The Journal of Symbolic Logic
Volume 80, Number 4, December 2015

SEMILATTICES AND THE RAMSEY PROPERTY

MIODRAG SOKIĆ

Abstract. We consider $\mathcal{S}$, the class of finite semilattices; $\mathcal{T}$, the class of finite treeable semilattices; and $\mathcal{T}_m$, the subclass of $\mathcal{T}$ which contains trees with branching bounded by m. We prove that $\mathcal{ES}$, the class of finite lattices with linear extensions, is a Ramsey class. We calculate Ramsey degrees for structures in $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{T}_m$. In addition to this we give a topological interpretation of our results and we apply our result to canonization of linear orderings on finite semilattices. In particular, we give an example of a Fraïssé class $\mathcal{K}$ which is not a Hrushovski class, and for which the automorphism group of the Fraïssé limit of $\mathcal{K}$ is not extremely amenable (with the infinite universal minimal flow) but is uniquely ergodic.

§1. Introduction. A semilattice can be considered as a relational or as a functional structure. As a relational structure, a semilattice is a poset with the property that every two elements have an infimum. As a functional structure, a semilattice contains only one binary operation which defines a partial ordering such that the infimum of any two elements is given by the binary operation. In this paper, we consider semilattices as a functional structures and we examine finite semilattices in two steps. In the first step, we consider semilattices with respect to the Ramsey property. In the second step, we examine semilattices from the point of view of topological dynamics. We denote by $\mathcal{S}$ the class of finite semilattices, by $\mathcal{T}$ the class of finite trees and by $\mathcal{T}_m$ the class of finite trees with branching bounded by m.

Which classes of finite structures have the Ramsey property is one the central questions of the Ramsey theory. When a given class does not have the Ramsey property we ask for the Ramsey degrees of its objects, see Section 2 for definitions. We recall that a class is a Ramsey class iff all of its objects have Ramsey degree equal to 1. In this paper, we prove that $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{T}_m$ are not Ramsey classes by calculating the Ramsey degrees for their objects. We recall that there are only two known Ramsey classes of functional structures: the class of finite Boolean algebras, see [6], and the class of finite vector spaces over a given finite field, see [8]. In addition to these two classes we know the Ramsey degrees for all objects in a given functional class only for the class of finite unary functions, see [22].

A connection between structural Ramsey theory and topological dynamics is given by the work of Kechris–Pestov–Todorčević in [11] and its generalization in [16]. This connection is based on Fraïssé theory, see [9]. Fraïssé classes $\mathcal{S}$, $\mathcal{T}$,

Received January 21, 2014.
2010 *Mathematics Subject Classification.* Primary: 05D10,06A12; Secondary: 37B05, 43A07.
Key words and phrases. semilattices, Ramsey property, linear orderings.

0022-4812/15/8004-0008
DOI:10.1017/jsl.2014.40

and $\mathcal{T}_m$ generates Fraïssé structures $\mathbb{S}$, $\mathbb{T}$, and $\mathbb{T}_m$ respectively. Some characteristics of model theoretic structures are obtained by examining their groups of automorphisms. In particular we consider groups $Aut(\mathbb{S})$, $Aut(\mathbb{T})$, and $Aut(\mathbb{T}_m)$ with the pointwise convergence topology. Let G be a topological group. A continuous action of G on a compact Hausdorff space X is called a G-flow. A G-flow X is minimal if for every $x \in X$ we have $X = \overline{\{gx : g \in G\}}$. It is a classical result in topological dynamics that every topological group G admits a unique, up to isomorphism, minimal G-flow, called the universal minimal G-flow, such that all other minimal G-flows are factors of it. If the universal minimal flow is a point then we say that the group is extremely amenable. In topological dynamics we ask for a concrete description of the universal minimal flows. In this paper, we give concrete descriptions of the universal minimal flows for $Aut(\mathbb{S})$, $Aut(\mathbb{T})$ and $Aut(\mathbb{T}_m)$. Moreover, these universal minimal flows are metrizable. We should contrast this with the fact that a universal minimal flow of a countable discrete group is never metrizable, see page 121 in [11]. A topological group G is amenable if every G-flow admits an invariant Borel probability measure. A G-flow is uniquely ergodic if it has a unique invariant probability measure. We say that a topological group G is uniquely ergodic if every minimal G-flow is uniquely ergodic. In this paper, we prove that $Aut(\mathbb{T})$ and $Aut(\mathbb{T}_m)$ are uniquely ergodic groups. Also we explain why $Aut(\mathbb{S})$ is a nonamenable group. Moreover, $\mathcal{T}$ and $\mathcal{T}_m$ are the first known examples of Fraïssé classes which are not Hrushovski classes such that the groups of automorphisms of their corresponding Fraïssé structures have nonisomorphic universal minimal flow and they are uniquely ergodic.

In Section 2 we give formal definitions and present the main results. In Section 5 we introduce more ordered classes of semilattices which are natural to consider and we recall basic concepts from Fraïssé theory. This is important so we can give a topological interpretation of our results in Section 7. In particular, we examine amenability, extreme amenability, and unique ergodicity of automorphism groups of certain countable structures. We give the first example of a Fraïssé class $\mathcal{K}$ with corresponding automorphism group of its Fraïssé limit such that $\mathcal{K}$ is not a Hrushovski class and the automorphism group has a nontrivial minimal flow and is uniquely ergodic, see Theorem 7.9. In Section 8 we consider natural orderings on semilattices.

§**2. Background.** A *meet semilattice* is a poset with the property that every two elements have an infimum, and a *join semilattice* is a poset with the property that every two elements have a supremum. A poset which is a meet semilattice or a join semilattice is called a *semilattice*. Each meet (join) semilattice $(A, \leq)$ defines a binary *meet (join) operation* $\circ$ with $a \circ b = \inf\{a, b\}$ $(\sup\{a, b\})$ for $a, b \in A$. Moreover for all $a, b, c \in A$ we have

$$a \circ (b \circ c) = (a \circ b) \circ c, a \circ b = b \circ a, a \circ a = a. \tag{2.1}$$

Also if $\circ$ is a binary operation on a set A which satisfies 2.1 for all $a, b, c \in A$ then on the set A we may define a partial ordering $\leq$ such that $(A, \leq)$ is:

(i): either a meet semilattice with $a \leq b \Leftrightarrow a \circ b = a$ and $\inf\{a, b\} = a \circ b$,
(ii): or a join semilattice with $a \leq b \Leftrightarrow a \circ b = b$ and $\sup\{a, b\} = a \circ b$.

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

It follows from this connection that there is a duality between meet and join semilattices, so in the rest of this paper we will consider only join semilattices and we will call them only semilattices instead of meet semilattices. Also we consider a semilattice as a pair $(A, \circ^A)$, where $\circ^A$ is a binary operation on A which satisfies 2.1 for all $a, b, c \in A$. We denote by $\leq^A$ a partial ordering on the set A such that $a \leq^A b \Leftrightarrow a \circ^A b = a$ for all $a, b \in A$, and we also write $(A, \circ^A, \leq^A)$ instead of $(A, \circ^A)$. In the rest of this paper we consider a semilattice as a structure with a binary function, denoted by $\circ$, and we define in the semilattice a partial ordering, denoted by $\leq$, using function $\circ$. We refer the reader for more details on lattices and semilattices to [18]. We denote by $\mathcal{S}$ the class of finite semilattices. Beside the class $\mathcal{S}$, we also consider few subclasses of the class $\mathcal{S}$.

We say that $(A, \circ^A) \in \mathcal{S}$ is a *treeable semilattice* if the induced poset $(A, \leq^A)$ is a tree, i.e., it has a minimum called the *root* and for each $a \in A$ the set $\{b \in A : b \leq^A a\}$ is linearly ordered with $\leq^A$. We denote by $\mathcal{T}$ the class of finite treeable semilattices and we say that structures in $\mathcal{T}$ are trees. For a tree $\mathbb{A} = (A, \leq^A)$ and $a, b \in A$ we say that b is an *immediate successor* of a if for all $c \in A$ we have $a \leq^A c \leq^A b \Rightarrow c = a$ or $c = b$. We denote the set of immediate successors of a in a tree $\mathbb{A}$ by $im_{\mathbb{A}}(a)$. We say that $a \in A$ is *terminal* if $im_{\mathbb{A}}(a) = \emptyset$ and the *height of* a is $ht_{\mathbb{A}}(a) = |\{b \in A : b < a\}|$. Tree $\mathbb{A}$ has a *constant branching* k if for every nonterminal $a \in A$ we have $k = |im_{\mathbb{A}}(a)|$. We say that the tree $\mathbb{A}$ is (h, k)-*balanced* if it has constant branching k and all terminal elements have the height h. We say that a given tree is *balanced* if it is (h, k)-balanced for some h and k. For $a \in A$ we denote by $\mathbb{A}(a) = \{b \in A : a \leq b\}$ the substructure of $\mathbb{A}$.

For a natural number $m \geq 1$ we denote by $\mathcal{T}_m$ the subclass of $\mathcal{T}$ that contains structures $\mathbb{A} = (A, \leq^A)$ from $\mathcal{T}$ with the property that $|im_{\mathbb{A}}(a)| \leq m$ for $a \in A$. In particular, $\mathcal{T}_1$ is the class of finite linearly ordered sets.

The analysis in [11] leads to consideration of the classes of finite semilattices and also the classes of finite ordered semilattices. Before we introduce classes of finite ordered semilattices we fix notation. The collection of all linear orderings on a nonempty set A we denote by $lo(A)$. A linear ordering $\preceq$ on a A is a *linear extension* of a partial ordering $\sqsubseteq$ on A if for all $a, b \in A$ we have $a \sqsubseteq b \Rightarrow a \preceq b$. We denote by $le(\sqsubseteq)$ the collection of all linear extensions of a partial ordering $\sqsubseteq$. We say that partial orderings $\leq$ and $\preceq$ on A are *opposite*, $\preceq = op(\leq)$ or $\leq = op(\preceq)$, if $a \leq b \Leftrightarrow b \preceq a$ for all $a, b \in A$. We denote the strict part of partial orderings $\leq, \preceq$, or $\sqsubseteq$ by $<, \prec$ or $\sqsubset$. For a partial ordering $\leq$ on a set A and subsets $B, C \subset A$ with $B \cap C = \emptyset$, we write $B < C$ if $b \leq c$ for all $b \in B$ and all $c \in C$. The cardinality of a given set A we denote by $|A|$ or $card(A)$. If a and b are incomparable with respect to the partial ordering $\leq$ then we write $nc(\leq, a, b)$.

By adding linear extensions to structures from $\mathcal{S}$ we obtain the class

$$\mathcal{ES} = \{(A, \circ^A, \preceq^A) : (A, \circ^A) \in \mathcal{S}, \preceq^A \in le(\leq^A)\}.$$

Let $(A, \circ^A, \preceq^A)$ be in $\mathcal{ES}$ such that $(A, \circ^A) \in \mathcal{T}$ and let $\mathbb{A} = (A, \leq^A)$. Then we say that $\preceq^A$ is a *convex ordering* with respect to $\leq^A$ if for all $a, b, c \in A$ with $a \circ^A b = c$, $a \neq c$, $b \neq c$ we have

$$a \preceq^A b \Leftrightarrow a' \preceq^A b',$$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

where $a', b' \in im_{\mathbb{A}}(c)$, $a' \leq^A a$, $b' \leq^A b$. We denote the set of all convex ordering with respect to $\leq^A$ by $co(\leq^A)$, and we consider the class

$$\mathcal{CT} = \{(A, \circ^A, \preceq^A) : (A, \circ^A) \in \mathcal{T}, \preceq^A \in co(\leq^A)\}.$$

In order to simplify exposition we introduce additional notation. Let $\mathbb{A}$ and $\mathbb{B}$ be two structures in a given signature L. If there is an embedding from $\mathbb{A}$ into $\mathbb{B}$ then we write $\mathbb{A} \hookrightarrow \mathbb{B}$, otherwise we write $\mathbb{A} \not\hookrightarrow \mathbb{B}$. If $\mathbb{A}$ is a substructure of $\mathbb{B}$ then we write $\mathbb{A} \leq \mathbb{B}$, and if $\mathbb{A}$ is isomorphic to $\mathbb{B}$ then we write $\mathbb{A} \cong \mathbb{B}$. We denote by $\binom{\mathbb{B}}{\mathbb{A}}$ the collection of all substructures of $\mathbb{B}$ which are isomorphic to $\mathbb{A}$. For a structure $\mathbb{K}$ we denote the class of all finite substructures that are isomorphic to a substructure of $\mathbb{K}$ by $Age(\mathbb{K})$. We denote by $|\mathbb{A}|$ the size of the underlying set of the structure $\mathbb{A}$. For a function $f : X \to Y$ and $A \subseteq X$, we write $f(A) = \{f(a) : a \in A\}$.

Let $\mathcal{K}$ be a class of finite structures in a signature L. If for natural numbers r and t, and structures $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{K}$ we have that for every coloring $c : \binom{\mathbb{C}}{\mathbb{A}} \to \{1, \dots, r\}$ there is $\mathbb{B}' \in \binom{\mathbb{C}}{\mathbb{B}}$ such that $|c(\binom{\mathbb{B}'}{\mathbb{A}})| \leq t$ then we write

$$\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_{r,t}.$$

In particular, for $t = 1$ we write $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_r$. We say that $\mathbb{A} \in \mathcal{K}$ has *Ramsey degree* t_0 in $\mathcal{K}$, denoted by $t_{\mathcal{K}}(\mathbb{A})$, if t_0 is the smallest natural number with the property that for any natural number r and any $\mathbb{B} \in \mathcal{K}$ there is $\mathbb{C} \in \mathcal{K}$ such that $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_{r,t_0}$. A structure $\mathbb{A} \in \mathcal{K}$ is a *Ramsey object* in $\mathcal{K}$ if $t_{\mathcal{K}}(\mathbb{A}) = 1$. We say that the class $\mathcal{K}$ is a *Ramsey class* or that $\mathcal{K}$ satisfies the *Ramsey property* (RP) if for all $\mathbb{A}$ in $\mathcal{K}$ we have $t_{\mathcal{K}}(\mathbb{A}) = 1$.

