mu\pi} e^{-\sigma} \sum_{p=0}^{\infty} (-1)^p B_p \sigma^{-p-\frac{1}{2}}, \quad \ldots\ldots(18)$$

the same result as II (87).

The form (18) holds so long as $-3\pi/2 < \arg \sigma < 3\pi/2$; in fact, when the path in (16) is taken as straight, it may be turned round so long as it does not pass through the point $t = -1$, *i.e.* so long as $-\pi < \arg \sigma < \pi$; and such a path will continue to serve so long as the real part of σt remains positive, *i.e.* for a further range of $\pm \pi/2$ for $\arg \sigma$.

13. The chief interest of the foregoing results being their relationship to the analogous theorems for the Bessel Functions, it may be interesting to write down the degenerate forms of some of the functions and integrals which appear in this analysis.

When $k = 0$, the equation for μ is (I, §§ 2, 6, 7)

$$\cos 2\mu\pi = \cos s\pi$$

or

$$2\mu = 2N \pm s,$$

where N is any integer.

We get the well known Bessel Function results by taking $2\mu = s$ simply; but the results obtained by giving N another value than 0 are not the same as these, and will be found worth looking at; they will not, however, be further considered here.

We take, then, $2\mu = s$.

Again, by I (43) and II (10), since $\phi(z)$ reduces to its first term, we have

$$\chi(z) = \sqrt{(\pi/2)} / 2^{2z}\, \Pi(z + \tfrac{1}{2}s)\, \Pi(z - \tfrac{1}{2}s),$$

so that

$$\chi(n + \mu) = \sqrt{(\pi/2)} / 2^{2n+s}\, \Pi(n + s)\, \Pi(n).$$

Thus $\displaystyle\sum_{n=-\infty}^{\infty} \chi(n + \mu)\, \sigma^{2n+2\mu}$

$$= \sum_{n=0}^{\infty} \sqrt{(\pi/2)}\, \sigma^{2n+s} / 2^{2n+s}\, \Pi(n + s)\, \Pi(n);$$

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and, by (2), when $k = 0$,

$$J(\mu, i\sigma) = e^{is\pi/2} \sum_{n=0}^{\infty} \sigma^{2n+s} / 2^{2n+s} \Pi(n+s) \Pi(n)$$
$$= J_s(i\sigma),$$

the Bessel Function; and therefore, by (2)

$$G(\mu, i\sigma) = \frac{\pi}{2 \sin s\pi} \{J_{-s}(i\sigma) - e^{-is\pi} J_s(i\sigma)\}$$
$$= G_s(i\sigma).$$

Next, when $k = 0$, by II, § 3,

$$B_p = \frac{(-1)^p}{2^p p!}(s^2 - \tfrac{1}{4})(s^2 - \tfrac{9}{4}) \dots \{s^2 - (p - \tfrac{1}{2})^2\}$$
$$= \frac{(-1)^p}{2^p p!} \frac{\Pi(s + p - \frac{1}{2})}{\Pi(s - p - \frac{1}{2})};$$

so that, by (3),

$$F(t) = \sum_{p=0}^{\infty} B_p \frac{(1-t)^{s - \frac{1}{2} + p}}{\Pi(s - \frac{1}{2} + p)}$$
$$= (1-t)^{s-\frac{1}{2}} \{1 - \tfrac{1}{2}(1-t)\}^{s-\frac{1}{2}} / \Pi(s - \tfrac{1}{2})$$
$$= (1 - t^2)^{s-\frac{1}{2}} / 2^{s-\frac{1}{2}} \Pi(s - \tfrac{1}{2}).$$

Also $\chi(\mu - 1) = \chi(\mu - 2) = \dots = 0$,
so that $\Omega(t) = 0$.

The path of fig. 3 closes; (13) and (14) become identical, giving

$$J_s(i\sigma) = i^{-s} \pi^{-\frac{3}{2}} 2^{-s-1} \Pi(-s - \tfrac{1}{2}) \sigma^s .$$
$$\int_0^{(-1-, 1+)} e^{-\sigma t} (1 - t^2)^{s - \frac{1}{2}} dt,$$

the well known result.

The continuation of $F(t)$ in (12), and the integral and asymptotic series for the Bessel Function $G_s(i\sigma)$ also follow at once.