It was proved that every one element lattice is a Ramsey object in the class of finite lattices, see [15] and [10]. Proposition 2.1 in [10] implies that every one element semilattice is a Ramsey object in $\mathcal{S}$, and $\mathcal{T}$, but it does not give any information about Ramsey objects in $\mathcal{ES}$ and $\mathcal{CT}$. In Section 3 we prove the following

THEOREM 2.1. *$\mathcal{ES}$ is a Ramsey class.*

Deuber in [4] considers trees as semilattices, i.e., structures with a binary function. For a given natural number $m > 1$, the class of (k, m)-balanced trees in $\mathcal{T}_m$ is a Ramsey class, see [4]. Moreover, the result in [4] implies that (k, m)-balanced trees are Ramsey objects in $\mathcal{T}_m$, but it does not tell us if there are some other Ramsey objects in $\mathcal{T}_m$. Note that the result in [4] does not imply immediately that balanced trees are Ramsey objects in $\mathcal{T}$. Using the concept of strong subtrees, see [13], we may obtain the same conclusion. If we make the restriction to the class of (k, m)-balanced trees in $\mathcal{T}_m$ but considering only embeddings that map terminal elements to terminal elements then we have a Ramsey class, see [23]. This result does not give any information about Ramsey objects in $\mathcal{T}$, $\mathcal{T}_m$ or $\mathcal{CT}$. In [24] it is proved that objects in $\mathcal{T}$ have finite Ramsey degree, but they are not calculated explicitly. Note that the results in [4] and [13] also imply that objects in $\mathcal{T}$ and $\mathcal{T}_m$ have finite Ramsey degree. We give a proof of Theorem 2.2 in the Appendix by modifying the approach in [4]. We point out that Theorem 2.2 was stated, in a slightly different form, in [7] on page 276 and it is credited to Leeb. We give a proof of Theorem 2.2 in the Appendix by modifying the approach in [4], but it can be also proved by using the approach from [23]. We decide to add this proof in the Appendix because we were unable to find a proof in the literature.

THEOREM 2.2. *$\mathcal{CT}$ is a Ramsey class.*

We also calculate Ramsey degrees for structures in $\mathcal{S}$ and $\mathcal{T}$, see Section 4.

THEOREM 2.3.

(i) *For $\mathbb{A} = (A, \circ^A, \leq^A)$ in $\mathcal{S}$ we have $t_{\mathcal{S}}(\mathbb{A}) = \frac{|\{\preceq^A : \preceq^A \in le(\leq^A)\}|}{|Aut(\mathbb{A})|}$.*

(ii) *For $\mathbb{A} = (A, \circ^A, \leq^A)$ in $\mathcal{T}$ we have $t_{\mathcal{T}}(\mathbb{A}) = \frac{|\{\preceq^A : \preceq^A \in co(\leq^A)\}|}{|Aut(\mathbb{A})|}$.*

Note that the chains are Ramsey objects in $\mathcal{S}$, but not the only Ramsey objects. Similarly for the class $\mathcal{T}$, we see that balanced trees are Ramsey objects, but not the only Ramsey objects.

For a natural number $m \geq 1$ we define a function $t_m : \mathcal{T}_m \to \mathbb{N}$ inductively on the size of the structures. Let $\mathbb{A} = (A, \circ^A, \leq^A)$ be a structure in $\mathcal{T}_m$. For $|\mathbb{A}| = 1$ we take $t_m(\mathbb{A}) = 1$. Suppose that $|\mathbb{A}| > 1$ and that $t_m(\mathbb{A}')$ is defined for all $\mathbb{A}' \in \mathcal{T}_m$ with $|\mathbb{A}'| < |\mathbb{A}|$. Let a be the root of the tree $(A, \leq^A)$ and let $im_{\mathbb{A}}(a) = \{a_1, \ldots, a_n\}$. Without loss of generality we may assume that the structures $\mathbb{A}(a_1), \ldots, \mathbb{A}(a_k)$ are mutually nonisomorphic and that for every $j > k$ there is a $i \leq k$ such that $\mathbb{A}(a_i) \cong \mathbb{A}(a_j)$. For $i \leq k$ we consider $n_i = |\{1 \leq j \leq n : \mathbb{A}(a_i) \cong \mathbb{A}(a_j)\}|$ and we define

$$t_m(\mathbb{A}) = \binom{m}{n} \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^{n} t_m(\mathbb{A}(a_i)).$$

In Section 6 we prove the following.

THEOREM 2.4. *For a natural number $m \geq 1$ and $\mathbb{A} \in \mathcal{T}_m$ we have $t_{\mathcal{T}_m}(\mathbb{A}) = t_m(\mathbb{A})$.*

For $\mathbb{A} \in \mathcal{T}_m$ we have that $t_{\mathcal{T}_m}(\mathbb{A}) = t_m(\mathbb{A}) = 1$ implies $m = n$, $k = 1$ and $t_m(\mathbb{A}(a_1)) = 1$. Using an induction on the height of the Ramsey objects in $\mathcal{T}_m$ we obtain that the Ramsey objects in $\mathcal{T}_m$ are exactly the (k, m)-balanced trees. We also give an alternative way to calculate the Ramsey degree for structures in $\mathcal{T}_2$ in Section 5.

§3. Proof of Theorem 2.1. Before we start with the proof we need to recall some basic facts about Boolean lattices. We consider a Boolean lattice as a structure with two binary operations $\circ$ and $\bullet$. Each Boolean lattice $(B, \circ^B, \bullet^B)$ comes with the induced partial ordering $\leq^A$ defined such that for all $a, b \in B$ we have

$$a \leq^A b \Leftrightarrow a \circ^A b = a \Leftrightarrow a \bullet^A b = b.$$

Note that $\circ^A$ gives infimum while $\bullet^A$ gives supremum for two elements with respect to $\leq^A$. We write $(B, \circ^B, \bullet^B, \leq^B)$ instead of $(B, \circ^B, \bullet^B)$ and we denote by $\mathcal{BL}$ the *class of finite Boolean lattices*. Let $\mathbb{B} = (B, \circ^B, \bullet^B, \leq^B)$ be a finite Boolean lattice. Then we denote by $0_{\mathbb{B}}$ and $1_{\mathbb{B}}$ the maximum and minimum in $\mathbb{B}$ with respect to $\leq^B$. An element $b \in B$ with the property that $b \neq 0_{\mathbb{B}}$, $b \neq 1_{\mathbb{B}}$ and that for all $a \in B$ we have $a \leq^B b \Rightarrow a = b$ or $a = 0_{\mathbb{B}}$ is called an *atom*. The set of all atoms in $\mathbb{B}$ is denoted by $atom(\mathbb{B})$. Let $atom(\mathbb{B}) = \{a_1, \ldots, a_k\}$. Then every $a \in B$ has a unique representation of the form $a = \epsilon_1 a_1 \bullet^B \cdots \bullet^B \epsilon_k a_k$, where $\epsilon_i \in \{0, 1\}$, and $\epsilon_i a_i = \begin{cases} a_i; \epsilon_i = 1, \\ 0_{\mathbb{B}}; \epsilon_i = 0. \end{cases}$ Suppose that $b \in B$ has the unique representation $b = \delta_1 a_1 \bullet^B \cdots \bullet^B \delta_k a_k$ with $\delta_i \in \{0, 1\}$ and let $\sqsubseteq^B$ be a linear ordering of the atoms $atom(\mathbb{B})$. Then we say that a linear ordering $\preceq^B$ is an *antilexicographical ordering* on $\mathbb{B}$ if for $a, b \in B$ we have

$$a \prec^B b \Leftrightarrow \epsilon_j < \delta_j \ ,$$

where a_j is the maximal element in $\{a_i : \epsilon_i \neq \delta_i\}$ with respect to $\sqsubseteq^B$. We denote by $lex(\leq^B)$ or $lex(\mathbb{B})$ the set of all antilexicographical orderings on $\mathbb{B}$. We denote by $\mathcal{LBL}$ the class of finite structures $(B, \circ^B, \bullet^B, \leq^B, \preceq^B)$ with the property that $(B, \circ^B, \bullet^B) \in \mathcal{BL}$ and $\preceq^B \in lex(\leq^B)$. Note that for every Boolean lattice $(B, \circ^B, \bullet^B)$, the structure $(B, \circ^B)$ is a semilattice. Since $lex(\leq^B) \subseteq le(\leq^B)$, we have that $(B, \circ^B, \bullet^B, \leq^B, \preceq^B) \in \mathcal{LBL}$ implies $(B, \circ^B, \leq^B, \preceq^B) \in \mathcal{ES}$.

Let $\mathbb{A} = (A, \circ^A, \leq^A, \preceq^A)$ be a structure from $\mathcal{ES}$ such that $|A| = k$ and $A = \{a_1 \prec^A a_2 \prec^A \cdots \prec^A a_k\}$. Let $\mathbb{B} = (B, \circ^B, \bullet^B, \leq^B, \preceq^B)$ be a structure from $\mathcal{LBL}$ such that $atom((B, \circ^B, \bullet^B)) = \{b_1 \prec^B b_2 \prec^B \cdots \prec^B b_k\}$. Then we consider a map $\varphi : A \to B$ given by

$$\varphi(a_i) = \bullet^B\{b_j : a_j \leq^A a_i\}.$$

We denote by $\varphi(\mathbb{A}, \mathbb{B})$ the substructure of $\mathbb{B}$ induced by the set $\varphi(A)$. Let $1 \leq i, j, l \leq k$. Then we have the following:

Fact 0: $\varphi(A)$ is closed under the operation $\circ^B$, i.e., $\varphi(a_i) \circ^B \varphi(a_j) \in \varphi(A)$.
Fact 1: φ preserves the linear ordering, i.e., $a_i \prec^A a_j \Leftrightarrow \varphi(a_i) \prec^B \varphi(a_j)$.
Fact 2: φ preserves the partial ordering, i.e., $a_i <^A a_j \Leftrightarrow \varphi(a_i) <^B \varphi(a_j)$.
Fact 3: φ preserves the binary operation, i.e., $a_i \circ^A a_j = a_l \Leftrightarrow \varphi(a_i) \circ^A \varphi(a_j) = \varphi(a_l)$.

Note that the map φ is good for defining some kind of canonical image of $\mathbb{A}$ into any Boolean lattice $\mathbb{B}$ with $|atom(\mathbb{B})| = |\mathbb{A}|$. Therefore, we will be able to use the dual Ramsey theorem.

Proof of Theorem 2.1. Let r be a natural number. Let $\mathbb{A} = (A, \circ^A, \leq^A, \preceq^A)$ and $\mathbb{B} = (B, \circ^B, \leq^B, \preceq^A)$ be structures from $\mathcal{ES}$ such that $\binom{\mathbb{B}}{\mathbb{A}} \neq \emptyset$, $|A| = k$ and $|B| = m$. Then we consider structures $\mathbb{A}_0, \mathbb{B}_0 \in \mathcal{BL}$ with k, and m atoms respectively. Using the Ramsey property for $\mathcal{BL}$, see [6], there is $\mathbb{C} = (C, \circ^C, \bullet^C, \leq^C)$ in $\mathcal{BL}$ such that

$$\mathbb{C}_0 \to (\mathbb{B}_0)^{\mathbb{A}_0}_r.$$

Now we consider a linear ordering $\preceq^C$ on C such that $(\mathbb{C}_0, \preceq^C) = (C, \circ^C, \bullet^C, \leq^C, \preceq^C) \in \mathcal{LBL}$. Then we have that $\mathbb{C} = (C, \circ^C, \leq^C, \preceq^C) \in \mathcal{ES}$ and we claim

$$\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_r.$$

To check this we consider a coloring $c : \binom{\mathbb{C}}{\mathbb{A}} \to \{1, \ldots, r\}$ and an induced coloring

$$c_0 : \binom{\mathbb{C}_0}{\mathbb{A}_0} \to \{1, \ldots, r\},$$
$$c_0(\mathbb{A}'_0) = c(\varphi(\mathbb{A}, (\mathbb{A}'_0, \preceq^C))),$$

where $(\mathbb{A}'_0, \preceq^C)$ is the substructure of $(\mathbb{C}_0, \preceq^C)$ given by $\mathbb{A}'_0$. By the choice of the structure $\mathbb{C}_0$ there is $\mathbb{B}'_0 \in \binom{\mathbb{C}_0}{\mathbb{B}_0}$ such that

$$c_0 \restriction \binom{\mathbb{B}'_0}{\mathbb{A}_0} = const.$$

It is enough to show that for $\mathbb{B}' = \varphi(\mathbb{B}, (\mathbb{B}'_0, \preceq^C))$ we have

$$c \restriction \binom{\mathbb{B}'}{\mathbb{A}} = const.$$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

Suppose that atoms of $\mathbb{B}_0'$ are linearly ordered according to $\preceq^C$ as follows $b_1 \prec^C \cdots \prec^C b_m$, and let $\mathbb{A}' \in \binom{\mathbb{B}'}{\mathbb{A}}$ have elements $a_1' \prec^C a_2' \prec^C \cdots \prec^C a_k'$. Then for each $1 \leq i \leq k$, we consider elements $a_1'', \ldots, a_k''$ in $\mathbb{B}_0'$ given by

$$a_i'' = \bullet^C\{b_l : b_l \leq^C a_i' \ \& \ (\forall t)[a_t' <^C a_i' \Rightarrow \rceil(b_l \leq^C a_t')]\}$$

for $1 \leq i \leq k$. Now we have that $a_1'', \ldots, a_k''$ are atoms of the Boolean lattice $\mathbb{A}_0'$ which is also sublattice of $\mathbb{B}_0'$. Moreover we have $\mathbb{A}' = \varphi(\mathbb{A}, (\mathbb{A}_0', \preceq^C))$, so our Ramsey property is verified. ⊣

§**4. Ordering property.** Let L be a signature and let $\preceq$ be a binary relational symbol such that $\preceq \notin L$. If $\mathbb{A}'$ is a structure in $L \cup \{\preceq\}$ and $\mathbb{A}$ is a structure in L obtained by dropping the interpretation of the symbol $\preceq$ in $\mathbb{A}'$ then we say that $\mathbb{A}$ is a *reduct* of $\mathbb{A}'$ or that $\mathbb{A}'$ is an *expansion* of $\mathbb{A}$ and we write $\mathbb{A} = \mathbb{A}'|L$. Let $\mathcal{K}'$ be a class of structures in L' and let $\mathcal{K}$ be a class of the structures in L. If $\mathcal{K} = \{\mathbb{A}|L : \mathbb{A} \in \mathcal{K}'\}$ then we say that $\mathcal{K}$ is a *reduct* of $\mathcal{K}'$ in L or that $\mathcal{K}'$ is an *expansion* of $\mathcal{K}$ in L' and we write $\mathcal{K} = \mathcal{K}'|L$. If $\preceq$ is interpreted as a linear ordering in each structure from $\mathcal{K}'$ and $\mathcal{K} = \mathcal{K}'|L$ then we say that $\mathcal{K}'$ is an *ordered expansion* of $\mathcal{K}$. If $\mathcal{K}'$ is an ordered expansion of $\mathcal{K}$ and for all $\mathbb{A}, \mathbb{B} \in \mathcal{K}$, every embedding $\Phi : \mathbb{A} \to \mathbb{B}$, and every $\mathbb{A}' \in \mathcal{K}'$ with $\mathbb{A} = \mathbb{A}'|L$ there is $\mathbb{B}' \in \mathcal{K}'$ such that $\mathbb{B} = \mathbb{B}'|L$ and Φ is also an embedding $\Phi : \mathbb{A}' \to \mathbb{B}'$ then we say that $\mathcal{K}'$ is a *reasonable expansion* of $\mathcal{K}$. Suppose that $\mathcal{K}'$ is an ordered expansion of $\mathcal{K}$. Then we say that the class $\mathcal{K}'$ has the *ordering property (OP)* with respect to $\mathcal{K}$ if for every $\mathbb{A} \in \mathcal{K}$ there is $\mathbb{B} \in \mathcal{K}$ such that for every $\mathbb{A}' \in \mathcal{K}'$ and $\mathbb{B}' \in \mathcal{K}'$ with the property that $\mathbb{A}'|L = \mathbb{A}$ and $\mathbb{B}'|L = \mathbb{B}$ we have $\mathbb{A}' \hookrightarrow \mathbb{B}'$. In this case we say that $\mathbb{B}$ *verifies OP* for $\mathbb{A}$. If for $\mathbb{A}' \in \mathcal{K}'$ there is $\mathbb{B} \in \mathcal{K}$ such that for every $\mathbb{B}' \in \mathcal{K}'$ with $\mathbb{B}'|L = \mathbb{B}$ we have $\mathbb{A}' \hookrightarrow \mathbb{B}'$, then we say that $\mathbb{B}$ *verifies OP* for $\mathbb{A}'$.

We point out that the verification that we have a reasonable expansion of a given class is straightforward and we will avoid such verifications. Also we point out that this property is important for the topological interpretation of our results.

For the purpose of examination of the OP for the class $\mathcal{ES}$ with respect to the class $\mathcal{S}$ we need to consider structures, see Figure 1, $\mathbb{P} = (P, \circ^P, \preceq^P)$, $\mathbb{R} = (R.\circ^R, \preceq^R)$, $\mathbb{P}_1 = (P_1, \circ^{P_1}, \preceq^{P_1})$ and $\mathbb{P}_2 = (P_2, \circ^{P_2}, \preceq^{P_2})$ from $\mathcal{ES}$ given by:

- $P = \{p, p_0, p_1\}$, $p_0 \circ^P p_1 = p$, $p \prec^P p_0 \prec^P p_1$.
- $R = \{r_0, r_1, r_2, r_3, r_4, r_5\}$, $r_0 <^R r_1 <^R r_3 <^R r_5$, $r_0 <^R r_2 <^R r_5$, $nc(\leq^R, r_2, r_1)$, $nc(\leq^R, r_2, r_3)$, $nc(\leq^R, r_4, r_3)$, $nc(\leq^R, r_4, r_5)$, $r_1 <^R r_4$, $r_2 <^R r_4$, $r_0 \prec^R r_1 \prec^R r_2 \prec^R r_3 \prec^R r_4 \prec^R r_5$.
- $\mathbb{P}_1 \leq \mathbb{R}$, $P_1 = \{r_1, r_3, r_4\}$.
- $\mathbb{P}_2 \leq \mathbb{R}$, $P_2 = \{r_2, r_4, r_5\}$.

Lemma 4.1. *$\mathcal{ES}$ satisfies OP with respect to $\mathcal{S}$.*

Proof. It is enough to show that for a given $\mathbb{A} = (A, \circ^A, \leq^A, \preceq^A)$ in $\mathcal{ES}$ there is $\mathbb{B} = (B, \circ^B, \leq^B)$ in $\mathcal{S}$ such that for every $\preceq^B$ with $(\mathbb{B}, \preceq^B)$ we have $\mathbb{A} \hookrightarrow (\mathbb{B}, \preceq^B)$. There is a structure $\mathbb{A}_0 \in \mathcal{ES}$ such that $\mathbb{A} \hookrightarrow \mathbb{A}_0$ and $\mathbb{R} \hookrightarrow \mathbb{A}_0$. By Theorem 2.1 there is $\mathbb{C} = (C, \circ^C, \leq^C, \preceq^C)$ in $\mathcal{ES}$ such that

$$\mathbb{C} \to (\mathbb{A}_0)^{\mathbb{P}}_2. \tag{4.1}$$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

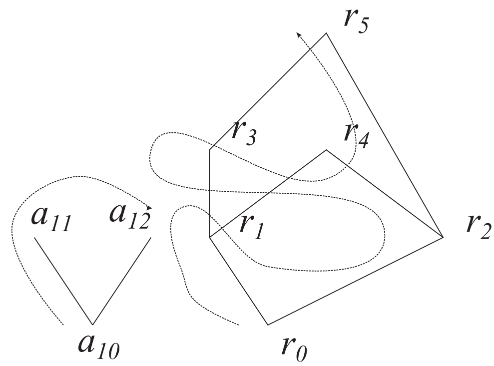

FIGURE 1. Structures $\mathbb{P}$ and $\mathbb{R}$: curved lines are linear orderings.

We claim that $(C, \circ^C, \leq^C)$ verifies OP for $\mathbb{A}$. Suppose that $\sqsubseteq^C$ is a linear ordering on C such that $(\mathbb{C}, \sqsubseteq^C) \in \mathcal{ES}$. Then we have a coloring

$$c : \binom{\mathbb{C}}{\mathbb{P}} \to \{0, 1\},$$
$$c(\mathbb{U}) = 1 \Leftrightarrow (\sqsubseteq^C \restriction U^2 = \preceq^C \restriction U^2),$$

where U is the underlying set of the structure $\mathbb{U}$. From 4.1 we obtain $\mathbb{A}_0' \in \binom{\mathbb{C}}{\mathbb{A}_0}$ such that $c \restriction \binom{\mathbb{A}_0'}{\mathbb{P}} = c_i$ for some $c_i \in \{0, 1\}$. Since $\mathbb{R} \hookrightarrow \mathbb{A}_0$, we may suppose without loss of generality that $\mathbb{R} \leq \mathbb{A}_0'$. Note that we have $\mathbb{P}_1 \cong \mathbb{P}_2 \cong \mathbb{P}$ and $r_3 \prec^C r_4 \prec^C r_5$. If $c_i = 0$ then $r_4 \sqsubset^C r_3$ and $r_5 \sqsubset^C r_4$, and consequently $r_5 \sqsubset^C r_3$. But this is a contradiction since $r_3 <^C r_5$. Therefore, we must have $c_i = 1$ which shows that $\sqsubseteq^C$ and $\preceq^C$ agree on the underlying set of $\mathbb{A}_0'$, so there is an embedding $\mathbb{A} \hookrightarrow (\mathbb{C}, \sqsubseteq^C)$. ⊣

LEMMA 4.2. *$\mathcal{CT}$ satisfies OP with respect to $\mathcal{T}$.*

PROOF. Let $\mathbb{A} = (A, \circ^A, \leq^A, \preceq^A)$ be a structure in $\mathcal{CT}$ and let $\mathbb{A}_0 = (A, \circ^A, \leq^A)$. Let $h_0 = \max\{ht_{\mathbb{A}_0}(a) : a \in A\}$ and let $k_0 = \max\{|im_{\mathbb{A}_0}(a)| : a \in A_0\}$. Let $\mathbb{B}_0$ be an (h_0, k_0)-balanced tree in $\mathcal{T}$. Then it is easy to see that for every linear ordering $\preceq^B$ with the property $(\mathbb{B}_0, \preceq^B) \in \mathcal{CT}$ we have $\mathbb{A} \hookrightarrow (\mathbb{B}_0, \preceq^B)$, so $\mathbb{B}_0$ verifies OP for $\mathbb{A}$. Consequently, we have that $\mathcal{CT}$ satisfies OP with respect to $\mathcal{T}$. ⊣

PROOF OF THEOREM 2.3. This follows from Proposition 10.5 in [11] applied to Theorem 2.1 and Lemma 4.1 in the first case and Theorem 2.2 and Lemma 4.2 in the second case. We point out that our symbol $t_{\mathcal{K}}(\mathbb{A})$ for Ramsey degrees has a different meaning in [11], Ramsey degrees are denoted by $t(\mathbb{A}, \mathcal{K})$ in [11]. ⊣

§5. More ordered classes. We assume that every class of structures in this paper is closed under taking isomorphic images. We say that a given class $\mathcal{K}$ of structures is *countable* if it contains, up to isomorphism, countably many nonisomorphic structures.

Let $\mathcal{K}$ be a class of finite structures in a signature L. We say that $\mathcal{K}$ satisfies the:

- *Hereditary property* (HP) if for all $\mathbb{A} \hookrightarrow \mathbb{B}$ and $\mathbb{B} \in \mathcal{K}$ then $\mathbb{A} \in \mathcal{K}$.
- *Joint embedding property* (JEP) if for all $\mathbb{A} \in \mathcal{K}$ and $\mathbb{B} \in \mathcal{K}$ there is $\mathbb{C} \in \mathcal{K}$ such that $\mathbb{A} \hookrightarrow \mathbb{C}$ and $\mathbb{B} \hookrightarrow \mathbb{C}$.

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

- *Amalgamation property* (AP) if for all $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{K}$ and all embeddings $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ there are $\mathbb{D} \in \mathcal{K}$ and embeddings $\bar{f} : \mathbb{B} \to \mathbb{D}$ and $\bar{g} : \mathbb{C} \to \mathbb{D}$ such that $\bar{f} \circ f = \bar{g} \circ g$.
- *Strong amalgamation property* (SAP) if for all $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{K}$ with the underlying sets A, B, C respectively and all embeddings $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ there are $\mathbb{D} \in \mathcal{K}$ and embeddings $\bar{f} : \mathbb{B} \to \mathbb{D}$ and $\bar{g} : \mathbb{C} \to \mathbb{D}$ such that $\bar{f} \circ f = \bar{g} \circ g$ and $\bar{f}(B) \cap \bar{g}(C) = \bar{f} \circ f(A) = \bar{g} \circ g(A)$.

A class $\mathcal{K}$ of finite structures in a countable signature L which is countable, contains structures of arbitrary large finite cardinality, and satisfies HP, JEP, and AP is called a *Fraïssé class*.

A structure $\mathbb{A}$ is *ultrahomogeneous* if for all isomorphisms $\phi : \mathbb{B} \to \mathbb{C}$ between its finite substructures there is an automorphism $\Phi : \mathbb{A} \to \mathbb{A}$ such that $\Phi \restriction \mathbb{B} = \phi$. A structure $\mathbb{A}$ is *locally finite* if all its finitely generated substructures are finite. A structure $\mathbb{A}$ is called a *Fraïssé structure* if it is infinite, countable, locally finite, and ultrahomogeneous. The following Theorem provides a connection between Fraïssé classes and Fraïssé structures.

Theorem 5.1 ([9]). *The connection between Fraïssé structures and classes is given in the following*:

(i): *If $\mathbb{A}$ is a Fraïssé structure then $Age(\mathbb{A})$ is a Fraïssé class.*
(ii): *If $\mathcal{A}$ is a Fraïssé class then there is a unique, up to isomorphism, Fraïssé structure $\mathbb{A}$ such that $Age(\mathbb{A}) = \mathcal{A}$.*

The structure $\mathbb{A}$ given by the second part of the previous Theorem is called a *Fraïssé limit* of the class $\mathcal{A}$, $\mathbb{A} = F\lim(\mathcal{A})$. There is a bijection between Fraïssé classes and Fraïssé structures given by:

$$\mathcal{A} \mapsto F\lim(\mathcal{A}), \mathbb{A} \mapsto Age(\mathbb{A}).$$

It can be proved by an easy but tedious argument, which we skip, that $\mathcal{S}$, $\mathcal{T}$, $\mathcal{ES}$, and $\mathcal{CT}$ are Fraïssé classes whose Fraïssé limits we denote by $\mathbb{S}$, $\mathbb{T}$, $\mathbb{ES}$ and $\mathbb{CT}$ respectively. Since $\mathcal{S}$ and $\mathcal{T}$ satisfy SAP and JEP then by Proposition 5.3 and Proposition 5.4 in [11] we have Fraïssé classes

$$\mathcal{OS} = \{(A, \circ^A, \preceq^A) : (A, \circ^A) \in \mathcal{S}, \preceq^A \in lo(A)\},$$
$$\mathcal{OT} = \{(A, \circ^A, \preceq^A) : (A, \circ^A) \in \mathcal{T}, \preceq^A \in lo(A)\},$$

with limits $\mathbb{OS}$ and $\mathbb{OT}$ respectively. At this point it is natural to consider the Fraïssé class

$$\mathcal{ET} = \{(A, \circ^A, \preceq^A) : (A, \circ^A) \in \mathcal{T}, \preceq^A \in le(\leq^A)\}.$$

It is straightforward to see that $\mathcal{OS}$ does not satisfy OP with respect to $\mathcal{S}$, and that $\mathcal{OT}$ and $\mathcal{ET}$ do not satisfy OP with respect to $\mathcal{T}$.

Lemma 5.2. *$\mathcal{OS}$, $\mathcal{OT}$ and $\mathcal{ET}$ are not Ramsey classes.*

Proof. We prove the claim for $\mathcal{OS}$ and $\mathcal{OT}$ by presenting a counterexample to the Ramsey property. This can be done by using the same counterexample for both classes. For this we use Lemma 4 in [20] which shows that the class of finite ordered posets is not a Ramsey class. We consider $\mathbb{A} = (A, \circ^A, \leq^A, \preceq^A)$ and $\mathbb{B} = (B, \circ^B, \leq^B, \preceq^B)$ from $\mathcal{OT} \subset \mathcal{OS}$ given by:

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

- $A = \{a, a_0, a_1\}, B = \{b, b_0, b_1, b_{00}, b_{11}\}$,
- $a = a_0 \circ^A a_1$,
- $b = b_0 \circ^B b_1 = b_{00} \circ^B b_{11}, b_0 = b_0 \circ^B b_{00}, b_1 = b_1 \circ^B b_{11}$,
- $a \prec^A a_0 \prec^A a_1$,
- $b \prec^B b_{00} \prec^B b_1 \prec^B b_0 \prec^B b_{11}$.

Suppose there is $\mathbb{C} = (C, \circ^C, \leq^C, \preceq^C)$ in $\mathcal{OT}$ or $\mathcal{OS}$ such that $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_2$. Let $\preceq^0 \in le(\leq^C)$. We consider the coloring

$$\chi : \binom{\mathbb{C}}{\mathbb{A}} \to \{0, 1\},$$
$$\chi(\mathbb{L}) = 1 \Leftrightarrow \preceq^C \restriction L^2 = \preceq^0 \restriction L^2,$$

where L is the underlying set of the structure $\mathbb{L}$. Without loss of generality we may assume that $\mathbb{B} \leq \mathbb{C}$. Considering each $\preceq' \in le(\leq^B)$ we obtain $\chi \restriction \binom{\mathbb{B}}{\mathbb{A}} \neq const$. This is in contradiction with $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_2$, so $\mathcal{OT}$ and $\mathcal{OS}$ are not Ramsey classes.

To show that $\mathcal{ET}$ is not a Ramsey class we consider again the structure $\mathbb{A}$, which is in $\mathcal{ET}$, and $\mathbb{U}_0 = (U_0, \circ^{U_0}, \leq^{U_0}, \preceq^{U_0})$ in $\mathcal{ET}$. $\mathbb{U}_0$ is a binary tree such that:

- $U_0 = \{u, u_0, u_1, u_{00}, u_{01}, u_{10}, u_{11}\}$,
- u is the root,
- $u_s \leq^{U_0} u_{s'} \Leftrightarrow s'$ is an end extension of s,
- $u \prec^{U_0} u_0 \prec^{U_0} u_1 \prec^{U_0} u_{00} \prec^{U_0} u_{10} \prec^{U_0} u_{01} \prec^{U_0} u_{11}$.

Suppose there is $\mathbb{V}_0 = (V_0, \circ^{V_0}, \leq^{V_0}, \preceq^{V_0})$ in $\mathcal{ET}$ such that $\mathbb{V}_0 \to (\mathbb{U}_0)^{\mathbb{A}}_2$. Then there is $\sqsubseteq^{V_0} \in lo(V_0)$ such that $(V_0, \circ^{V_0}, \leq^{V_0}, \sqsubseteq^{V_0}) \in \mathcal{CT}$ and there is a coloring

$$c : \binom{\mathbb{V}_0}{\mathbb{A}} \to \{0, 1\},$$
$$c(\mathbb{L}) = 1 \Leftrightarrow \preceq^{V_0} \restriction L^2 = \sqsubseteq^{V_0} \restriction L^2,$$

where L is the underlying set of the structure $\mathbb{L}$. We claim that there is no $\mathbb{U} \in \binom{\mathbb{V}_0}{\mathbb{U}_0}$ such that $c \restriction \binom{\mathbb{U}}{\mathbb{A}} = const$. Without loss of generality we may assume that $\mathbb{U} = \mathbb{U}_0$. Now we have two options, either $u_0 \sqsubset^{U_0} u_1$ or $u_1 \sqsubset^{U_0} u_0$. In the case that $u_0 \sqsubset^{U_0} u_1$ then we have $u_{00} \sqsubset^{U_0} u_{11}$ and $u_{01} \sqsubset^{U_0} u_{10}$, and also $u_{00} \prec^{U_0} u_{11}$ and $u_{10} \prec^{U_0} u_{01}$. If $\mathbb{A}_0$ and $\mathbb{A}_1$ are substructures of $\mathbb{U}$ given by the underlying sets $\{u, u_{00}, u_{11}\}$ and $\{u, u_{01}, u_{10}\}$ then we have $c(\mathbb{A}_0) = 1$ and $c(\mathbb{A}_1) = 0$, so we have contradiction with the starting assumption. In the case $u_1 \sqsubset^{U_0} u_0$ we obtain a contradiction in the similar way. ⊣

REMARK 5.3. Let $\mathcal{L}$ be the class of finite lattices. Let $\mathcal{OL}$ be the class of finite structures of the form $(\mathbb{A}, \leq^A)$, where $\mathbb{A} \in \mathcal{L}$ and $\leq^A \in lo(\mathbb{A})$. Using the same approach as in the proof of Lemma 5.2 we may show that $\mathcal{OL}$ is not a Ramsey class.

For a natural number $m \geq 1$ we have that $\mathcal{T}_m$ is a Fraïssé class which satisfies SAP and has the limit $\mathbb{T}_m$. In this case we may consider classes:

$$\mathcal{OT}_m = \{(A, \circ^A, \preceq^A) : (A, \circ^A) \in \mathcal{T}_m, \preceq^A \in lo(A)\},$$
$$\mathcal{ET}_m = \{(A, \circ^A, \preceq^A) \in \mathcal{ET} : (A, \circ^A) \in \mathcal{T}_m\},$$
$$\mathcal{CT}_m = \{(A, \circ^A, \preceq^A) \in \mathcal{CT} : (A, \circ^A) \in \mathcal{T}_m\}.$$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

Since $\mathcal{T}_m$ satisfies SAP we have that $\mathcal{OT}_m$ and $\mathcal{ET}_m$ are Fraïssé classes which satisfy SAP. We denote the limits of these classes by $\mathbb{OT}_m$ and $\mathbb{ET}_m$. On the other hand $\mathcal{CT}_m$ does not satisfy AP for $m > 1$. For $m = 2$ we consider also the class $\mathcal{BT}_2$ which contains structures $\mathbb{A} = (A, \circ^A, \preceq^A) \in \mathcal{OT}_2$ with the property that for every $a \in A$ with $im_{\mathbb{A}}(a) = \{a_1, a_2\}$ we have:

- $A_1 = \{a \in A : a_1 \leq^A a\}$ and $A_2 = \{a \in A : a_2 \leq^A a\}$ are intervals with respect to $\preceq^A$.
- Either $A_1 \prec^A a \prec^A A_2$ or $A_2 \prec^A a \prec^A A_1$.

By an easy but tedious argument we may see that $\mathcal{BT}_2$ is a Fraïssé class with limit $\mathbb{BT}_2$.

For $m = 1$, $\mathcal{ET}_1 = \mathcal{CT}_1$, so we can view $\mathcal{ET}_1$ as the class of finite ordered sets, which satisfies RP by the classical Ramsey theorem, see [19]. For $m = 1$, $\mathcal{OT}_1$ can be seen as the class of finite sets with two linear orderings, which satisfies RP, see Proposition 1 in [20].

Lemma 5.4.

(i) *For $m > 1$, the classes $\mathcal{OT}_m$, $\mathcal{ET}_m$ and $\mathcal{CT}_m$ do not satisfy RP.*
(ii) *$\mathcal{BT}_2$ satisfies RP.*

Proof.

(i) This is proved by using the same counterexamples as in the proof of Lemma 5.2.
(ii) This is done in the same way as the proof of Theorem 2.2 except the base of the induction. The reason for this follows from the fact that $\mathcal{T}_2$ is not closed under taking products. In this case for the base of induction we use the fact that strong subtrees form a Ramsey class, see [13] and [24], and the fact that strong subtrees are also substructures in the sense of treeable semilattices. We leave details of the proof for the reader, see also Section 6. ⊣

It is easy to see that for $m = 1$, $\mathcal{ET}_1 = \mathcal{CT}_1$ satisfies OP with respect to $\mathcal{T}_1$, while $\mathcal{OT}_1$ does not satisfy OP with respect to $\mathcal{T}_1$. We leave the following lemma as an easy exercise.

Lemma 5.5.

(i) *For $m > 1$, the classes $\mathcal{OT}_m$ and $\mathcal{ET}_m$ do not satisfy OP with respect to $\mathcal{T}_m$, while $\mathcal{CT}_m$ satisfies OP with respect to $\mathcal{T}_m$.*
(ii) *$\mathcal{BT}_2$ satisfies OP with respect to $\mathcal{T}_2$.*

Note that we may use Lemma 5.5 (ii), Theorem 5.4, and Proposition 10.5 in [11] to calculate Ramsey degrees for objects in $\mathcal{T}_2$. Instead of doing this we will present an approach in the following section which gives a calculation of the Ramsey degrees in $\mathcal{T}_m$ for all $m \geq 1$. The following Lemma shows that we cannot use order expansions of classes $\mathcal{T}_m$ in order to calculate Ramsey degrees.

Lemma 5.6. *For $m > 3$, there is no order expansion of the class $\mathcal{T}_m$ which satisfies RP.*

Proof. Suppose there is an ordered Ramsey expansion $\mathcal{K}$ of the class $\mathcal{T}_m$. First we need to consider structures $\mathbb{A}_1 = (A_1, \circ^{A_1}, \leq^{A_1}, \preceq^{A_1})$, $\mathbb{A}_2 = (A_2, \circ^{A_2}, \leq^{A_2}, \preceq^{A_2})$, $\mathbb{B}_1 = (B_1, \circ^{B_1}, \leq^{B_1}, \preceq^{B_1})$ and $\mathbb{B}_2 = (B_2, \circ^{B_2}, \leq^{B_2}, \preceq^{B_2})$, see Figure 2, given by:

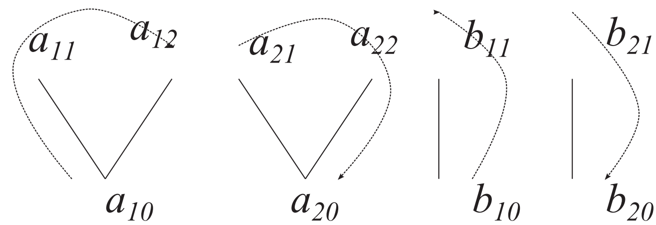

FIGURE 2. Structures $\mathbb{A}_1, \mathbb{A}_2, \mathbb{B}_1, \mathbb{B}_2$: curved lines are linear orderings.

- $\preceq^{A_1} \in lo(A_1)$, $\preceq^{A_2} \in lo(A_2)$, $\preceq^{B_1} \in lo(B_1)$, $\preceq^{B_2} \in lo(B_2)$,
- $(A_1, \circ^{A_1}, \leq^{A_1}), (A_2, \circ^{A_2}, \leq^{A_2}) \in \mathcal{T}_m$, $(B_1, \circ^{B_1}, \leq^{B_1}), (B_2, \circ^{B_2}, \leq^{B_2}) \in \mathcal{T}_m$,
- $A_1 = \{a_{10}, a_{11}, a_{12}\}$, $A_2 = \{a_{20}, a_{21}, a_{22}\}$, $B_1 = \{b_{10}, b_{11}\}$, $B_2 = \{b_{20}, b_{21}\}$,
- $a_{10} = a_{11} \circ^{A_1} a_{12}$, $a_{20} = a_{21} \circ^{A_2} a_{22}$, $b_{10} <^{B_1} b_{11}$, $b_{20} <^{B_2} b_{21}$,
- $a_{10} \prec^{A_1} a_{11} \prec^{A_1} a_{12}$, $a_{21} \prec^{A_2} a_{22} \prec^{A_2} a_{20}$, $b_{10} \prec^{B_1} b_{11}$, $b_{21} \prec^{B_2} b_{20}$.

Now we consider the structure $\mathbb{B} = (B, \circ^B, \leq^B)$ in $\mathcal{T}_m$ which is a tree such that all its terminal elements have height 1 and it has exactly m terminal elements. We denote its root by b and its terminal elements by $b_1, \ldots, b_m$. Let $\preceq^B \in lo(B)$ be such that $(\mathbb{B}, \preceq^B) \in \mathcal{K}$. Without loss of generality we may assume that $b_1 \prec^B b_2 \prec^B \cdots \prec^B b_m$. Then we have three cases: (1) $b \prec^B b_1$, (2) $b_m \prec^B b$, (3) $b_i \prec^B b \prec^B b_{i+1}$ for some $1 \leq i < m$. Considering these three cases we have two options $\mathbb{A}_1 \in \mathcal{K}$ or $\mathbb{A}_2 \in \mathcal{K}$. Then we have that $\mathbb{A}_1 \in \mathcal{K} \Rightarrow \mathbb{B}_1 \in \mathcal{K}$ and $\mathbb{A}_2 \in \mathcal{K} \Rightarrow \mathbb{B}_2 \in \mathcal{K}$. At this point we emphasize that this assumption is not valid for $m = 2$. We discuss the option $\mathbb{A}_1 \in \mathcal{K}$, (the other one is similar) by giving a counterexample to the Ramsey property. Let $\mathbb{C} \in \mathcal{K}$ be such that $\binom{\mathbb{C}}{\mathbb{A}_1} \neq \emptyset$. We construct a coloring $c : \binom{\mathbb{C}}{\mathbb{B}_1} \to \{1, \ldots, m\}$ by describing $c(\mathbb{P})$. Let $\varphi : \mathbb{B}_1 \to \mathbb{P}$ be the unique isomorphism. For each $c \in \mathbb{C}$ we fix a listing of the set $im_{\mathbb{C}}(c) = \{c_1, \ldots, c_k\}$, $k \leq m$. If $\varphi(b_{10}) = c$ and $\varphi(b_{11}) \in \mathbb{C}(c_i)$ then we take $c(\mathbb{P}) = i$. Now it is easy to see that $c \restriction \binom{\mathbb{R}}{\mathbb{B}_1} \neq const$ for any $\mathbb{R} \in \binom{\mathbb{C}}{\mathbb{A}_1}$. We obtain a similar conclusion by considering the case $\mathbb{A}_2 \in \mathcal{K}$. Therefore, $\mathcal{K}$ does not satisfy RP which is in contradiction with the starting assumption. ⊣

§6. Bounded branching. For a natural number $m \geq 1$, we consider a sequence $(R_{m,i})_{i=1}^m$ of binary relational symbols. In order to calculate Ramsey degrees for structures in $\mathcal{T}_m$ we consider the class $\mathcal{DT}_m$. This class contains structures of the form $(A, \circ^A, \leq^A, (R^A_{m,i})_{i=1}^m)$ such that:

- $(A, \circ^A, \leq^A) \in \mathcal{T}_m$,
- $R^A_{m,i}(a, b) \Rightarrow a <^A b$,
- $R^A_{m,i}(a, b), b \leq^A c \Rightarrow R^A_{m,i}(a, c)$,
- $R^A_{m,i}(a, b), b \circ^A c = a \Rightarrow \neg R^A_{m,i}(a, c)$,
- $a <^A b \Rightarrow (\exists i)[R^A_{m,i}(a, b)]$.

We explain how to visualize structures in $\mathcal{DT}_m$ by considering $(R_{m,i})_{i=1}^m$ as a collection of unary predicates assigned to each vertex of a given tree. Let $\mathbb{A}$ be a structure in $\mathcal{T}_m$. For every vertex a in $\mathbb{A}$ we add unary predicates on immediate successors of a such that: each successor is indicated by at least one predicate and different successors are indicated by different predicates.

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

It is easy to see that $\mathcal{DT}_m$ is a Fraïssé class with SAP and limit $\mathbb{DT}_m$. Note that for $m = 1$ we may identify $\mathcal{DT}_1$ with $\mathcal{T}_1$ which is a Ramsey class. For $m = 2$ we may see that $\mathcal{DT}_2$ is bidefinable with the class $\mathcal{BT}_2$ for which we have already proved RP. Therefore, the main point of Theorem 6.1 is in considering the case $m > 2$. We need to recall a definition of strong subtrees. Let $\mathbb{A} = (A, \leq^A)$ and $\mathbb{B} = (B, \leq^B)$ be trees. For a natural number $k \geq 0$ we denote the *k-level of the tree* $\mathbb{A}$ by $\mathbb{A}[k] = \{a \in A : ht_{\mathbb{A}}(a) = k\}$. We write $\mathbb{A}[T] = \bigcup\{\mathbb{A}[k] : k \in T\}$ for $T \subseteq \mathbb{N}$. We say that $\mathbb{A}$ is a *strong subtree* of $\mathbb{B}$ if there is $S \subseteq \mathbb{N}$ such that:

- $A \subseteq \mathbb{B}[S]$ and $A \cap \mathbb{B}[s] \neq \emptyset$ for all $s \in S$,
- if $s_1 < s_2$ are two successive elements in S and $a_1 \in A \cap \mathbb{B}[s_1]$ then every $b \in im_{\mathbb{B}}(a)$ has exactly one extension in $A \cap \mathbb{B}[s_2]$.

We point out that a strong subtree is also a substructure in the sense of treeable semilattices.

THEOREM 6.1. *For a natural number $m \geq 1$, the class $\mathcal{DT}_m$ satisfies RP.*

PROOF. Note that a given structure in $\mathcal{DT}_m$ can be embedded in a structure from $\mathcal{DT}_m$ which is a balanced tree with branching m. Therefore, in order to prove that $\mathcal{DT}_m$ is a Ramsey class it is enough to show that for every natural number $r > 1$, every $\mathbb{A} \in \mathcal{DT}_m$ and every $\mathbb{B} \in \mathcal{DT}_m$ which is a balanced tree with the constant branching m and property that $\binom{\mathbb{B}}{\mathbb{A}} \neq \emptyset$, there is a $\mathbb{C} \in \mathcal{DT}_m$ which is a balanced tree with the constant branching m and satisfies $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_r$. We prove this by an induction on $|\mathbb{A}|$ for $\mathbb{A} \in \mathcal{DT}_m$.

Base of induction $|\mathbb{A}| = 1$: This follows from the Ramsey property for strong subtrees, see [13] and Corollary 6.6 in [24].

Inductive step $k - 1 \mapsto k, k > 1$: We assume that the statement is correct for all $\mathbb{A} \in \mathcal{DT}_m$ whenever $|\mathbb{A}| < k$. Let $r > 1$ be a natural number and let $\mathbb{A} = (A, \circ^A, \leq^A, (R^A_{m,i})^m_{i=1})$ and $\mathbb{B} = (B, \circ^B, \leq^B, (R^B_{m,i})^m_{i=1})$ be structures from $\mathcal{DT}_m$ such that $\binom{\mathbb{B}}{\mathbb{A}} \neq \emptyset$ and $\mathbb{B}$ is a balanced tree with the constant branching m. By the base of the induction we choose $\mathbb{D}$, which is also (h, m)-balanced tree for some h, such that:

$$\mathbb{D} \to (\mathbb{B})^{\star}_r, \tag{6.1}$$

where $\star$ denotes the one element structure from $\mathcal{DT}_m$. We recursively construct a sequence $(\mathbb{B}_i)^h_{i=0}$ of structures from $\mathcal{DT}_m$ such that for all $0 \leq i < h$ we have:

- $\mathbb{B}_h = \mathbb{D}$.
- $\mathbb{B}_{i+1} \leq \mathbb{B}_i$.
- Each $\mathbb{B}_i$ is a balanced tree with the constant branching m.
- Trees $\mathbb{B}_i$ and $\mathbb{B}_{i-1}$ have the same elements of the height $0, 1, \dots, i - 1$.

We consider elements in $\mathbb{A}$, $a_1, \dots, a_l$ with height 1. Without loss of generality we may assume that $R^A_{m,q}(a, a_j) \Leftrightarrow q = j$, where a is the root of $\mathbb{A}$. We suppose that we have constructed the structure $\mathbb{B}_{i+1}$ and we proceed with the construction of the structure $\mathbb{B}_i$. Let b be an element in the tree $\mathbb{B}_{i+1}$ with the height i and let $b_1, \dots, b_l$ be the immediate successors of b with the property that $R^{B_{i+1}}_{m,q}(b, b_j) \Leftrightarrow q = j$. By the product Ramsey theorem for classes, see Theorem 2 in [21], and the inductive assumption there is a structure $\mathbb{B}_{i+1,0} \in \mathcal{DT}_m$, which is a balanced tree with constant branching m, such that:

$$\mathbb{B}_{i+1,0} \to (\mathbb{B}(b_1), \dots, \mathbb{B}(b_l))^{(\mathbb{A}(a_1), \dots, \mathbb{A}(a_l))}_r. \tag{6.2}$$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

Note that this is possible by the inductive assumption since $|\mathbb{A}(a_i)| < |\mathbb{A}|$. Now we take $\mathbb{B}_i$ to be such that:

- $\mathbb{B}_i$ and $\mathbb{B}_{i+1}$ agree on the first i levels.
- For each b in $\mathbb{B}_i$ with the height i we have $\mathbb{B}_i(b) \cong \mathbb{B}_{i+1,0}$.

This completes the construction of the sequence $(\mathbb{B}_i)_{i=0}^h$, and we claim that $\mathbb{B}_0 \to (\mathbb{B})_r^{\mathbb{A}}$. So, we need to consider a given coloring

$$c : \binom{\mathbb{B}_0}{\mathbb{A}} \to \{1, \dots, r\}.$$

For an element b in $\mathbb{B}_0$, we denote by $\binom{\mathbb{B}_0}{\mathbb{A}}_b$ the collection of all elements in $\binom{\mathbb{B}_0}{\mathbb{A}}$ with the minimal element equal to b. Then we have an induced coloring

$$c_b = c \restriction \binom{\mathbb{B}_0}{\mathbb{A}}_b.$$

Our choice of the sequence $(\mathbb{B}_i)_{i=0}^h$ gives us a sequence $(\mathbb{B}_i')_{i=0}^h$ of structures from $\mathcal{DT}_m$ such that for each $0 \leq i < h$ we have:

- $\mathbb{B}_0' = \mathbb{B}_0$.
- $\mathbb{B}_i' \cong \mathbb{B}_i$.
- $\mathbb{B}_i' \geq \mathbb{B}_{i+1}'$.
- For all $b \in \mathbb{B}_i'$ with the height i and all $\mathbb{A}', \mathbb{A}'' \in \binom{\mathbb{B}_i'}{\mathbb{A}}$ with the same root equal to b we have $c(\mathbb{A}') = c(\mathbb{A}'')$.

Now we explain how to obtain the sequence $(\mathbb{B}_i')_{i=0}^m$. Suppose that we have obtained $\mathbb{B}_{i-1}'$. For each b in $\mathbb{B}_{i-1}'$ with height i, from 6.2 we obtain $\mathbb{B}_{i,b}' \leq \mathbb{B}_i'(b)$, a balanced tree with constant branching m, such that all copies of $\mathbb{A}$ in $\mathbb{B}_{i,b}'$ are monochromatic and they have color com_b. Moreover, we have $\mathbb{B}_{i,b}' \cong \mathbb{B}_{i,0}$. Now we take $\mathbb{B}_i'$ to agree with $\mathbb{B}_{i-1}'$ on the first i many levels and for b with the height i we take $\mathbb{B}_i'(b) = \mathbb{B}_{i,b}'$.

In the end structure $\mathbb{B}_h'$ has the property that every two copies of $\mathbb{A}$ in $\mathbb{B}_h'$ with the same root have the same color. Therefore, we have an induced coloring

$$c' : \binom{\mathbb{B}_h'}{\star} \to \{1, \dots, r\},$$
$$c'(b) = com_b.$$

From 6.1 there is $\mathbb{B}' \in \binom{\mathbb{B}_h'}{\mathbb{B}}$ such that $c' \restriction \binom{\mathbb{B}'}{\star} = const$. Moreover this implies that $c \restriction \binom{\mathbb{B}'}{\mathbb{A}} = const$, so $\mathcal{DT}_m$ satisfies RP. ⊣

Lemma 6.2. *Let $m \geq 1$ be a natural number, and let $\mathbb{A} = (A, \circ^A, \leq^A)$ be a structure from $\mathcal{T}_m$. Then there is a structure $\mathbb{B} = (B, \circ^B, \leq^B)$ in $\mathcal{T}_m$ such that for every sequence of binary relations $(R_{m,i}^A)_{i=1}^m$ and $(R_{m,i}^B)_{i=1}^m$ on A and B respectively with $(\mathbb{A}, (R_{m,i}^A)_{i=1}^m) \in \mathcal{DT}_m$ and $(\mathbb{B}, (R_{m,i}^B)_{i=1}^m) \in \mathcal{DT}_m$ we have $(\mathbb{A}, (R_{m,i}^A)_{i=1}^m) \hookrightarrow (\mathbb{B}, (R_{m,i}^B)_{i=1}^m)$.*

Proof. If $h = \max\{ht_{\mathbb{A}}(a) : a \in A\}$ then it is enough to take $\mathbb{B}$ to be a (h, m)-balanced tree. ⊣

Proof of Theorem 2.4. For a structure $\mathbb{A} \in \mathcal{T}_m$ we denote by $\#(\mathbb{A})$ the number of mutually nonisomorphic structures $(\mathbb{A}, (R_{m,i}^A)_{i=1}^m) \in \mathcal{DT}_m$. From Theorem 6.1, Lemma 6.2, and results in [16], which are generalization of the results in [11], we have for $\mathbb{A} = (A, \circ^A, \leq^A)$ in $\mathcal{T}_m$ the following

$$t_{\mathcal{T}_m}(\mathbb{A}) = \frac{\#(\mathbb{A})}{|Aut(\mathbb{A})|}. \tag{6.3}$$

We show that $t_{\mathcal{T}_m}(\mathbb{A}) = t_m(\mathbb{A})$ by induction on $|\mathbb{A}|$. Clearly for $|\mathbb{A}| = 1$, the claim is satisfied. So suppose we have $t_{\mathcal{T}_m}(\mathbb{A}') = t_m(\mathbb{A}')$ for all $\mathbb{A}' \in \mathcal{T}_m$ with $|\mathbb{A}'| < |\mathbb{A}|$. Let a be the root of the tree $(A, \leq^A)$ and let $im_{\mathbb{A}}(a) = \{a_1, \ldots, a_n\}$. Then we have

$$\#(\mathbb{A}) = \binom{m}{n} n! \prod_{i=1}^{n} \#(\mathbb{A}(a_i)). \tag{6.4}$$

Without loss of generality we may assume that the structures $(\mathbb{A}(a_i))_{i=1}^{k}$ are mutually nonisomorphic and that for every $j > k$ there is $i \leq k$ such that $\mathbb{A}(a_i) \cong \mathbb{A}(a_j)$. Let $(n_i)_{i=1}^{k}$ be a sequence of natural numbers such that $n_i = |\{j \leq n : \mathbb{A}(a_i) \cong \mathbb{A}(a_j)\}|$. Then we have

$$|Aut(\mathbb{A})| = n_1! \cdots n_k! \prod_{i=1}^{n} |Aut(\mathbb{A}(a_i))|. \tag{6.5}$$

Now from 6.3, 6.4, 6.5, and the inductive assumption we obtain

$$\begin{aligned} t_{\mathcal{T}_m}(\mathbb{A}) &= \frac{\binom{m}{n} n! \prod_{i=1}^{n} \#(\mathbb{A}(a_i))}{n_1! \cdots n_k! \prod_{i=1}^{n} |Aut(\mathbb{A}(a_i))|} = \frac{\binom{m}{n} n!}{n_1! \cdots n_k!} \prod_{i=1}^{n} \frac{\#(\mathbb{A}(a_i))}{|Aut(\mathbb{A}(a_i))|} \\ &= \frac{\binom{m}{n} n!}{n_1! \cdots n_k!} \prod_{i=1}^{n} t_{\mathcal{T}_m}(\mathbb{A}(a_i)) = \frac{\binom{m}{n} n!}{n_1! \cdots n_k!} \prod_{i=1}^{n} t_m(\mathbb{A}(a_i)) = t_m(\mathbb{A}). \end{aligned}$$

⊣

§7. **Dynamics.** A continuous action $G \times X \to X$ of a topological group G on a compact Hausdorff space X is called a *G-flow*. A G-flow X is *minimal* if for every $x \in X$ we have $\overline{\{gx : g \in G\}} = X$. A G-flow Y is a *subflow* of G-flow X if Y is a G-invariant subset of X. Zorn's lemma implies that every G-flow contains a minimal subflow. Among minimal G-flows there is a maximal one which is called the *universal minimal G-flow* which is unique up to isomorphism, see [2]. If the universal minimal G-flow contains only one point then we say that G is an *extremely amenable group*. We assume that groups of automorphisms of a countable structures are equipped with pointwise convergence topology, see [3] for more details. Moreover, all such groups are closed subgroups of $\mathbb{S}_\infty$, the group of permutations of natural numbers with pointwise convergence topology. More details on closed subgroups of $\mathbb{S}_\infty$ can be found in [3].

By Theorem 6.1 in [11] and Theorem 2.1, Theorem 2.2, and Lemma 5.2 we have the following.

Corollary 7.1. *$Aut(\mathbb{OS})$ and $Aut(\mathbb{CT})$ are extremely amenable groups, while $Aut(\mathbb{ES})$, $Aut(\mathbb{ET})$, and $Aut(\mathbb{OT})$ are not extremely amenable groups.*

Let $\mathcal{K}$ be a Fraïssé class in a given signature L, let $\preceq$ be a binary relational symbol such that $\preceq \notin L$ and let $\mathcal{K}'$ be a Fraïssé class in $L \cup \{\preceq\}$. Let $\mathbb{K} = F\lim(\mathcal{K})$ and $\mathbb{K}' = F\lim(\mathcal{K}')$ be the corresponding Fraïssé limits, and let $G = Aut(\mathbb{K})$.

If $\mathcal{K}'$ is a reasonable expansion of $\mathcal{K}$ then we may assume without loss of generality that $\mathbb{K}$ and $\mathbb{K}'$ have the same underlying set, the set of natural numbers $\mathbb{N}$, and that $\mathbb{K} = \mathbb{K}'|L$. We denote by $\preceq_0$ the interpretation of the symbol $\preceq$ in $\mathbb{K}'$. We consider LO the set of linear orderings on $\mathbb{N}$ as the compact subset of $2^{\mathbb{N}^2}$, where $2 = \{0, 1\}$. Also we consider *logic action* of the group G on the set LO and define

$$X_{\mathcal{K}'} = \overline{G \cdot \preceq_0}.$$

In particular, $X_{\mathcal{K}'}$ is a G-flow. We use Theorem 7.4 and Theorem 7.5 in [11] with Theorem 2.1, Theorem 2.2, Lemma 5.2, Lemma 4.1, and Lemma 4.2 to obtain the following.

Corollary 7.2.

(i) *$X_{\mathcal{ES}}$ is the universal minimal $Aut(\mathbb{S})$ flow.*
(ii) *$X_{\mathcal{CT}}$ is the universal minimal $Aut(\mathbb{T})$ flow.*
(iii) *$X_{\mathcal{OS}}$ is not a minimal $Aut(\mathbb{S})$ flow.*
(iv) *$X_{\mathcal{ET}}$ and $X_{\mathcal{OT}}$ are not minimal $Aut(\mathbb{T})$ flows.*

In the case of the classes $\mathcal{T}_m$ and $\mathcal{DT}_m$ we have Fraïssé limits $\mathbb{T}_m$ and $\mathbb{DT}_m$ respectively. We may consider these two structures as structures with underlying set $\mathbb{N}$ and we write $\mathbb{DT}_m = (\mathbb{T}_m, (R_i)_{i=1}^m)$, where each R_i is a binary relation on $\mathbb{N}$, which can be seen as a subset of $2^{\mathbb{N}^2}$. Moreover we have logic action of the group $G = Aut(\mathbb{T}_m)$ on the space $\prod_{i=1}^m 2^{\mathbb{N}^2}$ and we consider the space

$$Y_{\mathcal{DT}_m} = \overline{G \cdot (R_1, \ldots, R_m)}.$$

Then from the results in [16], Theorem 6.1, and Lemma 6.2 we obtain the following.

Corollary 7.3.

(i) *$Aut(\mathbb{DT}_m)$ is an extremely amenable group.*
(ii) *$Y_{\mathcal{DT}_m}$ is the universal minimal $Aut(\mathbb{T}_m)$ flow.*

Let G be a given topological group. We recall that G is *amenable* if every G-flow has an invariant Borel probability measure. We say that a given G-flow is *uniquely ergodic* if it has a unique invariant measure. We say that G is *uniquely ergodic* if every minimal G-flow is uniquely ergodic. Recently it was shown that the automorphism group of the universal countable lattice is not amenable, see [12], and also in [25]. We point out that the completely same argument proves the following.

Corollary 7.4. *$Aut(\mathbb{S})$ is a nonamenable group.*

For a given structure $\mathbb{B} = (B, \circ^B)$ in $\mathcal{T}$ we denote by $\nabla(\mathbb{B})$ the cardinality of the set $\{\preceq \in lo(B) : (\mathbb{B}, \preceq) \in \mathcal{CT}\}$. It is easy to see that

$$\nabla(\mathbb{B}) = \prod\{|im_{\mathbb{B}}(b)|! : b \in B\},$$

where we assume that $0! = 1$. For $\mathbb{A}$ and $\mathbb{B}$, structures in $\mathcal{T}$, and a linear ordering $\preceq$ with the property that $\mathbb{A} \leq \mathbb{B}$ and $(\mathbb{A}, \preceq) \in \mathcal{CT}$ we write

$$N(\mathbb{A}, \preceq, \mathbb{B}) = \{\sqsubseteq : (\mathbb{B}, \sqsubseteq) \in \mathcal{CT} \ \& \sqsubseteq \restriction A^2 = \preceq\}.$$

For $\mathbb{B} = (B, \circ^B)$ in $\mathcal{T}_m$, we denote by $\nabla_m(\mathbb{B})$ the cardinality of the set $\{(R^B_{m,i})_{i=1}^m : (\mathbb{B}, (R^B_{m,i})_{i=1}^m) \in \mathcal{DT}_m\}$. It is easy to see that

$$\nabla_m(\mathbb{B}) = \prod\{\binom{m}{|im_{\mathbb{B}}(b)|}|im_{\mathbb{B}}(b)|! : b \in B\}.$$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

Let $m \geq 1$ be a given natural number. For $\mathbb{A}$ and $\mathbb{B}$, structures in $\mathcal{T}_m$, and a sequence of binary relations $(R^A_{m,i})^m_{i=1}$ with the property that $\mathbb{A} \leq \mathbb{B}$ and $(\mathbb{A}, (R^A_{m,i})^m_{i=1}) \in \mathcal{DT}_m$ we write

$$N_m(\mathbb{A}, (R^A_{m,i})^m_{i=1}, \mathbb{B}) = \{(R^B_{m,i})^m_{i=1} : (\mathbb{B}, (R^B_{m,i})^m_{i=1}) \in \mathcal{DT}_m \;\&\; (\forall i)[R^B_{m,i} \restriction A^2 = R^A_{m,i}]\}.$$

LEMMA 7.5. *Let $m \geq 1$ be a natural number and let $\mathcal{K} \in \{\mathcal{T}, \mathcal{T}_m\}$. Let $\mathbb{A} = (A, \circ^A)$ and $\mathbb{B} = (B, \circ^B)$ be structures in $\mathcal{K}$ such that $\mathbb{A} \leq \mathbb{B}$. Then we have:*

(i) *If $\mathcal{K} = \mathcal{T}$ and $\preceq$ is a linear ordering such that $(\mathbb{A}, \preceq) \in \mathcal{CT}$ then*

$$|N(\mathbb{A}, \preceq, \mathbb{B})| = \frac{\nabla(\mathbb{B})}{\nabla(\mathbb{A})}.$$

(ii) *If $\mathcal{K} = \mathcal{T}_m$ and $(R^A_{m,i})^m_{i=1}$ is a sequence of binary relations such that $(\mathbb{A}, (R^A_{m,i})^m_{i=1}) \in \mathcal{DT}_m$ then*

$$N_m(\mathbb{A}, (R^A_{m,i})^m_{i=1}, \mathbb{B}) = \frac{\nabla_m(\mathbb{B})}{\nabla_m(\mathbb{A})}.$$

PROOF. We prove the statement by an induction on $|\mathbb{A}|$.

(i) *Base of the induction* $|\mathbb{A}| = 1$: This follows from the fact that $N(\mathbb{A}, \preceq, \mathbb{B}) = \{\preceq \in lo(B) : (\mathbb{B}, \preceq) \in \mathcal{CT}\}$ and $\nabla(\mathbb{A}) = 1$.

Inductive step $k-1 \mapsto k, k > 1$: We assume that the statement is correct for all $\mathbb{A} \in \mathcal{T}$ with $|\mathbb{A}| < k$. Let a be the root of the tree $\mathbb{A}$, and let $im_{\mathbb{A}}(a) = \{a_1, \dots, a_l\}$ be linearly ordered by $\preceq$ as $a_1 \prec \cdots \prec a_l$. Let $b_i \in im_{\mathbb{B}}(a)$ be such that $b_i \leq^B a_i$ for $1 \leq i \leq l$. Let $\preceq_i$ be the restriction of the linear ordering $\preceq$ to subtree $\mathbb{A}(a_i)$. Then we have

$$\begin{aligned}
&|N(\mathbb{A}, \preceq, \mathbb{B})| = \\
&(\textstyle\prod\{|im_{\mathbb{B}}(b)|! : b \in B \;\&\; b \notin \mathbb{B}(a_i)\}) \times \\
&\binom{|im_{\mathbb{B}}(a)|}{l} \times (|im_{\mathbb{B}}(a)| - l)! \times (\textstyle\prod_{i=1}^{l} |N(\mathbb{A}(a_i), \preceq_i, \mathbb{B}(b_i))|) \\
&= (\textstyle\prod\{|im_{\mathbb{B}}(b)|! : b \in B \;\&\; b \notin \mathbb{B}(a_i)\}) \times \\
&\frac{|im_{\mathbb{B}}(a)|!}{l!} \times (\textstyle\prod_{i=1}^{l} |N(\mathbb{A}(a_i), \preceq_i, \mathbb{B}(b_i))|) \\
&= (\textstyle\prod\{|im_{\mathbb{B}}(b)|! : b \in B \;\&\; b \notin \mathbb{B}(a_i)\}) \times \\
&\frac{|im_{\mathbb{B}}(a)|!}{l!} \times \left(\textstyle\prod_{i=1}^{l} \frac{\nabla(\mathbb{B}(b_i))}{\nabla(\mathbb{A}(a_i))}\right) \\
&= (\textstyle\prod\{|im_{\mathbb{B}}(b)|! : b \in B \;\&\; b \notin \mathbb{B}(a_i)\}) \times \\
&|im_{\mathbb{B}}(a)|! \times (\textstyle\prod_{i=1}^{l} \nabla(\mathbb{B}(b_i))) \times (l! \times (\textstyle\prod_{i=1}^{l} \nabla(\mathbb{A}(a_i))))^{-1} = \nabla(\mathbb{B}) \times (\nabla(\mathbb{A}))^{-1}.
\end{aligned}$$

where the third equality follows from the inductive assumption since $|\mathbb{A}(a_i)| < |\mathbb{A}|$ for $1 \leq i \leq l$.

(ii) *Base of the induction* $|\mathbb{A}| = 1$: This follows from the fact that $N_m(\mathbb{A}, ((R^A_{m,i})^m_{i=1})^m_{i=1}, \mathbb{B}) = \{(R^B_{m,i})^m_{i=1} : (\mathbb{B}, (R^B_{m,i})^m_{i=1}) \in \mathcal{DT}_m\}$ and $\nabla_m(\mathbb{A}) = 1$ because each $R^A_{m,i}$ must be an empty relation for $1 \leq i \leq m$.

Inductive step $k-1 \mapsto k, k > 1$: We assume that the statement is correct for all $\mathbb{A} \in \mathcal{T}_m$ with $|\mathbb{A}| < k$. Let a be the root of the tree $\mathbb{A}$, and let

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

$im_{\mathbb{A}}(a) = \{a_1, \ldots, a_l\}$ such that $R^A_{m,i}(a, a_j) \Leftrightarrow i = j$. Let $b_i \in im_{\mathbb{B}}(a)$ be such that $b_i \leq^B a_i$ for $1 \leq i \leq l$. Let $R^{A,j}_{m,i}$ be the restriction of the relation $R^A_{m,i}$ to subtree $\mathbb{A}(a_j)$. Then we have

$$
\begin{aligned}
&|N_m(\mathbb{A}, (R^A_{m,i})^m_{i=1}, \mathbb{B})| = \\
&(\prod\{\binom{m}{|im_{\mathbb{B}}(b)|}|im_{\mathbb{B}}(b)|! : b \in B \;\&\; (\forall j)[b \notin \mathbb{B}(a_j)]\}) \times \\
&\binom{m-l}{|im_{\mathbb{B}}(a)|-l} \times (|im_{\mathbb{B}}(a)| - l)! \times (\prod_{j=1}^{l} |N_m(\mathbb{A}(a_j), (R^{A,j}_{m,i})^m_{i=1}, \mathbb{B}(b_j)|) \\
&= (\prod\{\binom{m}{|im_{\mathbb{B}}(b)|}|im_{\mathbb{B}}(b)|! : b \in B \;\&\; (\forall j)[b \notin \mathbb{B}(a_j)]\}) \times \\
&\frac{(m-l)!}{(m - |im_{\mathbb{B}}(a)|)!} \times (\prod_{j=1}^{l} |N_m(\mathbb{A}(a_j), (R^{A,j}_{m,i})^m_{i=1}, \mathbb{B}(b_j)|) \\
&= (\prod\{\binom{m}{|im_{\mathbb{B}}(b)|}|im_{\mathbb{B}}(b)|! : b \in B \;\&\; (\forall j)[b \notin \mathbb{B}(a_j)]\}) \times \\
&\frac{\binom{m}{|im_{\mathbb{B}}(a)|}|im_{\mathbb{B}}(a)|!}{\binom{m}{l} l!} \times \left(\prod_{j=1}^{l} \frac{\nabla_m(\mathbb{B}(b_j))}{\nabla_m(\mathbb{A}(a_j))}\right) \\
&= (\prod\{\binom{m}{|im_{\mathbb{B}}(b)|}|im_{\mathbb{B}}(b)|! : b \in B \;\&\; (\forall j)[b \notin \mathbb{B}(a_j)]\}) \times \\
&\binom{m}{|im_{\mathbb{B}}(a)|}|im_{\mathbb{B}}(a)|! \times (\prod_{j=1}^{l} \nabla_m(\mathbb{B}(b_j))) \times \\
&\left(\binom{m}{l} l! \times (\prod_{j=1}^{l} \nabla_m(\mathbb{A}(a_j)))\right)^{-1} = \nabla_m(\mathbb{B}) \times (\nabla_m(\mathbb{A}))^{-1}.
\end{aligned}
$$

where the third equality follows from the inductive assumption and properties of binomial coefficients. ⊣

From Proposition 8.1 in [1], Corollary 7.2, and Corollary 7.3 we may show that

PROPOSITION 7.6. *$Aut(\mathbb{T})$ and $Aut(\mathbb{T}_m)$ are amenable and uniquely ergodic groups.*

PROOF. We prove that the group $Aut(\mathbb{T})$ is amenable and uniquely ergodic, and a similar argument proves the statement for the group $Aut(\mathbb{T}_m)$. We consider

$$\mu : \mathcal{CT} \to [0, 1], \mu((\mathbb{A}, \preceq)) = \frac{1}{\nabla(\mathbb{A})}.$$

Lemma 7.5 (i) implies:

μ is "probability": For a given $\mathbb{A} \in \mathcal{T}$ we have $\sum\{\mu((\mathbb{A}, \preceq)) : (\mathbb{A}, \preceq) \in \mathcal{CT}\} = 1$.

μ is invariant: $(\mathbb{A}_1, \preceq_1) \cong (\mathbb{A}_2, \preceq_2) \Rightarrow \mathbb{A}_1 \cong \mathbb{A}_2 \Rightarrow \nabla(\mathbb{A}_1) = \nabla(\mathbb{A}_2) \Rightarrow \mu((\mathbb{A}_1, \preceq_1)) = \mu((\mathbb{A}_2, \preceq_2))$.

μ is consistent: For $\mathbb{A}_1 \leq \mathbb{A}_2$ and $(\mathbb{A}_1, \preceq_1) \in \mathcal{CT}$ we have

$$
\begin{aligned}
\mu((\mathbb{A}_1, \preceq_1)) &= \frac{1}{\nabla(\mathbb{A}_1)} = \frac{\nabla(\mathbb{A}_2)}{\nabla(\mathbb{A}_1)} \times \frac{1}{\nabla(\mathbb{A}_2)} = |N(\mathbb{A}_1, \preceq_1, \mathbb{A}_2)| \times \frac{1}{\nabla(\mathbb{A}_2)} \\
&= \sum\{\mu((\mathbb{A}_2, \preceq_2)) : (\mathbb{A}_1, \preceq_1) \leq (\mathbb{A}_2, \preceq_2)\}.
\end{aligned}
$$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

This shows that μ is a consistent random $\mathcal{CT}$-admissible ordering on $\mathcal{T}$, see page 23 in [1]. Now we show that μ is the unique consistent random $\mathcal{CT}$-admissible ordering on $\mathcal{T}$. Let $\mathbb{B} \in \mathcal{T}$ be a balanced tree and let $\preceq_1$and $\preceq_2$ be such that $(\mathbb{B}, \preceq_1)$ and $(\mathbb{B}, \preceq_2) \in \mathcal{CT}$. Then there is $\phi \in Aut(\mathbb{B})$ such that $\phi(\preceq_1) = \preceq_2$. Since μ is invariant we have $\mu((\mathbb{B}, \preceq_1)) = \mu((\mathbb{B}, \preceq_2)) = \frac{1}{\nabla(\mathbb{B})}$. Therefore, μ is uniquely defined on balanced trees in $\mathcal{CT}$. For a given $(\mathbb{A}, \preceq_0) \in \mathcal{CT}$ there is a balanced tree $\mathbb{B}$ such that $\mathbb{A} \leq \mathbb{B}$. Since μ is unique on balanced trees and consistent Lemma 7.5 (i) implies

$$\begin{aligned}\mu((\mathbb{A}, \preceq_0)) &= \sum\{\mu((\mathbb{A}_2, \preceq_2)) : (\mathbb{A}_1, \preceq_1) \leq (\mathbb{A}_2, \preceq_2)\}\frac{1}{\nabla(\mathbb{A})}\\ &= |N(\mathbb{A}_1, \preceq_1, \mathbb{A}_2)| \times \frac{1}{\nabla(\mathbb{A}_2)} = \frac{1}{\nabla(\mathbb{A}_1)}.\end{aligned}$$

Therefore, μ is unique and from Proposition 9.2 in [1] we obtain that $Aut(\mathbb{T})$ is amenable and uniquely ergodic. ⊣

Let $\mathcal{K}$ be a class of finite structures in a given signature L. We say that $\mathcal{K}$ is a *Hrushovski class* if for all $\mathbb{A} \in \mathcal{K}$ there is $\mathbb{B} \in \mathcal{K}$ such that any isomorphism $\phi : \mathbb{A}_1 \to \mathbb{A}_2$ between substructures of $\mathbb{A}$ can be extended to an isomorphism $\psi : \mathbb{B} \to \mathbb{B}$.

COROLLARY 7.7. *Let $m \geq 1$ be a natural number. Then classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{T}_m$ are not Hrushovski classes.*

PROOF. Let $\mathcal{K} \in \{\mathcal{S}, \mathcal{T}, \mathcal{T}_m\}$. We consider $\mathbb{A} = (A, \circ^A, \leq^A) \in \mathcal{K}$, where $A = \{a_1, a_2\}$ and $a_1 <^A a_2$. Note that a_1 and a_2 determines isomorphic one-element substructures of $\mathbb{A}$, so we have a partial isomorphism $\phi : \{a_1\} \to \{a_2\}$. Suppose that there is $\mathbb{B} = (B, \circ^B, \leq^B) \in \mathcal{K}$ such that ϕ can be extended to an isomorphism $\psi : \mathbb{B} \to \mathbb{B}$. Since $\mathbb{B}$ is a finite structure and ψ is an isomorphism we have

$$|\{b \in B : a_2 <^B b\}| = |\{b \in B : a_1 <^B b\}|.$$

Since $a_1 <^A a_2 \Rightarrow a_1 <^B a_2$ we have

$$|\{b \in B : a_2 <^B b\}| < |\{b \in B : a_1 <^B b\}|.$$

Clearly, this is a contradiction which shows that $\mathcal{K}$ is not a Hrushovski class. ⊣

The proof of Corollary 7.7 also implies the following.

REMARK 7.8. The class of finite lattices, the class of finite distributive lattices, the class of finite posets, the class of finite permutations the class of finite linearly ordered sets are not Hrushovski classes.

We point out that most of the known examples of the unique ergodic groups of the form $Aut(F\lim(\mathcal{K}))$ for which group $Aut(F\lim(\mathcal{K}))$ has infinite universal minimal flow are given for a Hrushovski class $\mathcal{K}$. If we take that $\mathcal{K}$ is the class of finite linearly ordered sets we obtain a non Hrushovski class, but $Aut(F\lim(\mathcal{K})) = Aut(\mathbb{Q})$ is an extremely amenable group. Therefore, we emphasize the following what is obtained from Proposition 7.6 and Corollary 7.7.

THEOREM 7.9. *Let $m \geq 1$ be a natural number. $Aut(\mathbb{T}) = Aut(F\lim(\mathcal{T}))$ and $Aut(\mathbb{T}_m) = Aut(F\lim(\mathcal{T}_m))$ are uniquely ergodic groups which are not extremely amenable and $\mathcal{T}$ and $\mathcal{T}_m$ are not Hrushovski classes.*

§8. Canonization. Canonical orderings on the combinatorial cubes are discussed in [14] and [17]. In this section we canonize linear orderings for semilattices.

Let $\mathbb{A} = (A, \circ^A (R^A_{m,i})^m_{i=1}) \in \mathcal{DT}_m$ and let $\preceq$ be a linear ordering on A. We say that $\preceq$ is *canonical on* $\mathbb{A}$ *for the class* $\mathcal{DT}_m$ if there is $\sqsubseteq \in lo(\{0, 1, \dots, m\})$ such that for $a, b \in A$ we have

$$\begin{aligned} a \prec b \Leftrightarrow &((a < b \;\&\; R^A_{m,i}(a, b) \;\&\; 0 \sqsubset i) \textit{ or} \\ &(b < a \;\&\; R^A_{m,i}(b, a) \;\&\; i \sqsubset 0) \textit{ or} \\ &(a \circ^A b = c \;\&\; R^A_{m,i}(c, a) \;\&\; R^A_{m,j}(c, b) \;\&\; i \sqsubset j)). \end{aligned}$$

Let $\mathbb{A} = (A, \circ^A, \leq^A)$ be a semilattice and let $\preceq$ be a linear ordering on A. Then we say that $\preceq$ is *canonical on* $\mathbb{A}$ *for the class* $\mathcal{K}$ if:

$\mathcal{K} = \mathcal{S}$: $\preceq \in le(\leq^A)$ or $op(\preceq) \in le(\leq^A)$.
$\mathcal{K} = \mathcal{T}$: $(\mathbb{A}, \preceq\restriction A^2) \in \mathcal{CT}$ or $(\mathbb{A}, op(\preceq\restriction A^2)) \in \mathcal{CT}$.
$\mathcal{K} = \mathcal{T}_m$: There is $(\mathbb{A}, (R^A_{m,i})^m_{i=1}) \in \mathcal{DT}_m$ such that $\preceq$ is canonical on $(\mathbb{A}, (R^A_{m,i})^m_{i=1})$ for the class $\mathcal{DT}_m$.

Note that a structure in $\mathcal{DT}_m$ which is also a (k, m)-balanced tree has $(m + 1)!$ nonamenable canonical orderings. On the other hand, a structure in $\mathcal{T}_m$ which is also a (k, m)-balanced tree has $(m + 1)$ nonamenable canonical orderings.

PROPOSITION 8.1. *Let $m \geq 1$ be a natural number and let $\mathcal{K} \in \{\mathcal{S}, \mathcal{T}, \mathcal{T}_m\}$.*

(i) *For $\mathbb{A} = (A, \circ^A (R^A_{m,i})^m_{i=1}) \in \mathcal{DT}_m$ there is $\mathbb{B} \in \mathcal{K}$ such that for every $\sqsubseteq \in lo(\mathbb{B})$ there is $\mathbb{C} \in \binom{\mathbb{B}}{\mathbb{A}}$ with the underlying set C such that $\sqsubseteq\restriction C^2$ is canonical on $\mathbb{C}$ for the class $\mathcal{DT}_m$.*

(ii) *For $\mathbb{A} = (A, \circ^A, \leq^A) \in \mathcal{K}$ there is $\mathbb{B} \in \mathcal{K}$ such that for every $\sqsubseteq \in lo(\mathbb{B})$ there is $\mathbb{C} = (C, \circ^C, \leq^C) \in \binom{\mathbb{B}}{\mathbb{A}}$ such that $\sqsubseteq\restriction C^2$ is canonical on $\mathbb{C}$ for the class $\mathcal{K}$.*

PROOF.

(i) We consider structure $\mathbb{Z} = (Z, \circ^Z, \leq^Z, (R^Z_{m,i})^m_{i=1})$ in $\mathcal{DT}_m$ such that $Z = \{z_0, z_1, \dots, z_m\}$, z_0 is the root, for $1 \leq i < j \leq m$ and $1 \leq s \leq m$ we have $z_0 = z_i \circ^Z z_j$ and $R^Z_{m,s}(z_0, z_i) \Leftrightarrow i = s$. Without loss of generality we may assume that $\mathbb{A}$ is a (k, m) balanced tree for some k. By Theorem 6.1, there is a $\mathbb{B} \in \mathcal{DT}_m$ such that $\mathbb{B} \to (\mathbb{A})^{\mathbb{Z}}_{(m+1)!}$. We show that $\mathbb{B}$ proves the statement by considering $\sqsubseteq \in lo(\mathbb{B})$ and coloring

$$c : \binom{\mathbb{B}}{\mathbb{Z}} \to lo(\{0, 1, \dots, m\}).$$

To describe the coloring c, we consider $\mathbb{U} \in \binom{\mathbb{B}}{\mathbb{Z}}$ with the root u and the unique isomorphism $\pi : \mathbb{Z} \to \mathbb{U}$ given by $\pi(z_0) = u$ and $R^U_{m,i}(u, \pi(z_i))$ for $1 \leq i \leq m$. Now we define $c(\mathbb{U})$ by

$$ic(\mathbb{U})j \Leftrightarrow \pi(i) \sqsubseteq \pi(j).$$

Then there is $\mathbb{C} \in \binom{\mathbb{B}}{\mathbb{A}}$ and there is $\trianglelefteq \in lo(\{0, 1, \dots, m\})$ such that $c \restriction \binom{\mathbb{C}}{\mathbb{Z}} = \trianglelefteq$. Clearly this means that $\sqsubseteq\restriction C^2$ is canonical.

(ii) We consider structures $\mathbb{P} = (P, \circ^P, \leq^P)$ and $\mathbb{R} = (R, \circ^R, \leq^R)$ given by $P = \{p_1, p_2\}$, $p_1 <^P p_2$, $R = \{r_0, r_1, r_2\}$ and $r_0 = r_1 \circ^R r_2$. We have $t_{\mathcal{K}}(\mathbb{P}) = 1$ only for $\mathcal{K} \in \{\mathcal{S}, \mathcal{T}_1, \mathcal{T}\}$ and $t_{\mathcal{K}}(\mathbb{R}) = 1$ only for $\mathcal{K} \in \{\mathcal{S}, \mathcal{T}_2, \mathcal{T}\}$.

$\mathcal{K} \in \{\mathcal{S}, \mathcal{T}_1\}$: Since $t_{\mathcal{K}}(\mathbb{P}) = 1$ we have a $\mathbb{B} \in \mathcal{K}$ such that $\mathbb{B} \to (\mathbb{A})^{\mathbb{P}}_2$. For a linear ordering $\sqsubseteq \in lo(\mathbb{B})$ we consider coloring

$$c : \binom{\mathbb{B}}{\mathbb{P}} \to \{0,1\},$$
$$c(\mathbb{V}) = 1 \Leftrightarrow \sqsubseteq \restriction V^2 = \leq^V,$$

where $\mathbb{V} = (V, \circ^V, \leq^V)$. Then there is $\mathbb{C} \in \binom{\mathbb{B}}{\mathbb{A}}$ and there is r such that $c \restriction \binom{\mathbb{C}}{\mathbb{P}} = r$. If $r = 1$ then we have $\sqsubseteq \restriction C^2 \in le(\leq^A)$ and otherwise we have $op(\sqsubseteq \restriction C^2) \in le(\leq^A)$.

$\mathcal{K} = \mathcal{T}$: From the fact that $t_{\mathcal{K}}(\mathbb{P}) = 1 = t_{\mathcal{K}}(\mathbb{R})$ we obtain structures $\mathbb{D}$, $\mathbb{B} \in \mathcal{K}$ such that $\mathbb{D} \to (\mathbb{A})^{\mathbb{R}}_2$ and $\mathbb{B} \to (\mathbb{D})^{\mathbb{P}}_2$. We consider $\sqsubseteq \in lo(\mathbb{B})$. From the first part of this proof there is $\mathbb{H} = (H, \circ^H, \leq^H) \in \binom{\mathbb{B}}{\mathbb{D}}$ such that $\sqsubseteq \restriction H^2 \in le(\leq^H)$ or $op(\sqsubseteq \restriction H^2) \in le(\leq^H)$. We discuss only the case $\sqsubseteq \restriction H^2 \in le(\leq^H)$ and the other case is similar. Let $\preceq^H \in lo(H)$ be such that $(\mathbb{H}, \preceq^H) \in \mathcal{CT}$. Then we have a coloring

$$c : \binom{\mathbb{H}}{\mathbb{R}} \to \{0,1\},$$
$$c(\mathbb{V}) = 1 \Leftrightarrow \sqsubseteq \restriction V^2 = \preceq^H \restriction V^2,$$

where $\mathbb{V} = (V, \circ^V, \leq^V)$. The coloring c is well-defined because there are only two linear extensions of $\leq^R$. Then there is $\mathbb{C} \in \binom{\mathbb{H}}{\mathbb{A}}$ and there is r such that $c \restriction \binom{\mathbb{C}}{\mathbb{R}} = r$. If $r = 1$ then we have $\sqsubseteq \restriction C^2 = \preceq^H \restriction C^2$ what implies $(\mathbb{C}, \sqsubseteq \restriction C^2) \in \mathcal{CT}$. If $r = 0$ then we consider structure $\mathbb{W} = (W, \circ^W, \leq^W) \in \mathcal{T}$ such that $W = \{w, w_0, w_1, w_{01}\}$, w is the root, $w_s <^W w_{s'}$ iff s' end extends s. Without loss of generality we may assume that $\binom{\mathbb{A}}{\mathbb{W}} \neq \emptyset$. In order to show that $(\mathbb{C}, \sqsubseteq \restriction C^2) \in \mathcal{CT}$ it is enough to show that $w_0 \sqsubset w_1 \sqsubset w_{01}$ is impossible. If this is the case then we have $w_{01} \prec^H w_1 \prec^H w_0$ what is in contradiction with the fact that $\preceq^H \in le(\leq^H)$.

$\mathcal{K} = \mathcal{T}_m$: This follows from the part (i) of this Proposition. ⊣

§**9. Appendix.** We prove by an induction on $|\mathbb{A}|$ that each $\mathbb{A} \in \mathcal{CT}$ is a Ramsey object in $\mathcal{CT}$.

Base of induction $|\mathbb{A}| = 1$: Since $|\mathbb{A}| = 1$, we may consider $\mathbb{A}$ also as a structure in $\mathcal{T}$. Let $r \geq 1$ be a given natural number and let $(\mathbb{B}, \leq^B) \in \mathcal{CT}$ be such that $\binom{(\mathbb{B}, \leq^B)}{\mathbb{A}} \neq \emptyset$. Without loss of generality we may assume that $\mathbb{B}$ is a balanced tree. Since the class $\mathcal{T}$ is closed under taking products and taking substructures, we have by Proposition 2.1 in [10] that one-element structures are Ramsey objects in $\mathcal{T}$. So there is $\mathbb{C} \in \mathcal{T}$ such that $\mathbb{C} \to (\mathbb{B})^{\mathbb{A}}_r$. Note that for a linear ordering $\leq^C$ with $(\mathbb{C}, \leq^C) \in \mathcal{CT}$ and some $\mathbb{D} \leq \mathbb{C}$ we have $(\mathbb{D}, \leq^D) \in \mathcal{CT}$ where $\leq^D$ is the restriction of $\leq^C$ on the underlying set of the structure $\mathbb{D}$. Then for a linear ordering $\leq^C$ with the property that $(\mathbb{C}, \leq^C) \in \mathcal{CT}$ we have $(\mathbb{C}, \leq^C) \to (\mathbb{B}, \leq^B)^{\mathbb{A}}_r$, so $\mathbb{A}$ is a Ramsey object in $\mathcal{CT}$. In the rest of the proof we denote by $\star$ the unique one-element structure in $\mathcal{CT}$.

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

Inductive step $k-1 \mapsto k, k > 1$: We assume that $\mathbb{A} \in \mathcal{T}$ is a Ramsey object in $\mathcal{T}$ when $|\mathbb{A}| < k$. Let $r > 1$ be a natural number and let $\mathbb{A} = (A, \circ^A, \preceq^A)$ and $\mathbb{B} = (B, \circ^B, \preceq^B)$ be structures from $\mathcal{CT}$ such that $\binom{\mathbb{B}}{\mathbb{A}} \neq \emptyset$. By the base of induction we choose $\mathbb{D}$ such that:

$$\mathbb{D} \to (\mathbb{B})^{\star}_r. \tag{9.1}$$

Without loss of generality we may assume that $\mathbb{D}$ is (m, k_0)-balanced tree for some m and k_0. We going backward construct a sequence $(\mathbb{B}_i)_{i=0}^m$ of structures from $\mathcal{CT}$ such that for all $0 < i \leq m$ we have:

- $\mathbb{B}_m = \mathbb{D}$.
- $\mathbb{B}_{i-1} \geq \mathbb{B}_i$.
- Trees $\mathbb{B}_i$ and $\mathbb{B}_{i-1}$ have the same elements of the height $0, 1, \dots, i-1$.
- Elements in $\mathbb{B}_i$ with the height $i-1$ have the same number of immediate successors equal to k_i.
- If b and b' are elements in $\mathbb{B}_i$ with the height i then $\mathbb{B}_i(b) \cong \mathbb{B}_i(b')$.

We suppose that we have constructed structure $\mathbb{B}_{i+1}$ and we proceed with the construction of the structure $\mathbb{B}_i$. Let $(b_j)_{j=1}^u$ be the list of all elements in $\mathbb{B}_{i+1}$ with the height i. Note that we can calculate u, but it is not of importance for our consideration. At this point we need to consider elements in $\mathbb{A}$ with the height $1 : a_1 \prec^A a_2 \prec^A \cdots \prec^A a_l$. By the classical Ramsey theorem there is a natural number v such that

$$v \to (u)^l_r. \tag{9.2}$$

Properties of $\mathbb{B}_{i+1}$ implies:

$$\mathbb{B}_{i+1}(b_1) \cong \mathbb{B}_{i+1}(b_2) \cong \cdots \cong \mathbb{B}_{i+1}(b_u) \cong \mathbb{B}_{i+1,0}.$$

Now we use the product Ramsey theorem for classes, see Theorem 2 in [21], to obtain recursively a sequence $(\mathbb{B}_{i+1,s})_{s=0}^{|\binom{v}{l}|}$ in $\mathcal{CT}$ such that for $0 < s \leq |\binom{v}{l}|$ we have:

$$\mathbb{B}_{i+1,s} \to (\mathbb{B}_{i+1,s-1}, \dots, \mathbb{B}_{i+1,s-1})_r^{(\mathbb{A}(a_1),\dots,\mathbb{A}(a_l))}. \tag{9.3}$$

Note that this is possible by the inductive assumption since $|\mathbb{A}(a_i)| < |\mathbb{A}|$. Without loss of generality we may assume that each $\mathbb{B}_{i+1,s}$ is a balanced tree. Now we take $\mathbb{B}_i$ to be such that:

- $\mathbb{B}_i$ and $\mathbb{B}_{i+1}$ have the same elements of the height at most i.
- Each b in $\mathbb{B}_i$ with the height i has v immediate successors.
- For each b in $\mathbb{B}_i$ with the height i we have $\mathbb{B}_i(b) \cong \mathbb{B}_{i+1,|\binom{v}{l}|}$.

Now we have finished construction of the sequence $(\mathbb{B}_i)_{i=0}^m$, and we claim that $\mathbb{B}_0 \to (\mathbb{B})^{\mathbb{A}}_r$. So, we need to consider a given coloring

$$c : \binom{\mathbb{B}_0}{\mathbb{A}} \to \{1, \dots, r\}.$$

Let b be an element in $\mathbb{B}_0$ and let $C = \{c_1 \prec^{B_m} \cdots \prec^{B_m} c_l\}$ be a subset of $im_{\mathbb{B}_0}(b)$. Recall that l is the number of elements in $\mathbb{A}$ with height 1. We denote by $\binom{\mathbb{B}_0}{\mathbb{A}}_{b,C}$

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

the collection of all elements in $\binom{\mathbb{B}_0}{\mathbb{A}}$ with the minimal element equal to b and such that they are contained in $\{b\} \cup \bigcup_{i=1}^{l} \mathbb{B}_0(c_i)$. Then we have an induced coloring

$$c_{b,C} = c \restriction \binom{\mathbb{B}_0}{\mathbb{A}}_{b,C}.$$

Our choice of the sequence $(\mathbb{B}_i)_{i=0}^m$ give us a sequence $(\mathbb{B}'_i)_{i=0}^m$ of structures from $\mathcal{CT}$ such that for each $0 < i \leq m$ we have:

- $\mathbb{B}'_0 \cong \mathbb{B}_0$.
- $\mathbb{B}'_i \cong \mathbb{B}_i$.
- $\mathbb{B}'_i \geq \mathbb{B}'_{i+1}$.
- For all $b \in \mathbb{B}'_i$ with the height i and all $\mathbb{A}', \mathbb{A}'' \in \binom{\mathbb{B}'_m}{\mathbb{A}}$ with the same root equal to b we have $c(\mathbb{A}') = c(\mathbb{A}'')$.

Now we explain how to obtain the sequence $(\mathbb{B}'_i)_{i=0}^m$. Suppose that we have obtained $\mathbb{B}'_{i-1}$ and we explain how to obtain $\mathbb{B}'_i$. Let $b \in \mathbb{B}'_{i-1}$ has the height i and let $(C_j)_{j=1}^w$ be the list of all l-subsets of $im_{\mathbb{B}'_{i-1}}(b)$. From 9.2 and 9.3 we may find $D \subset im_{\mathbb{B}'_{i-1}}(b)$ and a sequence of trees $(\mathbb{T}_d)_{d \in D}$ in $\mathcal{CT}$ such that for all $d, d' \in D$ we have:

- $\mathbb{T}_d \cong \mathbb{T}_{d'}$.
- $\mathbb{T}_d \leq \mathbb{B}'_{i-1}(d)$.
- Let $\mathbb{Q}_b$ be the substructure of $\mathbb{B}'_{i-1}$ given by the union of b and structures $(\mathbb{T}_d)_{d \in D}$. There is a constant con_b such that for every l-subset $C \subset D$ we have
$$c \restriction \binom{\mathbb{Q}_b}{\mathbb{A}}_{b,C} = con_b.$$

In this way for every $b \in \mathbb{B}'_{i-1}$ we find a tree $\mathbb{Q}_b$. Moreover we may assume that for $b \neq b'$ we have $\mathbb{Q}_b \cong \mathbb{Q}_{b'}$. Now we take that $\mathbb{B}'_i$ agrees with $\mathbb{B}'_{i-1}$ on the first i many levels and for each b with height i we take $\mathbb{B}'_i(b) = \mathbb{Q}_b$.

In particular structure $\mathbb{B}'_m$ has the property that every two copies of $\mathbb{A}$ in $\mathbb{B}'_m$ with the same root have the same color. Therefore we have an induced coloring

$$c' : \binom{\mathbb{B}'_m}{\star} \to \{1, \dots, r\},$$
$$c'(b) = com_b.$$

From 9.1 we obtain $\mathbb{B}' \in \binom{\mathbb{B}'_m}{\mathbb{B}}$ such that $c' \restriction \binom{\mathbb{B}'}{\star} = const$. Moreover this implies that $c \restriction \binom{\mathbb{B}'}{\mathbb{A}} = const$, so $\mathcal{CT}$ satisfies RP.

§10. **Acknowledgement.** The author is thankful to Alekos Kechris, Aleksandra Kwiatkowska, Slawek Solecki, and Stevo Todorčević for valuable comments and suggestions. The author is also thankful to the referee for valuable comments and suggestions.

REFERENCES

[1] O. Angel, A. S. Kechris, and R. Lyons, *Random orderings and unique ergodicity of automorphism groups*. arXiv:1208.2389.

[2] J. Auslander, ***Minimal flows and their extensions***, North Holland, Amsterdam, 1998.

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press

[3] H. Becker and A. S. Kechris, ***The descriptive set theory of Polish group actions***, Cambridge University Press, New York, 1996.

[4] W. Deuber, *A Generalization of Ramsey's theorem for regular trees*. ***Journal of Combinatorial Theory, Series B***, vol. 18 (1975), pp. 18–23.

[5] W. Deuber and B. Rothschild, *Categories without the Ramsey property*. ***Combinatorics*** (Proc. Fifth Hungarian Colloq., Keszthely, 1976), vol. 1, pp. 225–249, ***Colloquia Mathematica Societatis Janos Bolyai***, vol. 18, North-Holland, Amsterdam-New York, 1978.

[6] R. L. Graham and B. L. Rothschild, *Ramsey's theorem for n-parameter sets*. ***Transactions of the American Mathematical Society***, vol. 159 (1971) pp. 257–292.

[7] ———, *Some recent developments in Ramsey theory*. ***Combinatorics*** (Proc. Advanced Study Inst., Breukelen, 1974), ***Part 2: Graph theory; foundations, partitions and combinatorial geometry***, Mathematical Centre tracts, No. 56, Mathematisch Centrum, Amsterdam, pp. 61–76, 1974.

[8] R. L. Graham, K. Leeb, and B. L. Rothschild, *Ramsey's theorem for a class of categories*. ***Advances in Mathematics***, vol. 8 (1972), pp. 417–433.

[9] W. Hodges, ***Model theory***, Cambridge University Press, Cambridge, 1993.

[10] J. Ježek and J. Nešetřil, *Ramsey varieties*. ***European Journal of Combinatorics***, vol. 4 (1983), no. 2, pp. 143–147.

[11] A. S. Kechris, V. Pestov, and S. Todorčević, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*. ***Geometric and Functional Analysis***, vol. 15 (2005), pp. 106–189.

[12] M. Malicki, *The automorphism group of the random lattice in not amenable*. ***Fundamenta Mathematicae***, vol. 226 (2014), pp. 245–252.

[13] K. R. Milliken, *A Ramsey theorem for trees*. ***Journal of Combinatorial Theory, Series A***, vol. 26 (1979), no. 3, pp. 215–237.

[14] J. Nešetřil, H. J. Prömel, V. Rödl, and B. Voigt, *Note on canonizing ordering theorems for Hales Jewett structures*, ***Proceedings of the 11th winter school on abstract analysis*** (Železná Ruda, 1983), Rendiconti del Circolo Matematico di Palermo, vol. 2, Suppl. No. 3, pp. 191–196, 1984.

[15] J. Nešetřil and V. Rödl, *Combinatorial partitions of finite posets and lattices—Ramsey lattices*. ***Algebra Universalis***, vol. 19 (1984), no. 1, pp. 106–119.

[16] L. Nguyen Van The, *More on Kechris-Pestov-Todorcevic correspondence: Precompact Expansions*. ArXiv: 1201.1270v2, March 2012. Reviewed Nguyen Van Thé, L. *More on the Kechris-Pestov-Todorcevic correspondence: Precompact expansions*. ***Fundamenta Mathematicae***, vol. 222 (2013), no. 1, pp. 19–47.

[17] H. J. Prömel, *Some remarks on natural orders for combinatorial cubes*. Proceedings of the Oberwolfach Meeting "Kombinatorik" (1986), ***Discrete mathematics***, vol. 73 (1989), no. 1–2, pp. 189–198.

[18] S. Roman, ***Lattices and ordered sets***, Springer, New York, 2008.

[19] F. P. Ramsey, *On a Problem of Formal Logic*. ***Proceedings of the London Mathematical Society***, vol. S2-30, no. 1, p. 264.

[20] M. Sokić, *Ramsey property of finite posets*. ***Order***, vol. 29 (2012), no. 1, pp. 1–30.

[21] ———, *Ramsey property of finite posets II*. ***Order***, vol. 29 (2012), no. 1, pp. 31–47.

[22] ———, *Unary functions*, under review.

[23] S. Solecki, *Abstract Approach to Ramsey Theory and Ramsey Theorems for Finite Trees*, ***Asymptotic Geometric Analysis***, Fields Institute Communications, Springer, Berlin, 2013, pp. 313–340.

[24] S. Todorčević, ***Introduction to Ramsey spaces***. Annals of Mathematics Studies, vol. 174, Princeton University Press, New Jersey, 2010.

[25] A. Zucker, *Amenability and Unique Ergodicity of Automorphism Groups of Fraïssé Structures*, arXiv:1304.2839v2.

DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY
TORONTO ON, CANADA
E-mail: msokic@yorku.ca

https://doi.org/10.1017/jsl.2014.40 Published online by Cambridge University Press